This International Student Edition is for use outside of the U.S.

Intermediate Algebra

SIXTH EDITION

Julie Miller
Molly O'Neill
Nancy Hyde

Mc Graw Hill

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INTERMEDIATE ALGEBRA

This book is printed on acid-free paper.

1 2 3 4 5 6 7 8 9 LWI 26 25 24 23 22 21

ISBN 978-1-264-41875-6
MHID 1-264-41875-2

Cover Image: ©*WAYHOME studio/Shutterstock*

mheducation.com/highered

Letter from the Authors

Dear Colleagues,

Across the country, Developmental Math courses are in a state of flux, and we as instructors are at the center of it all. As many of our institutions are grappling with the challenges of placement, retention, and graduation rates, we are on the front lines with our students—supporting all of them in their educational journey.

Flexibility—No Matter Your Course Format!

The three of us each teach differently, as do many of our current users. The Miller/O'Neill/Hyde series is designed for successful use in a variety of course formats, both traditional and modern—classroom lecture settings, flipped classrooms, hybrid classes, and online-only classes.

Ease of Instructor Preparation

We've all had to fill in for a colleague, pick up a last-minute section, or find ourselves running across campus to yet a different course. The Miller/O'Neill/Hyde series is carefully designed to support instructors teaching in a variety of different settings and circumstances. Experienced, senior faculty members can draw from a massive library of static and algorithmic content found in ALEKS to meticulously build assignments and assessments sharply tailored to individual student needs. Newer instructors and part-time adjunct instructors, on the other hand, will find support through a wide range of digital resources and prebuilt assignments ready to go on Day One. With these tools, instructors with limited time to prepare for class can still facilitate successful student outcomes.

Many instructors want to incorporate discovery-based learning and groupwork into their courses but don't have time to write or find quality materials. Each section of the text has numerous discovery-based activities that we have tested in our own classrooms. These are found in the text and Student Resource Manual along with other targeted worksheets for additional practice and materials for a student portfolio.

Student Success—Now and in the Future

Too often our math placement tests fail our students, which can lead to frustration, anxiety, and often withdrawal from their education journey. We encourage you to learn more about ALEKS Placement, Preparation, and Learning (ALEKS PPL), which uses adaptive learning technology to place students appropriately. No matter the skills they come in with, the Miller/O'Neill/Hyde series provides resources and support that uniquely position them for success in that course and for their next course. Whether they need a brush-up on their basic skills, ADA supportive materials, or advanced topics to help them cross the bridge to the next level, we've created a support system for them.

We hope you are as excited as we are about the series and the supporting resources and services that accompany it. Please reach out to any of us with any questions or comments you have about our texts.

Julie Miller　　Molly O'Neill　　Nancy Hyde

About the Authors

Photo courtesy of Molly O'Neill

Julie Miller is from Daytona State College, where she taught developmental and upper-level mathematics courses for 20 years. Prior to her work at Daytona State College, she worked as a software engineer for General Electric in the area of flight and radar simulation. Julie earned a Bachelor of Science in Applied Mathematics from Union College in Schenectady, New York, and a Master of Science in Mathematics from the University of Florida. In addition to this textbook, she has authored textbooks for college algebra, trigonometry, and precalculus, as well as several short works of fiction and nonfiction for young readers.

"My father is a medical researcher, and I got hooked on math and science when I was young and would visit his laboratory. I can remember using graph paper to plot data points for his experiments and doing simple calculations. He would then tell me what the peaks and features in the graph meant in the context of his experiment. I think that applications and hands-on experience made math come alive for me, and I'd like to see math come alive for my students."

—*Julie Miller*

Molly O'Neill is also from Daytona State College, where she taught for 22 years in the School of Mathematics. She has taught a variety of courses from developmental mathematics to calculus. Before she came to Florida, Molly taught as an adjunct instructor at the University of Michigan–Dearborn, Eastern Michigan University, Wayne State University, and Oakland Community College. Molly earned a Bachelor of Science in Mathematics and a Master of Arts and Teaching from Western Michigan University in Kalamazoo, Michigan. Besides this textbook, she has authored several course supplements for college algebra, trigonometry, and precalculus and has reviewed texts for developmental mathematics.

"I differ from many of my colleagues in that math was not always easy for me. But in seventh grade I had a teacher who taught me that if I follow the rules of mathematics, even I could solve math problems. Once I understood this, I enjoyed math to the point of choosing it for my career. I now have the greatest job because I get to do math every day and I have the opportunity to influence my students just as I was influenced. Authoring these texts has given me another avenue to reach even more students."

—*Molly O'Neill*

Nancy Hyde served as a full-time faculty member of the Mathematics Department at Broward College for 24 years. During this time she taught the full spectrum of courses from developmental math through differential equations. She received a Bachelor of Science in Math Education from Florida State University and a Master's degree in Math Education from Florida Atlantic University. She has conducted workshops and seminars for both students and teachers on the use of technology in the classroom. In addition to this textbook, she has authored a graphing calculator supplement for *College Algebra*.

"I grew up in Brevard County, Florida, where my father worked at Cape Canaveral. I was always excited by mathematics and physics in relation to the space program. As I studied higher levels of mathematics I became more intrigued by its abstract nature and infinite possibilities. It is enjoyable and rewarding to convey this perspective to students while helping them to understand mathematics."

—*Nancy Hyde*

Dedication

To Our Students

Julie Miller ❧ Molly O'Neill ❧ Nancy Hyde

The Miller/O'Neill/Hyde Developmental Math Series

Julie Miller, Molly O'Neill, and Nancy Hyde originally wrote their developmental math series because students were entering their College Algebra course underprepared. The students were not mathematically mature enough to understand the concepts of math, nor were they fully engaged with the material. The authors began their developmental mathematics offerings with Intermediate Algebra to help bridge that gap. This in turn evolved into several series of textbooks from Prealgebra through Precalculus to help students at all levels before Calculus.

What sets all of the Miller/O'Neill/Hyde series apart is that they address course content through an author-created digital package that maintains a consistent voice and notation throughout the program. This consistency—in videos, PowerPoints, Lecture Notes, and Integrated Video and Study Guides—coupled with the power of ALEKS, ensures that students master the skills necessary to be successful in Developmental Math through Precalculus and prepares them for the Calculus sequence.

Developmental Math Series
The Developmental Math series is traditional in approach, delivering a purposeful balance of skills and conceptual development. It places a strong emphasis on conceptual learning to prepare students for success in subsequent courses.

Basic College Mathematics, Third Edition
Prealgebra, Third Edition
Prealgebra & Introductory Algebra, Second Edition
Beginning Algebra, Sixth Edition
Beginning & Intermediate Algebra, Sixth Edition
Intermediate Algebra, Sixth Edition
Developmental Mathematics: Prealgebra, Beginning Algebra, & Intermediate Algebra, Second Edition

The Miller/Gerken College Algebra/Precalculus Series
The Precalculus series serves as the bridge from Developmental Math coursework to future courses by emphasizing the skills and concepts needed for Calculus.

College Algebra with Corequisite Support, First Edition
College Algebra, Second Edition
College Algebra and Trigonometry, First Edition
Precalculus, First Edition

Acknowledgments

The author team most humbly would like to thank all the people who have contributed to this project and the Miller/O'Neill/Hyde Developmental Math series as a whole.

First and foremost, our utmost gratitude to Sarah Alamilla for her close partnership, creativity, and collaboration throughout this revision. Special thanks to our team of digital contributors for their thousands of hours of work: to Kelly Jackson, Jody Harris, Lizette Hernandez Foley, Lisa Rombes, Kelly Kohlmetz, and Leah Rineck for their devoted work. To Donna Gerken: thank you for the countless grueling hours working through spreadsheets to ensure thorough coverage of our content in ALEKS. To our digital authors, Linda Schott, Michael Larkin, and Alina Coronel: thank you for digitizing our content so it could be brought into ALEKS. We also offer our sincerest appreciation to the outstanding video talent: Alina Coronel, Didi Quesada, Tony Alfonso, and Brianna Ashley. So many students have learned from you! To Jennifer Blue, Carey Lange, John Murdzek, and Kevin Campbell: thank you so much for ensuring accuracy in our manuscripts.

We also greatly appreciate the many people behind the scenes at McGraw Hill without whom we would still be on page 1. To Megan Platt, our product developer: thank you for being our help desk and handling all things math, English, and editorial. To Brittney Merriman and Jennifer Morales, our portfolio managers and team leaders: thank you so much for leading us down this path. Your insight, creativity, and commitment to our project has made our job easier.

To the marketing team, Michele McTighe, Noah Evans, and Mary Ellen Rahn: thank you for your creative ideas in making our books come to life in the market. Thank you as well to Debbie McFarland, Justin Washington, and Sherry Bartel for continuing to drive our long-term content vision through their market development efforts. And many thanks to the team at ALEKS for creating its spectacular adaptive technology and for overseeing the quality control.

To the production team: Jane Mohr, David Hash, Lorraine Buczek, and Sandy Ludovissy—thank you for making the manuscript beautiful and for keeping the unruly authors on track. To our copyeditor Kevin Campbell and proofreader John Murdzek, who have kept a watchful eye over our manuscripts—the two of you are brilliant. To our compositor Manvir Singh and his team at Aptara, you've been a dream to work with. And finally, to Kathleen McMahon and Caroline Celano, thank you for supporting our projects for many years and for the confidence you've always shown in us.

Most importantly, we give special thanks to the students and instructors who use our series in their classes.

Julie Miller
Molly O'Neill
Nancy Hyde

Contents

To the Student

Take a deep breath and know that you aren't alone. Your instructor, fellow students, and we, your authors, are here to help you learn and master the material for this course and prepare you for future courses. You may feel like math just isn't your thing, or maybe it's been a long time since you've had a math class—that's okay!

We wrote the text and all the supporting materials with you in mind. Most of our students aren't really sure how to be successful in math, but we can help with that.

As you begin your class, we'd like to offer some specific suggestions:

1. **Attend class.** Arrive on time and be prepared. If your instructor has asked you to read prior to attending class—do it. How often have you sat in class and thought you understood the material, only to get home and realize you don't know how to get started? By reading and trying a couple of Skill Practice exercises, which follow each example, you will be able to ask questions and gain clarification from your instructor when needed.
2. **Be an *active* learner.** Whether you are at lecture, watching an author lecture or exercise video, or are reading the text, pick up a pencil and work out the examples given. Math is learned only by doing; we like to say, "Math is not a spectator sport." If you like a bit more guidance, we encourage you to use the Integrated Video and Study Guide. It was designed to provide structure and note-taking for lectures and while watching the accompanying videos.
3. **Schedule time to do some math every day.** Exercise, foreign language study, and math are three things that you must do every day to get the results you want. If you are used to cramming and doing all of your work in a few hours on a weekend, you should know that even mathematicians start making silly errors after an hour or so! Check your answers. Skill Practice exercises all have the answers at the bottom of that page. Odd-numbered exercises throughout the text have answers in the back of the text. If you didn't get it right, don't throw in the towel. Try again, revisit an example, or bring your questions to class for extra help.
4. **Prepare for quizzes and exams.** Each chapter has a set of Chapter Review Exercises at the end to help you integrate all of the important concepts. In addition, there is a detailed Chapter Summary and a Chapter Test. If you use ALEKS, use all of the tools available within the program to test your understanding.
5. **Use your resources.** This text comes with numerous supporting resources designed to help you succeed in this class and in your future classes. Additionally, your instructor can direct you to resources within your institution or community. Form a student study group. Teaching others is a great way to strengthen your own understanding, and they might be able to return the favor if you get stuck.

We wish you all the best in this class and in your educational journey!

Julie Miller   Molly O'Neill   Nancy Hyde

Student Guide to the Text

Clear, Precise Writing

Learning from our own students, we have written this text in simple and accessible language. Our goal is to keep you engaged and supported throughout your coursework.

Call-Outs

Just as your instructor will share tips and math advice in class, we provide call-outs throughout the text to offer tips and warn against common mistakes.

- Tip boxes offer additional insight into a concept or procedure.
- Avoiding Mistakes help fend off common student errors.
- For Review boxes positioned strategically throughout the text remind students of key skills relating to the current topic.

Examples

- Each example is step-by-step, with thorough annotation to the right explaining each step.
- Following each example is a similar **Skill Practice** exercise to give you a chance to test your understanding. You will find the answer at the bottom of the page—providing a quick check.

Exercise Sets

Each type of exercise is built so you can successfully learn the materials and show your mastery on exams.

- **Activities for discovery-based learning** appear before the exercise sets to walk students through the concepts presented in each section of the text.
- **Study Skills Exercises** integrate your studies of math concepts with strategies for helping you grow as a student overall.
- **Vocabulary and Key Concept Exercises** check your understanding of the language and ideas presented within the section.
- **Prerequisite Review** exercises keep fresh your knowledge of math content already learned by providing practice with concepts explored in previous sections.
- **Concept Exercises** assess your comprehension of the specific math concepts presented within the section.
- **Mixed Exercises** evaluate your ability to successfully complete exercises that combine multiple concepts presented within the section.
- **Expanding Your Skills** challenge you with advanced skills practice exercises around the concepts presented within the section.
- **Problem Recognition Exercises** appear in strategic locations in each chapter of the text. These will require you to distinguish between similar problem types and to determine what type of problem-solving technique to apply.
- **Technology Exercises** appear where appropriate.

End-of-Chapter Materials

The features at the end of each chapter are perfect for reviewing before test time.

- **Section-by-section summaries** provide references to key concepts, examples, and vocabulary.
- **Chapter Review Exercises** provide additional opportunities to practice material from the entire chapter.
- **Chapter tests** are an excellent way to test your complete understanding of the chapter concepts.

How Will Miller/O'Neill/Hyde Help Your Students *Get Better Results?*

Clarity, Quality, and Accuracy

Julie Miller, Molly O'Neill, and Nancy Hyde know what students need to be successful in mathematics. Better results come from clarity in their exposition, quality of step-by-step worked examples, and accuracy of their exercise sets; but it takes more than just great authors to build a textbook series to help students achieve success in mathematics. Our authors worked with a strong team of mathematics instructors from around the country to ensure that the clarity, quality, and accuracy you expect from the Miller/O'Neill/Hyde series was included in this edition.

Exercise Sets

Comprehensive sets of exercises are available for every student level. Julie Miller, Molly O'Neill, and Nancy Hyde worked with a board of advisors from across the country to offer the appropriate depth and breadth of exercises for your students. **Problem Recognition Exercises** were created to improve student performance while testing.

Practice exercise sets help students progress from skill development to conceptual understanding. Student tested and instructor approved, the Miller/O'Neill/Hyde exercise sets will help your students *get better results*.

- **Activities for Discovery-Based Learning**
- **Prerequisite Review Exercises**
- **Problem Recognition Exercises**
- **Skill Practice Exercises**
- **Study Skills Exercises**
- **Mixed Exercises**
- **Expanding Your Skills Exercises**
- **Vocabulary and Key Concepts Exercises**
- **Technology Exercises**

Step-By-Step Pedagogy

This text provides enhanced step-by-step learning tools to help students *get better results*.

- **For Review** tips placed in the margin guide students back to related prerequisite skills needed for full understanding of course-level topics.
- **Worked Examples** provide an "easy-to-understand" approach, clearly guiding each student through a step-by-step approach to master each practice exercise for better comprehension.
- **TIPs** offer students extra cautious direction to help improve understanding through hints and further insight.
- **Avoiding Mistakes** boxes alert students to common errors and provide practical ways to avoid them. Both of these learning aids will help students get better results by showing how to work through a problem using a clearly defined step-by-step methodology that has been class tested and student approved.

Get Better Results

Formula for Student Success

Step-by-Step Worked Examples

- Do you get the feeling that there is a disconnect between your students' class work and homework?
- Do your students have trouble finding worked examples that match the practice exercises?
- Do you prefer that your students see examples in the textbook that match the ones you use in class?

Miller/O'Neill/Hyde's *Worked Examples* offer a clear, concise methodology that replicates the mathematical processes used in the authors' classroom lectures.

Example 1 Determining the Order of a Matrix

Determine the order of each matrix.

a. $\begin{bmatrix} 2 & -4 & 1 \\ 5 & \pi & \sqrt{7} \end{bmatrix}$ **b.** $\begin{bmatrix} 1.9 \\ 0 \\ 7.2 \\ -6.1 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **d.** $[a \quad b \quad c]$

Solution:

a. This matrix has two rows and three columns. Therefore, it is a 2×3 matrix.

b. This matrix has four rows and one column. Therefore, it is a 4×1 matrix.
A matrix with one column is called a **column matrix**.

c. This matrix has three rows and three columns. Therefore, it is a 3×3 matrix.
A matrix with the same number of rows and columns is called a **square matrix**.

d. This matrix has one row and three columns. Therefore, it is a 1×3 matrix.
A matrix with one row is called a **row matrix**.

Skill Practice Determine the order of the matrix.

1. $\begin{bmatrix} -5 & 2 \\ 1 & 3 \end{bmatrix}$ **2.** $[4 \quad -8]$ **3.** $\begin{bmatrix} 5 \\ 10 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & -0.5 \\ -1 & 6 \end{bmatrix}$

Classroom Examples

To ensure that the classroom experience also matches the examples in the text and the practice exercises, we have included references to even-numbered exercises to be used as Classroom Examples. These exercises are highlighted in the Practice Exercises at the end of each section.

Example 5 Finding the *x*- and *y*-Intercepts of a Line

Given $2x + 4y = 8$, find the x- and y-intercepts. Then graph the equation.

Solution:

To find the x-intercept, substitute $y = 0$.

$$2x + 4y = 8$$
$$2x + 4(0) = 8$$
$$2x = 8$$
$$x = 4$$

The x-intercept is $(4, 0)$.

To find the y-intercept, substitute $x = 0$.

$$2x + 4y = 8$$
$$2(0) + 4y = 8$$
$$4y = 8$$
$$y = 2$$

The y-intercept is $(0, 2)$.

Quality Learning Tools

For Review Boxes

Throughout the text, just-in-time tips and reminders of prerequisite skills appear in the margin alongside the concepts for which they are needed. References to prior sections are given for cases where more comprehensive review is available earlier in the text.

FOR REVIEW

Recall that the sum of an expression and its opposite is zero. For example:

$4y + (-4y) = 0$

TIP and Avoiding Mistakes Boxes

TIP and **Avoiding Mistakes** boxes have been created based on the authors' classroom experiences—they have also been integrated into the **Worked Examples.** These pedagogical tools will help students get better results by learning how to work through a problem using a clearly defined step-by-step methodology.

Example 7 Simplifying a Radical Expression

Simplify. $\dfrac{7\sqrt{50}}{10}$

Solution:

$\dfrac{7\sqrt{50}}{10} = \dfrac{7\sqrt{25 \cdot 2}}{10}$ 25 is the greatest perfect square in the radicand.

$= \dfrac{7 \cdot 5\sqrt{2}}{10}$ Simplify the radical.

$= \dfrac{7 \cdot \overset{1}{\cancel{5}}\sqrt{2}}{\underset{2}{\cancel{10}}}$ Simplify the fraction to lowest terms.

$= \dfrac{7\sqrt{2}}{2}$

Avoiding Mistakes

The expression $\frac{7\sqrt{2}}{2}$ cannot be simplified further because one factor of 2 is in the radicand and the other is outside the radical.

Skill Practice Simplify.

9. $\dfrac{2\sqrt{300}}{30}$

Avoiding Mistakes Boxes:

Avoiding Mistakes boxes are integrated throughout the textbook to alert students to common errors and how to avoid them.

TIP: When solving a literal equation for a specified variable, there is sometimes more than one way to express your final answer. This flexibility often presents difficulty for students. Students may leave their answer in one form, but the answer given in the text may look different. Yet both forms may be correct. To know if your answer is equivalent to the form given in the text, you must try to manipulate it to look like the answer in the book, a process called *form fitting.*

The literal equation from Example 4 can be written in several different forms. The quantity $(2A - b_2h)/h$ can be split into two fractions.

$$b_1 = \frac{2A - b_2h}{h} = \frac{2A}{h} - \frac{b_2h}{h} = \frac{2A}{h} - b_2$$

TIP Boxes

Teaching tips are usually revealed only in the classroom. Not anymore! TIP boxes offer students helpful hints and extra direction to help improve understanding and provide further insight.

Get Better Results

Better Exercise Sets and Better Practice Yield Better Results

- Do your students have trouble with problem solving?
- Do you want to help students overcome math anxiety?
- Do you want to help your students improve performance on math assessments?

Problem Recognition Exercises

Problem Recognition Exercises present a collection of problems that look similar to a student upon first glance, but are actually quite different in the manner of their individual solutions. Students sharpen critical thinking skills and better develop their "solution recall" to help them distinguish the method needed to solve an exercise—an essential skill in mathematics.

Problem Recognition Exercises were tested in the authors' developmental mathematics classes and were created to improve student performance on tests.

Problem Recognition Exercises

Rational Equations vs. Expressions

1. **a.** Simplify. $\dfrac{3}{w-5}+\dfrac{10}{w^2-25}-\dfrac{1}{w+5}$

b. Solve. $\dfrac{3}{w-5}+\dfrac{10}{w^2-25}-\dfrac{1}{w+5}=0$

c. Identify each problem in parts (a) and (b) as either an equation or an expression.

2. **a.** Simplify. $\dfrac{x}{2x+4}+\dfrac{2}{3x+6}-1$

b. Solve. $\dfrac{x}{2x+4}+\dfrac{2}{3x+6}=1$

c. Identify each problem in parts (a) and (b) as either an equation or an expression.

For Exercises 3–20, first ask yourself whether the problem is an expression to simplify or an equation to solve. Then simplify or solve as indicated.

3. $\dfrac{2}{a^2+4a+3}+\dfrac{1}{a+3}$

4. $\dfrac{1}{c+6}+\dfrac{4}{c^2+8c+12}$

5. $\dfrac{7}{y^2-y-2}+\dfrac{1}{y+1}-\dfrac{3}{y-2}=0$

6. $\dfrac{3}{b+2}-\dfrac{1}{b-1}-\dfrac{5}{b^2+b-2}=0$

7. $\dfrac{x}{x-1}-\dfrac{12}{x^2-x}$

8. $\dfrac{3}{5t-20}+\dfrac{4}{t-4}$

Get Better Results

Student-Centered Applications

The Miller/O'Neill/Hyde Board of Advisors partnered with our authors to bring the *best applications* from every region in the country! These applications include real data and topics that are more relevant and interesting to today's student.

92. ^{99m}Tc is a radionuclide of technetium that is widely used in nuclear medicine. Although its half-life is only 6 hr, the isotope is continuously produced via the decay of its longer-lived parent ^{99}Mo (molybdenum-99), whose half-life is approximately 3 days. The ^{99}Mo generators (or "cows") are sold to hospitals in which the ^{99m}Tc can be "milked" as needed over a period of a few weeks. Once separated from its parent, the ^{99m}Tc may be chemically incorporated into a variety of imaging agents, each of which is designed to be taken up by a specific target organ within the body. Special cameras, sensitive to the gamma rays emitted by the technetium, are then used to record a "picture" (similar in appearance to an X-ray film) of the selected organ.

Suppose a technician prepares a sample of ^{99m}Tc-pyrophosphate to image the heart of a patient suspected of having had a mild heart attack. If the injection contains 10 millicuries (mCi) of ^{99m}Tc at 1:00 P.M., then the amount of technetium still present is given by

$$T(t) = 10e^{-0.1155t}$$

where $t > 0$ represents the time in hours after 1:00 P.M. and $T(t)$ represents the amount of ^{99m}Tc (in millicuries) still present.

a. How many millicuries of ^{99m}Tc will remain at 4:20 P.M. when the image is recorded? Round to the nearest tenth of a millicurie.

b. How long will it take for the radioactive level of the ^{99m}Tc to reach 2 mCi? Round to the nearest tenth of an hour.

Activities

Each section of the text ends with an activity that steps the student through the major concepts of the section. The purpose of the activities is to promote active, discovery-based learning for the student. The implementation of the activities is flexible for a variety of delivery methods. For face-to-face classes, the activities can be used to break up lecture by covering the exercises intermittently during the class. For the flipped classroom and hybrid classes, students can watch the videos and try the activities. Then, in the classroom, the instructor can go over the activities or have the students compare their answers in groups. For online classes, the activities provide great discussion questions.

Section 2.6 Activity

A.1. Given a set of ordered pairs, how can you determine whether the relation defines y as a function of x?

For Exercises A.2–A.3, consider the given relation.

a. Do any two ordered pairs have the same x value but different y values?

b. Is the relation a function?

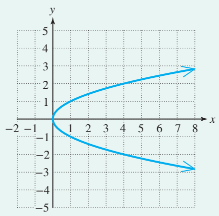
A.2. $\{(-5, 1), (3, 4), (-2, 6), (-5, 2), (0, -3)\}$

A.3. $\{(-1, 6), (2, 11), (8, 6), (-3, 1), (0.4, -0.5)\}$

A.4. **a.** For the graph given, draw a vertical line through the point (4, 2). Does the vertical line intersect the graph at any other point?

b. Does this graph define y as a function of x?

c. Using this example, explain how the vertical line test is used to determine if a graph defines y as a function of x.



A.5. Consider the equation $y = 2x + 1$.

a. If $x = 3$, what is the corresponding y value?

b. Write the result of part (a) as an ordered pair (x, y).

A.6. Consider the function defined by $f(x) = 2x + 1$.

a. Find $f(3)$. That is, evaluate the function for $x = 3$ by substituting 3 for x.

b. Write the result of part (a) as an ordered pair (x, y).

c. Refer to Exercise A.5 and compare the results.

A.7. Given $g(x) = \dfrac{4}{x-1}$, find the function values if possible.

a. $g(2)$ **b.** $g(-3)$

c. $g(0)$ **d.** $g(1)$

A.8. Refer to $g(x) = \dfrac{4}{x-1}$ from Exercise A.7.

a. What value(s) of x must be excluded from the domain of g? Why?

b. Write the domain of g in interval notation.

A.9. Given $h(x) = \sqrt{x+2}$, find the function values if possible.

a. $h(-1)$ **b.** $h(2)$

c. $h(7)$ **d.** $h(-6)$

Get Better Results

Additional Supplements

Lecture Videos Created by the Authors

Julie Miller began creating these lecture videos for her own students to use when they were absent from class. The student response was overwhelmingly positive, prompting the author team to create the lecture videos for their entire developmental math book series. In these videos, the authors walk students through the learning objectives using the same language and procedures outlined in the book. Students learn and review right alongside the author! Students can also access the written notes that accompany the videos.

Integrated Video and Study Workbooks

The Integrated Video and Study Workbooks were built to be used in conjunction with the Miller/O'Neill/Hyde Developmental Math series online lecture videos. These new video guides allow students to consolidate their notes as they work through the material in the book, and they provide students with an opportunity to focus their studies on particular topics that they are struggling with rather than entire chapters at a time. Each video guide contains written examples to reinforce the content students are watching in the corresponding lecture video, along with additional written exercises for extra practice. There is also space provided for students to take their own notes alongside the guided notes already provided. By the end of the academic term, the video guides will not only be a robust study resource for exams, but will serve as a portfolio showcasing the hard work of students throughout the term.

Dynamic Math Animations

The authors have constructed a series of animations to illustrate difficult concepts where static images and text fall short. The animations leverage the use of on-screen movement and morphing shapes to give students an interactive approach to conceptual learning. Some provide a virtual laboratory for which an application is simulated and where students can collect data points for analysis and modeling. Others provide interactive question-and-answer sessions to test conceptual learning.

Exercise Videos

The authors, along with a team of faculty who have used the Miller/O'Neill/Hyde textbooks for many years, have created exercise videos for designated exercises in the textbook. These videos cover a representative sample of the main objectives in each section of the text. Each presenter works through selected problems, following the solution methodology employed in the text.

The video series is available online as part of ALEKS 360. The videos are closed-captioned for the hearing impaired and meet the Americans with Disabilities Act Standards for Accessible Design.

Student Resource Manual

The *Student Resource Manual (SRM),* created by the authors, is a printable, electronic supplement available to students through ALEKS. Instructors can also choose to customize this manual and package it with their course materials. With increasing demands on faculty schedules, this resource offers a convenient means for both full-time and adjunct faculty to promote active learning and success strategies in the classroom.

This manual supports the series in a variety of different ways:

- Additional group activities developed by the authors to supplement what is already available in the text
- Discovery-based classroom activities written by the authors for each section
- Excel activities that not only provide students with numerical insights into algebraic concepts, but also teach simple computer skills to manipulate data in a spreadsheet

- Worksheets for extra practice written by the authors, including Problem Recognition Exercise Worksheets
- Lecture Notes designed to help students organize and take notes on key concepts
- Materials for a student portfolio

Annotated Instructor's Edition

In the *Annotated Instructor's Edition* (*AIE*), answers to all exercises appear adjacent to each exercise in a color used *only* for annotations. The *AIE* also contains Instructor Notes that appear in the margin. These notes offer instructors assistance with lecture preparation. In addition, there are Classroom Examples referenced in the text that are highlighted in the Practice Exercises. Also found in the *AIE* are icons within the Practice Exercises that serve to guide instructors in their preparation of homework assignments and lessons.

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Acknowledgments and Reviewers

Paramount to the development of this series was the invaluable feedback provided by the instructors from around the country who reviewed the manuscript or attended a market development event over the course of the several years the text was in development.

Maryann Faller, *Adirondack Community College*
Albert Miller, *Ball State University*
Debra Pearson, *Ball State University*
Patricia Parkison, *Ball State University*
Robin Rufatto, *Ball State University*
Melanie Walker, *Bergen Community College*
Robert Fusco, *Bergen Community College*
Latonya Ellis, *Bishop State Community College*
Ana Leon, *Bluegrass Community College & Technical College*
Kaye Black, *Bluegrass Community College & Technical College*
Barbara Elzey, *Bluegrass Community College & Technical College*
Cheryl Grant, *Bowling Green State University*
Beth Rountree, *Brevard College*
Juliet Carl, *Broward College*
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Michael Maltenfort, *Truman College*
Abdallah Shuaibi, *Truman College*
Marta Hidegkuti, *Truman College*
Sandra Wilder, *University of Akron*
Sandra Jovicic, *University of Akron*
Edward Migliore, *University of California–Santa Cruz*
Kelly Kohlmetz, *University of Wisconsin—Milwaukee*
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Claudia Martinez, *Valencia College*
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David Cooper, *Wake Tech Community College*
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Jacqui Fields, *Wake Tech Community College*
Jennifer Smeal, *Wake Tech Community College*
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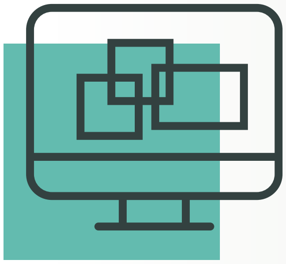

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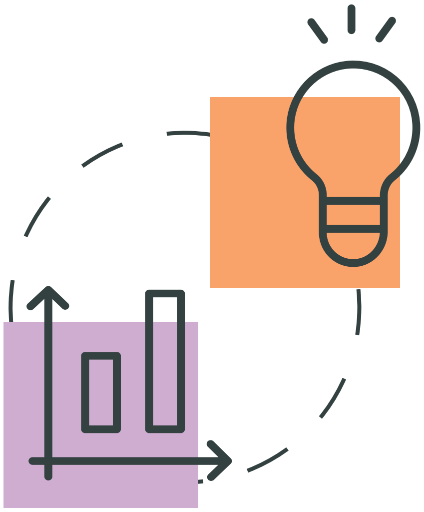

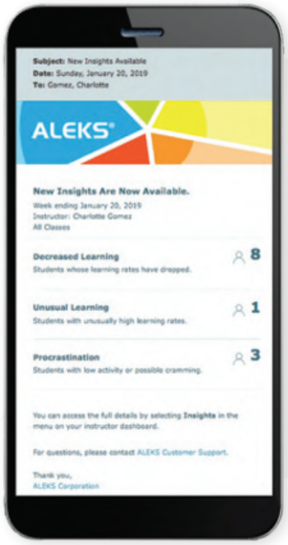

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Review of Basic Algebraic Concepts

R

CHAPTER OUTLINE

Mathematics and Consistency

Many of the activities we perform every day follow a natural order. For example, we would not put on our shoes before putting on our socks, nor would a doctor begin surgery before giving an anesthetic.

In mathematics, it is also necessary to follow a prescribed order of operations to simplify an algebraic expression. This is important, for example, because we would not want two different engineers working on a space probe to Mars to interpret a mathematical statement differently.

Suppose that the high temperature for a summer day near the equator of Mars is 20°C. To convert this to degrees Fahrenheit F, we would substitute 20 for C in the equation.

$$F = \frac{9}{5}C + 32 \xrightarrow{\text{Substitute 20 for } C} F = \frac{9}{5}(20) + 32$$

In this expression, the operation between $\frac{9}{5}$ and 20 is implied multiplication, and it is universally understood that multiplication is performed before addition. Thus,

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$$F = \frac{9}{5}(20) + 32 = 36 + 32 = 68. \qquad \text{The temperature in Fahrenheit is 68°F.}$$

If an engineer had erroneously added 20 and 32 first and then multiplied by $\frac{9}{5}$, a different temperature of 93.6°F would result. This illustrates the importance of a prescribed order for mathematical operations.

Section R.1 Sets of Numbers and Interval Notation

Concepts

1. The Set of Real Numbers
2. Inequalities
3. Interval Notation
4. Translations Involving Inequalities

1. The Set of Real Numbers

Algebra is a powerful mathematical tool that is used to solve real-world problems in science, business, and many other fields. We begin our study of algebra with a review of basic definitions and notations used to express algebraic relationships.

In mathematics, a collection of items (called elements) is called a **set**, and the set braces { } are used to enclose the elements of the set. For example, the set {a, e, i, o, u} represents the vowels in the English alphabet. The set {1, 3, 5, 7} represents the first four positive odd numbers. Another method to express a set is to *describe* the elements of the set by using **set-builder notation**. Consider the set {a, e, i, o, u} in set-builder notation.

Set-builder notation: $\{x \mid \overbrace{x \text{ is a vowel in the English alphabet}}^{\text{description of set}}\}$

"the set of" "all x" "such that" "x is a vowel in the English alphabet"

Consider the set {1, 3, 5, 7} in set-builder notation.

Set-builder notation: $\{x \mid \overbrace{x \text{ is an odd number between 0 and 8}}^{\text{description of set}}\}$

"the set of" "all x" "such that" "x is an odd number between 0 and 8"

Several sets of numbers are used extensively in algebra. The numbers you are familiar with in day-to-day calculations are elements of the set of **real numbers**. These numbers can be represented graphically on a horizontal number line with a point labeled as 0. Positive real numbers are graphed to the right of 0, and negative real numbers are graphed to the left. Each point on the number line corresponds to exactly one real number, and for this reason, the line is called the **real number line** (Figure R-1).

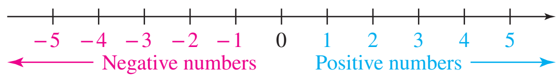


Figure R-1

Several sets of numbers are **subsets** (or part) of the set of real numbers. These are

The set of natural numbers

The set of whole numbers

The set of integers

The set of rational numbers

The set of irrational numbers

Natural Numbers, Whole Numbers, and Integers

The set of **natural numbers** is {1, 2, 3, . . .}.

The set of **whole numbers** is {0, 1, 2, 3, . . .}.

The set of **integers** is {. . . , −3, −2, −1, 0, 1, 2, 3, . . .}.

The set of rational numbers consists of all the numbers that can be defined as a ratio of two integers.

Rational Numbers

The set of **rational numbers** is $\left\{\frac{p}{q} \mid p \text{ and } q \text{ are integers and } q \text{ does not equal zero}\right\}$.

Example 1 Identifying Rational Numbers

Show that each number is a rational number by finding two integers whose ratio equals the given number.

a. $\frac{-4}{7}$ **b.** 8 **c.** $0.\overline{6}$ **d.** 0.87

Solution:

a. $\frac{-4}{7}$ is a rational number because it can be expressed as the ratio of the integers -4 and 7.

b. 8 is a rational number because it can be expressed as the ratio of the integers 8 and 1 $\left(8 = \frac{8}{1}\right)$. In this example we see that *an integer is also a rational number*.

c. $0.\overline{6}$ represents the repeating decimal 0.6666666 . . . and can be expressed as the ratio of 2 and 3 $\left(0.\overline{6} = \frac{2}{3}\right)$. In this example we see that *a repeating decimal is a rational number*.

d. 0.87 is the ratio of 87 and 100 $\left(0.87 = \frac{87}{100}\right)$. In this example we see that *a terminating decimal is a rational number*.

Skill Practice Show that the numbers are rational by writing them as a ratio of integers.

1. $\frac{-9}{8}$ **2.** 0 **3.** $0.\overline{3}$ **4.** 0.45

TIP: Any rational number can be represented by a terminating decimal or by a repeating decimal.

Some real numbers such as the number π (pi) cannot be represented by the ratio of two integers. In decimal form, an irrational number is a nonterminating, nonrepeating decimal. The value of π, for example, can be approximated as $\pi \approx 3.1415926535897932$. However, the decimal digits continue indefinitely with no pattern. Other examples of irrational numbers are the square roots of nonperfect squares, such as $\sqrt{3}$ and $\sqrt{10}$.

Irrational Numbers

The set of **irrational numbers** is a subset of the real numbers whose elements cannot be written as a ratio of two integers.

Note: An irrational number cannot be written as a terminating decimal or as a repeating decimal.

The set of real numbers consists of both the rational numbers and the irrational numbers. The relationships among the sets of numbers discussed thus far are illustrated in Figure R-2.

Answers

1. $\frac{-9}{8}$ **2.** $\frac{0}{1}$

3. $\frac{1}{3}$ **4.** $\frac{45}{100}$ or $\frac{9}{20}$

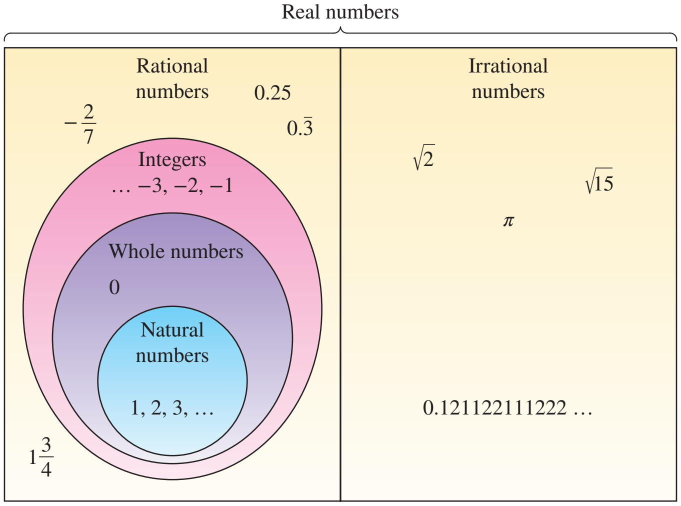


Figure R-2

Example 2 Classifying Numbers by Set

Check the set(s) to which each number belongs. The numbers may belong to more than one set.

	Natural Numbers	Whole Numbers	Integers	Rational Numbers	Irrational Numbers	Real Numbers
-6						
$\sqrt{23}$						
$-\frac{2}{7}$						
3						
$2.\overline{3}$						

Solution:

	Natural Numbers	Whole Numbers	Integers	Rational Numbers	Irrational Numbers	Real Numbers
-6			✓	✓		✓
$\sqrt{23}$					✓	✓
$-\frac{2}{7}$				✓		✓
3	✓	✓	✓	✓		✓
$2.\overline{3}$				✓		✓

Skill Practice

5. Check the set(s) to which each number belongs.

	1	0.47	$\sqrt{5}$	$-\frac{1}{2}$
Natural				
Whole				
Integer				
Rational				
Irrational				
Real				

Answer

5.

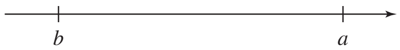
	1	0.47	$\sqrt{5}$	$-\frac{1}{2}$
Natural	✓			
Whole	✓			
Integer	✓			
Rational	✓	✓		✓
Irrational			✓	
Real	✓	✓	✓	✓

2. Inequalities

The relative value of two numbers can be compared by using the real number line. We say that a is less than b (written mathematically as $a < b$) if a lies to the left of b on the number line.

$a < b$

We say that a is greater than b (written mathematically as $a > b$) if a lies to the right of b on the number line.



$a > b$

From looking at the number line, note that $a > b$ is the same as $b < a$. Table R-1 summarizes the relational operators that compare two real numbers a and b.

Table R-1

Mathematical Expression	Translation	Other Meanings
$a < b$	a is less than b	b exceeds a b is greater than a
$a > b$	a is greater than b	a exceeds b b is less than a
$a \leq b$	a is less than or equal to b	a is at most b a is no more than b
$a \geq b$	a is greater than or equal to b	a is no less than b a is at least b
$a = b$	a is equal to b	
$a \neq b$	a is not equal to b	
$a \approx b$	a is approximately equal to b	

The symbols $<$, $>$, $\leq$, $\geq$, and $\neq$ are called inequality signs, and the expressions $a < b$, $a > b$, $a \leq b$, $a \geq b$, and $a \neq b$ are called **inequalities**.

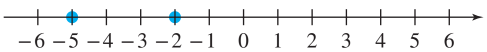
Example 3 Ordering Real Numbers

Fill in the blank with the appropriate inequality sign: $<$ or $>$

a. -2 ______ -5 **b.** $\frac{4}{7}$ ______ $\frac{3}{5}$ **c.** -1.3 ______ $-1.\overline{3}$

Solution:

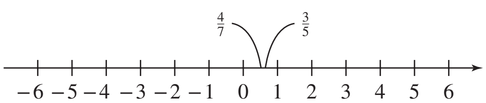
a. -2 __>__ -5



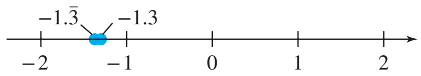
b. To compare $\frac{4}{7}$ and $\frac{3}{5}$, write the fractions as equivalent fractions with a common denominator.

$$\frac{4}{7} \cdot \frac{5}{5} = \frac{20}{35} \quad \text{and} \quad \frac{3}{5} \cdot \frac{7}{7} = \frac{21}{35}$$

Because $\frac{20}{35} < \frac{21}{35}$, then $\frac{4}{7}$ __<__ $\frac{3}{5}$



c. $-1.3 \underline{\ \ \geq\ \ } -1.33333\ldots$

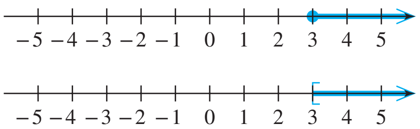


Skill Practice Fill in the blanks with the appropriate sign, < or >.

6. 2 ______ -12 **7.** $\frac{1}{4}$ ______ $\frac{2}{9}$ **8.** $-7.\overline{2}$ ______ -7.2

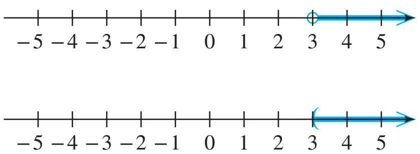
3. Interval Notation

The set $\{x \mid x \geq 3\}$ represents all real numbers greater than or equal to 3. This set can be illustrated graphically on the number line.



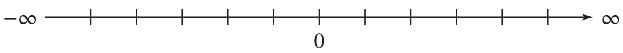
By convention, a closed circle ● or a square bracket [is used to indicate that an "endpoint" $(x = 3)$ *is included* in the set.

The set $\{x \mid x > 3\}$ represents all real numbers strictly greater than 3. This set can be illustrated graphically on the number line.



By convention, an open circle ○ or a parenthesis (is used to indicate that an "endpoint" $(x = 3)$ is *not* included in the set.

Notice that the sets $\{x \mid x \geq 3\}$ and $\{x \mid x > 3\}$ consist of an infinite number of elements that cannot all be listed. Another method to represent the elements of such sets is by using **interval notation**. To understand interval notation, first consider the real number line, which extends infinitely far to the left and right. The symbol ∞ is used to represent infinity. The symbol $-\infty$ is used to represent negative infinity.



To express a set of real numbers in interval notation, sketch the graph first, using the symbols ( ) or [ ]. Then use these symbols at the endpoints to define the interval.

Example 4 **Expressing Sets by Using Interval Notation**

Graph each set on the number line, and express the set in interval notation.

a. $\{x \mid x \geq -2\}$ **b.** $\{p \mid p > -2\}$

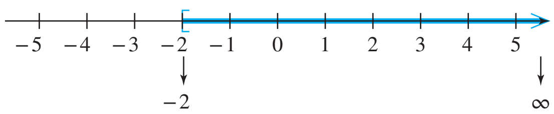
FOR REVIEW

Recall that an inequality may be written with the variable on either side of the inequality sign. The statement $x \geq -2$ is equivalent to $-2 \leq x$.

Solution:

a. Set-builder notation: $\{x \mid x \geq -2\}$

Graph:



Interval notation: $[-2, \infty)$

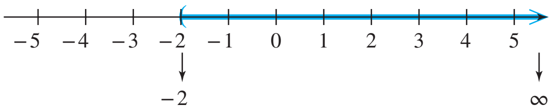
Answers

6. > **7.** > **8.** <

The graph of the set $\{x \mid x \geq -2\}$ "begins" at -2 and extends infinitely far to the right. The corresponding interval notation "begins" at -2 and extends to ∞. Notice that a square bracket [is used at -2 for both the graph and the interval notation to include $x = -2$. *A parenthesis is always used at* ∞ *and at* $-\infty$ *because there is no endpoint*.

b. Set-builder notation: $\{p \mid p > -2\}$

Graph:



Interval notation: $(-2, \infty)$

Skill Practice Graph each set, and express the set in interval notation.

9. $\{w \mid w \geq -7\}$ **10.** $\{x \mid x < 0\}$

In general, we use the following guidelines when applying interval notation.

Using Interval Notation

- The endpoints used in interval notation are always written from left to right. That is, the smaller number is written first, followed by a comma, followed by the larger number.
- Parentheses) or (indicate that an endpoint is *excluded* from the set.
- Square brackets] or [indicate that an endpoint is *included* in the set.
- Parentheses are always used with ∞ or $-\infty$.

Example 5 Expressing Sets by Using Interval Notation

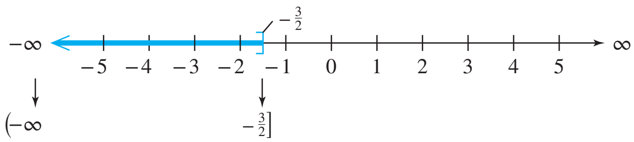
Graph each set on the number line, and express the set in interval notation.

a. $\{z \mid z \leq -\frac{3}{2}\}$ **b.** $\{x \mid -4 < x \leq 2\}$

Solution:

a. Set-builder notation: $\{z \mid z \leq -\frac{3}{2}\}$

Graph:



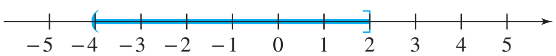
Interval notation: $(-\infty, -\frac{3}{2}]$

The graph of the set $\{z \mid z \leq -\frac{3}{2}\}$ extends infinitely far to the left. Interval notation is always written from left to right. Therefore, $-\infty$ is written first, followed by a comma, and then followed by the right-hand endpoint $-\frac{3}{2}$.

b. The inequality $-4 < x \leq 2$ means that x is greater than -4 and also less than or equal to 2. More concisely, we can say that x represents the real numbers *between* -4 and 2, including the endpoint, 2.

Set-builder notation: $\{x \mid -4 < x \leq 2\}$

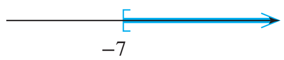
Graph:



Interval notation: $(-4, 2]$

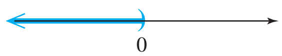
Answers

9.



$[-7, \infty)$

10.



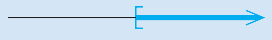
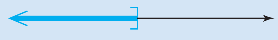
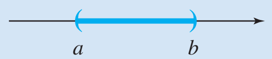
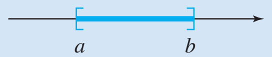
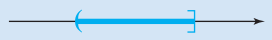
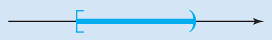
$(-\infty, 0)$

Skill Practice Graph the set on the number line, and express the set in interval notation.

11. $\{w \mid w \geq -\frac{5}{3}\}$ **12.** $\{y \mid -7 \leq y < 4\}$

Table R-2 summarizes interval notation.

Table R-2

Interval Notation	Graph	Interval Notation	Graph
(a, ∞)		$[a, \infty)$	
$(-\infty, a)$		$(-\infty, a]$	
(a, b)		$[a, b]$	
$(a, b]$		$[a, b)$	

4. Translations Involving Inequalities

In Table R-1, we learned that phrases such as *at least*, *at most*, *no more than*, *no less than*, and *between* can be translated into mathematical terms by using inequality signs.

Example 6 Translating Inequalities

The intensity of a hurricane is often defined according to its maximum sustained winds, for which wind speed is measured to the nearest mile per hour. Translate the italicized phrases into mathematical inequalities.

a. A tropical storm is updated to hurricane status if the sustained wind speed, w, *is at least 74 mph*.

b. Hurricanes are categorized according to intensity by the Saffir-Simpson scale. On a scale of 1 to 5, a category 5 hurricane is the most destructive. A category 5 hurricane has sustained winds, w, *exceeding 155 mph*.

c. A category 4 hurricane has sustained winds, w, *of at least 131 mph but no more than 155 mph*.

Solution:

a. $w \geq 74$ mph **b.** $w > 155$ mph **c.** $131 \text{ mph} \leq w \leq 155$ mph

Skill Practice Translate the italicized phrase to a mathematical inequality.

13. The gas mileage, m, for an economy car is *at least 30 mpg*.

14. The gas mileage, m, for a motorcycle is *more than 45 mpg*.

15. The gas mileage, m, for an SUV is *at least 10 mpg, but no more than 20 mpg*.

Answers

11. 

$[-\frac{5}{3}, \infty)$

12. 

$[-7, 4)$

13. $m \geq 30$
14. $m > 45$
15. $10 \leq m \leq 20$

Section R.1 Activity

For Exercises A.1–A.6, refer to set A. $A = \left\{-5, -2.\overline{5}, \sqrt{7}, 0, \frac{7}{4}, 4, -2.5\right\}$

A.1. Which elements from A are natural numbers?

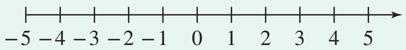
A.2. Which elements from A are whole numbers?

A.3. Which elements from A are integers?

A.4. Which elements from A are rational numbers?

A.5. Which elements from A are irrational numbers?

A.6. Plot the elements from A on the number line.



Inequality statements surround us in day-to-day life. In Exercises A.7–A.9, let x represent the unknown quantity, and write a mathematical inequality to represent the given statement.

A.7. A child must be at least 44 inches tall to ride Space Mountain.

A.8. According to the posted speed limit on a country road, a driver traveling at most 35 mph will not get a speeding ticket.

A.9. To preregister to vote in the United States, a person must be at least 16, but less than 18 years old.

A.10. **a.** Graph the inequality $x < 1$.

b. Graph the inequality $x \leq 1$.

c. Explain when to use parentheses, (or), versus brackets [or], when graphing an inequality.

d. Write the inequality $x < 1$ in interval notation.

e. Write the inequality $x \leq 1$ in interval notation.

f. Explain when to use parentheses, (or), versus brackets [or], when using interval notation.

For Exercises A.11–A.12, write the set in interval notation.

A.11. $\{x \mid -4 < x \leq -1\}$

A.12. $\left\{x \,\middle|\, \frac{3}{2} \leq x\right\}$

Practice Exercises Section R.1

Study Skills Exercises

Mindset plays an important role in your approach to learning mathematics. Mindset consists of our thoughts, beliefs, and attitudes about our abilities based on lifetime experiences. There are two types of mindsets: fixed mindsets and growth mindsets. People with a fixed mindset believe that they are born with a certain amount of intelligence that cannot be changed despite their actions. On the other hand, a person with a growth mindset believes that intelligence is dynamic and can be increased with effort and learning. What type of mindset do you have? Think about the following questions:

- Have you said to yourself, "I'm just not good at math"?
- Do you believe you lack the necessary skills to understand math?
- Can you recall an experience that has positively impacted your self-confidence in mathematics?

Prerequisite Review

For Exercises R.1–R.4, fill in the blank with <, >, or =.

R.1. $-3.85 \; \square \; -3.84$ **R.2.** $-59.7 \; \square \; -59.8$ **R.3.** $\frac{11}{7} \; \square \; \frac{8}{5}$ **R.4.** $\frac{7}{11} \; \square \; \frac{5}{8}$

R.5. **a.** Write the first six digits to the right of the decimal point for the repeating decimal $0.8\overline{35}$.
b. Round $0.8\overline{35}$ to the thousandths place.
c. Round $0.8\overline{35}$ to the hundredths place.

R.6. **a.** Write the first six digits to the right of the decimal point for the repeating decimal $0.\overline{265}$.
b. Round $0.\overline{265}$ to the tenths place.
c. Round $0.\overline{265}$ to the ten-thousandths place.

R.7. Order the numbers from least to greatest.
a. $2.45, 2.\overline{45}, 2.4\overline{5}, 2.44999$
b. $-2.45, -2.\overline{45}, -2.4\overline{5}, -2.44999$

R.8. Order the numbers from least to greatest.
a. $3.1735, 3.173499, 3.1\overline{7}, 3.2$
b. $-3.1735, -3.173499, -3.1\overline{7}, -3.2$

Vocabulary and Key Concepts

1. **a.** In mathematics, a well-defined collection of elements is called a ____________.

b. The statements $a < b$, $a > b$, and $a \neq b$ are examples of ____________.

c. The statement $a < b$ is read as "____________."

d. The statement $c \geq d$ is read as "____________________."

e. The statement $5 \neq 6$ is read as "____________."

f. The symbol ∞ represents ____________ and $-\infty$ represents ____________.

g. The set of real numbers greater than 5 can be written in set-builder notation as ____________ and in ____________ notation as $(5, \infty)$.

h. The interval $(-2, 5]$ (includes/excludes) the value -2 and (includes/excludes) the value 5.

i. When expressing interval notation, use a (parenthesis/bracket) with infinity.

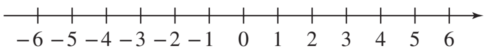
Concept 1: The Set of Real Numbers

2. Determine the two consecutive integers between which the given number is located on the number line.

a. $\frac{19}{4}$ **b.** $-\frac{2}{3}$ **c.** -4.6 **d.** π

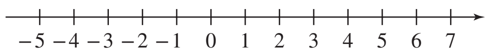
3. Plot the numbers on the number line.

$$\{1.7, \pi, -5, 4.\overline{2}\}$$



4. Plot the numbers on the number line.

$$\left\{1\frac{1}{2}, 0, -3, -\frac{1}{2}, \frac{3}{4}\right\}$$



For Exercises 5–10, show that each number is a rational number by finding a ratio of two integers equal to the given number. **(See Example 1.)**

5. -10 **6.** $7\frac{3}{4}$ **7.** $-\frac{3}{5}$

8. -0.1 **9.** 0 **10.** 0.35

11. Check the sets to which each number belongs. **(See Example 2.)**

	Real Numbers	Irrational Numbers	Rational Numbers	Integers	Whole Numbers	Natural Numbers
5						
$-\sqrt{9}$						
-1.7						
$\frac{1}{2}$						
$\sqrt{7}$						
$\frac{0}{4}$						
$0.\overline{2}$						

12. Check the sets to which each number belongs.

	Real Numbers	Irrational Numbers	Rational Numbers	Integers	Whole Numbers	Natural Numbers
$\frac{6}{8}$						
$1\frac{1}{2}$						
π						
0						
$-0.\overline{8}$						
$\frac{8}{2}$						
$4.\overline{2}$						

Concept 2: Inequalities

For Exercises 13–20, fill in the blanks with the appropriate symbol: $<$ or $>$. **(See Example 3.)**

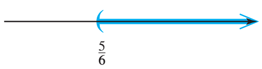
13. -9 ___ -1 **14.** 0 ___ -6 **15.** $0.1\overline{5}$ ___ 0.15 **16.** $-2.\overline{5}$ ___ -2.5

17. $\frac{5}{3}$ ___ $\frac{10}{7}$ **18.** $-\frac{21}{5}$ ___ $-\frac{17}{4}$ **19.** $-\frac{5}{8}$ ___ $-\frac{1}{8}$ **20.** $-\frac{13}{15}$ ___ $-\frac{17}{12}$

Concept 3: Interval Notation

For Exercises 21–28, express the set in interval notation.

21. 

22. 

23. 

24. 

25. 

26. 

27. 

28. 

For Exercises 29–46, graph the sets and express each set in interval notation. **(See Examples 4–5.)**

29. $\{x \mid x > -1\}$

30. $\{x \mid x < 3\}$

31. $\{y \mid -2 \geq y\}$

32. $\{z \mid -4 \leq z\}$

33. $\{w \mid w < \frac{9}{2}\}$

34. $\{p \mid p \geq -\frac{7}{3}\}$

35. $\{x \mid -2.5 < x \leq 4.5\}$

36. $\{x \mid -6 \leq x < 0\}$

37. All real numbers less than -3.

38. All real numbers greater than 2.34.

39. All real numbers greater than $\frac{5}{2}$.

40. All real numbers less than $\frac{4}{7}$.

41. All real numbers not less than 2.

42. All real numbers no more than 5.

43. All real numbers between -4 and 4.

44. All real numbers between -7 and -1.

45. All real numbers between -3 and 0, inclusive.

46. All real numbers between -1 and 6, inclusive.

For Exercises 47–54, write an expression in words that describes the set of numbers given by each interval. (Answers may vary.)

47. $(-\infty, -4)$

48. $[2, \infty)$

49. $(-2, 7]$

50. $(-3.9, 0)$

51. $[-180, 90]$

52. $(3.2, \infty)$

53. $(-\infty, \infty)$

54. $(-\infty, -1]$

Concept 4: Translations Involving Inequalities

For Exercises 55–64, write the expressions as an inequality. **(See Example 6.)**

55. The age, a, to get in to see a certain movie is at least 18 years old.

56. Winston is a cat that was picked up at the Humane Society. His age, a, at the time was no more than 2 years.

57. The cost, c, to have dinner at Jack's Café is at most \$25.

58. The number of hours, h, that Katlyn spent studying was no less than 40.

59. The wind speed, s, for an F-5 tornado is no less than 261 mph.

60. The high temperature, t, for a certain December day in Albany is at most 26°F.

61. After a summer drought, the total rainfall, r, for June, July, and August was no more than 4.5 in.

62. Jessica works for a networking firm. Her salary, s, is at least \$85,000 per year.

63. To play in a certain division of a tennis tournament, a player's age, a, must be at least 18 years but not more than 25 years.

64. The average age, a, of students at Central Community College is estimated to be between 25 years and 29 years.

The following chart defines the ranges for normal blood pressure, high normal blood pressure, and high blood pressure (*hypertension*). All values are measured in millimeters of mercury (mm Hg). (Source: American Heart Association.)

Normal	Systolic less than 130	Diastolic less than 85
High normal	Systolic 130–139, inclusive	Diastolic 85–89, inclusive
Hypertension	Systolic 140 or greater	Diastolic 90 or greater

For Exercises 65–68, write an inequality using the variable p that represents each condition.

65. Normal systolic blood pressure

66. Diastolic pressure in hypertension

67. High normal range for systolic pressure

68. Systolic pressure in hypertension

A pH scale determines whether a solution is acidic or alkaline. The pH scale runs from 0 to 14, with 0 being the most acidic and 14 being the most alkaline. A pH of 7 is neutral (distilled water has a pH of 7).

For Exercises 69–72, write the pH ranges as inequalities and label the substances as acidic or alkaline.

69. Lemon juice: 2.2 through 2.4, inclusive

70. Eggs: 7.6 through 8.0, inclusive

71. Carbonated soft drinks: 3.0 through 3.5, inclusive

72. Milk: 6.6 through 6.9, inclusive

Operations on Real Numbers

Section R.2

Concepts

1. Opposite and Absolute Value
2. Addition and Subtraction of Real Numbers
3. Multiplication and Division of Real Numbers
4. Exponential Expressions
5. Square Roots
6. Order of Operations
7. Evaluating Formulas

1. Opposite and Absolute Value

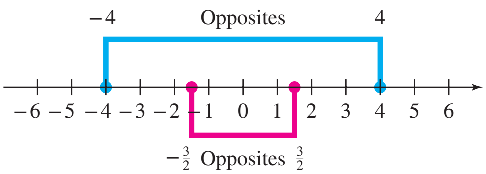
Several key definitions are associated with the set of real numbers and constitute the foundation of algebra. Two important definitions are the opposite of a real number and the absolute value of a real number.

Opposite of a Real Number

Two numbers that are the same distance from 0 but on opposite sides of 0 on the number line are called **opposites** of each other.

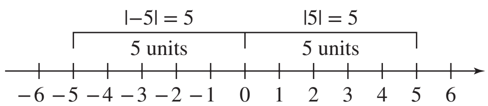
Symbolically, we denote the opposite of a real number a as $-a$.

The numbers -4 and 4 are opposites of each other. Similarly, the numbers $\frac{3}{2}$ and $-\frac{3}{2}$ are opposites.



The **absolute value** of a real number a, denoted $|a|$, is the distance between a and 0 on the number line. *Note:* The absolute value of any real number is *nonnegative*.

For example: $|5| = 5$
and
$|-5| = 5$



Example 1 **Evaluating Absolute Value Expressions**

Simplify the expressions.

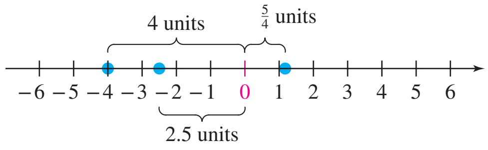
a. $|-2.5|$ **b.** $\left|\frac{5}{4}\right|$ **c.** $-|-4|$

Solution:

a. $|-2.5| = 2.5$

b. $\left|\frac{5}{4}\right| = \frac{5}{4}$

c. $-|-4| = -(4) = -4$



Skill Practice Simplify.

1. $|-9.2|$ **2.** $\left|\frac{7}{6}\right|$ **3.** $-|-2|$

The absolute value of a number a is its distance from zero on the number line. The definition of $|a|$ may also be given algebraically depending on whether a is negative or nonnegative.

FOR REVIEW

Recall that two numbers that are added are called **addends**. The result of the addition is called the **sum**.

Absolute Value of a Real Number

Let a be a real number. Then

1. If a is nonnegative (that is, if $a \geq 0$), then $|a| = a$.
2. If a is negative (that is, if $a < 0$), then $|a| = -a$

This definition states that if a is positive or zero, then $|a|$ equals a itself. If a is a negative number, then $|a|$ equals the opposite of a. For example,

$|9| = 9$ Because 9 is positive, $|9|$ equals the number 9 itself.

$|-7| = 7$ Because -7 is negative, $|-7|$ equals the opposite of -7, which is 7.

2. Addition and Subtraction of Real Numbers

Addition of Real Numbers

- To add two numbers with the *same sign*, add their absolute values and apply the common sign to the sum.
- To add two numbers with *different signs*, subtract the smaller absolute value from the larger absolute value. Then apply the sign of the number having the larger absolute value.

Answers

1. 9.2 **2.** $\frac{7}{6}$ **3.** -2

Example 2 Adding Real Numbers

Perform the indicated operations.

a. $-2 + (-6)$ **b.** $-10.3 + 13.8$ **c.** $\frac{5}{6} + \left(-1\frac{1}{4}\right)$

Solution:

a. $-2 + (-6)$ First find the absolute value of the addends.
$|-2| = 2$ and $|-6| = 6$

$= -(2 + 6)$ Add their absolute values and apply the common sign. In this case, the common sign is negative.

Common sign is negative.

$= -8$ The sum is -8.

b. $-10.3 + 13.8$ First find the absolute value of the addends.
$|-10.3| = 10.3$ and $|13.8| = 13.8$

The absolute value of 13.8 is greater than the absolute value of -10.3. Therefore, the sum is positive.

$= +(13.8 - 10.3)$ Subtract the smaller absolute value from the larger absolute value.

Apply the sign of the number with the larger absolute value.

$= 3.5$

c. $\frac{5}{6} + \left(-1\frac{1}{4}\right)$

$= \frac{5}{6} + \left(-\frac{5}{4}\right)$ Write $-1\frac{1}{4}$ as a fraction.

$= \frac{5 \cdot 2}{6 \cdot 2} + \left(-\frac{5 \cdot 3}{4 \cdot 3}\right)$ The least common denominator (LCD) is 12. Write each fraction with the LCD.

$= \frac{10}{12} + \left(-\frac{15}{12}\right)$ Find the absolute value of the addends.
$\left|\frac{10}{12}\right| = \frac{10}{12}$ and $\left|-\frac{15}{12}\right| = \frac{15}{12}$

The absolute value of $-\frac{15}{12}$ is greater than the absolute value of $\frac{10}{12}$. Therefore, the sum is negative.

$= -\left(\frac{15}{12} - \frac{10}{12}\right)$ Subtract the smaller absolute value from the larger absolute value.

Apply the sign of the number with the larger absolute value.

$= -\frac{5}{12}$

FOR REVIEW

The LCD of two fractions is the product of unique prime factors from each denominator. Each factor is raised to the highest power to which it occurs. From Example 2(c),

$6 = 2^1 \cdot 3^1$
$4 = 2^2$

The LCD of $\frac{5}{6}$ and $-\frac{5}{4}$ is $2^2 \cdot 3^1 = 12$.

Skill Practice Perform the indicated operations.

4. $-4 + (-1)$ **5.** $-2.6 + 1.8$ **6.** $-1 + \left(-\frac{3}{7}\right)$

Answers

4. -5 **5.** -0.8 **6.** $-\frac{10}{7}$

Subtraction of real numbers is defined in terms of the addition process. To subtract two real numbers, add the opposite of the second number to the first number.

Subtraction of Real Numbers

If a and b are real numbers, then $a - b = a + (-b)$

Example 3 Subtracting Real Numbers

Perform the indicated operations.

a. $-13 - 5$ **b.** $2.7 - (-3.8)$ **c.** $\frac{5}{2} - 4\frac{2}{3}$

Solution:

a. $-13 - 5$

$= -13 + (-5)$ Add the opposite of the second number to the first number.

$= -18$ Add.

b. $2.7 - (-3.8)$

$= 2.7 + (3.8)$ Add the opposite of the second number to the first number.

$= 6.5$ Add.

c. $\frac{5}{2} - 4\frac{2}{3}$

$= \frac{5}{2} + \left(-4\frac{2}{3}\right)$ Add the opposite of the second number to the first number.

$= \frac{5}{2} + \left(-\frac{14}{3}\right)$ Write the mixed number as a fraction.

$= \frac{5 \cdot 3}{2 \cdot 3} + \left(-\frac{14 \cdot 2}{3 \cdot 2}\right)$ The least common denominator is 6.

$= \frac{15}{6} + \left(-\frac{28}{6}\right)$ Get a common denominator and add.

$= -\frac{13}{6}$ or $-2\frac{1}{6}$

FOR REVIEW

Write a mixed number as an improper fraction as follows. Multiply the whole number by the denominator of the fraction. Then add the numerator. Write the result over the denominator.

$$4\frac{2}{3} = \frac{4(3) + 2}{3}$$
$$= \frac{12 + 2}{3} = \frac{14}{3}$$

Skill Practice Subtract.

7. $-9 - 8$ **8.** $1.1 - (-4.2)$ **9.** $\frac{1}{6} - 2\frac{1}{4}$

Answers

7. -17 **8.** 5.3 **9.** $-2\frac{1}{12}$ or $-\frac{25}{12}$

3. Multiplication and Division of Real Numbers

The sign of the product of two real numbers is determined by the signs of the factors.

Multiplication of Real Numbers

1. The product of two real numbers with the *same* sign is *positive*.
2. The product of two real numbers with *different* signs is *negative*.
3. The product of any real number and zero is *zero*.

Example 4 Multiplying Real Numbers

Multiply the real numbers.

a. $(2)(-5.1)$ **b.** $-\frac{2}{3} \cdot \frac{9}{8}$ **c.** $\left(-3\frac{1}{3}\right)\left(-\frac{3}{10}\right)$

Solution:

a. $(2)(-5.1)$

$= -10.2$ *Different* signs. The product is negative.

b. $-\frac{2}{3} \cdot \frac{9}{8}$

$= -\frac{18}{24}$ *Different* signs. The product is negative.

$= -\frac{3}{4}$ Simplify to lowest terms.

c. $\left(-3\frac{1}{3}\right)\left(-\frac{3}{10}\right)$

$= \left(-\frac{10}{3}\right)\left(-\frac{3}{10}\right)$ Write the mixed number as a fraction.

$= \frac{30}{30}$ *Same* signs. The product is positive.

$= 1$ Simplify to lowest terms.

TIP: A number and its reciprocal have the same sign. For example:

$$\left(-\frac{10}{3}\right)\left(-\frac{3}{10}\right) = 1 \quad \text{and} \quad 3 \cdot \frac{1}{3} = 1$$

Skill Practice Multiply.

10. $(-5)(2.2)$ **11.** $\frac{5}{7} \cdot \left(-\frac{14}{15}\right)$ **12.** $\left(-5\frac{1}{4}\right)\left(-\frac{8}{3}\right)$

Notice from Example 4(c) that $\left(-\frac{10}{3}\right)\left(-\frac{3}{10}\right) = 1$. If the product of two numbers is 1, then the numbers are **reciprocals**. That is, the reciprocal of a real *number a* is $\frac{1}{a}$. Furthermore, $a \cdot \frac{1}{a} = 1$.

Answers

10. -11 **11.** $-\frac{2}{3}$ **12.** 14

Recall that subtraction of real numbers was defined in terms of addition. In a similar way, division of real numbers can be defined in terms of multiplication.

Procedure to Divide Real Numbers

To divide two real numbers, multiply the first number by the reciprocal of the second number. For example:

$$10 \div 5 = 2 \quad \text{or equivalently} \quad 10 \cdot \frac{1}{5} = 2$$

Multiply

Reciprocal

Because division of real numbers can be expressed in terms of multiplication, the sign rules that apply to multiplication also apply to division.

$$\left.\begin{aligned} 10 \div 2 &= 10 \cdot \frac{1}{2} = 5 \\ -10 \div (-2) &= -10 \cdot \left(-\frac{1}{2}\right) = 5 \end{aligned}\right\}$$ Dividing two numbers of the same sign produces a *positive* quotient.

$$\left.\begin{aligned} 10 \div (-2) &= 10 \cdot \left(-\frac{1}{2}\right) = -5 \\ -10 \div 2 &= -10 \cdot \frac{1}{2} = -5 \end{aligned}\right\}$$ Dividing two numbers of opposite signs produces a *negative* quotient.

Division of Real Numbers

Assume that a and b are real numbers such that $b \neq 0$.

1. If a and b have the *same* sign, then the quotient $\frac{a}{b}$ is *positive*.
2. If a and b have *different* signs, then the quotient $\frac{a}{b}$ is *negative*.
3. $\dfrac{0}{b} = 0$.
4. $\dfrac{b}{0}$ is undefined.

The relationship between multiplication and division can be used to investigate properties 3 and 4 from the preceding box. For example,

$\dfrac{0}{6} = 0$ Because $6 \cdot 0 = 0$ ✓

$\dfrac{6}{0}$ is *undefined* Because there is no number that when multiplied by 0 will equal 6

Note: The quotient of 0 and 0 *cannot be determined*. Evaluating an expression of the form $\frac{0}{0} = ?$ is equivalent to asking, "What number times zero will equal 0?" That is, $(0)(?) = 0$. Any real number will satisfy this requirement; however, expressions involving $\frac{0}{0}$ are usually discussed in advanced mathematics courses.

Example 5 Dividing Real Numbers

Divide the real numbers. Write the answer as a fraction or whole number.

a. $\dfrac{-42}{7}$ **b.** $\dfrac{-96}{-144}$ **c.** $3\dfrac{1}{10} \div \left(-\dfrac{2}{5}\right)$ **d.** $\dfrac{-8}{-7}$

Solution:

a. $\dfrac{-42}{7} = -6$ *Different* signs. The quotient is negative.

b. $\dfrac{-96}{-144} = \dfrac{2}{3}$ *Same* signs. The quotient is positive. Simplify.

c. $3\dfrac{1}{10} \div \left(-\dfrac{2}{5}\right)$

$= \dfrac{31}{10}\left(-\dfrac{5}{2}\right)$ Write the mixed number as an improper fraction, and multiply by the reciprocal of the second number.

$= \dfrac{31}{\cancel{10}_{2}}\left(-\dfrac{\cancel{5}^{1}}{2}\right)$

$= -\dfrac{31}{4}$ *Different* signs. The quotient is negative.

d. $\dfrac{-8}{-7} = \dfrac{8}{7}$ *Same* signs. The quotient is positive. Because 7 does not divide into 8 evenly, the answer can be left as a fraction.

TIP: Multiplication may be used to check a division problem.

$$\frac{-42}{7} = -6$$

Check: $(7)(-6) = -42$ ✓

Avoiding Mistakes

If the numerator and denominator of a fraction have opposite signs, then the quotient will be negative. Therefore, a fraction has the same value whether the negative sign is written in the numerator, in the denominator, or in front of the fraction.

$$-\frac{31}{4} = \frac{-31}{4} = \frac{31}{-4}$$

Skill Practice Divide.

13. $\dfrac{42}{-2}$ **14.** $\dfrac{-28}{-4}$ **15.** $-\dfrac{2}{3} \div 4$ **16.** $\dfrac{-1}{-2}$

4. Exponential Expressions

To simplify the process of repeated multiplication, exponential notation is often used. For example, the quantity $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ can be written as 3^5 (3 to the fifth power).

Definition of b^n

Let b represent any real number and n represent a positive integer. Then

$$b^n = \underbrace{b \cdot b \cdot b \cdot b \cdot \ldots b}_{n \text{ factors of } b}$$

b^n is read as "b to the nth power."

b is called the **base** and n is called the **exponent**, or **power**.

b^2 is read as "b *squared*," and b^3 is read as "b *cubed*."

Answers

13. -21 **14.** 7
15. $-\dfrac{1}{6}$ **16.** $\dfrac{1}{2}$

Example 6 Evaluating Exponential Expressions

Simplify the expression.

a. 5^3 **b.** $(-2)^4$ **c.** -2^4 **d.** $\left(-\frac{1}{3}\right)^3$

Solution:

a. $5^3 = 5 \cdot 5 \cdot 5$
$= 125$

The base is 5, and the exponent is 3.

b. $(-2)^4 = (-2)(-2)(-2)(-2)$
$= 16$

The base is -2, and the exponent is 4. The exponent 4 applies to the entire contents of the parentheses.

c. $-2^4 = -[2 \cdot 2 \cdot 2 \cdot 2]$
$= -16$

The base is 2, and the exponent is 4. Because no parentheses enclose the negative sign, the exponent applies to only 2.

TIP: The quantity -2^4 can also be interpreted as $-1 \cdot 2^4$.

$$-2^4 = -1 \cdot 2^4 = -1 \cdot (2 \cdot 2 \cdot 2 \cdot 2) = -16$$

d. $\left(-\frac{1}{3}\right)^3 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$
$= -\frac{1}{27}$

The base is $-\frac{1}{3}$, and the exponent is 3.

Skill Practice Simplify.

17. 2^3 **18.** $(-10)^2$ **19.** -10^2 **20.** $\left(-\frac{3}{4}\right)^3$

5. Square Roots

The inverse operation to squaring a number is to find its square roots. For example, finding a square root of 9 is equivalent to asking, "What number when squared equals 9?" One obvious answer is 3, because $(3)^2 = 9$. However, -3 is also a square root of 9 because $(-3)^2 = 9$. For now, we will focus on the **principal square root**, which is always taken to be nonnegative.

The symbol $\sqrt{}$, called a **radical sign**, is used to denote the principal square root of a number. Therefore, the principal square root of 9 can be written as $\sqrt{9}$. The expression $\sqrt{64}$ represents the principal square root of 64.

Answers

17. 8 **18.** 100

19. -100 **20.** $-\frac{27}{64}$

Example 7 Evaluating Square Roots

Evaluate the expressions, if possible.

a. $\sqrt{81}$ **b.** $\sqrt{\frac{25}{64}}$ **c.** $\sqrt{-16}$ **d.** $-\sqrt{16}$

Solution:

a. $\sqrt{81} = 9$ because $(9)^2 = 81$

b. $\sqrt{\frac{25}{64}} = \frac{5}{8}$ because $\left(\frac{5}{8}\right)^2 = \frac{25}{64}$

c. $\sqrt{-16}$ is *not a real number* because no real number when squared will be negative.

d. $-\sqrt{16} = -4$ because $-\sqrt{16} = -(\sqrt{16}) = -4$.

Skill Practice Evaluate the expressions, if possible.

21. $\sqrt{25}$ **22.** $\sqrt{\frac{49}{100}}$ **23.** $\sqrt{-4}$ **24.** $-\sqrt{9}$

Example 7(c) illustrates that the square root of a negative number is not a real number because no real number when squared will be negative.

Square Root of a Negative Number

Let a be a negative real number. Then $\sqrt{a}$ is not a real number.

6. Order of Operations

When algebraic expressions contain numerous operations, it is important to use the proper **order of operations**. Parentheses ( ), brackets [ ], and braces { } are used for grouping numbers and algebraic expressions. It is important to recognize that operations must be done first within parentheses and other grouping symbols.

Order of Operations

Step 1 First, simplify expressions within parentheses and other grouping symbols. These include absolute value bars, fraction bars, and radicals. If embedded parentheses are present, start with the innermost parentheses.

Step 2 Evaluate expressions involving exponents, radicals, and absolute values.

Step 3 Perform multiplication or division in the order in which they occur from left to right.

Step 4 Perform addition or subtraction in the order in which they occur from left to right.

Answers

21. 5 **22.** $\frac{7}{10}$

23. Not a real number

24. -3

Example 8 Applying the Order of Operations

Simplify the expression. $10 - [2 - 4(6 - 8)]^2 + \sqrt{16 - 7}$

Solution:

Avoiding Mistakes

Don't try to perform too many steps at once. Taking a shortcut may result in a careless error. For each step rewrite the entire expression, changing only the operation being evaluated.

$10 - [2 - 4(6 - 8)]^2 + \sqrt{16 - 7}$

$= 10 - [2 - 4(-2)]^2 + \sqrt{9}$ Simplify inside the innermost parentheses and inside the radical.

$= 10 - [2 + 8]^2 + \sqrt{9}$ Simplify within square brackets. Perform multiplication before addition or subtraction.

$= 10 - [10]^2 + \sqrt{9}$

$= 10 - 100 + 3$ Simplify the exponential expression and the radical.

$= -90 + 3$ Perform addition or subtraction in the order in which they appear from left to right.

$= -87$

Skill Practice Simplify the expression.

25. $36 \div 2^2 \cdot 3 - [(18 - 5) \cdot 2 + 6]$

Example 9 Applying the Order of Operations

Simplify the expression. $\dfrac{|(-3)^3 + (5^2 - 3)|}{-15 \div (-3)(2)}$

Solution:

$\dfrac{|(-3)^3 + (5^2 - 3)|}{-15 \div (-3)(2)}$ Simplify numerator and denominator separately.

$= \dfrac{|(-3)^3 + (25 - 3)|}{5(2)}$ *Numerator:* Simplify within the inner parentheses.

Denominator: Perform division and multiplication (left to right).

$= \dfrac{|(-3)^3 + (22)|}{10}$ *Numerator:* Simplify inner parentheses. *Denominator:* Multiply.

$= \dfrac{|-27 + 22|}{10}$ Simplify exponent.

$= \dfrac{|-5|}{10}$ Add within the absolute value.

$= \dfrac{5}{10}$ or $\dfrac{1}{2}$ Evaluate the absolute value and simplify.

Skill Practice Simplify the expression.

26. $\dfrac{-|5 - 7| + 11}{(-1 - 2)^2}$

Answers

25. -5 **26.** 1

7. Evaluating Formulas

An algebraic expression or formula involves operations on numbers and variables. A **variable** is a letter that may represent any numerical value. To evaluate an expression or formula, we substitute known values of the variables into the expression. Then we follow the order of operations.

It is important to note that some formulas from geometry use Greek letters (such as π) and some use variables with subscripts. A **subscript** is a number or letter written to the right of and below a variable. For example, the area of a trapezoid is given by $A = \frac{1}{2}(b_1 + b_2)h$. The values b_1 and b_2 (read as "b sub 1" and "b sub 2") represent two different bases of the trapezoid (Figure R-3).

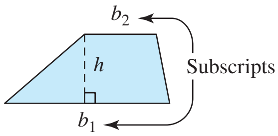


Figure R-3

Example 10 Evaluating a Formula

A homeowner in North Carolina wants to buy protective film for a trapezoid-shaped window. The film will adhere to shattered glass in the event that the glass breaks during a bad storm. Find the area of the window whose dimensions are given in Figure R-4.

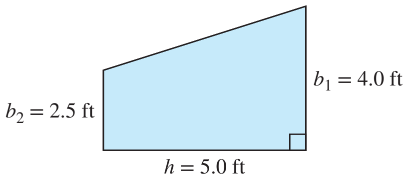


Figure R-4

Solution:

$$A = \frac{1}{2}(b_1 + b_2)h$$

$$= \frac{1}{2}(4.0\text{ ft} + 2.5\text{ ft})(5.0\text{ ft})$$ Substitute $b_1 = 4.0$ ft, $b_2 = 2.5$ ft, and $h = 5.0$ ft.

$$= \frac{1}{2}(6.5\text{ ft})(5.0\text{ ft})$$ Simplify inside parentheses.

$$= 16.25\text{ ft}^2$$ Multiply from left to right.

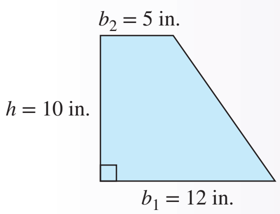
The area of the window is 16.25 ft^2.

TIP: Subscripts should not be confused with *superscripts,* which are written above a variable. Superscripts are used to denote powers.

$$b_2 \neq b^2$$

Skill Practice

27. Use the formula given in Example 10 to find the area of the trapezoid.



Answer

27. The area is 85 in.^{2}

Section R.2 Activity

A.1. **a.** What does the notation $|-6|$ mean?
b. Simplify $|-6|$.
c. Simplify $|-10|$.
d. Add. $-6 + (-10)$
e. In general, how do you add two numbers with the same sign?

A.2. **a.** Simplify $|-23|$.
b. Simplify $|4|$.
c. Add. $-23 + 4$
d. In general, how to you add two numbers with different signs?

A.3. **a.** What is the opposite of -8?
b. Write the statement $2 - (-8)$ as an equivalent addition statement.
c. Simplify $2 - (-8)$.
d. In general, a subtraction statement $a - b$ is equivalent to what addition statement?

For Exercises A.4–A.6, write an equivalent addition statement and simplify.

	Original Expression	Equivalent Addition Expression	Result
A.4.	$-4.2 - 12.6$		
A.5.	$\frac{7}{8} - \frac{11}{6}$		
A.6.	$-1\frac{3}{4} - \left(-5\frac{1}{2}\right)$		

A.7. Consider the sum. $(-3) + (-3) + (-3) + (-3)$
a. Is this sum positive or negative?
b. Write a multiplication statement that represents this sum.
c. In general, the product of two numbers with opposite signs is (choose one: positive/negative). By contrast, the product of two numbers of the same sign is (choose one: positive/negative)

A.8. **a.** What is the reciprocal of -2?
b. Write the statement $12 \div (-2)$ as an equivalent multiplication statement.
c. Simplify $12 \div (-2)$.
d. In general for $b \neq 0$, a division statement $a \div b$ is equivalent to what multiplication statement?
e. Because a division statement can be written in terms of multiplication, the same sign rules that apply to the product of two real numbers apply to the quotient of two real numbers.
- Thus, the quotient of two numbers of the same sign is (choose one: positive/negative).
- The quotient of two numbers of different signs is (choose one: positive/negative).

A.9. **a.** Fill in the blanks. $-15 \div (-3) = \square$ because $-3 \cdot \square = -15$
b. Fill in the blanks. $0 \div 4 = \square$ because $4 \cdot \square = 0$
c. Explain why $4 \div 0$ is undefined.

For Exercises A.10–A.17, perform the indicated operations.

A.10. **a.** $-20 + 4$ **b.** $-20 - 4$ **c.** $-20 \cdot (4)$ **d.** $-\frac{20}{4}$

A.11. **a.** $-\frac{5}{6}+\left(-\frac{10}{3}\right)$ **b.** $-\frac{5}{6}-\left(-\frac{10}{3}\right)$ **c.** $-\frac{5}{6}\cdot\left(-\frac{10}{3}\right)$ **d.** $-\frac{5}{6}\div\left(-\frac{10}{3}\right)$

A.12. **a.** $-4.2 \div 0$
b. $0 \div (-4.2)$

A.13. **a.** $-\frac{3}{4}\cdot\left(-\frac{4}{3}\right)$
b. $-\frac{3}{4}+\frac{3}{4}$

A.14. **a.** 3^2
b. 3^3
c. 3^4

A.15. **a.** $(-3)^2$
b. $(-3)^3$
c. $(-3)^4$

A.16. **a.** -5^2
b. $(-5)^2$
c. -5^3
d. $(-5)^3$

A.17. **a.** $\sqrt{16}$
b. $-\sqrt{16}$
c. $\sqrt{-16}$

For Exercises A.18–A.19, divide your paper into two columns vertically. Use the order of operations to simplify the expression in the left column. In the right column, write a description of each step.

A.18. $52 \div 26 - 3(1-6)^2$ Simplify the expression here...	Explain each step here...
A.19. $\dfrac{\lvert -15+3^2 \rvert}{\sqrt{10^2-8^2}}$ Simplify the expression here...	Explain each step here...

Practice Exercises

Section R.2

Study Skills Exercise

Reading comprehension in a mathematics course is essential. Reading comprehension can range from understanding mathematical operations and symbols to solving lengthy word problems. When reading mathematics, it is not unusual to find multiple symbolic representations for mathematical expressions. For example,

- Write four different ways to express 5×4 using different notations.

You will also find different ways to express a mathematical statement in words. For example,

- Write three different English statements that represent $8 - 5$.

Prerequisite Review

For Exercises R.1–R.8, simplify the expressions.

R.1. **a.** $\frac{4}{5} - \frac{2}{3}$ **b.** $\frac{4}{5} + \frac{2}{3}$

R.2. **a.** $\frac{9}{4} - \frac{7}{6}$ **b.** $\frac{9}{4} + \frac{7}{6}$

R.3. **a.** $\frac{5}{6} \cdot \frac{3}{10}$ **b.** $\frac{5}{6} \div \frac{3}{10}$

R.4. **a.** $\frac{5}{14} \cdot \frac{35}{2}$ **b.** $\frac{5}{14} \div \frac{35}{2}$

R.5. **a.** $2\frac{2}{3} + 1\frac{1}{6}$ **b.** $2\frac{2}{3} - 1\frac{1}{6}$

R.6. **a.** $4\frac{7}{8} + 3\frac{3}{4}$ **b.** $4\frac{7}{8} - 3\frac{3}{4}$

R.7. **a.** $\left(3\frac{3}{5}\right) \cdot \left(2\frac{1}{2}\right)$ **b.** $\left(3\frac{3}{5}\right) \div \left(2\frac{1}{2}\right)$

R.8. **a.** $\left(5\frac{1}{3}\right) \cdot \left(2\frac{1}{6}\right)$ **b.** $\left(5\frac{1}{3}\right) \div \left(2\frac{1}{6}\right)$

Vocabulary and Key Concepts

1. **a.** Two numbers that are the same distance from 0 but on opposite sides of 0 on the number line are called ____________.

b. The absolute value of a real number, a, is denoted by _____ and is the distance between a and _____ on the number line.

c. Given the expression b^n, the value b is called the ____________ and ____________ is called the exponent or power.

d. The symbol $\sqrt{}$ is called a ____________ sign and is used to find the principal ____________ root of a nonnegative real number.

e. If a is a nonzero real number, then the reciprocal of a is ____________. The product of a number and its reciprocal is ____________.

f. If either a or b is zero then $ab =$ ____________.

g. If $a = 0$ and $b \neq 0$, then $\frac{a}{b} =$ ____________ and $\frac{b}{a}$ is ____________.

2. If a and b are both negative, then $a + b$ will be (positive/negative) and ab will be (positive/negative).

3. If $a < 0$ and $b > 0$, and if $|a| < |b|$, then the sign of $a + b$ will be (positive/negative) and the sign of $\frac{a}{b}$ will be (positive/negative).

4. The expression $a - b = a +$ _____. If $a > 0$ and $b < 0$, then the sign of $a - b$ is ____________.

For Exercises 5–6, fill in the blank with $<$, $>$, or $=$.

5. If $a < 0$ and $b < 0$, then $ab \square 0$.

6. If $a < 0$ and $b < 0$, then $\frac{a}{b} \square 0$.

Concept 1: Opposite and Absolute Value

7. If the absolute value of a number can be thought of as its distance from zero, explain why an absolute value can never be negative.

8. If a number is negative, then its *opposite* will be

a. Positive. **b.** Negative.

9. If a number is negative, then its *reciprocal* will be

a. Positive. **b.** Negative.

10. If a number is negative, then its *absolute value* will be

a. Positive. **b.** Negative.

11. Complete the table. **(See Example 1.)**

Number	Opposite	Reciprocal	Absolute Value
6			
	$-\frac{1}{11}$		
		$-\frac{1}{8}$	
	$\frac{13}{10}$		
0			
		$-0.\overline{3}$	

12. Complete the table.

Number	Opposite	Reciprocal	Absolute Value
-9			
	$\frac{2}{3}$		
		14	
-1			
		Undefined	
		$2\frac{1}{9}$	

For Exercises 13–20, fill in the blank with the appropriate symbol ($<$, $>$, $=$). **(See Example 1.)**

13. $-|6|$ ______ $|-6|$

14. $-(-5)$ ______ $-|-5|$

15. $|-4|$ ______ $|4|$

16. $-|2|$ ______ (-2)

17. $-|-1|$ ______ 1

18. -3 ______ $-|-7|$

19. $|2 + (-5)|$ ______ $|2| + |-5|$

20. $|4 + 3|$ ______ $|4| + |3|$

Concept 2: Addition and Subtraction of Real Numbers

For Exercises 21–36, add or subtract as indicated. **(See Examples 2–3.)**

21. $-8 + 4$

22. $3 + (-7)$

23. $-12 + (-7)$

24. $-5 + (-11)$

25. $-17 - (-10)$

26. $-14 - (-2)$

27. $5 - (-9)$

28. $8 - (-4)$

29. $-6.3 - 15.8$

30. $-21.9 - 4.7$

31. $1.5 - 9.6$

32. $4.8 - 10$

33. $\frac{2}{3} + \left(-2\frac{1}{3}\right)$

34. $-\frac{4}{7} + \left(1\frac{4}{7}\right)$

35. $-\frac{5}{9} - \frac{14}{15}$

36. $-6 - \frac{2}{9}$

Concept 3: Multiplication and Division of Real Numbers

For Exercises 37–52, perform the indicated operation. **(See Examples 4–5.)**

37. $4(-8)$

38. $-21(3)$

39. $\frac{2}{9} \cdot \frac{12}{7}$

40. $\left(-\frac{5}{9}\right) \cdot \left(-1\frac{7}{11}\right)$

41. $\frac{-6}{-10}$

42. $\frac{-15}{-24}$

43. $-2\frac{1}{4} \div \frac{5}{8}$

44. $-\frac{2}{3} \div \left(-1\frac{5}{7}\right)$

45. $7 \div 0$

46. $\frac{1}{16} \div 0$

47. $0 \div (-3)$

48. $0 \div 11$

49. $(-1.2)(-3.1)$

50. $(4.6)(-2.25)$

51. $\frac{-5}{-11}$

52. $\frac{-3}{-13}$

Concept 4: Exponential Expressions

For Exercises 53–60, evaluate the expression. **(See Example 6.)**

53. 4^3

54. -2^3

55. -7^2

56. -3^4

57. $(-7)^2$

58. $(-5)^2$

59. $\left(\frac{5}{3}\right)^3$

60. $\left(\frac{10}{9}\right)^2$

Concept 5: Square Roots

For Exercises 61–68, evaluate the expression, if possible. **(See Example 7.)**

61. $\sqrt{9}$ **62.** $\sqrt{1}$ **63.** $\sqrt{-4}$ **64.** $\sqrt{-36}$

65. $\sqrt{\frac{1}{4}}$ **66.** $\sqrt{\frac{9}{4}}$ **67.** $-\sqrt{49}$ **68.** $-\sqrt{100}$

Concept 6: Order of Operations

For Exercises 69–96, simplify by using the order of operations. **(See Examples 8–9.)**

69. $5 + 3^3$ **70.** $10 - 2^4$ **71.** $5 \cdot 2^3$ **72.** $12 \div 2^2$

73. $(2 + 3)^2$ **74.** $(4 - 1)^3$ **75.** $2^2 + 3^2$ **76.** $4^3 - 1^3$

77. $6 + 10 \div 2 \cdot 3 - 4$ **78.** $12 \div 3 \cdot 4 - 18$ **79.** $4^2 - (5 - 2)^2 \cdot 3$ **80.** $5 - 3(8 \div 4)^2$

81. $2 - 5(9 - 4\sqrt{25})^2$ **82.** $5^2 - (\sqrt{9} + 4 \div 2)$ **83.** $\left(-\frac{3}{5}\right)^2 - \frac{3}{5} \cdot \frac{5}{9} + \frac{7}{10}$ **84.** $\frac{1}{2} - \left(\frac{2}{3} \div \frac{5}{9}\right) + \frac{5}{6}$

85. $1.75 \div 0.25 - (1.25)^2$ **86.** $5.4 - (0.3)^2 \div 0.09$ **87.** $\frac{\sqrt{10^2 - 8^2}}{3^2}$ **88.** $\frac{\sqrt{16 - 7} + 3^2}{\sqrt{16} - \sqrt{4}}$

89. $-|-11 + 5| + |7 - 2|$ **90.** $-|-8 - 3| - (-8 - 3)$

91. $25 - 2[(7 - 3)^2 \div 4] + \sqrt{18 - 2}$ **92.** $\sqrt{29 - 2^2} + [8 - 3(6 - 2)] \div 4 \cdot 5$

93. $\frac{|(10 - 7) - 2^3|}{6 - 16 \div 8 \cdot 3}$ **94.** $\frac{|-12 - (7 - 3^2)^2|}{40 - 6^2 - 8 \div 2}$

95. $\left(\frac{1}{2}\right)^2 + \left(\frac{6 - 4}{5}\right)^2 + \left(\frac{5 + 2}{10}\right)^2$ **96.** $\left(\frac{2^3}{2^3 + 1}\right)^2 \div \left[\frac{8 - (-2)}{3^2}\right]^2$

For Exercises 97–98, find the average of the set of data values by adding the values and dividing by the number of values.

97. Find the average low temperature for a week in January in St. John's, Newfoundland. Round to the nearest tenth of a degree.

Day	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.	Sun.
Low temperature	−18°C	−16°C	−20°C	−11°C	−4°C	−3°C	1°C

98. Find the average high temperature for a week in January in St. John's, Newfoundland. Round to the nearest tenth of a degree.

Day	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.	Sun.
High temperature	−2°C	−6°C	−7°C	0°C	1°C	8°C	10°C

Concept 7: Evaluating Formulas

99. The formula $C = \frac{5}{9}(F - 32)$ converts temperatures in the Fahrenheit scale to the Celsius scale. Find the equivalent Celsius temperature for each Fahrenheit temperature.

a. 77°F **b.** 212°F **c.** 32°F **d.** −40°F

100. The formula $F = \frac{9}{5}C + 32$ converts Celsius temperatures to Fahrenheit temperatures. Find the equivalent Fahrenheit temperature for each Celsius temperature.

a. −5°C **b.** 0°C **c.** 37°C **d.** −40°C

The equation $G_E = \frac{1}{22}c + \frac{1}{30}h$ represents the amount of gasoline used (in gal) for an economy car to drive c miles in the city and h miles on the highway. The equation $G_T = \frac{1}{12}c + \frac{1}{18}h$ represents the amount of gasoline used for a truck to make the same trip. Use these formulas for Exercises 101–102.

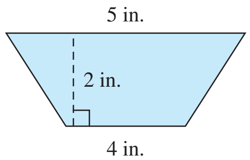
101. Determine the amount of gas used by an economy car that travels 33 mi in the city and 80 mi on the highway.

102. Determine the amount of gas used by a truck that travels 33 mi in the city and 80 mi on the highway.

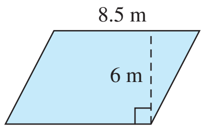
Use an appropriate formula from geometry to answer Exercises 103–112.

For Exercises 103–106, find the area. **(See Example 10.)**

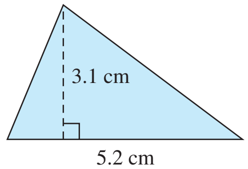
103. Trapezoid



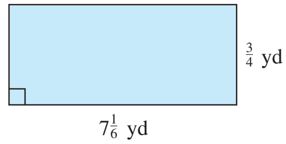
104. Parallelogram



105. Triangle

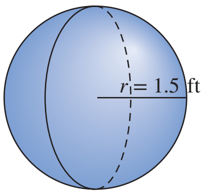


106. Rectangle

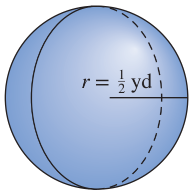


For Exercises 107–112, find the volume. (Use the π key on your calculator, and round the final answer to one decimal place.)

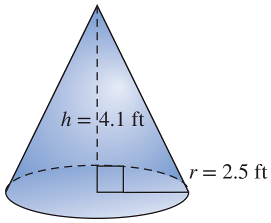
107. Sphere



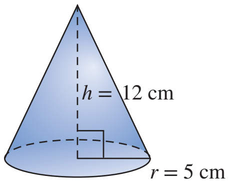
108. Sphere



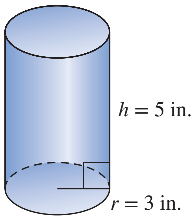
109. Right circular cone



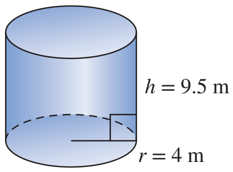
110. Right circular cone



111. Right circular cylinder



112. Right circular cylinder



Technology Connections

113. Which expression when entered into a graphing calculator will yield the correct value of $\dfrac{12}{6-2}$?

12/6 – 2 or 12/(6 – 2)

114. Which expression when entered into a graphing calculator will yield the correct value of $\dfrac{24-6}{3}$?

(24 – 6)/3 or 24 – 6/3

115. Verify your solution to Exercise 87 by entering the expression into a graphing calculator:

$(\sqrt{\ }(10^2 - 8^2))/3^2$

116. Verify your solution to Exercise 88 by entering the expression into a graphing calculator:

$(\sqrt{\ }(16 - 7) + 3^2)/(\sqrt{\ }(16) - \sqrt{\ }(4))$

Section R.3 Simplifying Algebraic Expressions

Concepts

1. Recognizing Terms, Factors, and Coefficients

A **term** is a constant or the product of a constant and one or more variables. An algebraic expression is a single term or a sum of two or more terms. For example, the expression

$$-6x^2 + 5xyz - 11 \quad \text{or} \quad -6x^2 + 5xyz + (-11)$$

consists of the terms $-6x^2$, $5xyz$, and -11.

The terms $-6x^2$ and $5xyz$ are **variable terms**, and the term -11 is called a **constant term**. It is important to distinguish between a term and the **factors** within a term. For example, the quantity $5xyz$ is one term, but the values 5, x, y, and z are factors within the term. The constant factor in a term is called the numerical coefficient or simply **coefficient** of the term. In the terms $-6x^2$, $5xyz$, and -11, the coefficients are -6, 5, and -11, respectively. A term containing only variables such as xy has a coefficient of 1.

Terms are called ***like* terms** if they each have the same variables and the corresponding variables are raised to the same powers. For example:

Like Terms			*Unlike* Terms			
$-6t$	and	$4t$	$-6t$	and	$4s$	(different variables)
$1.8ab$	and	$-3ab$	$1.8xy$	and	$-3x$	(different variables)
$\frac{1}{2}c^2d^3$	and	c^2d^3	$\frac{1}{2}c^2d^3$	and	c^2d	(different powers)
4	and	6	$4p$	and	4	(different variables)

Example 1 Identifying Terms, Factors, Coefficients, and *Like* Terms

a. List the terms of the expression. $-4x^2 - 7x + \frac{2}{3}$

b. Identify the coefficient of the term. yz^3

c. Identify the pair of *like* terms. $16b, 4b^2$ or $\frac{1}{2}c, -\frac{1}{6}c$

Solution:

a. The terms of the expression $-4x^2 - 7x + \frac{2}{3}$ are $-4x^2$, $-7x$, and $\frac{2}{3}$.

b. The term yz^3 can be written as $1yz^3$; therefore, the coefficient is 1.

c. $\frac{1}{2}c, -\frac{1}{6}c$ are *like* terms because they have the same variable raised to the same power.

Skill Practice Given: $-2x^2 + 5x + \dfrac{1}{2} - y^2$

1. List the terms of the expression.
2. Which term is the constant term?
3. Identify the coefficient of the term $-y^2$.

Answers

1. $-2x^2$, $5x$, $\dfrac{1}{2}$, $-y^2$

2. $\dfrac{1}{2}$ 3. -1

2. Properties of Real Numbers

Simplifying algebraic expressions requires several important properties of real numbers that are stated in Table R-3. Assume that a, b, and c represent real numbers or real-valued algebraic expressions.

Table R-3

Property Name	Algebraic Representation	Example	Description/Notes
Commutative property of addition	$a + b = b + a$	$5 + 3 = 3 + 5$	The order in which two real numbers are added or multiplied does not affect the result.
Commutative property of multiplication	$a \cdot b = b \cdot a$	$(5)(3) = (3)(5)$	
Associative property of addition	$(a + b) + c = a + (b + c)$	$(2 + 3) + 7 = 2 + (3 + 7)$	The manner in which real numbers are grouped under addition or multiplication does not affect the result.
Associative property of multiplication	$(a \cdot b)c = a(b \cdot c)$	$(2 \cdot 3)7 = 2(3 \cdot 7)$	
Distributive property of multiplication over addition	$a(b + c) = ab + ac$	$3(5 + 2) = 3 \cdot 5 + 3 \cdot 2$	A factor outside the parentheses is multiplied by each term inside the parentheses.
Identity property of addition	0 is the identity element for addition because $a + 0 = 0 + a = a$	$5 + 0 = 0 + 5 = 5$	Any number added to the identity element 0 will remain unchanged.
Identity property of multiplication	1 is the identity element for multiplication because $a \cdot 1 = 1 \cdot a = a$	$5 \cdot 1 = 1 \cdot 5 = 5$	Any number multiplied by the identity element 1 will remain unchanged.
Inverse property of addition	a and $(-a)$ are additive inverses because $a + (-a) = 0$ and $(-a) + a = 0$	$3 + (-3) = 0$	The sum of a number and its additive inverse (opposite) is the identity element 0.
Inverse property of multiplication	a and $\frac{1}{a}$ are multiplicative inverses because $a \cdot \dfrac{1}{a} = 1$ and $\dfrac{1}{a} \cdot a = 1$ (provided $a \neq 0$)	$5 \cdot \frac{1}{5} = 1$	The product of a number and its multiplicative inverse (reciprocal) is the identity element 1.

The properties of real numbers are used to multiply algebraic expressions. To multiply a term by an algebraic expression containing more than one term, we apply the distributive property of multiplication over addition.

Example 2 Applying the Distributive Property

Apply the distributive property.

a. $4(2x + 5)$ **b.** $-(-3.4q + 5.7r)$

c. $-3(a + 2b - 5c)$ **d.** $-\frac{2}{3}\left(-9x + \frac{3}{8}y - 5\right)$

Solution:

a. $4(2x + 5)$

$= 4(2x) + 4(5)$ Apply the distributive property.

$= 8x + 20$ Simplify, using the associative property of multiplication.

b. $-(-3.4q + 5.7r)$ The negative sign preceding the parentheses can be interpreted as a factor of -1.

$= -1(-3.4q + 5.7r)$

$= -1(-3.4q) + (-1)(5.7r)$ Apply the distributive property.

$= 3.4q - 5.7r$

TIP: When applying the distributive property, a negative factor preceding the parentheses will change the signs of the terms within the parentheses.

$$-3(a + 2b - 5c)$$
$$-3a - 6b + 15c$$

c. $-3(a + 2b - 5c)$

$= -3(a) + (-3)(2b) + (-3)(-5c)$ Apply the distributive property.

$= -3a - 6b + 15c$ Simplify.

d. $-\frac{2}{3}\left(-9x + \frac{3}{8}y - 5\right)$

$= -\frac{2}{3}(-9x) + \left(-\frac{2}{3}\right)\left(\frac{3}{8}y\right) + \left(-\frac{2}{3}\right)(-5)$ Apply the distributive property.

$= \frac{18}{3}x - \frac{6}{24}y + \frac{10}{3}$ Simplify.

$= 6x - \frac{1}{4}y + \frac{10}{3}$ Simplify to lowest terms.

Skill Practice Apply the distributive property.

4. $10(30y - 40)$ **5.** $-(7t - 1.6s + 9.2)$

6. $-2(4x - 3y - 6)$ **7.** $-\frac{1}{2}(-4a + 7)$

Answers

4. $300y - 400$
5. $-7t + 1.6s - 9.2$
6. $-8x + 6y + 12$
7. $2a - \frac{7}{2}$

Notice that the parentheses are removed after the distributive property is applied. Sometimes this is referred to as clearing parentheses.

Two terms can be added or subtracted only if they are *like* terms. To add or subtract *like* terms, we use the distributive property, as shown in Example 3.

Example 3 Using the Distributive Property to Add and Subtract *Like* Terms

Add and subtract as indicated.

a. $-8x + 3x$ **b.** $4.75y^2 - 9.25y^2 + y^2$

Solution:

a. $-8x + 3x$

$= (-8 + 3)x$	Apply the distributive property.
$= (-5)x$	Simplify.
$= -5x$	

b. $4.75y^2 - 9.25y^2 + y^2$

$= 4.75y^2 - 9.25y^2 + 1y^2$	Notice that y^2 is interpreted as $1y^2$.
$= (4.75 - 9.25 + 1)y^2$	Apply the distributive property.
$= (-3.5)y^2$	Simplify.
$= -3.5y^2$	

Skill Practice Combine *like* terms.

8. $-4y + 7y$ **9.** $a^2 - 6.2a^2 + 2.8a^2$

Although the distributive property is used to add and subtract *like* terms, it is tedious to write each step. Observe that adding or subtracting *like* terms is a matter of combining the coefficients and leaving the variable factors unchanged. This can be shown in one step. This shortcut will be used throughout the text. For example:

$$4w + 7w = 11w \qquad 8ab^2 + 10ab^2 - 5ab^2 = 13ab^2$$

3. Simplifying Expressions

Clearing parentheses and combining *like* terms are important tools for simplifying algebraic expressions. This is demonstrated in Example 4.

Example 4 Clearing Parentheses and Combining *Like* Terms

Simplify by clearing parentheses and combining *like* terms.

a. $4 - 3(2x - 8) - 1$ **b.** $-(3s - 11t) - 5(2t + 8s) - 10s$

Solution:

a. $4 - 3(2x - 8) - 1$

$= 4 - 6x + 24 - 1$	Apply the distributive property.
$= -6x + 4 + 24 - 1$	Group *like* terms.
$= -6x + 27$	Combine *like* terms.

Answers

8. $3y$ **9.** $-2.4a^2$

b. $-(3s - 11t) - 5(2t + 8s) - 10s$

$= -3s + 11t - 10t - 40s - 10s$ Apply the distributive property.

$= -3s - 40s - 10s + 11t - 10t$ Group *like* terms.

$= -53s + t$ Combine *like* terms.

Skill Practice Simplify by clearing parentheses and combining *like* terms.

10. $7 - 2(3x - 4) - 5$ **11.** $-(6z - 10y) - 4(3z + y) - 8y$

Example 5 Clearing Parentheses and Combining *Like* Terms

Simplify by clearing parentheses and combining *like* terms.

a. $2[1.5x + 4.7(x^2 - 5.2x) - 3x]$ **b.** $-\frac{1}{3}(3w - 6) - \left(\frac{1}{4}w + 4\right)$

TIP: By using the commutative property of addition, the expression $-51.88x + 9.4x^2$ can also be written as $9.4x^2 + (-51.88x)$ or simply $9.4x^2 - 51.88x$. Although the expressions are all equal, it is customary to write the terms in descending order of the powers of the variable.

Solution:

a. $2[1.5x + 4.7(x^2 - 5.2x) - 3x]$

$= 2[1.5x + 4.7x^2 - 24.44x - 3x]$ Apply the distributive property to the inner parentheses.

$= 2[1.5x - 24.44x - 3x + 4.7x^2]$ Group *like* terms.

$= 2[-25.94x + 4.7x^2]$ Combine *like* terms.

$= -51.88x + 9.4x^2$ Apply the distributive property.

$= 9.4x^2 - 51.88x$

b. $-\frac{1}{3}(3w - 6) - \left(\frac{1}{4}w + 4\right)$

$= -\frac{3}{3}w + \frac{6}{3} - \frac{1}{4}w - 4$ Apply the distributive property.

$= -w + 2 - \frac{1}{4}w - 4$ Simplify fractions.

$= -\frac{4}{4}w - \frac{1}{4}w + 2 - 4$ Group *like* terms and find a common denominator.

$= -\frac{5}{4}w - 2$ Combine *like* terms.

Skill Practice Simplify by clearing parentheses and combining *like* terms.

12. $4[1.4a + 2.2(a^2 - 6a)] - 5.1a^2$ **13.** $-\frac{1}{2}(4p - 1) - \frac{5}{2}(p - 2)$

Answers

10. $-6x + 10$ **11.** $-2y - 18z$

12. $3.7a^2 - 47.2a$ **13.** $-\frac{9}{2}p + \frac{11}{2}$

Section R.3 Activity

A.1. **a.** Simplify $-3 + 9$. **b.** Simplify $9 + (-3)$.
c. Which property of real numbers do these expressions illustrate?

A.2. **a.** Simplify $-10 \cdot (-3)$. **b.** Simplify $-3 \cdot (-10)$.
c. Which property of real numbers do these expressions illustrate?

A.3. Describe the commutative properties of addition and multiplication.

A.4. **a.** Simplify $-2 + (3 + 8)$. **b.** Simplify $(-2 + 3) + 8$.
c. Which property of real numbers do these expressions illustrate?

A.5. **a.** Simplify $4 \cdot (-3 \cdot 5)$. **b.** Simplify $[4 \cdot (-3)] \cdot 5$.
c. Which property of real numbers do these expressions illustrate?

A.6. Describe the associative properties of addition and multiplication.

A.7. **a.** What happens when you add 0 to any real number?
b. What is the identity element of addition?

A.8. **a.** What happens when you multiply any real number by 1?
b. What is the identity element of multiplication?

A.9. **a.** Give the opposite and reciprocal of -5.
b. What happens when you add a number and its opposite, such as $-5 + 5$?
c. What happens when you multiply a number by its reciprocal, such as $-5 \cdot \left(-\frac{1}{5}\right)$?
d. Another name for the opposite of a real number is __________ inverse.
e. Another name for the reciprocal of a real number is __________ inverse.

A.10. **a.** Simplify $-3(4 + 6)$. **b.** Simplify $-3(4) + (-3)(6)$.
c. Which property of real numbers do these expressions illustrate?

A.11. Consider the expression $4a^2 - ab + 6b^2 - 7$.
a. List the terms of the expression.
b. List the coefficients of each term.
c. Identify the constant term.

For Exercises A.12–A.13, simplify the expression by combining *like* terms.

A.12. $-8x - 6x + x$

A.13. $\frac{4}{3}a^2 + \frac{1}{10}a - \frac{5}{6}a^2 + \frac{1}{5}a$

For Exercises A.14–A.15, simplify the expression by clearing parentheses (applying the distributive property).

A.14. $-(8b + 5c - 3d)$

A.15. $-\frac{1}{2}(4x^2 + 6x - 10)$

For Exercises A.16–A.17, simplify the expression by clearing parentheses and combining *like* terms.

A.16. $12m - 3(m + 2n) + 11n$

A.17. $5[2 + 3(c^2 - 5) - (4c^2 - 6c)]$

Section R.3 Practice Exercises

Prerequisite Review

For Exercises R.1–R.16, simplify the expressions.

R.1. **a.** $-4 + 6$ **b.** $6 + (-4)$

R.2. **a.** $-9 + (-2)$ **b.** $-2 + (-9)$

R.3. **a.** $-5 \cdot (-8)$ **b.** $-8 \cdot (-5)$

R.4. **a.** $-4 \cdot 3$ **b.** $3 \cdot (-4)$

R.5. **a.** $3 - 9$ **b.** $9 - 3$

R.6. **a.** $4 - 12$ **b.** $12 - 4$

R.7. **a.** $-18 \div 9$ **b.** $9 \div (-18)$

R.8. **a.** $-18 \div (-6)$ **b.** $-6 \div (-18)$

R.9. **a.** $5(1 + 3)$ **b.** $5 \cdot 1 + 5 \cdot 3$

R.10. **a.** $3(9 + 1)$ **b.** $3 \cdot 9 + 3 \cdot 1$

R.11. **a.** $-8 + 8$ **b.** $-8 \cdot \left(-\frac{1}{8}\right)$

R.12. **a.** $12 + (-12)$ **b.** $12 \cdot \left(\frac{1}{12}\right)$

R.13. **a.** $-13 + (6 + 10)$ **b.** $(-13 + 6) + 10$

R.14. **a.** $[9 + (-7)] + 12$ **b.** $9 + (-7 + 12)$

R.15. **a.** $(-4 \cdot 2) \cdot 6$ **b.** $-4 \cdot (2 \cdot 6)$

R.16. **a.** $10 \cdot [-3 \cdot 5]$ **b.** $[10 \cdot (-3)] \cdot 5$

Vocabulary and Key Concepts

1. **a.** Given the expression $8x + cd - 3y + 90$, the terms $8x$, cd, and $-3y$ are variable terms, whereas 90 is a ___________ term.

b. The constant factor in a term is called the ___________.

c. Given the expression x, the value of the coefficient is _____, and the exponent is ______.

d. Terms that have the same variables, with corresponding variables raised to the same powers, are called ___________ terms.

2. Write an expression with three terms, coefficients –3, 2, and 4 and using the variables a, b, and c.

3. **a.** Apply the commutative property of addition: $x + -7 =$ __________

b. Apply the commutative property of multiplication: $y \cdot (-4) =$ __________

4. **a.** Apply the associative property of addition: $-3 + (9 + x) =$ __________

b. Apply the associative property of multiplication: $3(5t) =$ __________

5. **a.** Fill in the blank: $5 \cdot \square = 5$

b. Fill in the blank: $5 \cdot \square = 1$

6. **a.** Fill in the blank: $5 \cdot \square = 0$

b. Fill in the blank: $5 \cdot \square = -5$

7. **a.** Fill in the blank: $-3 + \square = 0$

b. Fill in the blank: $\frac{1}{6} + \square = 0$

8. Fill in the blank: $3(4 + 8) = 3 \cdot \square + 3 \cdot \square$

Concept 1: Recognizing Terms, Factors, and Coefficients

For Exercises 9–12:

a. Determine the number of terms in the expression.

b. Identify the constant term.

c. List the coefficients of each term, separated by commas. **(See Example 1.)**

9. $2x^3 - 5xy + 6$

10. $a^2 - 4ab - b^2 + 8$

11. $pq - 7 + q^2 - 4q + p$

12. $7x - 1 + 3xy$

Concept 2: Properties of Real Numbers

For Exercises 13–30, match each expression with the appropriate property.

13. $3 + \frac{1}{2} = \frac{1}{2} + 3$

14. $7.2(4 + 1) = 7.2(4) + 7.2(1)$

15. $10 + 0 = 10$

16. $7 \cdot 1 = 7$

17. $(6 + 8) + 2 = 6 + (8 + 2)$

18. $(4 + 19) + 7 = (19 + 4) + 7$

19. $6 \cdot \frac{1}{6} = 1$

20. $2 + (-2) = 0$

21. $9(4 \cdot 12) = (9 \cdot 4)12$

22. $\left(\frac{1}{4} + 2\right)20 = 5 + 40$

23. $42 \cdot 1 = 42$

24. $4 \cdot \frac{1}{4} = 1$

25. $(13 \cdot 41)6 = (41 \cdot 13)6$

26. $6(x + 3) = 6x + 18$

27. $8 + (-8) = 0$

28. $21 + 0 = 21$

29. $3(y + 10) = 3(10 + y)$

30. $5(3 \cdot 7) = (5 \cdot 3)7$

a. Commutative property of addition

b. Associative property of multiplication

c. Distributive property of multiplication over addition

d. Commutative property of multiplication

e. Associative property of addition

f. Identity property of addition

g. Identity property of multiplication

h. Inverse property of addition

i. Inverse property of multiplication

For Exercises 31–42, clear parentheses by applying the distributive property. **(See Example 2.)**

31. $2(x - 3y + 8)$

32. $5(-2a + 4b - 9c)$

33. $-10(4s - 9t - 3)$

34. $-4(-8x + 6y + 3z)$

35. $-(-7w + 5z)$

36. $-(-22a - 17b)$

37. $-\frac{1}{5}\left(-\frac{5}{2}a + 10b - 8\right)$

38. $-\frac{3}{4}\left(6x - 4y + \frac{4}{9}\right)$

39. $3(2.6x - 4.1)$

40. $5(-7.2y + 2.3)$

41. $2(7c - 8) - 5(6d - f)$

42. $-2(-3q + r) - 7(5s + 2t)$

Concept 3: Simplifying Expressions

For Exercises 43–80, clear parentheses and combine *like* terms. **(See Examples 3–5.)**

43. $8y - 2x + y + 5y$

44. $-9a + a - b + 5a$

45. $4p^2 - 2p + 3p - 6 + 2p^2$

46. $6q - 9 + 3q^2 - q^2 + 10$

47. $2p - 7p^2 - 5p + 6p^2$

48. $5a^2 - 2a - 7a^2 + 6a + 4$

49. $m - 4n^3 + 3 + 5n^3 - 9$

50. $x + 2y^3 - 2x - 8y^3$

51. $5ab + 2ab + 8a$

52. $-6m^2n - 3mn^2 - 2m^2n$

53. $14xy^2 - 5y^2 + 2xy^2$

54. $9uv + 3u^2 + 5uv + 4u^2$

55. $8(x - 3) + 1$

56. $-4(b + 2) - 3$

57. $-2(c + 3) - 2c$

58. $4(z - 4) - 3z$

59. $-(10w - 1) + 9 + w$

60. $-(2y + 7) - 4 + 3y$

61. $-9 - 4(2 - z) + 1$

62. $3 + 3(4 - w) - 11$

63. $4(2s - 7) - (s - 2)$

64. $2(t-3)-(t-7)$

65. $-3(-5+2w)-8w+2(w-1)$

66. $5-(-4t-7)-t-9$

67. $8x-4(x-2)-2(2x+1)-6$

68. $6(y-2)-3(2y-5)-3$

69. $\frac{1}{2}(4-2c)+5c$

70. $\frac{2}{3}(3d+6)-4d$

71. $3.1(2x+2)-4(1.2x-1)$

72. $4.5(5-y)+3(1.9y+1)$

73. $2\left[5\left(\frac{1}{2}a+3\right)-(a^2+a)+4\right]$

74. $-3\left[3\left(b-\frac{2}{3}\right)-2(b+4)-6b^2\right]$

75. $(2y-5)-2(y-y^2)-3y$

76. $-(x+6)+3(x^2+1)+2x$

77. $2.2\{4-8[6x-1.5(x+4)-6]+7.5x\}$

78. $-3.2-\{6.1y-4[9-(2y+2.5)]+7y\}$

79. $\frac{1}{8}(24n-16m)-\frac{2}{3}(3m-18n-2)+\frac{2}{3}$

80. $\frac{1}{5}(25a-20b)-\frac{4}{7}(21a-14b+2)+\frac{1}{7}$

Expanding Your Skills

81. What is the identity element for addition? Use it in an example.

82. What is the identity element for multiplication? Use it in an example.

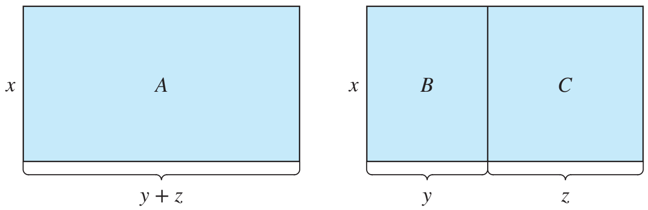
83. What is another name for a multiplicative inverse?

84. What is another name for an additive inverse?

85. Is the operation of subtraction commutative? If not, give an example.

86. Is the operation of division commutative? If not, give an example.

87. Given the rectangular regions:



a. Write an expression for the area of region A. (Do not simplify.)

b. Write an expression for the area of region B.

c. Write an expression for the area of region C.

d. Add the expressions for the areas of regions B and C.

e. Show that the area of region A is equal to the sum of the areas of regions B and C. What property of real numbers does this illustrate?

Chapter R Summary

Section R.1 Sets of Numbers and Interval Notation

Key Concepts

Natural numbers: $\{1, 2, 3, \ldots\}$
Whole numbers: $\{0, 1, 2, 3, \ldots\}$
Integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers: $\left\{\frac{p}{q} \,\middle|\, p \text{ and } q \text{ are integers and } q \neq 0\right\}$

Rational numbers include all terminating and repeating decimals.

Irrational numbers: A subset of the real numbers whose elements cannot be written as a ratio of two integers.

Irrational numbers cannot be written as repeating or terminating decimals.

Real numbers: $\{x \mid x \text{ is rational or } x \text{ is irrational}\}$

$a < b$	"a is less than b"
$a > b$	"a is greater than b"
$a \le b$	"a is less than or equal to b"
$a \ge b$	"a is greater than or equal to b"
$a < x < b$	"x is between a and b"

Examples

Example 1

Some rational numbers are:
$\frac{1}{7}, 0.5, 0.\overline{3}$

Some irrational numbers are:
$\sqrt{7}, \sqrt{2}, \pi$

Example 2

Set-Builder Notation	Interval Notation	Graph
$\{x \mid x > 3\}$	$(3, \infty)$	
$\{x \mid x \ge 3\}$	$[3, \infty)$	
$\{x \mid x < 3\}$	$(-\infty, 3)$	
$\{x \mid x \le 3\}$	$(-\infty, 3]$	

Section R.2 Operations on Real Numbers

Key Concepts

The **reciprocal** of a number $a \neq 0$ is $\frac{1}{a}$.
The **opposite** of a number a is $-a$.
The **absolute value** of a, denoted $|a|$, is its distance from zero on the number line.

$|a| = a \quad \text{if } a \ge 0$
$|a| = -a \quad \text{if } a < 0$

Addition of Real Numbers

Same Signs: Add the absolute values of the numbers, and apply the common sign to the sum.

Unlike Signs: Subtract the smaller absolute value from the larger absolute value. Then apply the sign of the number having the larger absolute value.

Examples

Example 1

The reciprocal of -5 is $-\frac{1}{5}$.

The opposite of -5 is 5.

The absolute value of -5 is 5.

Example 2

$-3 + (-4) = -7$
$-5 + 7 = 2$

Subtraction of Real Numbers

Add the opposite of the second number to the first number.
$a - b = a + (-b)$

Example 3

$7 - (-5) = 7 + (5) = 12$

Multiplication and Division of Real Numbers

Same Signs: Product or quotient is positive.
Opposite Signs: Product or quotient is negative.

The product of any real number and 0 is 0.
The quotient of 0 and a nonzero number is 0.
The quotient of a nonzero number and 0 is undefined.

Example 4

$(-3)(-4) = 12$ $\qquad \dfrac{-15}{-3} = 5$

$(-2)(5) = -10$ $\qquad \dfrac{6}{-12} = -\dfrac{1}{2}$

$(-7)(0) = 0$ $\qquad 0 \div 9 = 0$

$-3 \div 0$ is undefined

Exponents and Radicals

$b^4 = b \cdot b \cdot b \cdot b$ (b is the **base**, 4 is the **exponent**)
$\sqrt{b}$ is the **principal square root** of b ($\sqrt{\ }$ is the radical sign).

Example 5

$6^3 = 6 \cdot 6 \cdot 6 = 216$

$\sqrt{100} = 10$

Order of Operations

1. Simplify expressions within parentheses and other grouping symbols first.
2. Evaluate expressions involving exponents, radicals, and absolute values.
3. Perform multiplication or division in order from left to right.
4. Perform addition or subtraction in order from left to right.

Example 6

$$\begin{aligned}
&10 - 5(3 - 1)^2 + \sqrt{16} \\
&= 10 - 5(2)^2 + \sqrt{16} \\
&= 10 - 5(4) + 4 \\
&= 10 - 20 + 4 \\
&= -10 + 4 \\
&= -6
\end{aligned}$$

Section R.3 Simplifying Algebraic Expressions

Key Concepts

A **term** is a constant or the product or quotient of a constant and one or more variables.

- A **variable term** contains at least one variable.
- A **constant term** has no variable.

The **coefficient** of a term is the numerical factor of the term.

***Like* terms** have the same variables, and the corresponding variables are raised to the same powers.

Examples

Example 1

$-2x$ Variable term has coefficient -2.

x^2y Variable term has coefficient 1.

6 Constant term has coefficient 6.

Example 2

$4ab^3$ and $2ab^3$ are *like* terms.

Distributive Property of Multiplication over Addition

$a(b + c) = ab + ac$

Example 3

$2(x + 4y) = 2x + 8y$

$-(a + 6b - 5c) = -a - 6b + 5c$

Two terms can be added or subtracted if they are *like* terms. First clear parentheses before adding or subtracting *like* terms.

Example 4

$-4d + 12d + d$

$= 9d$

Example 5

$-2[w - 4(w - 2)] + 3$

$= -2[w - 4w + 8] + 3$

$= -2[-3w + 8] + 3$

$= 6w - 16 + 3$

$= 6w - 13$

Chapter R Review Exercises

Section R.1

1. Find a number that is a whole number but not a natural number.

For Exercises 2–3, answers may vary.

2. List three rational numbers that are not integers.

3. List five integers, two of which are not whole numbers.

For Exercises 4–9, write an expression in words that describes the set of numbers given by each interval. (Answers may vary.)

4. $(7, 16)$ **5.** $(0, 2.6]$

6. $[-6, -3]$ **7.** $(8, \infty)$

8. $(-\infty, 13]$ **9.** $(-\infty, \infty)$

For Exercises 10–12, graph each set and write the set in interval notation.

10. $\{x \mid x < 2\}$

11. $\{x \mid 0 \leq x\}$

12. $\{x \mid -1 < x < 5\}$

13. True or false? $x < 3$ is equivalent to $3 > x$

Section R.2

For Exercises 14–15, find the opposite, reciprocal, and absolute value.

14. -8 **15.** $\frac{4}{9}$

For Exercises 16–17, simplify the exponents and the radicals.

16. $4^2, \sqrt{4}$ **17.** $25^2, \sqrt{25}$

For Exercises 18–33, perform the indicated operations.

18. $6 + (-8)$ **19.** $(-2) - (-5)$

20. $8(-2.7)$ **21.** $(-1.1)(7.41)$

22. $\frac{5}{8} \div \left(-\frac{13}{40}\right)$ **23.** $\left(-\frac{1}{4}\right) \div \left(-\frac{11}{16}\right)$

24. $2\frac{2}{5} - \left(1\frac{1}{10}\right)^2$ **25.** $4\frac{2}{3} - 3\left(1\frac{1}{6}\right)$

26. $\frac{2 - 4(3 - 7)}{-4 - 5(1 - 3)}$ **27.** $\frac{12(2) - 8}{4(-3) + 2(5)}$

28. $24 \div 8 \cdot 2$ **29.** $40 \div 5 \cdot 6$

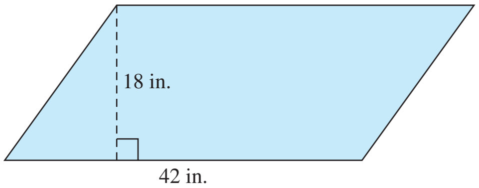
30. $3^2 + 2(|-10 + 5| \div 5)$

31. $-91 + \sqrt{4}\left(\sqrt{25} - 13\right)^2$

32. $\frac{3(3 - 8)^2}{|8 - 3^2|}$ **33.** $\frac{4(5 - 2)^2}{|3 - 7 - 5|}$

34. Given $h = \frac{1}{2}gt^2 + v_0t + h_0$, find h if $g = -32$, $v_0 = 64$, $h_0 = 256$, and $t = 4$.

35. Find the area of a parallelogram with base 42 in. and height 18 in.



Section R.3

For Exercises 36–39, apply the distributive property and simplify.

36. $3(x + 5y)$

37. $\frac{1}{2}(x + 8y - 5)$

38. $-(-4x + 10y - z)$

39. $-(13a - b - 5c)$

For Exercises 40–43, clear parentheses if necessary, and combine *like* terms.

40. $5 - 6q + 13q - 19$

41. $18p + 3 - 17p + 8p$

42. $7 - 3(y + 4) - 3y$

43. $\frac{3}{4}(8x - 4) + \frac{1}{2}(6x + 4)$

For Exercises 44–45, answers may vary.

44. Write an example of the commutative property of addition.

45. Write an example of the associative property of multiplication.

Chapter R Test

1. a. List the integers between -5 and 2, inclusive.

b. List three rational numbers between 1 and 2. (Answers may vary.)

2. Write the opposite, reciprocal, and absolute value for each number.

a. $-\frac{1}{2}$ **b.** 4 **c.** 0

3. Explain the difference between the interval $[4, \infty)$ and $(4, \infty)$.

4. Answer true or false: The set $\{x \mid x \geq 5\}$ is the same as $\{x \mid 5 \leq x\}$

For Exercises 5–6, graph the inequality and express the set in interval notation.

5. $\{y \mid y < -\frac{4}{3}\}$

6. $\{p \mid 12 \leq p\}$

For Exercises 7–8, write each English phrase as an algebraic statement.

7. x is no more than 5.

8. p is at least 7.

For Exercises 9–12, simplify the expression.

9. $|-8| - 4(2 - 3)^2 \div \sqrt{4}$

10. $\dfrac{-6^2 - 10^2}{-1 + 3^2}$

11. $\left(-\frac{1}{6} + \sqrt{\frac{4}{9}}\right)^2$

12. $-8 \div 3 \cdot 2$

13. Given $z = \dfrac{x - \mu}{\sigma/\sqrt{n}}$, find z when $n = 16$, $x = 18$, $\sigma = 1.8$, and $\mu = 17.5$. (Round the answer to 1 decimal place.)

For Exercises 14–16, simplify the expressions.

14. $5b + 2 - 7b + 6 - 14$

15. $-3(4 - x) + 9(x - 1) - 5(2x - 4)$

16. $\frac{1}{2}(2x - 1) - \left(3x - \frac{3}{2}\right)$

For Exercises 17–20, answer true or false.

17. $(x + y) + 2 = 2 + (x + y)$ is an example of the associative property of addition.

18. $(2 \cdot 3) \cdot 5 = (3 \cdot 2) \cdot 5$ is an example of the commutative property of multiplication.

19. $\left(\frac{1}{2}x + \frac{1}{3}\right)6 = 3x + 2$ is an example of the distributive property of multiplication over addition.

20. $(10 + y) + z = 10 + (y + z)$ is an example of the associative property of addition.

Linear Equations and Inequalities in One Variable

1

CHAPTER OUTLINE

Mathematics in Construction

The construction of the Golden Gate Bridge in San Francisco from 1932 to 1937 is perhaps one of the greatest architectural achievements of the twentieth century. The total length of the bridge including the approaches spans 1.7 miles and has a total weight of 887,000 tons. The bridge has two main cables that pass over the tops of two large towers. Together these cables contain 80,000 miles of steel wire—enough to circle the earth three times.

During the construction of the bridge, workers had to contend with foggy conditions, strong ocean currents, and powerful winds sweeping in from the Pacific Ocean. Eleven men lost their lives during construction.

The design and implementation of an engineering project of this magnitude requires solving a variety of different types of equations. In this chapter, we present **linear equations in one variable** and the manner in which to solve them.

ventdusud/Getty Images

Section 1.1 Linear Equations in One Variable

Concepts

1. Linear Equations in One Variable
2. Solving Linear Equations
3. Clearing Fractions and Decimals
4. Conditional Equations, Contradictions, and Identities

1. Linear Equations in One Variable

An **equation** is a statement that indicates that two quantities are equal. The following are equations.

$$x = -4 \qquad p + 3 = 11 \qquad -2z = -20$$

All equations have an equal sign. Furthermore, notice that the equal sign separates the equation into two parts, the left-hand side and the right-hand side. A **solution to an equation** is a value of the variable that makes the equation a true statement. Substituting a solution to an equation for the variable makes the right-hand side equal to the left-hand side.

Equation	Solution	Check	
$p + 3 = 11$	8	$p + 3 = 11$ $\downarrow$ $8 + 3 = 11$ ✓	Substitute 8 for p. Right-hand side equals left-hand side.
$-2z = -20$	10	$-2z = -20$ $\downarrow$ $-2(10) = -20$ ✓	Substitute 10 for z. Right-hand side equals left-hand side.

The **solution set** to an equation is the set of *all* solutions to an equation. We write the solution set using set braces. For example:

Equation		Solution set
$p + 3 = 11$	$\xrightarrow{\text{This equation has one solution.}}$	$\{8\}$
$w^2 = 16$	$\xrightarrow{\text{This equation has two solutions.}}$	$\{4, -4\}$

Throughout this text we will learn to recognize and solve several different types of equations, but in this chapter, we will focus on the specific type of equation called a linear equation in one variable.

Definition of a Linear Equation in One Variable

Let a, b, and c be real numbers such that $a \neq 0$. A **linear equation in one variable** is an equation that can be written in the form

$$ax + b = c$$

Note: A linear equation in one variable is often called a first-degree equation because the variable x has an implied exponent of 1.

Examples	Notes
$3x + 5 = 20$	$a = 3,\ b = 5,\ c = 20$
$-2x - 4 = 6$ can be written as $-2x + (-4) = 6$	$a = -2,\ b = -4,\ c = 6$
$6x + 7 - 5x = 11$ can be written as $x + 7 = 11$	$a = 1,\ b = 7,\ c = 11$

2. Solving Linear Equations

To solve a linear equation, the goal is to simplify the equation to isolate the variable. Each step used in simplifying an equation results in an equivalent equation. *Equivalent equations* have the same solution set. For example, the equations $2x + 3 = 7$ and $2x = 4$ are equivalent because $\{2\}$ is the solution set for both equations.

To solve an equation, we may use the addition, subtraction, multiplication, and division properties of equality. These properties state that adding, subtracting, multiplying, or dividing the same quantity on each side of an equation results in an equivalent equation.

Addition and Subtraction Properties of Equality

Let a, b, and c represent real numbers.

Addition property of equality: If $a = b$, then $a + c = b + c$.

*Subtraction property of equality: If $a = b$, then $a - c = b - c$.

*The subtraction property of equality follows directly from the addition property, because subtraction is defined in terms of addition.

$$\begin{aligned} \text{If} \quad a + (-c) &= b + (-c) \\ \text{then,} \quad a - c &= b - c \end{aligned}$$

Multiplication and Division Properties of Equality

Let a, b, and c represent real numbers with $c \neq 0$.

Multiplication property of equality: If $a = b$, then $a \cdot c = b \cdot c$.

*Division property of equality: If $a = b$, then $\dfrac{a}{c} = \dfrac{b}{c}$.

*The division property of equality follows directly from the multiplication property, because division is defined as multiplication by the reciprocal.

$$\begin{aligned} \text{If} \quad a \cdot \frac{1}{c} &= b \cdot \frac{1}{c} \\ \text{then,} \quad \frac{a}{c} &= \frac{b}{c} \end{aligned}$$

Example 1 Solving a Linear Equation

Solve the equation. $12 + x = 40$

Solution:

$$\begin{aligned} 12 + x &= 40 \\ 12 - 12 + x &= 40 - 12 && \text{To isolate } x, \text{ subtract 12 from both sides.} \\ x &= 28 && \text{Simplify.} \end{aligned}$$

Check: $12 + x = 40$

$12 + (28) \stackrel{?}{=} 40$

$40 \stackrel{?}{=} 40$ ✓ True

The solution set is $\{28\}$.

Skill Practice Solve the equation.

1. $x - 5 = -11$

Example 2 Solving Linear Equations

Solve each equation.

a. $-\frac{3}{5}p = \frac{4}{15}$ **b.** $4 = \frac{w}{2.2}$ **c.** $-x = 6$

Solution:

a.

$$-\frac{3}{5}p = \frac{4}{15}$$

$$\left(-\frac{5}{3}\right)\left(-\frac{3}{5}p\right) = \left(-\frac{5}{3}\right)\left(\frac{4}{15}\right)$$ To isolate p, multiply both sides by the reciprocal of $-\frac{3}{5}$.

$$p = \left(-\frac{\overset{1}{\cancel{5}}}{3}\right)\left(\frac{4}{\underset{3}{\cancel{15}}}\right)$$ Multiply fractions.

$$p = -\frac{4}{9}$$ The value $-\frac{4}{9}$ checks in the original equation.

The solution set is $\left\{-\frac{4}{9}\right\}$.

b.

$$4 = \frac{w}{2.2}$$

$$2.2(4) = 2.2\left(\frac{w}{2.2}\right)$$ To isolate w, multiply both sides by 2.2.

$$8.8 = w$$ The value 8.8 checks in the original equation.

The solution set is $\{8.8\}$.

c.

$$-x = 6$$

$$-1(-x) = -1(6)$$ To isolate x, multiply both sides by -1.

$$x = -6$$ The value -6 checks in the original equation.

The solution set is $\{-6\}$.

Skill Practice Solve the equations.

2. $-\frac{6}{5}y = -\frac{3}{5}$ **3.** $5 = \frac{t}{16}$ **4.** $-a = -2$

Answers

1. $\{-6\}$ **2.** $\left\{\frac{1}{2}\right\}$

3. $\{80\}$ **4.** $\{2\}$

For more complicated linear equations, several steps are required to isolate the variable. These steps are listed below.

Solving a Linear Equation in One Variable

Step 1 Simplify both sides of the equation.
- Clear parentheses.
- Consider clearing fractions or decimals (if any are present) by multiplying both sides of the equation by a common denominator of all terms.
- Combine *like* terms.

Step 2 Use the addition or subtraction property of equality to collect the variable terms on one side of the equation.

Step 3 Use the addition or subtraction property of equality to collect the constant terms on the other side of the equation.

Step 4 Use the multiplication or division property of equality to make the coefficient of the variable term equal to 1.

Step 5 Check your answer and write the solution set.

Example 3 Solving a Linear Equation

Solve the linear equation and check the answer.

$$3x + 1 = -7$$

Solution:

$$3x + 1 = -7$$

$$3x + 1 - 1 = -7 - 1 \qquad \text{Subtract 1 from both sides.}$$

$$3x = -8 \qquad \text{Combine } like \text{ terms.}$$

$$\frac{3x}{3} = \frac{-8}{3} \qquad \text{To isolate } x\text{, divide both sides of the equation by 3.}$$

$$x = -\frac{8}{3} \qquad \text{Simplify.}$$

Check: $3x + 1 \stackrel{?}{=} -7$

$$3\left(-\frac{8}{3}\right) + 1 \stackrel{?}{=} -7$$

$$\cancel{3}\left(-\frac{8}{\cancel{3}}\right) + 1 \stackrel{?}{=} -7$$

$$-8 + 1 \stackrel{?}{=} -7$$

$$-7 \stackrel{?}{=} -7 \checkmark \quad \text{True}$$

The solution set is $\left\{-\frac{8}{3}\right\}$.

Skill Practice Solve the linear equation and check the answer.

5. $5x - 19 = -23$

FOR REVIEW

Recall that it is important to use parentheses when making a substitution.

$$3x + 1 = -7$$

$$3(\ \ ) + 1 = -7$$

$$3\left(-\frac{8}{3}\right) + 1 = -7$$

Answer

5. $\left\{-\frac{4}{5}\right\}$

Example 4 Solving a Linear Equation

Solve the linear equation and check the answer.

$$11z + 2 = 5(z - 2)$$

Solution:

$11z + 2 = 5(z - 2)$	
$11z + 2 = 5z - 10$	Apply the distributive property to clear parentheses.
$11z - 5z + 2 = 5z - 5z - 10$	Subtract $5z$ from both sides.
$6z + 2 = -10$	Combine *like* terms.
$6z + 2 - 2 = -10 - 2$	Subtract 2 from both sides.
$6z = -12$	Combine *like* terms.
$\dfrac{6z}{6} = \dfrac{-12}{6}$	To isolate z, divide both sides of the equation by 6.
$z = -2$	Simplify.

<u>Check</u>: $11z + 2 = 5(z - 2)$

$$11(-2) + 2 \stackrel{?}{=} 5(-2 - 2)$$

$$-22 + 2 \stackrel{?}{=} 5(-4)$$

$$-20 \stackrel{?}{=} -20 \checkmark \quad \text{True}$$

The solution set is $\{-2\}$.

Skill Practice Solve the equations.

6. $7 + 2(y - 3) = 6y + 3$

Example 5 Solving a Linear Equation

Solve the equation. $-3(x - 4) + 2 = 7 - (x + 1)$

Solution:

$-3(x - 4) + 2 = 7 - (x + 1)$	
$-3x + 12 + 2 = 7 - x - 1$	Clear parentheses.
$-3x + 14 = -x + 6$	Combine *like* terms.
$-3x + x + 14 = -x + x + 6$	Add x to both sides of the equation.
$-2x + 14 = 6$	Combine *like* terms.
$-2x + 14 - 14 = 6 - 14$	Subtract 14 from both sides.

Answer

6. $\left\{\dfrac{1}{2}\right\}$

$$-2x = -8 \qquad \text{Combine } like \text{ terms.}$$

$$\frac{-2x}{-2} = \frac{-8}{-2} \qquad \text{To isolate } x \text{, divide both sides by } -2.$$

$$x = 4 \qquad \text{Simplify. The solution checks in the original equation.}$$

The solution set is $\{4\}$.

Skill Practice Solve the equation.

7. $4(2t + 2) - 6(t - 1) = 6 - t$

Example 6 **Solving a Linear Equation**

Solve the equation. $-4[y - 3(y - 5)] = 2(6 - 5y)$

Solution:

$-4[y - 3(y - 5)] = 2(6 - 5y)$	
$-4[y - 3y + 15] = 12 - 10y$	Clear parentheses.
$-4[-2y + 15] = 12 - 10y$	Combine *like* terms.
$8y - 60 = 12 - 10y$	Clear parentheses.
$8y + 10y - 60 = 12 - 10y + 10y$	Add $10y$ to both sides of the equation.
$18y - 60 = 12$	Combine *like* terms.
$18y - 60 + 60 = 12 + 60$	Add 60 to both sides of the equation.
$18y = 72$	
$\frac{18y}{18} = \frac{72}{18}$	To isolate y, divide both sides by 18.
$y = 4$	The solution checks.

The solution set is $\{4\}$.

Skill Practice Solve the equation.

8. $3[p + 2(p - 2)] = 4(p - 3)$

3. Clearing Fractions and Decimals

When an equation contains fractions or decimals, it is sometimes helpful to clear the fractions and decimals. This is accomplished by multiplying both sides of the equation by the least common denominator (LCD) of all terms within the equation. This is demonstrated in Example 7.

Answers

7. $\left\{-\frac{8}{3}\right\}$ **8.** $\{0\}$

Example 7 **Solving a Linear Equation by Clearing Fractions**

Solve the equation. $\dfrac{1}{4}w + \dfrac{1}{3}w - 1 = \dfrac{1}{2}(w - 4)$

Solution:

$$\frac{1}{4}w + \frac{1}{3}w - 1 = \frac{1}{2}(w - 4)$$

$$\frac{1}{4}w + \frac{1}{3}w - 1 = \frac{1}{2}w - 2 \qquad \text{Clear parentheses.}$$

$$12 \cdot \left(\frac{1}{4}w + \frac{1}{3}w - 1\right) = 12 \cdot \left(\frac{1}{2}w - 2\right) \qquad \text{Multiply both sides of the equation by the LCD of all terms. In this case, the LCD is 12.}$$

$$\frac{\overset{3}{\cancel{12}}}{1} \cdot \frac{1}{\cancel{4}_1}w + \frac{\overset{4}{\cancel{12}}}{1} \cdot \frac{1}{\cancel{3}_1}w + \frac{12}{1} \cdot \left(-\frac{1}{1}\right) = \frac{\overset{6}{\cancel{12}}}{1} \cdot \frac{1}{\cancel{2}_1}w + \frac{12}{1} \cdot \left(-\frac{2}{1}\right) \qquad \text{Apply the distributive property.}$$

$$3w + 4w - 12 = 6w - 24$$

$$7w - 12 = 6w - 24$$

$$w - 12 = -24 \qquad \text{Subtract } 6w.$$

$$w = -12 \qquad \text{The solution checks.}$$

The solution set is $\{-12\}$.

FOR REVIEW

Recall that an integer can be written as a fraction with a denominator of 1.

$$12 = \frac{12}{1}$$

Skill Practice Solve the equation by first clearing the fractions.

9. $\dfrac{3}{4}a + \dfrac{1}{2} = \dfrac{2}{3}a + \dfrac{1}{3}$

Example 8 **Solving a Linear Equation by Clearing Fractions**

Solve. $\dfrac{x - 2}{5} - \dfrac{x - 4}{2} = 2 + \dfrac{x + 4}{10}$

Solution:

$$\frac{x - 2}{5} - \frac{x - 4}{2} = \frac{2}{1} + \frac{x + 4}{10} \qquad \text{The LCD of all terms in the equation is 10.}$$

$$10\left(\frac{x - 2}{5} - \frac{x - 4}{2}\right) = 10\left(\frac{2}{1} + \frac{x + 4}{10}\right) \qquad \text{Multiply both sides by 10.}$$

$$\frac{\overset{2}{\cancel{10}}}{1} \cdot \left(\frac{x - 2}{\cancel{5}_1}\right) - \frac{\overset{5}{\cancel{10}}}{1} \cdot \left(\frac{x - 4}{\cancel{2}_1}\right) = \frac{10}{1} \cdot \left(\frac{2}{1}\right) + \frac{\overset{1}{\cancel{10}}}{1} \cdot \left(\frac{x + 4}{\cancel{10}_1}\right) \qquad \text{Apply the distributive property.}$$

$$2(x - 2) - 5(x - 4) = 20 + 1(x + 4) \qquad \text{Clear fractions.}$$

TIP: Clearing fractions is an application of the multiplication property of equality. We are multiplying both sides of the equation by the same number.

Answer

9. $\{-2\}$

$2x - 4 - 5x + 20 = 20 + x + 4$	Apply the distributive property.
$-3x + 16 = x + 24$	Simplify both sides of the equation.
$-4x + 16 = 24$	Subtract x from both sides.
$-4x = 8$	Subtract 16 from both sides. Then divide by -4.
$x = -2$	The value -2 checks in the original equation.

The solution set is $\{-2\}$.

Skill Practice Solve the equation.

10. $\dfrac{1}{8} - \dfrac{x+3}{4} = \dfrac{3x-2}{2}$

The same procedure used to clear fractions in an equation can be used to clear decimals.

Example 9 Solving a Linear Equation by Clearing Decimals

Solve the equation. $0.55x - 0.6 = 2.05x$

Solution:

Recall that any terminating decimal can be written as a fraction. Therefore, the equation $0.55x - 0.6 = 2.05x$ is equivalent to

$$\frac{55}{100}x - \frac{6}{10} = \frac{205}{100}x$$

A convenient common denominator for all terms in this equation is 100. Multiplying both sides of the equation by 100 will have the effect of "moving" the decimal point 2 places to the right.

$100(0.55x - 0.6) = 100(2.05x)$	Multiply both sides by 100 to clear decimals.
$55x - 60 = 205x$	
$-60 = 150x$	Subtract $55x$ from both sides.
$-\dfrac{60}{150} = x$	To isolate x, divide both sides by 150.
$x = -\dfrac{2}{5} = -0.4$	Simplify the fraction. The solution checks.

The solution set is $\{-0.4\}$.

Skill Practice Solve the equation by first clearing decimals.

11. $2.2x + 0.5 = 1.6x + 0.2$

Answers

10. $\left\{\dfrac{3}{14}\right\}$ **11.** $\{-0.5\}$

4. Conditional Equations, Contradictions, and Identities

The solution to a linear equation is the value of x that makes the equation a true statement. While linear equations have one unique solution, some equations have no solution, and others have infinitely many solutions.

I. Conditional Equations

An equation that is true for some values of the variable but false for other values is called a **conditional equation**. The equation $x + 4 = 6$ is a conditional equation because it is true on the *condition* that $x = 2$. For other values of x, the statement $x + 4 = 6$ is false. The solution set is $\{2\}$.

II. Contradictions

Some equations have no solution, such as $x + 1 = x + 2$. There is no value of x that when increased by 1 will equal the same value increased by 2. If we tried to solve the equation by subtracting x from both sides, we get the contradiction $1 = 2$.

$$\begin{aligned} x + 1 &= x + 2 \\ x - x + 1 &= x - x + 2 \\ 1 &= 2 \quad \text{(contradiction)} \end{aligned}$$

This indicates that the equation has no solution. An equation that has no solution is called a **contradiction**. The solution set for a contradiction is the empty set. The **empty set** is the set with no elements and is denoted by $\{\ \}$ or $\varnothing$.

III. Identities

An equation that is true for all real numbers is called an **identity**. For example, consider the equation $x + 4 = x + 4$. Because the left- and right-hand sides are *identical*, any real number substituted for x will result in equal quantities on both sides. If we solve the equation, we get the identity $4 = 4$. In such a case, the solution set is the set of real numbers.

$$\begin{aligned} x + 4 &= x + 4 \\ x - x + 4 &= x - x + 4 \\ 4 &= 4 \quad \text{(identity)} \qquad \text{The solution set is the set of real numbers.} \end{aligned}$$

In set-builder notation, we have $\{x \mid x \text{ is a real number}\}$.

Example 10 Identifying Conditional Equations, Contradictions, and Identities

Identify each equation as a conditional equation, a contradiction, or an identity. Then give the solution set.

a. $3[x - (x + 1)] = -2$

b. $5(3 + c) + 2 = 2c + 3c + 17$

c. $4x - 3 = 17$

Solution:

a. $3[x - (x + 1)] = -2$

$3[x - x - 1] = -2$ Clear parentheses.

$3[-1] = -2$ Combine *like* terms.

$-3 = -2$ Contradiction

This equation is a contradiction. The solution set is { }.

b. $5(3 + c) + 2 = 2c + 3c + 17$

$15 + 5c + 2 = 5c + 17$ Clear parentheses and combine *like* terms.

$5c + 17 = 5c + 17$ Identity

$0 = 0$

This equation is an identity. The solution set is $\{c \mid c \text{ is a real number}\}$.

c. $4x - 3 = 17$

$4x = 20$

$x = 5$

This equation is a conditional equation. The solution set is $\{5\}$.

TIP: Interval notation can also be used to express the set of real numbers, $(-\infty, \infty)$.

Skill Practice Identify each equation as a conditional equation, an identity, or a contradiction. Then give the solution set.

12. $2(-5x - 1) = 2x - 12x + 6$

13. $2(3x - 1) = 6(x + 1) - 8$

14. $4x + 1 - x = 6x - 2$

Answers

12. Contradiction; { }

13. Identity; $\{x \mid x \text{ is a real number}\}$

14. Conditional equation; {1}

Section 1.1 Activity

For Exercise A.1, solve the equations and indicate the operation used to isolate x.

A.1. **a.** $x - 4 = 12$

b. $x + 4 = 12$

c. $4x = 12$

d. $\dfrac{x}{4} = 12$

A.2. Explain the steps you would take to solve the equation $2x - 3 = 7$.

For Exercises A.3–A.5, divide your paper into three columns. Solve the equation in the left column. Explain each step in the middle column. Check the answer and write the solution set in the right column.

Solve and show all steps	Explain in words	Check
A.3 $-5x + 6 = 41$		$-5x + 6 = 41$ $-5(\quad) + 6 \stackrel{?}{=} 41$

A.3. $-5x + 6 = 41$

A.4. $-11t + 4 = -8t - 14$

A.5. $5 - 3(m + 4) = -1 - 2(m - 3) + 1$

A.6. Given the equation $\frac{1}{3}x - \frac{5}{2} = 1 + \frac{1}{4}x$,

- **a.** Which value(s) can be used to clear the fractions?
  2, 3, 4, 6, 12, 15, 18, 21, 24, 28, 36, 40, 48, 60
- **b.** Multiply. $12 \cdot \left(\frac{1}{3}x\right)$
- **c.** Multiply. $12 \cdot \left(-\frac{5}{2}\right)$
- **d.** Multiply. $12 \cdot (1)$
- **e.** Multiply. $12 \cdot \left(\frac{1}{4}x\right)$
- **f.** Solve the equation by first clearing fractions.

A.7. Given the equation $-3.9 - 1.2x = 7.84 - x$,

- **a.** Which value(s) can be used to clear the decimals?
  10, 100, 1000, 10,000, 100,000
- **b.** Multiply. $100 \cdot (-3.9)$
- **c.** Multiply. $100 \cdot (-1.2x)$
- **d.** Multiply. $100 \cdot (7.84)$
- **e.** Multiply. $100 \cdot (-x)$
- **f.** Solve the equation.

For Exercises A.8–A.10, divide your paper into three columns. Solve the equation in the left column, explain each step in the middle column, and check the potential solution in the right column.

Equation	Explain	Check
$\frac{2}{5}(x + 5) - \frac{1}{3}(2x - 1) = 1$		$\frac{2}{5}[(\square) + 5] - \frac{1}{3}[2(\square) - 1] \stackrel{?}{=} 1$

A.8. $\frac{2}{5}(x + 5) - \frac{1}{3}(2x - 1) = 1$

A.9. $0.04(5x + 2) + 0.12(x - 3) = 0.04$

A.10. $\frac{3w + 1}{10} - \frac{w + 2}{4} = \frac{1}{5}$

A.11. Explain why the equation $x + 3 = x + 5$ is a contradiction. Write the solution set.

A.12. Explain why the equation $3x + 7 = 3(x + 2) + 1$ is an identity. Write the solution set.

A.13. The equation $4x - 1 = x + 14$ is true under the condition that $x =$ ______. Therefore, this equation is a conditional equation. Write the solution set.

Practice Exercises

Section 1.1

Study Skills Exercise

Class notes serve as a summary of key concepts that can be used later to prepare for tests. Look over the notes that you took today. Do you understand what you wrote? Identify any material or challenging problems that you need to revisit. Add any comments to make your notes clearer to you, or rewrite the notes in your own words. If there were any rules, definitions, or formulas, highlight them so that they can be easily found when studying for the test.

Remember that note-taking skills are learned by practice.

Prerequisite Review

For Exercises R.1–R.8, simplify the expression.

R.1. $\frac{1}{2} \cdot 2t$

R.2. $-4 \cdot \left(-\frac{1}{4}x\right)$

R.3. $n - 2.1 + 2.1$

R.4. $m + 4.4 - 4.4$

R.5. $-2y + 7y - 4 + 6y$

R.6. $2p - 9 - 4p + p$

R.7. $-3 - 2(4 - x) + 8(3x - 4)$

R.8. $6 - (5m - 2) + 2(9 - 5m)$

For Exercises R.9–R.12, identify the least common denominator of the fractions.

R.9. $-\frac{3}{5}, \frac{7}{10}$, and $\frac{2}{15}$

R.10. $\frac{3}{4}, -\frac{2}{3}$, and $\frac{5}{18}$

R.11. $\frac{17}{100}, \frac{3}{1000}$, and $\frac{37}{10,000}$

R.12. $3, \frac{1}{10}$, and $\frac{51}{1000}$

For Exercises R.13–R.16, evaluate the expression for the given value of x.

R.13. $-8x + 5$ for $x = 3$

R.14. $7 - 3x$ for $x = -2$

R.15. $8 - 2(x - 4) + 5x$ for $x = -6$

R.16. $4x - 7 - 3(2x + 1)$ for $x = 5$

Vocabulary and Key Concepts

1. **a.** An __________ is a statement that indicates that two quantities are equal.

b. A __________ to an equation is a value of the variable that makes the equation a true statement.

c. An equation that can be written in the form $ax + b = c$ $(a \neq 0)$ is called a __________ equation in one variable.

d. A linear equation is also called a __________ -degree equation because the degree on the variable is 1.

e. The set of all solutions to an equation is called the __________ __________.

f. Two equations are equivalent if they have the same __________ set.

2. A __________ equation is true for some values of the variable, but false for other values.

3. An equation that has no solution is called a __________.

4. The set containing no elements is called the __________ and is denoted by __________.

5. An equation that has all real numbers as its solution set is called an __________.

6. Which equation(s) are linear?

a. $\frac{1}{x} + \frac{1}{4} = \frac{1}{3}$ **b.** $\frac{1}{2}x + \frac{1}{4} = \frac{1}{3}$

c. $-4x + 9 = 1$ **d.** $-4x^2 + 9 = 1$

Concept 1: Linear Equations in One Variable

For Exercises 7–12, label the equation as linear or nonlinear.

7. $2x + 1 = 5$ **8.** $10 = x + 6$ **9.** $x^2 + 7 = 9$

10. $3 + x^3 - x = 4$ **11.** $-3 = x$ **12.** $5.2 - 7x = 0$

13. Use substitution to determine which value is the solution to $2x - 1 = 5$.

a. 2 **b.** 3 **c.** 0 **d.** -1

14. Use substitution to determine which value is the solution to $2y - 3 = -2$.

a. 1 **b.** $\frac{1}{2}$ **c.** 0 **d.** $-\frac{1}{2}$

Concept 2: Solving Linear Equations

For Exercises 15–44, solve the equation and check the solution. **(See Examples 1–6.)**

15. $x + 7 = 19$ **16.** $-3 + y = -28$ **17.** $-x = 2$ **18.** $-t = \frac{3}{4}$

19. $-\frac{7}{8} = -\frac{5}{6}z$ **20.** $-\frac{12}{13} = \frac{4}{3}b$ **21.** $\frac{a}{5} = -8$ **22.** $\frac{x}{8} = \frac{1}{2}$

23. $2.53 = -2.3t$ **24.** $-4.8 = 6.1 + y$ **25.** $p - 2.9 = 3.8$ **26.** $-4.2a = 4.494$

27. $6q - 4 = 62$ **28.** $2w - 15 = 15$ **29.** $4y - 17 = 35$ **30.** $6z - 25 = 83$

31. $-b - 5 = 2$ **32.** $6 = -y + 1$ **33.** $3(x - 6) = 2x - 5$ **34.** $13y + 4 = 5(y - 4)$

35. $6 - (t + 2) = 5(3t - 4)$ **36.** $1 - 5(p + 2) = 2(p + 13)$

37. $6(a + 3) - 10 = -2(a - 4)$ **38.** $8(b - 2) + 3b = -9(b - 1)$

39. $-2[5 - (2z + 1)] - 4 = 2(3 - z)$ **40.** $3[w - (10 - w)] = 7(w + 1)$

41. $6(-y + 4) - 3(2y - 3) = -y + 5 + 5y$ **42.** $13 + 4w = -5(-w - 6) + 2(w + 1)$

43. $14 - 2x + 5x = -4(-2x - 5) - 6$ **44.** $8 - (p + 2) + 6p + 7 = p + 13$

Concept 3: Clearing Fractions and Decimals

For Exercises 45–56, solve the equations. **(See Examples 7–9.)**

45. $\frac{2}{3}x - \frac{1}{6} = -\frac{5}{12}x + \frac{3}{2} - \frac{1}{6}x$

46. $-\frac{1}{2}y + 4 = -\frac{9}{10}y + \frac{2}{5}$

47. $\frac{1}{5}(p - 5) = \frac{3}{5}p + \frac{1}{10}p + 1$

48. $\frac{5}{6}(q + 2) = -\frac{7}{9}q - \frac{1}{3} + 2$

49. $\frac{3x - 7}{2} + \frac{3 - 5x}{3} = \frac{3 - 6x}{5}$

50. $\frac{2y - 4}{5} = \frac{5y + 13}{4} + \frac{y}{2}$

51. $\frac{4}{3}(2q + 6) - \frac{5q - 6}{6} - \frac{q}{3} = 0$

52. $\frac{-3a + 9}{15} - \frac{2a - 5}{5} - \frac{a + 2}{10} = 0$

53. $6.3w - 1.5 = 4.8$

54. $0.2x + 53.6 = x$

55. $0.75(m - 2) + 0.25m = 0.5$

56. $0.4(n + 10) + 0.6n = 2$

Concept 4: Conditional Equations, Contradictions, and Identities

57. What is a conditional equation?

58. Explain the difference between a contradiction and an identity.

For Exercises 59–64, identify the equation as a conditional equation, a contradiction, or an identity. Then give the solution set. **(See Example 10.)**

59. $4x + 1 = 2(2x + 1) - 1$

60. $3x + 6 = 3x$

61. $-11x + 4(x - 3) = -2x - 12$

62. $5(x + 2) - 7 = 3$

63. $2x - 4 + 8x = 7x - 8 + 3x$

64. $-7x + 8 + 4x = -3(x - 3) - 1$

Mixed Exercises

For Exercises 65–96, solve the equations.

65. $-5b + 9 = -71$

66. $-3x + 18 = -66$

67. $16 = -10 + 13x$

68. $15 = -12 + 9x$

69. $10c + 3 = -3 + 12c$

70. $2w + 21 = 6w - 7$

71. $12b - 8b - 8 + 13 = 4b + 6 - 1$

72. $4z + 2 - 3z + 5 = 3 + z + 4$

73. $5(x - 2) - 2x = 3x + 7$

74. $2x + 3(x - 5) = 15$

75. $\frac{c}{2} - \frac{c}{4} + \frac{3c}{8} = 1$

76. $\frac{d}{5} - \frac{d}{10} + \frac{5d}{20} = \frac{7}{10}$

77. $0.75(8x - 4) = \frac{2}{3}(6x - 9)$

78. $-\frac{1}{2}(4z - 3) = -z$

79. $7(p + 2) - 4p = 3p + 14$

80. $6(z - 2) = 3z - 8 + 3z$

81. $4[3 + 5(3 - b) + 2b] = 6 - 2b$

82. $\frac{1}{3}(x + 3) - \frac{1}{6} = \frac{1}{6}(2x + 5)$

83. $3 - \frac{3}{4}x = 3\left(3 - \frac{1}{4}x\right)$

84. $\frac{9}{5} - 8w = 8\left(\frac{3}{5} - w\right)$

85. $\frac{5}{4} + \frac{y - 3}{8} = \frac{2y + 1}{2}$

86. $\frac{2}{3} - \frac{x + 2}{6} = \frac{5x - 2}{2}$

87. $\frac{2y - 9}{10} + \frac{3}{2} = y$

88. $\frac{2}{3}x - \frac{5}{6}x - 3 = \frac{1}{2}x - 5$

89. $0.48x - 0.08x = 0.12(260 - x)$

90. $0.07w + 0.06(140 - w) = 90$

91. $0.5x + 0.25 = \frac{1}{3}x + \frac{5}{4}$

92. $0.2b + \frac{1}{3} = \frac{7}{15}$

93. $0.3b - 1.5 = 0.25(b + 2) + 0.05b$

94. $0.7(a - 1) = 0.25 + 0.7a$

95. $-\frac{7}{8}y + \frac{1}{4} = \frac{1}{2}\left(5 - \frac{3}{4}y\right)$

96. $5x - (8 - x) = 2[-4 - (3 + 5x) - 13]$

Expanding Your Skills

97. A power company charges \$0.12 per kilowatt-hour (kWh) and \$14.89 in monthly taxes. The monthly charge C (in \$) is given by $C = 0.12h + 14.89$ where h is the number of kilowatt-hours used. If a family's bill comes to \$137.77, determine the number of kilowatt-hours used.

98. For a student's first semester at college, the college charges \$105 per credit-hour plus a one-time \$50 admissions fee. The cost C (in \$) for the first semester is given by $C = 105h + 50$ where h is the number of credit-hours taken. If a student is billed \$1415, how many credit-hours is she taking?

99. **a.** Simplify the expression. $-2(y - 1) + 3(y + 2)$

b. Solve the equation. $-2(y - 1) + 3(y + 2) = 0$

c. Explain the difference between simplifying an expression and solving an equation.

100. **a.** Simplify the expression. $4w - 8(2 + w)$

b. Solve the equation. $4w - 8(2 + w) = 0$

c. Explain the difference between simplifying an expression and solving an equation.

Problem Recognition Exercises

Equations Versus Expressions

For Exercises 1–20, identify each exercise as an expression or an equation. Then simplify the expressions and solve the equations.

1. $4x - 2 + 6 - 8x$

2. $-3y - 3 - 4y + 8$

3. $7b - 1 = 2b + 4$

4. $10t + 2 = 2 - 7t$

5. $4(a - 8) - 7(2a + 1)$

6. $10(2x + 3) - 8(5 - x)$

7. $7(2 - w) = 5w + 8$

8. $15(3 - 2y) = 21 + 2y$

9. $2(3x - 4) - 4(5x + 1) = -8x + 7$

10. $6(2 - 3a) - 2(8a + 3) = -12a - 19$

11. $\frac{1}{2}v + \frac{3}{5} - \frac{2}{3}v - \frac{7}{10}$

12. $-\frac{7}{8}t - \frac{4}{3}u - \frac{5}{4}t + \frac{11}{6}u$

13. $20x - 8 + 7x + 28 = 27x - 9$

14. $7 + 8w - 12 = 3w - 8 + 5w$

15. $\frac{5}{6}y - \frac{7}{8} = \frac{1}{2}y + \frac{3}{4}$

16. $\frac{4}{5} + 3z = \frac{1}{2}z + 1$

17. $0.29c + 4.495 - 0.12c$

18. $0.45k - 1.67 + 0.89 - 1.456k$

19. $0.125(2p - 8) = 0.25(p - 4)$

20. $0.5u + 1.2 - 0.74u = 0.8 - 0.24u + 0.4$

Applications of Linear Equations in One Variable

Section 1.2

Concepts

1. Introduction to Problem Solving
2. Applications Involving Consecutive Integers
3. Applications Involving Percents and Rates
4. Applications Involving Principal and Interest
5. Applications Involving Mixtures
6. Applications Involving Distance, Rate, and Time

1. Introduction to Problem Solving

One of the important uses of algebra is to develop mathematical models for understanding real-world phenomena. To solve an application problem, relevant information must be extracted from the wording of a problem and then translated into mathematical symbols. This is a skill that requires practice. The key is to stick with it and not to get discouraged.

Problem-Solving Flowchart for Word Problems

Step 1	Read the problem carefully.	• Familiarize yourself with the problem. Estimate the answer, if possible.
Step 2	Assign labels to unknown quantities.	• Identify the unknown quantity or quantities. Let x represent one of the unknowns. Draw a picture and write down relevant formulas.
Step 3	Write a verbal model.	• Write an equation in *words*.
Step 4	Write a mathematical equation.	• Replace the verbal model with a mathematical equation using x or another variable.
Step 5	Solve the equation.	• Solve for the variable, using the steps for solving linear equations.
Step 6	Interpret the results and write the final answer in words.	• Once you've obtained a numerical value for the variable, recall what it represents in the context of the problem. Can this value be used to determine other unknowns in the problem? Write an answer to the word problem in *words*.

Avoiding Mistakes

Once you have reached a solution to a word problem, verify that it is reasonable in the context of the problem.

To write an English statement as an algebraic expression, review the list of key terms given in Table 1-1.

Table 1-1

Addition: $a + b$ • the **sum** of a and b • a plus b • b added to a • b more than a • a increased by b • the total of a and b	Subtraction: $a - b$ • the **difference** of a and b • a minus b • b subtracted from a • a decreased by b • b less than a
Multiplication: $a \cdot b$ • the **product** of a and b • a times b • a multiplied by b	Division: $a \div b, \frac{a}{b}$ • the **quotient** of a and b • a divided by b • b divided into a • the ratio of a and b • a over b • a per b

Example 1 Translating and Solving a Linear Equation

The sum of two numbers is 39. One number is 3 less than twice the other. What are the numbers?

Solution:

Step 1: Read the problem carefully.

Step 2: Let x represent one number.

Let $2x - 3$ represent the other number.

Step 3: (One number) + (other number) = 39

Step 4: Replace the verbal model with a mathematical equation.

$$\begin{aligned}(\text{One number}) + (\text{other number}) &= 39\\ x \quad + \quad (2x-3) &= 39\end{aligned}$$

Step 5: Solve for x.

$$\begin{aligned}x + (2x - 3) &= 39\\ 3x - 3 &= 39\\ 3x &= 42\\ \frac{3x}{3} &= \frac{42}{3}\\ x &= 14\end{aligned}$$

Step 6: Interpret your results. Refer to step 2.

One number is x: $\longrightarrow$ 14

The other number is $2x - 3$:

$2(14) - 3 \longrightarrow$ 25

Answer: The numbers are 14 and 25.

Skill Practice

1. One number is 5 more than 3 times another number. The sum of the numbers is 45. Find the numbers.

2. Applications Involving Consecutive Integers

The word *consecutive* means "following one after the other in order."

- The numbers $-2, -1, 0, 1, 2$, and so on are examples of consecutive integers. Notice that two consecutive integers differ by 1. Therefore, if x represents an integer, then $x + 1$ represents the next consecutive integer.
- The numbers 2, 4, 6, 8, and so on are consecutive *even* integers. Consecutive even integers differ by 2. Therefore, if x represents an even integer, then $x + 2$ represents the next consecutive even integer.
- The numbers 15, 17, 19, and so on are consecutive *odd* integers. Consecutive odd integers also differ by 2. Therefore, if x represents an odd integer, then $x + 2$ represents the next consecutive odd integer.

Answer

1. The numbers are 10 and 35.

Example 2 Solving a Linear Equation Involving Consecutive Integers

Three times the sum of two consecutive odd integers is 516. Find the integers.

Solution:

Step 1: Read the problem carefully.

Step 2: Label the unknowns:

Let x represent the first odd integer.

Then $x + 2$ represents the next odd integer.

Step 3: Write an equation in words.

3[(first odd integer) + (second odd integer)] = 516

$$3[x + (x + 2)] = 516$$ **Step 4:** Write a mathematical equation.

$$3(2x + 2) = 516$$ **Step 5:** Solve for x.

$$6x + 6 = 516$$

$$6x = 510$$

$$x = 85$$

Step 6: Interpret your results.

The first odd integer is x: → 85

The second odd integer is $x + 2$:

$85 + 2$ → 87

Answer: The integers are 85 and 87.

Avoiding Mistakes

After completing a word problem, it is always a good idea to check that the answer is reasonable. Notice that 85 and 87 are consecutive odd integers, and three times their sum is 3(85 + 87), which equals 516.

Skill Practice

2. Four times the sum of three consecutive integers is 264. Find the integers.

3. Applications Involving Percents and Rates

In many real-world applications, percents are used to represent rates.

- The sales tax rate for a certain county is 6%.
- An ice cream machine is discounted 20%.
- A real estate sales broker receives a $4\frac{1}{2}\%$ commission on sales.
- A savings account earns 7% simple interest.

The following models are used to compute sales tax, commission, and simple interest. In each case the value is found by multiplying the base by the percentage.

Sales tax = (cost of merchandise)(tax rate)

Commission = (dollars in sales)(commission rate)

Simple interest = (principal)(annual interest rate)(time in years)

→ $I = Prt$

Answer

2. The integers are 21, 22, and 23.

Example 3 Solving a Percent Application

A woman invests \$5000 in an account that earns $5\frac{1}{4}\%$ simple interest. If the money is invested for 3 years (yr), how much money is in the account at the end of the 3-yr period?

Solution:

Let x represent the total money in the account. Label the variables.

$P = \$5000$ (principal amount invested)

$r = 0.0525$ (interest rate)

$t = 3$ (time in years)

The total amount of money includes principal plus interest.

(Total money) = (principal) + (interest) Verbal model

$x = P + Prt$ Mathematical equation

$x = \$5000 + (\$5000)(0.0525)(3)$ Substitute for P, r, and t.

$x = \$5000 + \$787.50$

$x = \$5787.50$ Solve for x.

The total amount of money in the account is \$5787.50. Interpret the results.

FOR REVIEW

Remember to use the decimal form of a percent when the number is used in a calculation. To convert a percent to decimal form, divide by 100 and drop the % symbol.

$$5\tfrac{1}{4}\% = 5.25\% = \frac{5.25}{100} = 0.0525$$

Skill Practice

3. Markos earned \$340 in 1 yr on an investment that paid a 4% dividend. Find the amount of money invested.

As consumers, we often encounter situations in which merchandise has been marked up or marked down from its original cost. It is important to note that percent increase and percent decrease are based on the original cost. For example, suppose a microwave oven originally priced at \$305 is marked down 20%.

The discount is determined by 20% of the original price: (0.20)(\$305) = \$61.00. The new price is \$305.00 − \$61.00 = \$244.00.

Example 4 Solving a Percent Increase Application

A college bookstore uses a standard markup of 40% on all books purchased wholesale from the publisher. If the bookstore sells a calculus book for \$179.20, what was the original wholesale cost?

Answer

3. \$8500

Solution:

Let x = original wholesale cost. Label the variables.

The selling price of the book is based on the original cost of the book plus the bookstore's markup.

(Selling price) = (original price) + (markup) Verbal model

(Selling price) = (original price) + (original price · markup rate)

$$179.20 = x + (x)(0.40)$$ Mathematical equation

$$179.20 = x + 0.40x$$

$$179.20 = 1.40x$$ Combine *like* terms.

$$\frac{179.20}{1.40} = x$$

$$x = 128$$ Simplify.

The original wholesale cost of the textbook was \$128.00. Interpret the results.

Skill Practice

4. An online bookstore gives a 20% discount on paperback books. Find the original price of a book that has a selling price of \$5.28 after the discount.

4. Applications Involving Principal and Interest

Example 5 Solving an Investment Growth Application

Miguel had \$10,000 to invest in two different mutual funds. One was a relatively safe bond fund that averaged 4% return on his investment at the end of 1 yr. The other fund was a riskier stock fund that averaged 7% return in 1 yr. If at the end of the year Miguel's portfolio grew to \$10,625 (\$625 above his \$10,000 investment), how much money did Miguel invest in each fund?

Solution:

This type of word problem is sometimes categorized as a mixture problem. Miguel is "mixing" his money between two different investments. We have to determine how the money was divided to earn \$625.

The information in this problem can be organized in a chart. (*Note:* There are two sources of money: the amount invested and the amount earned.)

	4% Bond Fund	7% Stock Fund	Total
Amount invested (\$)	x	$(10{,}000 - x)$	10,000
Amount earned (\$)	$0.04x$	$0.07(10{,}000 - x)$	625

Because the amount of principal is unknown for both accounts, we can let x represent the amount invested in the bond fund. If Miguel spends x dollars in the bond fund, then he has $(10{,}000 - x)$ left over to spend in the stock fund. The return for each fund is found by multiplying the principal and the percent growth rate.

Answer

4. \$6.60

To establish a mathematical model, we know that the total return ($625) must equal the earnings from the bond fund plus the earnings from the stock fund:

(Earnings from bond fund) + (earnings from stock fund) = (total earnings)

$$0.04x \quad + \quad 0.07(10{,}000 - x) \quad = \quad 625$$

$0.04x + 0.07(10{,}000 - x) = 625$	Mathematical equation
$4x + 7(10{,}000 - x) = 62{,}500$	Multiply by 100 to clear decimals.
$4x + 70{,}000 - 7x = 62{,}500$	
$-3x + 70{,}000 = 62{,}500$	Combine *like* terms.
$-3x = -7500$	Subtract 70,000 from both sides.
$\dfrac{-3x}{-3} = \dfrac{-7500}{-3}$	
$x = 2500$	Solve for x and interpret the results.

The amount invested in the bond fund is $2500.
The amount invested in the stock fund is $\$10{,}000 - x$, or $\$10{,}000 - \$2500 = \$7500$.

FOR REVIEW

Recall that a word problem should be checked in the context of the problem.

Principal:
$\$2500 + \$7500 = \$10{,}000$ ✓

Interest:
$0.04(\$2500) + 0.07(\$7500)$
$= \$100 + \$525$
$= \$625$ ✓

Skill Practice

5. Jonathan borrowed $4000 in two loans. One loan charged 7% interest, and the other charged 1.5% interest. After 1 yr, Jonathan paid $225 in interest. Find the amount borrowed in each loan.

5. Applications Involving Mixtures

Example 6 **Solving a Mixture Application**

How many liters (L) of a 40% antifreeze solution must be added to 4 L of a 10% antifreeze solution to produce a 35% antifreeze solution?

Solution:

The given information is illustrated in Figure 1-1.

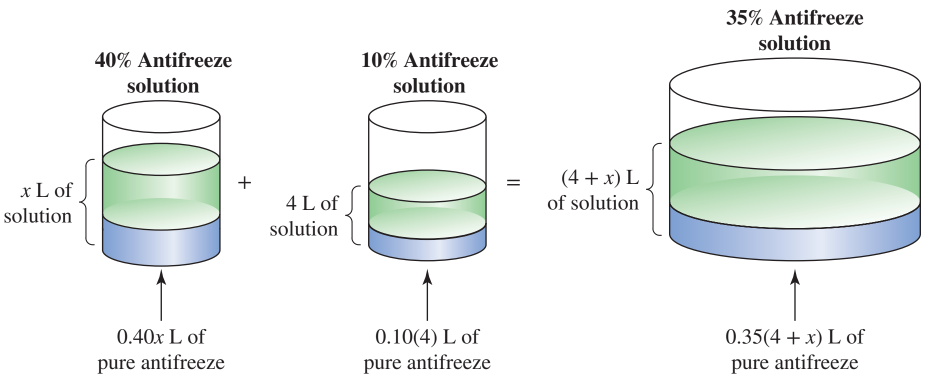


Figure 1-1

TIP: To understand the role of the concentration rate within a mixture problem, consider this example. Suppose you have 30 gal of a 10% antifreeze mixture. How much pure antifreeze is in the mixture?

pure antifreeze = 0.10(30 gal)
= 3 gal

Multiply the concentration rate by the amount of mixture.

Answer

5. $3000 was borrowed at 7% interest, and $1000 was borrowed at 1.5% interest.

The information can also be organized in a table.

	40% Antifreeze	10% Antifreeze	Final Solution: 35% Antifreeze
Number of liters of solution	x	4	$(4 + x)$
Number of liters of pure antifreeze	$0.40x$	$0.10(4)$	$0.35(4 + x)$

Notice that an algebraic equation is obtained from the second row of the table relating the number of liters of pure antifreeze in each container.

$$\begin{array}{ccccc} \begin{pmatrix}\text{Pure antifreeze}\\ \text{from solution 1}\end{pmatrix} & + & \begin{pmatrix}\text{pure antifreeze}\\ \text{from solution 2}\end{pmatrix} & = & \begin{pmatrix}\text{pure antifreeze}\\ \text{in the final solution}\end{pmatrix} \\ 0.40x & + & 0.10(4) & = & 0.35(4 + x) \end{array}$$

$0.40x + 0.10(4) = 0.35(x + 4)$	Mathematical equation
$0.4x + 0.4 = 0.35x + 1.4$	Apply the distributive property.
$0.4x - 0.35x + 0.4 = 0.35x - 0.35x + 1.4$	Subtract $0.35x$ from both sides.
$0.05x + 0.4 = 1.4$	
$0.05x + 0.4 - 0.4 = 1.4 - 0.4$	Subtract 0.4 from both sides.
$0.05x = 1.0$	
$\dfrac{0.05x}{0.05} = \dfrac{1.0}{0.05}$	Divide both sides by 0.05.
$x = 20$	

Therefore, 20 L of a 40% antifreeze solution is needed.

Skill Practice

6. Find the number of ounces (oz) of 30% alcohol solution that must be mixed with 10 oz of a 70% solution to obtain a solution that is 40% alcohol.

Avoiding Mistakes

To check, verify that the amount of pure antifreeze from the two solutions equals the amount of antifreeze in the final solution.

$$\begin{array}{rr} & 0.40(20\text{ L}) = 8\text{ L} \\ + & 0.10(4\text{ L}) = 0.4\text{ L} \\ \hline & 0.35(24\text{ L}) = 8.4\text{ L} \end{array}$$

6. Applications Involving Distance, Rate, and Time

The fundamental relationship among the variables distance, rate, and time is given by

$$\text{Distance} = (\text{rate})(\text{time}) \quad \text{or} \quad d = rt$$

For example, a motorist traveling 65 mph (miles per hour) for 3 hr (hours) will travel a distance of

$$d = \left(\frac{65\text{ mi}}{\cancel{\text{hr}}}\right)(3\ \cancel{\text{hr}}) = 195\text{ mi}$$

Answer

6. 30 oz of the 30% solution is needed.

Example 7 Solving a Distance, Rate, Time Application

A hiker can hike 1 mph faster downhill to Moose Lake than she can hike uphill back to the campsite. If it takes her 3 hr to hike to the lake and 4.5 hr to hike back, what is her speed hiking back to the campsite?

Digital Stock/Corbis

Solution:

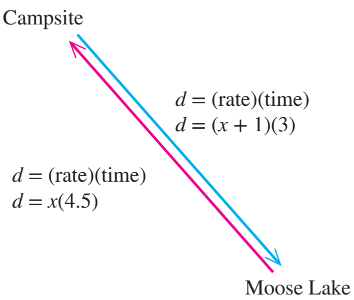
The information given in the problem can be organized in a table.

	Distance (mi)	Rate (mph)	Time (hr)
Trip to the lake		$x + 1$	3
Return trip		x	4.5

Column 2: Let the rate of the return trip be represented by x. Then the trip to the lake is 1 mph faster and can be represented by $x + 1$.

Column 3: The times hiking to and from the lake are given in the problem.

Column 1: To express the distance, we use the relationship $d = rt$. That is, multiply the quantities in the second and third columns.



	Distance (mi)	Rate (mph)	Time (hr)
Trip to the lake	$3(x + 1)$	$x + 1$	3
Return trip	$4.5x$	x	4.5

To create a mathematical model, note that the distances to and from the lake are equal. Therefore,

(Distance to lake) = (return distance) Verbal model

$3(x + 1) = 4.5x$ Mathematical equation

$3x + 3 = 4.5x$ Apply the distributive property.

$3x - 3x + 3 = 4.5x - 3x$ Subtract $3x$ from both sides.

$3 = 1.5x$

$\dfrac{3}{1.5} = \dfrac{1.5x}{1.5}$ Divide by 1.5 to isolate the variable.

$2 = x$ Solve for x.

The hiker's speed on the return trip to the campsite is 2 mph.

Avoiding Mistakes

Notice that the distance to (and from) the lake is 9 mi.

Distance:

4.5 hr(2 mph) = 9 mi

Verify that $d = rt$ to and from the lake.

9 mi = (3 mph)(3 hr) ✓

9 mi = (2 mph)(4.5 hr) ✓

Skill Practice

7. During a bad rainstorm, Jody drove 15 mph slower on a trip to her mother's house than she normally would when the weather is clear. If a trip to her mother's house takes 3.75 hr in clear weather and 5 hr in a bad storm, what is her normal driving speed during clear weather?

Answer

7. Jody normally drives 60 mph.

Section 1.2 Activity

A.1. **a.** If x represents the smallest of three consecutive integers, write an expression to represent each of the next two consecutive integers.

b. If x represents the *largest* of three consecutive integers, write an expression to represent each of the previous two consecutive integers.

A.2. Consider three consecutive odd integers. Suppose that the sum of the first integer and 2 times the second integer is 54 more than the third integer. To find the values of the three integers, follow these steps.

a. Read the problem again.

b. Let x represent the first integer. Write an expression to represent each of the next two consecutive odd integers.

c. Write an equation in words to represent this scenario.

d. Write a mathematical equation in terms of x to represent this scenario.

e. Solve the equation from part (d).

f. Interpret the answer from part (e) and give the values of the three integers.

A.3. **a.** A sofa originally costs \$1200 but is marked down by 25%. What is the amount of the discount? What is the new price?

b. A recliner originally costs x dollars but is marked down by 15%. Write an expression in terms of x for the amount of the discount. Write an expression in terms of x for the new price.

A.4. A repurposed video game console is marked down 40% from its original price. If the new price is \$216 (before tax), find the original price by following these steps.

a. Read the problem again.

b. Let x represent ______________________________.

c. Write an equation in words to represent this scenario.

d. Write a mathematical equation in terms of x to represent this scenario.

e. Solve the equation from part (d).

f. Interpret the results in words using proper units of measurement and verify that your solution makes sense in the context of the problem.

A.5. **a.** How much simple interest is earned on \$6000 at 7% for 1 yr?

b. Write an expression that represents the amount of simple interest earned on x dollars at 7% for 1 yr.

A.6. Jon borrowed a total of \$12,000. He borrowed part of the money from a bank that charges 7% simple interest. He borrowed the rest of the money from his brother who only charged him 2% interest. At the end of the year, he paid back the principal along with \$415 in interest. Determine the amount of money he borrowed from each source.

a. Read the problem again.

b. Let x represent ____.

c. Complete the table.

	7% Loan	2% Loan	Total
Amount borrowed (\$)			
Interest owed (\$)			

d. Write an equation that represents this scenario—namely, the interest owed on the 7% loan plus the interest owed on the 2% loan equals the total interest.

e. Solve the equation from part (d).

f. Interpret the results in words using proper units of measurement and verify that your solution makes sense in the context of the problem.

A.7. **a.** Consider 50 mL of a mixture that is 10% acid. How much of the mixture is pure acid? How much of the mixture is something other than acid?

b. A chemist has x milliliters of a 20% acid mixture. Write an expression that represents the amount of pure acid in the mixture.

A.8. How many ounces of a 10% acid solution must be mixed with 6 L of a 25% acid solution to produce a 20% acid solution?

a. Read the problem again.

b. Let x represent ____.

c. Complete the table.

	10% Acid Solution	25% Acid Solution	20% Acid Solution
Amount of solution (L)			
Amount of pure acid (L)			

d. Write an equation that represents

$$\begin{pmatrix}\text{The amount of} \\ \text{pure acid in} \\ \text{the 10\% solution}\end{pmatrix} + \begin{pmatrix}\text{the amount of} \\ \text{pure acid in} \\ \text{the 25\% solution}\end{pmatrix} = \begin{pmatrix}\text{the amount of} \\ \text{pure acid in} \\ \text{the 20\% solution}\end{pmatrix}$$

e. Solve the equation from part (d).

f. Interpret the results in words using proper units of measurement and verify that your solution makes sense in the context of the problem.

A.9. **a.** If a hiker walks 5 mph for 2 hr, how far does she walk?

b. If a hiker walks x miles per hour for 2 hr, write an expression for the distance walked.

A.10. Kesha can walk downhill from her car to a campsite. For the return trip uphill back to her car, she walks 2 mph slower. It takes 1.5 hr to walk to the campsite and 3 hr to walk back to the car. Determine the speeds at which Kesha walks to the campsite and back to the car. Then find the distance between the campsite and the car.

a. Read the problem again.

b. Let x represent: ____.

c. Complete the table.

	Distance (mi)	Rate (mph)	Time (hr)
Walking to the campsite			
Walking back to the car			

d. Write a mathematical equation that represents the fact that the distance Kesha walks to the campsite equals the distance she walks back.

e. Solve the equation from part (d).

f. Interpret the results in words using proper units of measurement, and verify that your solution makes sense in the context of the problem.

Practice Exercises

Section 1.2

Study Skills Exercise

Writing mathematically is the skill of translating words to a mathematical expression or expressing a mathematical concept in words. Translating words to a mathematical expression helps us to solve real-world application problems. *Sum, difference, product,* and *quotient* are important words that indicate the operations of addition, subtraction, multiplication, and division. However, there are other ways that we can express these operations in words.

- Use three different wordings to represent the expression $-6 + 10$.
- Use three different wordings to represent $-7.5 - 8.1$.
- Use three different wordings to represent $\frac{1}{2} \cdot \left(-\frac{4}{5}\right)$.
- Use three different wordings to represent $-24 \div 6$.

Prerequisite Review

R.1. Identify the smallest positive integer that could be used to clear fractions in the equation.

$$\frac{2}{3}x - \frac{4}{5} = 1 - \frac{1}{6}x$$

R.2. Identify the smallest power of 10 that could be used to clear decimals in the equation.

$$0.06x + 0.24(5000 - x) = 1020$$

For Exercises R.3–R.4, solve the equation.

R.3. $0.06x + 0.24(5000 - x) = 1020$

R.4. $0.3(6000 + x) + 0.5x = 3400$

R.5. Compute a 7% sales tax on a picture frame that costs \$36.

R.6. Compute the commission that a realtor makes if her commission is 3% on a house that sells for \$340,000.

R.7. Tickets to a movie cost \$15 each.
- **a.** How much do 8 tickets cost?
- **b.** How much do x tickets cost?

R.8. T-shirts at a concert cost \$25 each.
- **a.** A teacher orders 30 shirts for a class. How much do 30 shirts cost?
- **b.** How much do y shirts cost?

R.9. An acid solution is 20% acid and 80% water.
- **a.** How much acid is in 20 mL of solution?
- **b.** How much acid is in $40 - x$ milliliters of solution?

R.10. A mixture of nuts is 40% peanuts.
- **a.** How many pounds of peanuts are in a 2-lb bag of the mixture?
- **b.** How many pounds of peanuts are in $10 - x$ pounds of the mixture?

R.11. A plane travels at 450 mph.
- **a.** If the plane travels for 2 hr, how far will it fly?
- **b.** If the plane travels for $t + 2$ hours, how far will it fly?

R.12. A boat travels at 12 mph.
- **a.** If the boat travels for 0.4 hr, how far will it go?
- **b.** If the boat travels for $t - 1$ hours, how far will it go?

Vocabulary and Key Concepts

1. a. Integers that follow one after the other without “gaps” are called ____________ integers.

b. The integers -2, 0, 2, and 4 are examples of consecutive ____________ integers, and the integers -3, -1, 1, and 3 are examples of consecutive ____________ integers.

c. Two consecutive integers differ by ____________, two consecutive odd integers differ by ____________, and two consecutive even integers differ by ____________.

d. If x represents the smaller of two consecutive integers, then the expression ____________ represents the next greater consecutive integer.

e. If x represents the smaller of two consecutive odd integers, then the expression ____________ represents the next greater consecutive odd integer.

f. If x represents the smallest of three consecutive even integers, then the expressions ____________ and ____________ represent the next two greater consecutive even integers.

g. Simple interest is interest computed on the principal invested or borrowed. Simple interest is computed as $I =$ ____________ where P is the principal, r is the annual ____________ rate, and t is the time in years.

h. If \$5000 is borrowed at 6.5% simple interest for 4 yr, then the amount of simple interest is ____________.

2. If $d = rt$, then $r = \frac{\square}{\square}$ and $t = \frac{\square}{\square}$.

Concept 1: Introduction to Problem Solving

3. If x represents a number, write an expression for 5 more than the number.

4. If n represents a number, write an expression for 10 less than the number.

5. If t represents a number, write an expression for 7 less than twice the number.

6. If y represents a number, write an expression for 4 more than 3 times the number.

7. The larger of two numbers is 3 more than twice the smaller. The difference of the larger number and the smaller number is 8. Find the numbers. **(See Example 1.)**

8. One number is 3 less than another. Their sum is 15. Find the numbers.

9. The sum of 3 times a number and 2 is the same as the difference of the number and 4. Find the number.

10. Twice the sum of a number and 3 is the same as 1 subtracted from the number. Find the number.

Concept 2: Applications Involving Consecutive Integers

11. The sum of two consecutive page numbers in a book is 223. Find the page numbers. **(See Example 2.)**

12. The sum of the numbers on two consecutive raffle tickets is 808,455. Find the numbers on the tickets.

13. The sum of two consecutive odd integers is -148. Find the two integers.

14. The sum of three consecutive integers is -57. Find the integers.

15. Three times the smaller of two consecutive even integers is the same as -146 minus 4 times the larger integer. Find the integers.

16. Four times the smaller of two consecutive odd integers is the same as 73 less than 5 times the larger. Find the integers.

17. Two times the sum of three consecutive odd integers is the same as 23 more than 5 times the largest integer. Find the integers.

18. Five times the smallest of three consecutive even integers is 10 more than twice the largest. Find the integers.

Concept 3: Applications Involving Percents and Rates

19. Belle had the choice of taking out a 4-yr car loan at 8.5% simple interest or a 5-yr car loan at 7.75% simple interest. If she borrows \$15,000, how much interest would she pay for each loan? Which option will require less interest? **(See Example 3.)**

20. Robert can take out a 3-yr loan at 8% simple interest or a 2-yr loan at $8\frac{1}{2}$% simple interest. If he borrows \$7000, how much interest will he pay for each loan? Which option will require less interest?

21. An account executive earns \$600 per month plus a 3% commission on sales. The executive's goal is to earn \$2400 this month. How much must she sell to achieve this goal?

22. A salesperson earns \$50 a day plus 12% commission on sales over \$200. If her daily earnings are \$76.88, how much money in merchandise did she sell?

Rob Melnychuk/Getty Images

23. J. W. is an artist and sells his pottery each year at a local Renaissance Festival. He keeps track of his sales and the 8.05% sales tax he collects by making notations in a ledger. Every evening he checks his records by counting the total money in his cash drawer. After a day of selling pottery, the cash totaled \$1293.38. How much is from the sale of merchandise and how much is sales tax?

24. Wayne County has a sales tax rate of 7%. How much does Mike's used car cost before tax if the total cost of the car *plus tax* is \$13,888.60?

25. The price of a swimsuit after a 20% markup is \$43.08. What was the price before the markup? **(See Example 4.)**

26. The price of a used textbook after a 35% markdown is \$29.25. What was the original price?

Concept 4: Applications Involving Principal and Interest

27. Tony has a total of \$12,500 in two accounts. One account pays 2% simple interest per year and the other pays 5% simple interest. If he earned \$370 in interest in the first year, how much did he invest in each account? **(See Example 5.)**

28. Lillian had \$15,000 invested in two accounts, one paying 9% simple interest and one paying 10% simple interest. How much was invested in each account if the interest after 1 yr is \$1432?

29. Jason borrowed \$18,000 in two loans. One loan charged 11% simple interest and the other charged 6% simple interest. After 1 yr, Jason paid a total of \$1380. Find the amount borrowed in each loan.

30. Amanda borrowed \$6000 from two sources: her parents and a credit union. Her parents charged 3% simple interest and the credit union charged 8% simple interest. If after 1 yr, Amanda paid \$255 in interest, how much did she borrow from her parents, and how much did she borrow from the credit union?

31. Donna invested money in two accounts: one paying 4% simple interest and the other paying 3% simple interest. She invested \$4000 more in the 4% account than in the 3% account. If she received \$720 in interest at the end of 1 yr, how much did she invest in each account?

32. Mr. Hall had some money in his bank earning 4.5% simple interest. He had \$5000 more deposited in a credit union earning 6% simple interest. If his total interest for 1 yr was \$1140, how much did he deposit in each account?

Concept 5: Applications Involving Mixtures

33. Ahmed mixes two plant fertilizers. How much fertilizer with 15% nitrogen should be mixed with 2 oz of fertilizer with 10% nitrogen to produce a fertilizer that is 14% nitrogen? **(See Example 6.)**

34. How much 8% saline solution should Kent mix with 80 cc (cubic centimeters) of an 18% saline solution to produce a 12% saline solution?

35. Jacque has 3 L of a 50% antifreeze mixture. How much 75% mixture should be added to get a mixture that is 60% antifreeze?

36. One fruit punch has 40% fruit juice and another is 70% fruit juice. How much of the 40% punch should be mixed with 10 gal of the 70% punch to create a fruit punch that is 45% fruit juice?

37. How many liters of an 18% alcohol solution must be added to a 10% alcohol solution to get 20 L of a 15% alcohol solution?

38. How many milliliters of a 2.5% bleach solution must be mixed with a 10% bleach solution to produce 600 mL of a 5% bleach solution?

39. Ronald has a 12% solution of the fertilizer Super Grow. How much pure Super Grow should he add to the mixture to get 32 oz of a 17.5% concentration?

40. How many ounces of water must be added to 20 oz of an 8% salt solution to make a 2% salt solution?

Concept 6: Applications Involving Distance, Rate, and Time

41. An airplane travels 60 mph faster from Atlanta to Fort Lauderdale than it travels on the return trip from Fort Lauderdale to Atlanta. If it takes 2 hr from Atlanta to Fort Lauderdale and 2.5 hr for the return trip, determine the speed of each trip. **(See Example 7.)**

42. A woman can hike 1 mph faster down a trail to Archuletta Lake than she can on the return trip uphill. It takes her 3 hr to get to the lake and 6 hr to return. What is her speed hiking down to the lake?

43. Two cars are 192 mi apart and travel toward each other on the same road. They meet in 2 hr. One car travels 4 mph faster than the other. What is the average speed of each car?

44. Two cars are 190 mi apart and travel toward each other along the same road. They meet in 2 hr. One car travels 5 mph slower than the other car. What is the average speed of each car?

Mixed Exercises

45. The sum of two integers is 30. Ten times one integer is 5 times the other integer. Find the integers. (*Hint:* If one number is x, then the other number is $30 - x$.)

46. The sum of two integers is 10. Three times one integer is 3 less than 8 times the other integer. Find the integers. (*Hint:* If one number is x, then the other number is $10 - x$.)

47. An older model of smartphone is marked down to \$89.55 when newer technology comes on the market. If this is the new price after a 55% markdown, determine the original price.

48. A hardcover book is marked down to \$15.60 once the book comes out in paperback. If this is the new price after a 35% markdown, determine the original price.

49. Two boats traveling the same direction leave a harbor at noon. After 3 hr they are 60 mi apart. If one boat travels twice as fast as the other, find the average speed of each boat.

50. Two canoes travel down a river, starting at 9:00. One canoe travels twice as fast as the other. After 3.5 hr, the canoes are 5.25 mi apart. Find the average speed of each canoe.

51. Ms. Riley deposited some money in an account paying 5% simple interest and twice that amount in an account paying 6% simple interest. If the total interest from the two accounts is \$765 for 1 yr, how much was deposited in each account?

52. Sienna put some money an account earning 4.2% simple interest. She deposited twice that amount in another account paying 4% simple interest. After 1 yr her total interest was \$488. How much did Sienna deposit in the 4% account?

53. Two different teas are mixed to make a blend that will be sold at a fair. Black tea sells for \$2.20 per pound and green tea sells for \$3.00 per pound. How much of each should be used to obtain 4 lb of a blend selling for \$2.50?

54. A nut mixture consists of almonds and cashews. Almonds are \$4.98 per pound, and cashews are \$6.98 per pound. How many pounds of each type of nut should be mixed to produce 16 lb selling for \$5.73 per pound?

55. After a recent crash of the housing market, the median price of a new home (including land) in the United States was \$202,100. This represents a drop of 6% from the previous year. What was the median price the previous year?

John Lund/Drew Kelly/Blend Images

56. According to the Bureau of Labor Statistics, the median weekly salary for an individual with a bachelor's degree is \$1026 per week. This represents an increase of 35% over the salary of an individual with an associate's degree. What is the median weekly salary of an individual with an associate's degree?

Applications to Geometry and Literal Equations

Section 1.3

Concepts

1. Applications Involving Geometry
2. Literal Equations

1. Applications Involving Geometry

The word problems presented in Examples 1 and 2 involve the use of geometric formulas.

Example 1 **Solving an Application Involving Perimeter**

The length of a rectangular corral is 2 ft more than 3 times the width. The corral is situated such that one of its shorter sides is adjacent to a barn and does not require fencing. If the total amount of fencing is 774 ft, find the dimensions of the corral.

Solution:

Read the problem and draw a sketch (Figure 1-2).

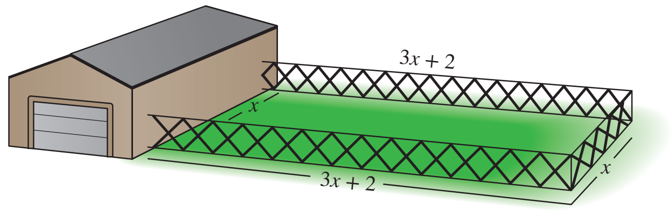


Figure 1-2

TIP: In Example 1, the length of the field is given in terms of the width. Therefore, we let x represent the width.

Let x represent the width. Label variables.

Let $3x + 2$ represent the length.

To write a verbal model, we might consider using the formula for the perimeter of a rectangle. However, the formula $P = 2l + 2w$ incorporates all four sides of the rectangle. The formula must be modified to include only one factor of the width.

$$\begin{pmatrix}\text{Distance around}\\ \text{three sides}\end{pmatrix} = \begin{pmatrix}\text{2 times}\\ \text{the length}\end{pmatrix} + \begin{pmatrix}\text{1 times}\\ \text{the width}\end{pmatrix} \qquad \text{Verbal model}$$

$$774 = 2(3x + 2) + x \qquad \text{Mathematical equation}$$

$774 = 2(3x + 2) + x$	Solve for x.
$774 = 6x + 4 + x$	Apply the distributive property.
$774 = 7x + 4$	Combine *like* terms.
$770 = 7x$	Subtract 4 from both sides.
$110 = x$	Divide by 7 on both sides.
$x = 110$	

Because x represents the width, the width of the corral is 110 ft. The length is given by

$3x + 2$ or $3(110) + 2 = 332$ Interpret the results.

The width of the corral is 110 ft, and the length is 332 ft.

Avoiding Mistakes

To check the answer to Example 1, verify that the three sides add to 774 ft.

110 ft + 332 ft + 332 ft

= 774 ft ✓

Skill Practice

1. The length of Karen's living room is 2 ft longer than the width. The perimeter is 80 ft. Find the length and width.

Recall some important facts involving angles.

- Two angles are **complementary** if the sum of their measures is 90°.
- Two angles are **supplementary** if the sum of their measures is 180°.
- The sum of the measures of the angles within a triangle is 180°.

Example 2 Solving an Application Involving Angles

Two angles are complementary. One angle measures 10° less than 4 times the other angle. Find the measure of each angle (Figure 1-3).

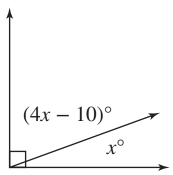


Figure 1-3

Answer

1. The length is 21 ft, and the width is 19 ft.

Solution:

Let x represent the measure of one angle.

Let $4x - 10$ represent the measure of the other angle.

Recall that two angles are complementary if the sum of their measures is 90°. Therefore, a verbal model is

(One angle) + (the complement of the angle) = 90°	Verbal model
$x + (4x - 10) = 90$	Mathematical equation
$5x - 10 = 90$	Solve for x.
$5x = 100$	
$x = 20$	

If $x = 20$, then $4x - 10 = 4(20) - 10 = 70$. The two angles are 20° and 70°.

Skill Practice

2. Two angles are supplementary, and the measure of one angle is 16° less than 3 times the other. Find the measure of each angle.

Avoiding Mistakes

From Example 2, notice that the sum of the measures of the angles 20° + 70° is 90° as expected.

2. Literal Equations

Literal equations are equations that contain several variables. A formula is a literal equation with a specific application. For example, the perimeter P of a rectangle can be found by the formula $P = 2l + 2w$. In this equation, P is expressed in terms of the length l and the width w. However, in science and other branches of applied mathematics, formulas may be more useful in alternative forms.

For example, the formula $P = 2l + 2w$ can be manipulated to solve for either l or w:

Solve for l		**Solve for w**	
$P = 2l + 2w$		$P = 2l + 2w$	
$P - 2w = 2l$	Subtract $2w$.	$P - 2l = 2w$	Subtract $2l$.
$\dfrac{P - 2w}{2} = l$	Divide by 2.	$\dfrac{P - 2l}{2} = w$	Divide by 2.
$l = \dfrac{P - 2w}{2}$		$w = \dfrac{P - 2l}{2}$	

To solve a literal equation for a specified variable, use the addition, subtraction, multiplication, and division properties of equality.

Example 3 **Applying a Literal Equation**

Buckingham Fountain is one of Chicago's most familiar landmarks. With 133 jets spraying a total of 14,000 gal of water per minute, Buckingham Fountain is one of the world's largest fountains. The circumference of the fountain is approximately 880 ft.

a. The circumference of a circle is given by $C = 2\pi r$. Solve the equation for r.

b. Use the equation from part (a) to find the radius and diameter of the fountain. Use 3.14 for π and round to the nearest foot.

Answer

2. 49° and 131°

Solution:

Steve Allen/Brand X Pictures/Getty Images

a. $C = 2\pi r$

$\frac{C}{2\pi} = \frac{\cancel{2\pi} r}{\cancel{2\pi}}$ To isolate r, divide both sides by 2π.

$\frac{C}{2\pi} = r$

$r = \frac{C}{2\pi}$

b. $r \approx \frac{880 \text{ ft}}{2(3.14)}$ Substitute 880 ft for C and 3.14 for π.

$\approx 140 \text{ ft}$

The radius is approximately 140 ft. The diameter is twice the radius ($d = 2r$). Therefore, the diameter is 280 ft.

Skill Practice The formula to compute the surface area S of a sphere is given by $S = 4\pi r^2$.

3. Solve the equation for π.

4. A sphere has a surface area of 113 in.^{2} and a radius of 3 in. Use the formula found in part (a) to approximate π. Round to two decimal places.

Example 4 Solving a Literal Equation

The formula to find the area of a trapezoid is given by $A = \frac{1}{2}(b_1 + b_2)h$, where b_1 and b_2 are the lengths of the parallel sides and h is the height. (See Figure 1-4.)

Solve this formula for b_1.

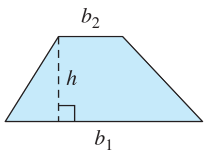


Figure 1-4

Solution:

$A = \frac{1}{2}(b_1 + b_2)h$ The goal is to isolate b_1.

$2A = 2 \cdot \frac{1}{2}(b_1 + b_2)h$ Multiply by 2 to clear fractions.

$2A = (b_1 + b_2)h$ Apply the distributive property.

$2A = b_1h + b_2h$

$2A - b_2h = b_1h$ Subtract b_2h from both sides.

$\frac{2A - b_2h}{h} = \frac{b_1\cancel{h}}{\cancel{h}}$ Divide by h.

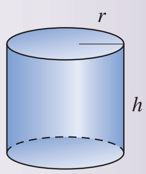
$\frac{2A - b_2h}{h} = b_1$

FOR REVIEW

Recall that small characters to the right of and below a variable are called **subscripts**. Subscript notation is used to designate different but related variables. In Example 4, b_1 and b_2 are different variables representing the two distinct bases in the trapezoid.

Skill Practice

5. The formula for the volume of a right circular cylinder is $V = \pi r^2 h$. Solve for h.



Answers

3. $\pi = \frac{S}{4r^2}$ **4.** 3.14 **5.** $h = \frac{V}{\pi r^2}$

TIP: When solving a literal equation for a specified variable, there is sometimes more than one way to express your final answer. This flexibility often presents difficulty for students. Students may leave their answer in one form, but the answer given in the text may look different. Yet both forms may be correct. To know if your answer is equivalent to the form given in the text, you must try to manipulate it to look like the answer in the book, a process called *form fitting*.

The literal equation from Example 4 can be written in several different forms. The quantity $(2A - b_2h)/h$ can be split into two fractions.

$$b_1 = \frac{2A - b_2h}{h} = \frac{2A}{h} - \frac{b_2h}{h} = \frac{2A}{h} - b_2$$

Example 5 Solving a Literal Equation

Given $-2x + 3y = 5$, solve for y.

Solution:

$$-2x + 3y = 5$$

$$3y = 2x + 5 \quad \text{Add } 2x \text{ to both sides.}$$

$$\frac{3y}{3} = \frac{2x + 5}{3} \quad \text{Divide by 3 on both sides.}$$

$$y = \frac{2x + 5}{3} \quad \text{or} \quad y = \frac{2}{3}x + \frac{5}{3}$$

Skill Practice Solve for y.

6. $5x + 2y = 11$

Sometimes the variable we want to isolate may appear in more than one term in a literal equation. In such a case, isolate all terms with that variable on one side of the equation. Then apply the distributive property as demonstrated in Example 6.

Example 6 Solving a Literal Equation

Solve the equation for x. $\quad ax - 3 = cx + 7$

Solution:

$$ax - 3 = cx + 7$$

$$ax - cx = 10$$

Collect the terms containing x on one side of the equation. Collect the remaining terms on the other side.

The variable x appears twice in the equation. To isolate x, we want x to appear in only one term. To accomplish this, we apply the distributive property in reverse.

$$x(a - c) = 10$$

Apply the distributive property. The variable x now appears one time in the equation.

$$\frac{x\cancel{(a - c)}}{\cancel{(a - c)}} = \frac{10}{(a - c)}$$

Divide both sides by $(a - c)$.

$$x = \frac{10}{a - c}$$

TIP: Applying the distributive property in reverse is called *factoring*. Factoring will be studied in detail in a later chapter.

Answer

6. $y = \frac{-5x + 11}{2}$ or $y = -\frac{5}{2}x + \frac{11}{2}$

Answer

7. $t = \dfrac{5}{m - n}$

Skill Practice Solve for t.

7. $mt + 4 = nt + 9$

Section 1.3 Activity

A.1. How are complementary angles related?

A.2. How are supplementary angles related?

A.3. What is the sum of the measures of the angles of a triangle?

A.4. Write a formula for the perimeter P of a rectangle of length l and width w.

For Exercises A.5–A.7, follow these steps to solve the application.

- Read the problem.
- Label the variable(s).
- Write a verbal model.
- Write an equation to represent the verbal model.
- Solve the equation.
- Interpret the results in words using the proper units of measurement, and verify that your solution makes sense in the context of the problem.

A.5. The length of a children's rectangular playground is 40 ft longer than the width. The perimeter is 400 ft. Find the length and width.

A.6. The measure of the largest angle in a triangle is three times the measure of the smallest angle. The middle angle is 5° more than the smallest angle. Find the measure of each angle.

A.7. Two angles are supplementary. The measure of one angle is 4° less than 7 times the measure of the other angle. Find the measure of each angle.

A.8. Two traffic cameras are set up 5 mi apart along a highway. The cameras use digital image recognition to identify vehicles as they pass each camera station. A certain vehicle passes the second camera 4 min $\left(\dfrac{1}{15}\text{ hr}\right)$ after passing the first camera.

a. Note that the relationship $d = rt$ indicates that distance d equals the product of the rate r times the time t. Solve the equation $d = rt$ for r.

b. Using the formula from part (a), substitute 5 mi for d and $\frac{1}{15}$ hr for t. What was the average speed of the vehicle over the 5-mi stretch of highway?

c. If the posted speed limit is 60 mph, was the vehicle speeding at some point over this part of the highway?

d. Use this example to explain why a formula such as $d = rt$ might be valuable when it is written in a different form with a different variable isolated.

A.9 **a.** Solve the equation for y. $2y + 7 = 5$

b. Solve the equation for y. $cy + d = m$

c. Explain the similarities for solving each equation.

A.10 **a.** Solve the equation for y. $-5x - 4y = 8$

b. Is there more than one way to express the final answer? Explain.

Practice Exercises

Section 1.3

Study Skills Exercise

With word problems, important mathematical relationships are often hidden within the text of the problem. This requires that you read the narrative and understand the vocabulary and key ideas. For example:

One acute angle in a right triangle is 20° more than the other acute angle. Find the measure of each acute angle.

- To extract the information necessary to solve this problem, you need to be familiar with key vocabulary. In this case, what is meant by an "acute angle"? What is meant by "right triangle"?
- As you read a word problem, be thinking of key mathematical formulas or theorems that might apply to the given problem. In this case, how are three angles within a triangle related? How are the two acute angles related in a right triangle?

Prerequisite Review

R.1. **a.** Determine the complement of a 71° angle.
b. Determine the supplement of a 71° angle.

R.2. **a.** Determine the complement of a 14° angle.
b. Determine the supplement of a 14° angle.

R.3. Two angles in a triangle are 42° and 61°. What is the measure of the third angle?

R.4. In a right triangle, one of the acute angles is 44°. What is the measure of the other acute angle?

R.5. **a.** Give a formula for the perimeter P of a rectangle of length l and width w.
b. Find the perimeter of a rectangle of length 12 ft and width 8 ft.

R.6. **a.** Give a formula for the area A of a rectangle of length l and width w.
b. Find the area of a rectangle of length 6.4 cm and width 2 cm.

Concept 1: Applications Involving Geometry

For Exercises 1–12, use an appropriate geometry formula.

1. A volleyball court is twice as long as it is wide. If the perimeter is 177 ft, find the dimensions of the court. **(See Example 1.)**

2. The length of a rectangular picture frame is 4 in. less than twice the width. The perimeter is 112 in. Find the length and the width.

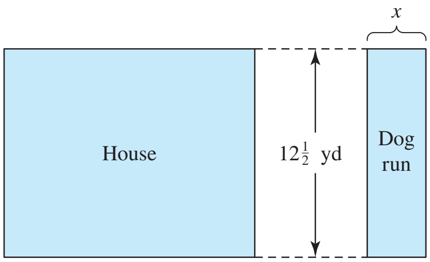
3. The lengths of the sides of a triangle are given by three consecutive even integers. The perimeter is 24 m. What is the length of each side?

4. A triangular garden has sides that can be represented by three consecutive integers. If the perimeter of the garden is 15 ft, what are the lengths of the sides?

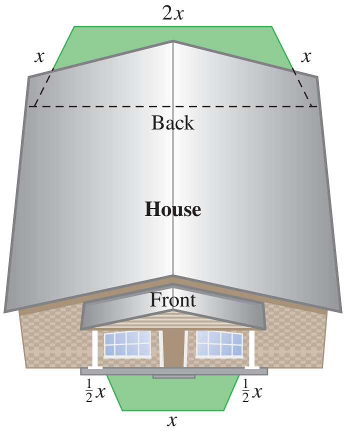
5. Raoul would like to build a rectangular dog run in the rear of his backyard, away from the house. The width of the yard is $12\frac{1}{2}$ yd, and Raoul wants an area of 100 yd^2 for his dog.

a. Find the dimensions of the dog run.

b. How much fencing would Raoul need to enclose the dog run?



6. Joanne wants to plant a flower garden in her backyard in the shape of a trapezoid, adjacent to her house (see the figure). She also wants a front yard garden in the same shape, but with sides one-half as long. What should the dimensions be for each garden if Joanne has only a total of 60 ft of fencing?



7. George built a rectangular pen for his rabbit such that the length is 7 ft less than twice the width. If the perimeter is 40 ft, what are the dimensions of the pen?

8. Antoine wants to put edging in the form of a square around a tree in his front yard. He has enough money to buy 18 ft of edging. Find the dimensions of the square that will use all the edging.

9. The measures of two angles in a triangle are equal. The third angle measures 2 times the sum of the equal angles. Find the measures of the three angles.

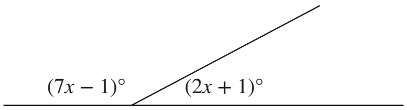
10. The smallest angle in a triangle is one-half the measure of the largest. The middle angle measures 25° less than the largest. Find the measures of the three angles.

11. Two angles are complementary. One angle is 5 times as large as the other angle. Find the measure of each angle. **(See Example 2.)**

12. Two angles are supplementary. One angle measures 12° less than 3 times the other. Find the measure of each angle.

In Exercises 13–20, solve for x, and then find the measure of each angle.

13.



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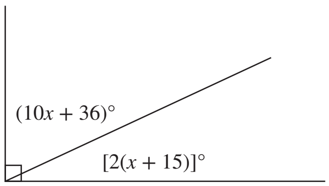

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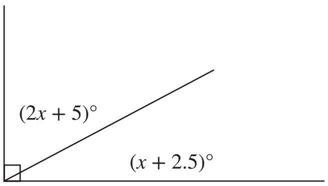

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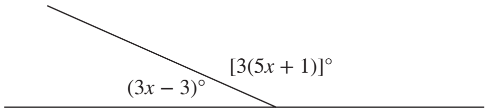

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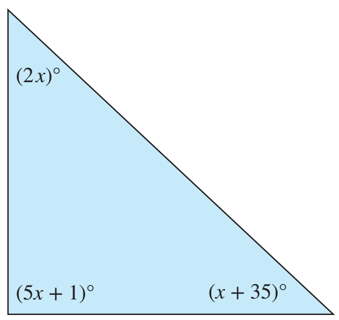

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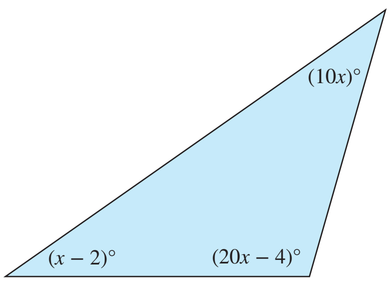

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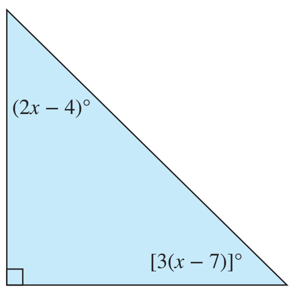

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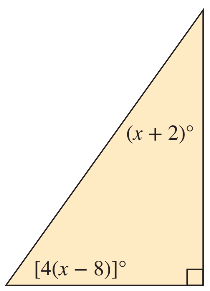


Concept 2: Literal Equations

21. Recently the winner of the Indianapolis 500 car race finished with the time of 3 hr 5 min 56 sec ($\approx$ 3.099 hr). **(See Example 3.)**

a. The relationship among the variables distance, rate, and time is given by $d = rt$. Solve for r.

b. Determine the average rate of speed if the total distance for the race is 500 mi. Round to one decimal place.

22. Recently the winner of the Daytona 500 car race finished with an average speed of 157.5 mph.

a. The relationship among the variables distance, rate, and time is given by $d = rt$. Solve for t.

b. Determine the total time of the race if the race is 500 mi. Round to one decimal place.

23. The amount of simple interest earned or borrowed is the product of the principal, the annual interest rate, and the time invested (in years). This is given by $I = Prt$.

a. Solve $I = Prt$ for t.

b. Determine the amount of time necessary for the interest on \$5000 invested at 4% to reach \$1400.

24. The force of an object is equal to its mass times the acceleration, or $F = ma$.

a. Solve $F = ma$ for m.

b. The force on an object is 24.5 N (newtons), and the acceleration is 9.8 m/sec^2. Find the mass of the object (the answer will be in kilograms).

For Exercises 25–42, solve for the indicated variable. **(See Example 4.)**

25. $A = lw$ for l

26. $C_1 = \frac{5}{2}R$ for R

27. $I = Prt$ for P

28. $a + b + c = P$ for b

29. $W = K_2 - K_1$ for K_1

30. $y = mx + b$ for x

31. $F = \frac{9}{5}C + 32$ for C

32. $C = \frac{5}{9}(F - 32)$ for F

33. $K = \frac{1}{2}mv^2$ for v^2

34. $I = Prt$ for r

35. $v = v_0 + at$ for a

36. $a^2 + b^2 = c^2$ for b^2

37. $w = p(v_2 - v_1)$ for v_2

38. $A = lw$ for w

39. $ax + by = c$ for y

40. $P = 2L + 2W$ for L

41. $V = \dfrac{1}{3}Bh$ for B

42. $V = \dfrac{1}{3}\pi r^2 h$ for h

When we study linear equations in two variables, it will be necessary to change equations from the form $Ax + By = C$ to $y = mx + b$. For Exercises 43–54, express each equation in the form $y = mx + b$ by solving for y. **(See Example 5.)**

43. $3x + y = 6$

44. $x + y = -4$

45. $5x - 4y = 20$

46. $-4x - 5y = 25$

47. $-6x - 2y = 13$

48. $5x - 7y = 15$

49. $3x - 3y = 6$

50. $2x - 2y = 8$

51. $9x + \frac{4}{3}y = 5$

52. $4x - \frac{1}{3}y = 5$

53. $-x + \frac{2}{3}y = 0$

54. $x - \frac{1}{4}y = 0$

In statistics, the z-score formula $z = \frac{x - \mu}{\sigma}$ is used in studying probability. Use this formula for Exercises 55–56.

55. **a.** Solve $z = \frac{x - \mu}{\sigma}$ for x.

b. Find x when $z = 2.5$, $\mu = 100$, and $\sigma = 12$.

56. **a.** Solve $z = \frac{x - \mu}{\sigma}$ for μ.

b. Find μ when $x = 150$, $z = 2.5$, and $\sigma = 16$.

57. Which expressions are equivalent to $\frac{-5}{x - 3}$?

a. $-\frac{5}{x - 3}$ **b.** $\frac{5}{3 - x}$ **c.** $\frac{5}{-x + 3}$

58. Which expressions are equivalent to $\frac{z - 1}{-2}$?

a. $\frac{1 - z}{2}$ **b.** $-\frac{z - 1}{2}$ **c.** $\frac{-z + 1}{2}$

59. Which expressions are equivalent to $\frac{-x - 7}{y}$?

a. $-\frac{x + 7}{y}$ **b.** $\frac{x + 7}{-y}$ **c.** $\frac{-x - 7}{-y}$

60. Which expressions are equivalent to $\frac{-3w}{-x - y}$?

a. $-\frac{3w}{-x - y}$ **b.** $\frac{3w}{x + y}$ **c.** $-\frac{-3w}{x + y}$

For Exercises 61–69, solve for the indicated variable. **(See Example 6.)**

61. $6t - rt = 12$ for t

62. $5 = 4a + ca$ for a

63. $ax + 5 = 6x + 3$ for x

64. $cx - 4 = dx + 9$ for x

65. $A = P + Prt$ for P

66. $A = P + Prt$ for r

67. $T = mg - mf$ for m

68. $T = mg - mf$ for f

69. $ax + by = cx + z$ for x

Section 1.4 Linear Inequalities in One Variable

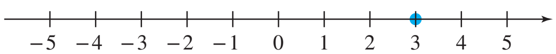
Concepts

1. **Solving Linear Inequalities**
2. **Applications of Inequalities**

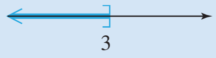
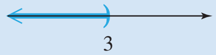
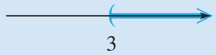
1. Solving Linear Inequalities

Previously, we learned how to solve linear equations and their applications. In this section, we will learn the process of solving linear *inequalities*. A **linear inequality** in one variable, x, is defined as any relationship of the form: $ax + b < c$, $ax + b \leq c$, $ax + b > c$, or $ax + b \geq c$, where $a \neq 0$.

The solution to the equation $x = 3$ can be graphed as a single point on the number line.



Now consider the related *inequalities*.

Inequality	Solutions	Graph	Interval Notation	Set-Builder Notation
$x \le 3$	All real numbers less than or equal to 3.		$(-\infty, 3]$	$\{x \mid x \le 3\}$
$x < 3$	All real numbers less than 3.		$(-\infty, 3)$	$\{x \mid x < 3\}$
$x \ge 3$	All real numbers greater than or equal to 3.		$[3, \infty)$	$\{x \mid x \ge 3\}$
$x > 3$	All real numbers greater than 3.		$(3, \infty)$	$\{x \mid x > 3\}$

Avoiding Mistakes

Recall that parentheses (or) indicate that an endpoint to an interval is not included in the interval. Brackets [or] indicate that an endpoint is included.

The solution sets to these inequalities have an infinite number of elements that cannot all be listed. Therefore, the solution set can be shown in a graph, or expressed in interval notation or in set-builder notation.

The addition and subtraction properties of equality indicate that a value added to or subtracted from both sides of an equation results in an equivalent equation. The same is true for inequalities.

Addition and Subtraction Properties of Inequality

Let a, b, and c represent real numbers.

*Addition property of inequality: If $a < b$ then $a + c < b + c$

*Subtraction property of inequality: If $a < b$ then $a - c < b - c$

*These properties may also be stated for $a \le b$, $a > b$, and $a \ge b$.

Example 1 Solving a Linear Inequality

Solve the inequality.

$$3x - 7 > 2(x - 4) - 1$$

Solution:

$3x - 7 > 2(x - 4) - 1$

$3x - 7 > 2x - 8 - 1$ Apply the distributive property.

$3x - 7 > 2x - 9$ Combine *like* terms.

$3x - 2x - 7 > 2x - 2x - 9$ Subtract $2x$ from both sides.

$x - 7 > -9$

$x - 7 + 7 > -9 + 7$ Add 7 to both sides.

$x > -2$

Set-builder notation: $\{x \mid x > -2\}$ 

Interval notation: $(-2, \infty)$

FOR REVIEW

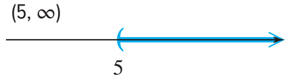
The first step in solving a linear inequality is the same as the first step in solving a linear equation. Simplify both sides by applying the distributive property and combining *like* terms.

Skill Practice Solve the inequality.

1. $4(2x - 1) > 7x + 1$

Answer

1. $\{x \mid x > 5\}$
$(5, \infty)$

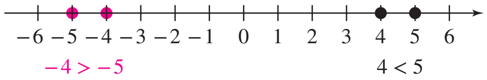


Multiplying both sides of an equation by the same nonzero quantity results in an equivalent equation. However, the same is not always true for an inequality. If you multiply or divide an inequality by a *negative* quantity, the direction of the inequality symbol must be *reversed*.

For example, consider multiplying or dividing the inequality $4 < 5$ by -1.

Multiply/divide by -1: $4 < 5$

$-4 > -5$



The number 4 lies to the left of 5 on the number line. However, -4 lies to the right of -5. Changing the signs of two numbers changes their relative position on the number line. This is stated formally in the multiplication and division properties of inequality.

Multiplication and Division Properties of Inequality

Let a, b, and c represent real numbers.

*If c is *positive* and $a < b$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

*If c is *negative* and $a < b$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

The second statement indicates that if both sides of an inequality are multiplied or divided by a negative quantity, the inequality sign must be *reversed*.

*These properties may also be stated for $a \le b$, $a > b$, and $a \ge b$.

Example 2 Solving a Linear Inequality

Solve the inequality.

$$-2x - 5 < 2$$

Solution:

$$-2x - 5 < 2$$

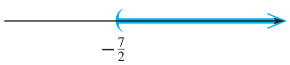
$$-2x - 5 + 5 < 2 + 5 \qquad \text{Add 5 to both sides.}$$

$$-2x < 7$$

$$\frac{-2x}{-2} > \frac{7}{-2} \qquad \text{Divide by } -2 \text{ (\textit{reverse} the inequality sign).}$$

$$x > -\frac{7}{2} \quad \text{or} \quad x > -3.5$$

Set-builder notation: $\{x \mid x > -\frac{7}{2}\}$



Interval notation: $(-\frac{7}{2}, \infty)$

Avoiding Mistakes

Do not forget to reverse the direction of the inequality sign when multiplying or dividing by a negative number.

TIP: The inequality $-2x - 5 < 2$ could have been solved by isolating x on the right-hand side of the inequality. This creates a positive coefficient on the x term and eliminates the need to divide by a negative number.

$$-2x - 5 < 2$$

$-5 < 2x + 2$	Add $2x$ to both sides.
$-7 < 2x$	Subtract 2 from both sides.
$\dfrac{-7}{2} < \dfrac{2x}{2}$	Divide by 2 (because 2 is positive, do *not* reverse the inequality sign).
$-\dfrac{7}{2} < x$	(Note that the inequality $-\frac{7}{2} < x$ is equivalent to $x > -\frac{7}{2}$.)

Skill Practice Solve the inequality.

2. $-4x - 12 \geq 20$

Example 3 Solving a Linear Inequality

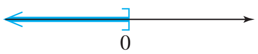
Solve the inequality.

$$-6(x - 3) \geq 2 - 2(x - 8)$$

Solution:

$-6(x - 3) \geq 2 - 2(x - 8)$	
$-6x + 18 \geq 2 - 2x + 16$	Apply the distributive property.
$-6x + 18 \geq 18 - 2x$	Combine *like* terms.
$-6x + 2x + 18 \geq 18 - 2x + 2x$	Add $2x$ to both sides.
$-4x + 18 \geq 18$	
$-4x + 18 - 18 \geq 18 - 18$	Subtract 18 from both sides.
$-4x \geq 0$	
$\dfrac{-4x}{-4} \leq \dfrac{0}{-4}$	Divide by -4 (*reverse* the inequality sign).
$x \leq 0$	

Set-builder notation: $\{x \mid x \leq 0\}$



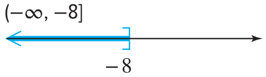
Interval notation: $(-\infty, 0]$

Skill Practice Solve the inequality.

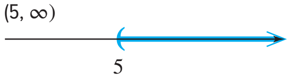
3. $5(3x + 1) < 4(5x - 5)$

Answers

2. $\{x \mid x \leq -8\}$
$(-\infty, -8]$



3. $\{x \mid x > 5\}$
$(5, \infty)$



Example 4 Solving a Linear Inequality

Solve the inequality. $\dfrac{-5x+2}{-3} > x+2$

Solution:

$$\frac{-5x+2}{-3} > x+2$$

$$-3\left(\frac{-5x+2}{-3}\right) < -3(x+2) \qquad \text{Multiply by } -3 \text{ to clear fractions (\textit{reverse} the inequality sign).}$$

$$-5x+2 < -3x-6$$

$$-2x+2 < -6 \qquad \text{Add } 3x \text{ to both sides.}$$

$$-2x < -8 \qquad \text{Subtract 2 from both sides.}$$

$$\frac{-2x}{-2} > \frac{-8}{-2} \qquad \text{Divide by } -2 \text{ (the inequality sign is reversed \textit{again}).}$$

$$x > 4 \qquad \text{Simplify.}$$

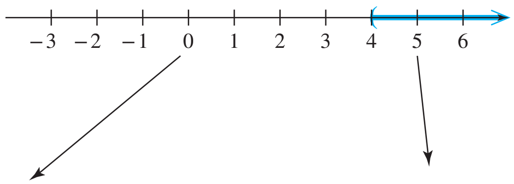
Set-builder notation: $\{x \mid x > 4\}$

Interval notation: $(4, \infty)$

Skill Practice Solve the inequality.

4. $\dfrac{x+1}{-3} \geq -x+1$

In Example 4, the inequality sign was reversed twice: once for multiplying the inequality by -3 and once for dividing by -2. If you are in doubt about whether you have the inequality sign in the correct direction, you can check your final answer by using the **test point method**. That is, pick a point in the proposed solution set, and verify that it makes the original inequality true. Furthermore, any test point picked outside the solution set should make the original inequality false.



Pick $x = 0$ as a test point

$$\frac{-5x+2}{-3} > x+2$$

$$\frac{-5(0)+2}{-3} \stackrel{?}{>} (0)+2$$

$$\frac{2}{-3} \stackrel{?}{>} 2 \qquad \text{False}$$

Pick $x = 5$ as a test point

$$\frac{-5x+2}{-3} > x+2$$

$$\frac{-5(5)+2}{-3} \stackrel{?}{>} (5)+2$$

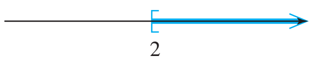
$$\frac{-23}{-3} \stackrel{?}{>} 7$$

$$7\tfrac{2}{3} \stackrel{?}{>} 7 \qquad \text{True}$$

Because a test point to the right of $x = 4$ makes the inequality true, we have shaded the correct part of the number line.

Answer

4. $\{x \mid x \geq 2\}$
$[2, \infty)$



2. Applications of Inequalities

Example 5 Solving a Linear Inequality Application

Beth received grades of 97, 82, 89, and 99 on her first four algebra tests. To earn an "A" in the course, she needs an average of 90 or more. What scores can she receive on the fifth test to earn an "A"?

Solution:

Let x represent the score on the fifth test.

The average of the five tests is given by $\dfrac{97 + 82 + 89 + 99 + x}{5}$

To earn an A, we have:

(Average of test scores) ≥ 90 Verbal model

$\dfrac{97 + 82 + 89 + 99 + x}{5} \geq 90$ Mathematical equation

$\dfrac{367 + x}{5} \geq 90$ Simplify the numerator.

$5\left(\dfrac{367 + x}{5}\right) \geq 5(90)$ Clear fractions.

$367 + x \geq 450$ Simplify.

$x \geq 83$

To earn an "A," Beth would have to score at least 83 on her fifth test.

Skill Practice

5. Jamie is a salesman who works on commission, so his salary varies from month to month. To qualify for an automobile loan, his monthly salary must average at least \$2100 for 6 months. His salaries for the past 5 months have been \$1800, \$2300, \$1500, \$2200, and \$2800. What amount does he need to earn in the last month to qualify for the loan?

FOR REVIEW

To compute the average (or mean) of a set of data, first add the values in the data set. Then divide by the number of values in the data set.

Example 6 Solving a Linear Inequality Application

The water level in a retention pond in northern California is 7.2 ft. During a time of drought, the water level decreases at a rate of 0.05 ft/day. The water level L (in ft) is given by $L = 7.2 - 0.05d$, where d is the number of days after the drought begins (Figure 1-5). For which days after the beginning of the drought will the water level be less than 6 ft?

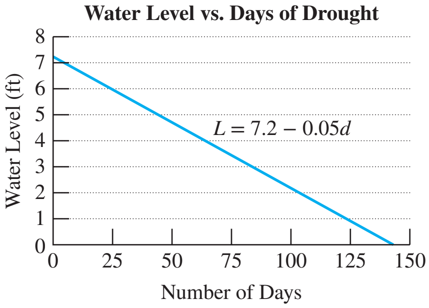


Figure 1-5

Answer

5. Jamie's salary must be at least \$2000.

Solution:

We require that $L < 6$.

$$\begin{aligned} \overbrace{7.2 - 0.05d}^{L} &< 6 && \text{Substitute } 7.2 - 0.05d \text{ for } L. \\ 7.2 - 7.2 - 0.05d &< 6 - 7.2 && \text{Subtract 7.2 from both sides.} \\ -0.05d &< -1.2 \\ \frac{-0.05d}{-0.05} &> \frac{-1.2}{-0.05} && \text{Divide both sides by } -0.05 \text{ and reverse the direction of the inequality sign.} \\ d &> 24 \end{aligned}$$

The water level in the pond will be less than 6 ft after day 24 of the drought.

Skill Practice

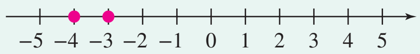
6. The population of Alaska has steadily increased since 1950 according to the equation $P = 10t + 117$, where t represents the number of years after 1950 and P represents the population in thousands. For what years since 1950 was the population less than 417 thousand people?

Answer

6. The population was less than 417 thousand for $t < 30$. This corresponds to the years before 1980.

Section 1.4 Activity

A.1. Refer to the number line to fill in the blanks with $<$ or $>$.



a. $3 \square 4$ **b.** $-3 \square -4$

A.2. From the number line, we know that $3 < 4$. What is the effect of multiplying both sides of the inequality by -1?

A.3. The following inequalities can each be solved in one step. Solve the inequality. Graph the solution set and write the solution set in interval notation.

a. $-3x \geq 12$

b. $-3 + x \geq 12$

c. $-\dfrac{x}{3} > 12$

d. $x + 3 > 12$

A.4. Which inequalities in Exercise A.3 required that the direction of the inequality sign be reversed? Why?

For Exercises A.5–A.7, solve the inequality. Write the solution set in interval notation.

A.5. $-6x + 4 \leq 4x - 12$ **A.6.** $-5(2x - 4) + 6x > 3(x + 2) - 7$

A.7. $\dfrac{5x + 6}{-2} > \dfrac{3x - 4}{4}$

A.8. Greg has the following test grades for four of his chemistry tests: 92, 84, 76, 96. The final exam in the class counts as two test grades. What score(s) would Greg need to earn an "A" in the class if the criterion for an "A" is an overall average of 90 or better?

Practice Exercises — Section 1.4

Study Skills Exercise

How you respond when you are faced with tough times and adversity is a personal characteristic called "grit." Grit is defined as the sustained perseverance and passion for long-term goals. Achievement and success are not the result of intelligence and IQ alone. Rather, they are a combination of both skill *and* grit.

Reflect on the following questions. Write your answers in your math notebook. Revisit your responses at any point during the course when you find yourself challenged or inclined to give up.

- What are your long-term career goals?
- How is this course relevant to your future career?
- What are your strengths?
- How can your strengths be used to help you succeed in this course?

Prerequisite Review

For Exercises R.1–R.6, complete the table.

	Set-Builder Notation	Interval Notation	Graph
R.1.	$\{x \mid x > 5\}$		
R.2.	$\{x \mid x \le -2\}$		
R.3.		$(-3, 6]$	
R.4.		$[0, 4)$	
R.5.			4
R.6.			10

For Exercises R.7–R.8, determine whether the given value of x makes the statement true or false.

R.7. $3(x + 4) + 1 \le x - 5$

a. $x = 0$ **b.** $x = -10$ **c.** $x = -9$

R.8. $-5x - 3(4 - x) > 2$

a. $x = -14$ **b.** $x = 7$ **c.** $x = -7$

Vocabulary and Key Concepts

1. **a.** A relationship of the form $ax + b > c$ or $ax + b < c$ $(a \neq 0)$ is called a ____________ ____________ in one variable.

b. When solving an inequality, the direction of the inequality sign must be reversed when multiplying or dividing both sides of the inequality by a ____________ number.

2. Which inequality statement represents the set of real numbers greater than 2, $x > 2$ or $2 < x$?

Concept 1: Solving Linear Inequalities

For Exercises 3–4, solve the equation or inequality.

		Set Notation	Interval Notation	Graph
3. a.	$-2x + 4 = 10$			
b.	$-2x + 4 < 10$			
c.	$-2x + 4 > 10$			
4. a.	$-4x + 2 = -6$			
b.	$-4x + 2 < -6$			
c.	$-4x + 2 > -6$			

For Exercises 5–40, solve the inequality and graph the solution set. Write the solution set in (a) set-builder notation and (b) interval notation. **(See Examples 1–4.)**

5. $2y + 6 \leq 4$

6. $3y + 11 > 5$

7. $-2x - 5 \leq -25$

8. $-4z - 2 > -22$

9. $6z + 3 > 16$

10. $8w - 2 \leq 13$

11. $-8 > \frac{2}{3}t$

12. $-4 \leq \frac{1}{5}p$

13. $\frac{3}{4}(8y - 9) < 3$

14. $\frac{2}{5}(2x - 1) > 10$

15. $0.8a - 0.5 \leq 0.3a - 11$

16. $0.2w - 0.7 < 0.4 - 0.9w$

17. $-5x + 7 < 22$

18. $-3w - 6 > 9$

19. $-\frac{5}{6}x \leq -\frac{3}{4}$

20. $-\frac{3}{2}y > -\frac{21}{16}$

21. $\frac{3p - 1}{-2} > 5$

22. $\frac{3k - 2}{-5} \leq 4$

23. $0.2t + 1 > 2.4t - 10$

24. $20 \leq 8 - \frac{1}{3}x$

25. $3 - 4(y + 2) \leq 6 + 4(2y + 1)$

26. $1 + 4(b - 2) < 2(b - 5) + 4$

27. $7.2k - 5.1 \geq 5.7$

28. $6h - 2.92 \leq 16.58$

29. $\frac{3}{4}x - 8 \leq 1$

30. $-\frac{2}{5}a - 3 > 5$

31. $-1.2b - 0.4 \geq -0.4b$

32. $-0.4t + 1.2 < -2$

33. $-\frac{3}{4}c - \frac{5}{4} \geq 2c$

34. $-\frac{2}{3}q - \frac{1}{3} > \frac{1}{2}q$

35. $4 - 4(y - 2) < -5y + 6$

36. $6 - 6(k - 3) \geq -4k + 12$

37. $-6(2x + 1) < 5 - (x - 4) - 6x$

38. $2(4p + 3) - p \leq 5 + 3(p - 3)$

39. $6a - (9a + 1) - 3(a - 1) \geq 2$

40. $8(q + 1) - (2q + 1) + 5 > 12$

Concept 2: Applications of Inequalities

41. Nadia received quiz grades of 80, 86, 73, and 91. **(See Example 5.)**

a. What grade would she need to make on the fifth quiz to get a "B" or better, that is, an average of at least 80?

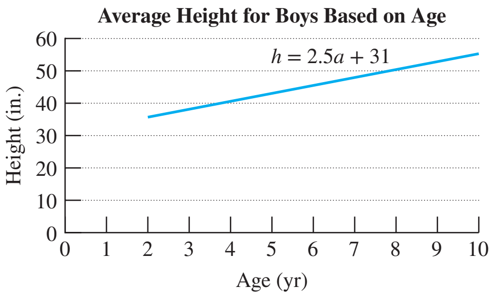
b. Is it possible for Nadia to get an "A" average for her quizzes (at least 90)?

42. Ty received test grades of 78, 75, 71, 83, and 73.

a. What grade would he need to make on the sixth test to get a "C" or better, that is, an average of at least 75?

b. Is it possible for Ty to get a "B" or better for his test average (at least 80)?

For Exercises 43–46, use the graph that shows the average height for boys based on age. Let a represent a boy's age (in years) and let h represent his height (in inches). **(See Example 6.)**



43. Determine the age range for which the average height of boys is at least 51 in.

44. Determine the age range for which the average height of boys is greater than or equal to 41 in.

45. Determine the age range for which the average height of boys is no more than 46 in.

46. Determine the age range for which the average height of boys is at most 53.5 in.

47. Nolvia sells copy machines, and her salary is \$25,000 plus a 4% commission on sales. The equation $S = 25{,}000 + 0.04x$ represents her salary S in dollars in terms of her total sales x in dollars.

a. How much money in sales does Nolvia need to earn a salary that exceeds \$40,000?

b. How much money in sales does Nolvia need to earn a salary that exceeds \$80,000?

c. Why is the money in sales required to earn a salary of \$80,000 more than twice the money in sales required to earn a salary of \$40,000?

48. The amount of money A in a savings account depends on the principal P, the interest rate r, and the time in years t that the money is invested. The equation $A = P + Prt$ shows the relationship among the variables for an account earning simple interest. If an investor deposits \$5000 at $6\frac{1}{2}\%$ simple interest, the account will grow according to the formula $A = 5000 + 5000(0.065)t$.

a. How many years will it take for the investment to exceed \$10,000? (Round to the nearest tenth of a year.)

b. How many years will it take for the investment to exceed \$15,000? (Round to the nearest tenth of a year.)

49. The revenue R for selling x fleece jackets is given by the equation $R = 49.95x$. The cost C (in dollars) to produce x jackets is $C = 2300 + 18.50x$. Find the number of jackets that the company needs to sell to produce a profit. (*Hint:* A profit occurs when revenue exceeds cost.)

50. The revenue R for selling x mountain bikes is $R = 249.95x$. The cost C (in dollars) to produce x bikes is $C = 56{,}000 + 140x$. Find the number of bikes that the company needs to sell to produce a profit.


Creatas Images/Getty Images

Expanding Your Skills

For Exercises 51–54, assume $a > b$. Determine which inequality sign ($>$ or $<$) should be inserted to make a true statement. Assume $a \neq 0$ and $b \neq 0$.

51. $a + c$ ______ $b + c$, for $c > 0$

52. $a + c$ ______ $b + c$, for $c < 0$

53. ac ______ bc, for $c < 0$

54. ac ______ bc, for $c > 0$

Compound Inequalities

Section 1.5

Concepts

1. Union and Intersection of Sets
2. Solving Compound Inequalities: And
3. Solving Inequalities of the Form $a < x < b$
4. Solving Compound Inequalities: Or
5. Applications of Compound Inequalities

1. Union and Intersection of Sets

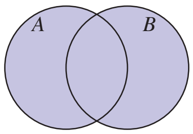
We have already learned how to graph linear inequalities and to express the solution sets in interval notation and in set-builder notation. Now we will solve **compound inequalities** that involve the union or intersection of two or more inequalities.

Definition of *A* Union *B* and *A* Intersection *B*

The **union** of sets A and B, denoted $A \cup B$, is the set of elements that belong to set A or to set B or to both sets A and B.

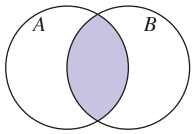
The **intersection** of two sets A and B, denoted $A \cap B$, is the set of elements common to both A and B.

The concepts of the union and intersection of two sets are illustrated in Figures 1-6 and 1-7.



$A \cup B$
A union B
The elements in A *or* B *or* both

Figure 1-6



$A \cap B$
A intersection B
The elements in A *and* B

Figure 1-7

Example 1 Finding the Union and Intersection of Sets

Given the sets: $A = \{a, b, c, d, e, f\}$ $B = \{a, c, e, g, i, k\}$ $C = \{g, h, i, j, k\}$

Find: **a.** $A \cup B$ **b.** $A \cap B$ **c.** $A \cap C$

Solution:

a. $A \cup B = \{a, b, c, d, e, f, g, i, k\}$ The union of A and B includes all the elements of A along with all the elements of B. Notice that the elements a, c, and e are not listed twice.

b. $A \cap B = \{a, c, e\}$ The intersection of A and B includes only those elements that are common to both sets.

c. $A \cap C = \{\ \}$ (the empty set) Because A and C share no common elements, the intersection of A and C is the empty set (also called the null set).

TIP: The empty set is denoted by the symbol $\{\ \}$ or by the symbol $\varnothing$.

Skill Practice Given: $A = \{r, s, t, u, v, w\}$ $B = \{s, v, w, y, z\}$ $C = \{x, y, z\}$

Find: **1.** $B \cup C$ **2.** $A \cap B$ **3.** $A \cap C$

Answers

1. $\{s, v, w, x, y, z\}$ **2.** $\{s, v, w\}$ **3.** $\{\ \}$

Example 2 Finding the Union and Intersection of Sets

Given the sets: $A = \{x \mid x < 3\}$ $B = \{x \mid x \geq -2\}$ $C = \{x \mid x \geq 5\}$

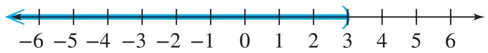
Graph the following sets. Then express each set in interval notation.

a. $A \cap B$ **b.** $A \cup C$

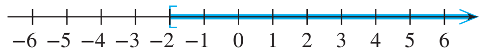
Solution:

It is helpful to visualize the graphs of individual sets on the number line before taking the union or intersection.

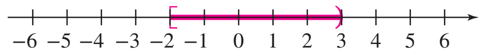
a. Graph of $A = \{x \mid x < 3\}$



Graph of $B = \{x \mid x \geq -2\}$



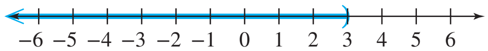
Graph of $A \cap B$ (the "overlap")



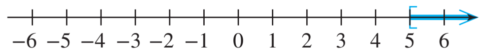
Interval notation: $[-2, 3)$

Note that the set $A \cap B$ represents the real numbers greater than or equal to -2 and less than 3. This relationship can be written more concisely as a compound inequality: $-2 \leq x < 3$. We can interpret this inequality as "x is between -2 and 3, including $x = -2$."

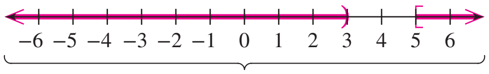
b. Graph of $A = \{x \mid x < 3\}$



Graph of $C = \{x \mid x \geq 5\}$



Graph of $A \cup C$



$A \cup C$ includes all elements from set A along with the elements from set C.

Interval notation: $(-\infty, 3) \cup [5, \infty)$

Skill Practice Given the sets: $A = \{x \mid x \geq -1\}$, $B = \{x \mid x < 4\}$, and $C = \{x \mid x \geq 9\}$, determine the union or intersection and express the answer in interval notation.

4. $A \cap B$ **5.** $B \cup C$

FOR REVIEW

When writing a set of real numbers using interval notation, recall the following rules.

- A parenthesis (or) means that an endpoint is *not* included in the set.
- A square bracket [or] means that an endpoint *is* included.
- Use parentheses for ∞ and $-\infty$.

In Example 3, we find the union and intersection of sets expressed in interval notation.

Example 3 Finding the Union and Intersection of Two Intervals

Find the union or intersection as indicated. Write the answer in interval notation.

a. $(-\infty, -2) \cup [-4, 3)$ **b.** $(-\infty, -2) \cap [-4, 3)$

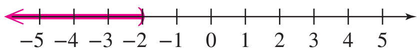
Solution:

a. $(-\infty, -2) \cup [-4, 3)$

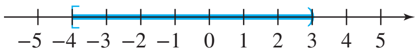
To find the union, graph each interval separately. The union is the collection of real numbers that lie in the first interval, the second interval, or both intervals.

Answers

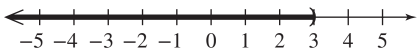
4. $[-1, 4)$ **5.** $(-\infty, 4) \cup [9, \infty)$



$(-\infty, -2)$



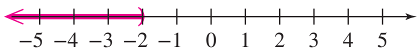
$[-4, 3)$



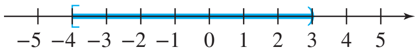
The union is $(-\infty, 3)$.

The union consists of all real numbers in the red interval along with the real numbers in the blue interval: $(-\infty, 3)$

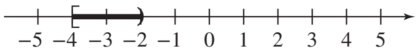
b. $(-\infty, -2) \cap [-4, 3)$



$(-\infty, -2)$



$[-4, 3)$



The intersection is the "overlap" of the two intervals: $[-4, -2)$.

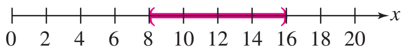
The intersection is $[-4, -2)$.

Skill Practice Find the union or intersection. Write the answer in interval notation.

6. $(-\infty, -5] \cup (-7, 0)$ **7.** $(-\infty, -5] \cap (-7, 0)$

2. Solving Compound Inequalities: And

The solution to two inequalities joined by the word *and* is the intersection of their solution sets. For example, to play in a golf tournament for juniors, a child's age x must be at least 8 yr and not more than 16 yr. This is translated as $x \geq 8$ and $x \leq 16$. The word "and" joins the two inequalities and implies that we want the intersection of the individual solution sets.



Solving a Compound Inequality: And

Step 1 Solve and graph each inequality separately.

Step 2 If the inequalities are joined by the word *and*, find the intersection of the two solution sets.

Step 3 Express the solution set in interval notation or in set-builder notation.

As you work through the examples in this section, remember that multiplying or dividing an inequality by a negative factor reverses the direction of the inequality sign.

Example 4 Solving a Compound Inequality: And

Solve the compound inequality.

$$-2x < 6 \quad \text{and} \quad x + 5 \leq 7$$

Solution:

$-2x < 6$	and	$x + 5 \leq 7$	Solve each inequality separately.
$\dfrac{-2x}{-2} > \dfrac{6}{-2}$	and	$x \leq 2$	Reverse the first inequality sign.
$x > -3$	and	$x \leq 2$	

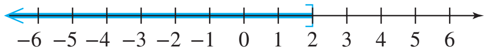
FOR REVIEW

Multiplying or dividing an inequality by a negative factor changes the signs within the inequality. As a result, the direction of the inequality sign must be reversed. For example, $2 < 5$, but $-2 > -5$.

Answers

6. $(-\infty, 0)$ **7.** $(-7, -5]$

$\{x \mid x > -3\}$



$\{x \mid x \le 2\}$

Identify the intersection of the solution sets: $\{x \mid -3 < x \le 2\}$

The solution is $\{x \mid -3 < x \le 2\}$, or equivalently in interval notation, $(-3, 2]$.

Skill Practice Solve the compound inequality.

8. $5x + 2 \ge -8$ and $-4x > -24$

Example 5 Solving a Compound Inequality: And

Solve the compound inequality.

$$4.4a + 3.1 < -12.3 \quad \text{and} \quad -2.8a + 9.1 < -6.3$$

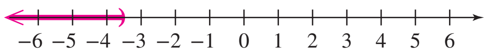
Solution:

$4.4a + 3.1 < -12.3$ and $-2.8a + 9.1 < -6.3$

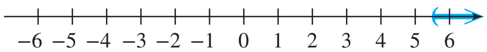
$4.4a < -15.4$ and $-2.8a < -15.4$ Solve each inequality separately.

$\dfrac{4.4a}{4.4} < \dfrac{-15.4}{4.4}$ and $\dfrac{-2.8a}{-2.8} > \dfrac{-15.4}{-2.8}$ Reverse the second inequality sign.

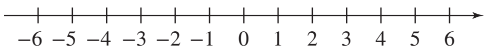
$a < -3.5$ and $a > 5.5$



$\{a \mid a < -3.5\}$



$\{a \mid a > 5.5\}$



The intersection of the solution sets is the empty set: { }

There are no real numbers that are simultaneously less than -3.5 and greater than 5.5. There is no solution.

The solution set is { }.

Skill Practice Solve the compound inequality.

9. $3.2y - 2.4 > 16.8$ and $-4.1y \ge 8.2$

Answers

8. $\{x \mid -2 \le x < 6\}$; $[-2, 6)$

9. { }

Example 6 **Solving a Compound Inequality: And**

Solve the compound inequality.

$$-\frac{2}{3}x \le 6 \quad \text{and} \quad -\frac{1}{2}x < 1$$

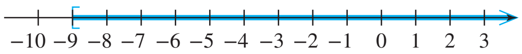
Solution:

$$-\frac{2}{3}x \le 6 \qquad \text{and} \qquad -\frac{1}{2}x < 1$$

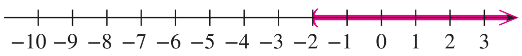
$$-\frac{3}{2}\left(-\frac{2}{3}x\right) \ge -\frac{3}{2}(6) \qquad \text{and} \qquad -2\left(-\frac{1}{2}x\right) > -2(1)$$

Solve each inequality separately.

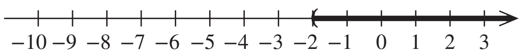
$$x \ge -9 \qquad \text{and} \qquad x > -2$$



$\{x \mid x \ge -9\}$



$\{x \mid x > -2\}$



Identify the intersection of the solution sets: $\{x \mid x > -2\}$

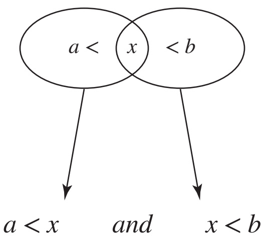
The solution set is $\{x \mid x > -2\}$, or in interval notation, $(-2, \infty)$.

Skill Practice Solve the compound inequality.

10. $-\frac{1}{4}z < \frac{5}{8}$ and $\frac{1}{2}z + 1 \ge 3$

3. Solving Inequalities of the Form $a < x < b$

An inequality of the form $a < x < b$ is a type of compound inequality, one that defines two simultaneous conditions on x.



The solution set to the compound inequality $a < x < b$ is the *intersection* of the solution sets to the inequalities $a < x$ and $x < b$.

Example 7 **Solving an Inequality of the Form $a < x < b$**

Solve the inequality. $\quad -4 < 3x + 5 \le 10$

Solution:

$$-4 < 3x + 5 \le 10$$

$$-4 < 3x + 5 \qquad \text{and} \qquad 3x + 5 \le 10$$

Set up the intersection of two inequalities.

FOR REVIEW

Recall that the expression $a < x$ is equivalent to $x > a$. For example, $3 < x$ is equivalent to $x > 3$.

Answer

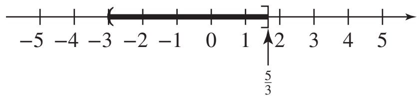
10. $\{z \mid z \ge 4\}$; $[4, \infty)$

$$-9 < 3x \quad \text{and} \quad 3x \le 5$$ Solve each inequality.

$$\frac{-9}{3} < \frac{3x}{3} \quad \text{and} \quad \frac{3x}{3} \le \frac{5}{3}$$

$$-3 < x \quad \text{and} \quad x \le \frac{5}{3}$$ Recall $-3 < x$ is the same as $x > -3$.

$$-3 < x \le \frac{5}{3}$$ Identify the intersection of the solution sets.



The solution is $\{x \mid -3 < x \le \frac{5}{3}\}$, or equivalently in interval notation, $(-3, \frac{5}{3}]$.

Skill Practice Solve the inequality.

11. $-6 \le 2x - 5 < 1$

To solve an inequality of the form $a < x < b$, we can also work with the inequality as a "three-part" inequality and isolate x. This is demonstrated in Example 8.

Example 8 Solving an Inequality of the Form $a < x < b$

Solve the inequality. $\quad 2 \ge \dfrac{p-2}{-3} \ge -1$

Solution:

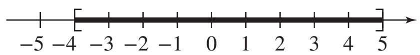
$$2 \ge \frac{p-2}{-3} \ge -1$$ Isolate the variable in the middle part.

$$-3(2) \le -3\left(\frac{p-2}{-3}\right) \le -3(-1)$$ Multiply all three parts by -3. Remember to reverse the inequality signs.

$$-6 \le p - 2 \le 3$$ Simplify.

$$-6 + 2 \le p - 2 + 2 \le 3 + 2$$ Add 2 to all three parts to isolate p.

$$-4 \le p \le 5$$



The solution is $\{p \mid -4 \le p \le 5\}$, or equivalently in interval notation, $[-4, 5]$.

Skill Practice Solve the inequality.

12. $8 > \dfrac{t+4}{-2} > -5$

4. Solving Compound Inequalities: Or

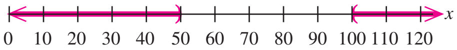
In Examples 9 and 10, we solve compound inequalities that involve inequalities joined by the word "or." In such a case, the solution to the compound inequality is the union of the solution sets of the individual inequalities.

Answers

11. $\{x \mid -\frac{1}{2} \le x < 3\}$; $[-\frac{1}{2}, 3)$

12. $\{t \mid -20 < t < 6\}$; $(-20, 6)$

For example, a resting heart rate x is potentially abnormal if it is below 50 beats per minute or above 100 beats per minute.



Solving a Compound Inequality: Or

Step 1 Solve and graph each inequality separately.

Step 2 If the inequalities are joined by the word *or*, find the union of the two solution sets.

Step 3 Express the solution set in interval notation or in set-builder notation.

Example 9 Solving a Compound Inequality: Or

Solve the compound inequality. $-3y - 5 > 4$ or $4 - y \leq 6$

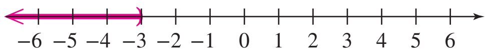
Solution:

$-3y - 5 > 4$ or $4 - y \leq 6$

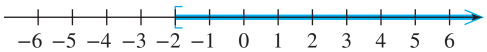
$-3y > 9$ or $-y \leq 2$ — Solve each inequality separately.

$\dfrac{-3y}{-3} < \dfrac{9}{-3}$ or $\dfrac{-y}{-1} \geq \dfrac{2}{-1}$ — Reverse the inequality signs.

$y < -3$ or $y \geq -2$



$\{y \mid y < -3\}$



$\{y \mid y \geq -2\}$

Identify the union of the solution sets: $\{y \mid y < -3 \text{ or } y \geq -2\}$

The solution is $\{y \mid y < -3 \text{ or } y \geq -2\}$, or equivalently in interval notation, $(-\infty, -3) \cup [-2, \infty)$.

Skill Practice Solve the compound inequality.

13. $-10t - 8 \geq 12$ or $3t - 6 > 3$

Answer

13. $\{t \mid t \leq -2 \text{ or } t > 3\}$; $(-\infty, -2] \cup (3, \infty)$

Example 10 Solving a Compound Inequality: Or

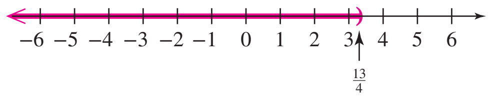
Solve the compound inequality. $4x + 3 < 16$ or $-2x < 3$

Solution:

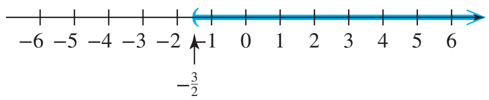
$$4x + 3 < 16 \quad \text{or} \quad -2x < 3$$

$$4x < 13 \quad \text{or} \quad x > -\frac{3}{2}$$ Solve each inequality separately.

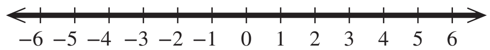
$$x < \frac{13}{4} \quad \text{or} \quad x > -\frac{3}{2}$$



$\{x \mid x < \frac{13}{4}\}$



$\{x \mid x > -\frac{3}{2}\}$



Identify the union of the solution sets.

The union of the solution sets is $\{x \mid x \text{ is a real number}\}$, or equivalently, $(-\infty, \infty)$.

Skill Practice Solve the compound inequality.

14. $x - 7 > -2$ or $-6x > -48$

5. Applications of Compound Inequalities

Compound inequalities are used in many applications, as shown in Examples 11 and 12.

Example 11 Translating Compound Inequalities

The normal level of thyroid-stimulating hormone (TSH) for adults ranges from 0.4 to 4.8 microunits per milliliter (μU/mL), inclusive. Let x represent the amount of TSH measured in microunits per milliliter.

a. Write an inequality representing the normal range of TSH.

b. Write a compound inequality representing abnormal TSH levels.

Solution:

a. $0.4 \leq x \leq 4.8$ **b.** $x < 0.4$ or $x > 4.8$

Skill Practice The length of a normal human pregnancy, w, is from 37 to 41 weeks, inclusive.

15. Write an inequality representing the normal length of a pregnancy.

16. Write a compound inequality representing an abnormal length for a pregnancy.

Answers

14. $\{x \mid x \text{ is a real number}\}$; $(-\infty, \infty)$

15. $37 \leq w \leq 41$

16. $w < 37$ or $w > 41$

TIP:

- In mathematics, the word "between" means strictly between two values. That is, the endpoints are *excluded*.
  Example: x is between 4 and 10 $\Rightarrow$ (4, 10).
- If the word "inclusive" is added to the statement, then we *include* the endpoints.
  Example: x is between 4 and 10, inclusive $\Rightarrow$ [4, 10].

Example 12 Translating and Solving a Compound Inequality

The sum of a number and 4 is between −5 and 12. Find all such numbers.

Solution:

Let x represent a number.

$-5 < x + 4 < 12$	Translate the inequality.
$-5 - 4 < x + 4 - 4 < 12 - 4$	Subtract 4 from all three parts of the inequality.
$-9 < x < 8$	

The number may be any real number between −9 and 8: $\{x \mid -9 < x < 8\}$.

Skill Practice

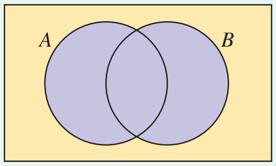
17. The sum of twice a number and 11 is between 21 and 31. Find all such numbers.

Answer

17. Any real number between 5 and 10: $\{x \mid 5 < x < 10\}$

Section 1.5 Activity

A.1. Consider sets A and B defined as $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$. Place the elements of sets A and B in the Venn diagram.



A.2. a. The ________ of set A and set B, denoted by ________, is the set of elements that belong to set A, or to set B, or to both sets A and B.

b. Given $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$, find $A \cup B$. Refer to the Venn diagram in Exercise A.1.

A.3. a. The ________ of set A and set B, denoted by ________, is the set of elements common to both A and B.

b. Given $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9\}$, find $A \cap B$. Refer to the Venn diagram in Exercise A.1.

A.4. Consider the sets $C = \{x \mid x < 2\}$ and $D = \{x \mid x \geq -3\}$.

a. Graph C.

b. Graph D.

c. Graph $C \cup D$.

d. Write $C \cup D$ in interval notation.

e. Graph $C \cap D$.

f. Write $C \cap D$ in interval notation.

A.5. **a.** Solve the inequality $-4x - 1 < 19$ and write the solution set in interval notation.

b. Solve the inequality $2x + 5 \geq 3$ and write the solution set in interval notation.

c. Solve the compound inequality and write the solution set in interval notation.

$$-4x - 1 < 19 \quad \text{and} \quad 2x + 5 \geq 3$$

d. Solve the compound inequality and write the solution set in interval notation.

$$-4x - 1 < 19 \quad \text{or} \quad 2x + 5 \geq 3$$

e. The solution set to a compound inequality joined by the conjunction *and* is the (choose one: union/intersection) of the solution sets to the individual inequalities. (Refer to part [c].)

f. The solution set to a compound inequality joined by the conjunction *or* is the (choose one: union/intersection) of the solution sets to the individual inequalities. (Refer to part [d].)

A.6. **a.** Write the inequality $-3 \leq \frac{1}{2}x - 5 < 1$ as two separate inequalities.

b. Solve the inequality $-3 \leq \frac{1}{2}x - 5 < 1$ and write the solution set in interval notation.

A.7. The normal number of red blood cells for human blood is between 4,200,000 and 5,900,000 cells per cubic millimeter (per mm^3), inclusive. Let x represent the number of red blood cells per cubic millimeter.

a. Write a compound inequality representing the normal range of red blood cells.

b. Write a compound inequality representing abnormal levels of red blood cells.

Section 1.5 Practice Exercises

Study Skills Exercise

Memorization techniques can be used to move mathematical concepts, formulas, and properties from your short-term memory to your long-term memory. Students can use these techniques in a math course to retain information for problem-solving. Self-testing is one such technique. Self-testing is the process where students write review questions and key concepts on study cards. On one side of the card, pose a practice problem or concept. On the other side, write the answer with an explanation. Repeated review of these cards helps build proficiency in the subject matter.

Write down the key concepts covered in this section and create five example problems to help you review for the next test.

Prerequisite Review

For Exercises R.1–R.4, write the set in interval notation.

R.1. $\{x \mid 8 > x\}$

R.2. $\{x \mid -4 \leq x\}$

R.3. $\{x \mid -9 < x \leq -6\}$

R.4. $\{x \mid 0 \leq x < 2\}$

For Exercises R.5–R.10, solve the inequality. Write the solution set in interval notation.

R.5. $-5(x-3) - 2 \leq 43$

R.6. $-3(x-4) + 8 < -10$

R.7. $-9 \leq 2x - 5 < 7$

R.8. $-15 < 3x + 6 \leq 12$

R.9. $\frac{2}{3}y - 1 \geq 11$

R.10. $4 - \frac{3}{4}t > 16$

Vocabulary and Key Concepts

1. **a.** The ______ of two sets A and B, denoted by ______, is the set of elements that belong to A or B or both A and B.

b. The __________ of two sets A and B, denoted by ______, is the set of elements common to both A and B.

c. The solution set to the compound inequality $x < c$ and $x > d$ is the (union/intersection) of the solution sets of the individual inequalities.

d. The compound inequality $a < x$ and $x < b$ can be written as the three-part inequality __________.

e. The solution set to the compound inequality $x < a$ or $x > b$ is the (union/intersection) of the solution sets of the individual inequalities.

Concept 1: Union and Intersection of Sets

For Exercises 2–6, refer to the Venn diagram and sets A, B, and C. Find the union or intersection as indicated.

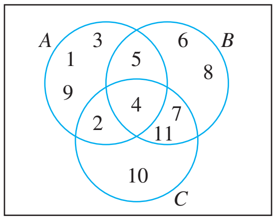
2. $A \cap B$

3. $A \cap C$

4. $B \cap C$

5. $B \cup C$

6. $A \cup C$



7. Given $M = \{-3, -1, 1, 3, 5\}$ and $N = \{-4, -3, -2, -1, 0\}$ **(See Example 1.)**

List the elements of the following sets.

a. $M \cap N$ **b.** $M \cup N$

8. Given $P = \{a, b, c, d, e, f, g, h, i\}$ and $Q = \{a, e, i, o, u\}$

List the elements of the following sets.

a. $P \cap Q$ **b.** $P \cup Q$

For Exercises 9–20, refer to the sets A, B, C, and D. Determine the union or intersection as indicated. Express the answer in interval notation, if possible. **(See Example 2.)**

$A = \{x \mid x < -4\}$, $B = \{x \mid x > 2\}$, $C = \{x \mid x \geq -7\}$, $D = \{x \mid 0 \leq x < 5\}$

9. $A \cap C$

10. $B \cap C$

11. $A \cup B$

12. $A \cup D$

13. $A \cap B$

14. $A \cap D$

15. $B \cup C$

16. $B \cup D$

17. $C \cap D$

18. $B \cap D$

19. $C \cup D$

20. $A \cup C$

For Exercises 21–26, find the intersection and union of sets as indicated. Write the answers in interval notation. **(See Example 3.)**

21. **a.** $(-2, 5) \cap [-1, \infty)$
b. $(-2, 5) \cup [-1, \infty)$

22. **a.** $(-\infty, 4) \cap [-1, 5)$
b. $(-\infty, 4) \cup [-1, 5)$

23. **a.** $\left(-\frac{5}{2}, 3\right) \cap \left(-1, \frac{9}{2}\right)$
b. $\left(-\frac{5}{2}, 3\right) \cup \left(-1, \frac{9}{2}\right)$

24. **a.** $(-3.4, 1.6) \cap (-2.2, 4.1)$
b. $(-3.4, 1.6) \cup (-2.2, 4.1)$

25. **a.** $(-4, 5] \cap (0, 2]$
b. $(-4, 5] \cup (0, 2]$

26. **a.** $[-1, 5) \cap (0, 3)$
b. $[-1, 5) \cup (0, 3)$

Concept 2: Solving Compound Inequalities: And

For Exercises 27–36, solve the compound inequality and graph the solution. Write the answer in interval notation. **(See Examples 4–6.)**

27. $y - 7 \geq -9$ and $y + 2 \leq 5$

28. $a + 6 > -2$ and $5a < 30$

29. $2t + 7 < 19$ and $5t + 13 > 28$

30. $5p + 2p \geq -21$ and $-9p + 3p \geq -24$

31. $2.1k - 1.1 \leq 0.6k + 1.9$ and $0.3k - 1.1 < -0.1k + 0.9$

32. $0.6w + 0.1 > 0.3w - 1.1$ and $2.3w + 1.5 \geq 0.3w + 6.5$

33. $\frac{2}{3}(2p - 1) \geq 10$ and $\frac{4}{5}(3p + 4) \geq 20$

34. $\frac{5}{2}(a + 2) < -6$ and $\frac{3}{4}(a - 2) < 1$

35. $-2 < -x - 12$ and $-14 < 5(x - 3) + 6x$

36. $-8 \geq -3y - 2$ and $3(y - 7) + 16 > 4y$

Concept 3: Solving Inequalities of the Form $a < x < b$

37. Write $-4 \le t < \frac{3}{4}$ as two separate inequalities.

38. Write $-2.8 < y \le 15$ as two separate inequalities.

39. Explain why $6 < x < 2$ has no solution.

40. Explain why $4 < t < 1$ has no solution.

41. Explain why $-5 > y > -2$ has no solution.

42. Explain why $-3 > w > -1$ has no solution.

For Exercises 43–54, solve the compound inequality and graph the solution set. Write the answer in interval notation. **(See Examples 7–8.)**

43. $0 \le 2b - 5 < 9$

44. $-6 < 3k - 9 \le 0$

45. $-1 < \dfrac{a}{6} \le 1$

46. $-3 \le \dfrac{1}{2}x < 0$

47. $-\dfrac{2}{3} < \dfrac{y-4}{-6} < \dfrac{1}{3}$

48. $\dfrac{1}{3} > \dfrac{t-4}{-3} > -2$

49. $5 \le -3x - 2 \le 8$

50. $-1 < -2x + 4 \le 5$

51. $12 > 6x + 3 \ge 0$

52. $-4 \ge 2x - 5 > -7$

53. $-0.2 < 2.6 + 7t < 4$

54. $-1.5 < 0.1x \le 8.1$

Concept 4: Solving Compound Inequalities: Or

For Exercises 55–64, solve the compound inequality and graph the solution set. Write the answer in interval notation. **(See Examples 9–10.)**

55. $2y - 1 \ge 3$ or $y < -2$

56. $x < 0$ or $3x + 1 \ge 7$

57. $1 > 6z - 8$ or $8z - 6 \le 10$

58. $22 > 4t - 10$ or $7 > 2t - 5$

59. $5(x - 1) \ge -5$ or $5 - x \le 11$

60. $-p + 7 \ge 10$ or $3(p - 1) \le 12$

61. $\frac{5}{3}v \le 5$ or $-v - 6 < 1$

62. $\frac{3}{8}u + 1 > 0$ or $-2u \ge -4$

63. $0.5w + 5 < 2.5w - 4$ or $0.3w \le -0.1w - 1.6$

64. $1.25a + 3 \le 0.5a - 6$ or $2.5a - 1 \ge 9 - 1.5a$

Mixed Exercises

For Exercises 65–74, solve the compound inequality. Write the answer in interval notation.

65. **a.** $3x - 5 < 19$ and $-2x + 3 < 23$

b. $3x - 5 < 19$ or $-2x + 3 < 23$

66. **a.** $0.5(6x + 8) > 0.8x - 7$ and $4(x + 1) < 7.2$

b. $0.5(6x + 8) > 0.8x - 7$ or $4(x + 1) < 7.2$

67. **a.** $8x - 4 \ge 6.4$ or $0.3(x + 6) \le -0.6$

b. $8x - 4 \ge 6.4$ and $0.3(x + 6) \le -0.6$

68. **a.** $-2r + 4 \le -8$ or $3r + 5 \le 8$

b. $-2r + 4 \le -8$ and $3r + 5 \le 8$

69. $-4 \le \frac{2 - 4x}{3} < 8$

70. $-1 < \frac{3 - x}{2} \le 0$

71. $5 \ge -4(t - 3) + 3t$ or $6 < 12t + 8(4 - t)$

72. $3 > -(w - 3) + 4w$ or
$-5 \ge -3(w - 5) + 6w$

73. $\frac{-x + 3}{2} > \frac{4 + x}{5}$ or $\frac{1 - x}{4} > \frac{2 - x}{3}$

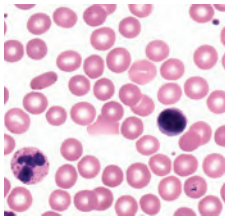
74. $\frac{y - 7}{-3} < \frac{1}{4}$ or $\frac{y + 1}{-2} > -\frac{1}{3}$

Concept 5: Applications of Compound Inequalities

75. The normal number of white blood cells for human blood is between 4800 and 10,800 cells per cubic millimeter, inclusive. Let x represent the number of white blood cells per cubic millimeter. **(See Example 11.)**

a. Write an inequality representing the normal range of white blood cells per cubic millimeter.

b. Write a compound inequality representing abnormal levels of white blood cells per cubic millimeter.



Al Telser/McGraw Hill

76. Normal hemoglobin levels in human blood for adult males are between 13 and 16 grams per deciliter (g/dL), inclusive. Let x represent the level of hemoglobin measured in grams per deciliter.

a. Write an inequality representing normal hemoglobin levels for adult males.

b. Write a compound inequality representing abnormal levels of hemoglobin for adult males.

77. A polling company estimates that a certain candidate running for office will receive between 44% and 48% of the votes. Let x represent the percentage of votes for this candidate.

a. Write a strict inequality representing the expected percentage of votes for this candidate.

b. Write a compound inequality representing the percentage of votes that would fall outside the polling company's prediction.

78. A machine is calibrated to cut a piece of wood between 2.4 in. thick and 2.6 in. thick. Let x represent the thickness of the wood after it is cut.

a. Write a strict inequality representing the expected range of thickness of the wood after it has been cut.

b. Write a compound inequality representing the thickness of wood that would fall outside the normal range for this machine.

79. Twice a number is between -3 and 12. Find all such numbers. **(See Example 12.)**

80. The difference of a number and 6 is between 0 and 8. Find all such numbers.

81. One plus twice a number is either greater than 5 or less than -1. Find all such numbers.

82. One-third of a number is either less than -2 or greater than 5. Find all such numbers.

83. Amy knows from reading her syllabus in intermediate algebra that the average of her chapter tests accounts for 80% (0.8) of her overall course grade. She also knows that the final exam counts as 20% (0.2) of her grade. Suppose that the average of Amy's chapter tests is 92%.

a. Determine the range of grades that she would need on her final exam to get an "A" in the class. (Assume that a grade of "A" is obtained if Amy's overall average is 90% or better.)

b. Determine the range of grades that Amy would need on her final exam to get a "B" in the class. (Assume that a grade of "B" is obtained if Amy's overall average is at least 80% but less than 90%.)

84. Robert knows from reading his syllabus in intermediate algebra that the average of his chapter tests accounts for 60% (0.6) of his overall course grade. He also knows that the final exam counts as 40% (0.4) of his grade. Suppose that the average of Robert's chapter tests is 89%.

a. Determine the range of grades that he would need on his final exam to get an "A" in the class. (Assume that a grade of "A" is obtained if Robert's overall average is 90% or better.)

b. Determine the range of grades that Robert would need on his final exam to get a "B" in the class. (Assume that a grade of "B" is obtained if Robert's overall average is at least 80% but less than 90%.)

85. The average high and low temperatures for Vancouver, British Columbia, in January are 5.6°C and 0°C, respectively. The formula relating Celsius temperatures to Fahrenheit temperatures is given by $C = \frac{5}{9}(F - 32)$. Convert the inequality $0.0° \le C \le 5.6°$ to an equivalent inequality using Fahrenheit temperatures.

Steve Allen/Brand X Pictures/Getty Images

86. For a day in July, the temperature in Austin, Texas, ranged from 20°C to 29°C. The formula relating Celsius temperatures to Fahrenheit temperatures is given by $C = \frac{5}{9}(F - 32)$. Convert the inequality $20° \le C \le 29°$ to an equivalent inequality using Fahrenheit temperatures.

Section 1.6 Absolute Value Equations

Concepts

1. Solving Absolute Value Equations
2. Solving Equations Containing Two Absolute Values

1. Solving Absolute Value Equations

An equation of the form $|x| = a$ is called an **absolute value equation**. The solution includes all real numbers whose absolute value equals a. For example, the solutions to the equation $|x| = 4$ are 4 as well as -4, because $|4| = 4$ and $|-4| = 4$. A geometric interpretation of the absolute value of a number is its distance from zero on the number line (Figure 1-8). Therefore, the solutions to the equation $|x| = 4$ are the values of x that are 4 units away from zero.

$$|x| = 4$$

$$x = -4 \quad \text{or} \quad x = 4$$

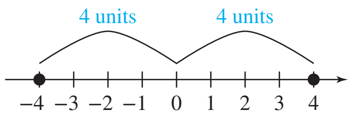


Figure 1-8

> **Solving Absolute Value Equations of the Form $|x| = a$**
>
> If a is a real number, then
>
> - If $a \geq 0$, the solutions to the equation $|x| = a$ are $x = a$ and $x = -a$.
> - If $a < 0$, there is no solution to the equation $|x| = a$.

To solve an absolute value equation of the form $|x| = a$ $(a \geq 0)$, rewrite the equation as $x = a$ or $x = -a$.

Example 1 Solving Absolute Value Equations

Solve the absolute value equations.

a. $|x| = 5$ **b.** $|w| - 2 = 12$ **c.** $|p| = 0$ **d.** $|x| = -6$

Solution:

a. $|x| = 5$ The equation is in the form $|x| = a$, where $a = 5$.

$x = 5$ or $x = -5$ Rewrite the equation as $x = a$ or $x = -a$.

The solution set is $\{5, -5\}$.

b. $|w| - 2 = 12$ Isolate the absolute value to write the equation in the form $|x| = a$.

$|w| = 14$

$w = 14$ or $w = -14$ Rewrite the equation as $w = a$ or $w = -a$.

The solution set is $\{14, -14\}$.

c. $|p| = 0$

$p = 0$ or $p = -0$

Rewrite as two equations. Notice that the second equation $p = -0$ is the same as the first equation. Intuitively, $p = 0$ is the only number whose absolute value equals 0.

The solution set is $\{0\}$.

d. $|x| = -6$

No solution, { }

The equation is of the form $|x| = a$, but a is negative. There is no number whose absolute value is negative.

Skill Practice Solve the absolute value equations.

1. $|y| = 7$ **2.** $|v| + 6 = 10$ **3.** $|w| + 3 = 3$ **4.** $|z| = -12$

We have solved absolute value equations of the form $|x| = a$. Notice that x can represent any algebraic quantity. For example, to solve the equation $|2w - 3| = 5$, we still rewrite the absolute value equation as two equations. In this case, we set the quantity $2w - 3$ equal to 5 and to -5, respectively.

$$|2w - 3| = 5$$

$$2w - 3 = 5 \quad \text{or} \quad 2w - 3 = -5$$

Solving an Absolute Value Equation

Step 1 Isolate the absolute value. That is, write the equation in the form $|x| = a$, where a is a real number.

Step 2 If $a < 0$, there is no solution.

Step 3 Otherwise, if $a \geq 0$, rewrite the absolute value equation as $x = a$ or $x = -a$.

Step 4 Solve the individual equations from step 3.

Step 5 Check the answers in the original absolute value equation.

Example 2 Solving an Absolute Value Equation

Solve the equation. $|2w - 3| = 5$

Solution:

$|2w - 3| = 5$ — The equation is already in the form $|x| = a$, where $x = 2w - 3$.

$2w - 3 = 5$ or $2w - 3 = -5$ — Rewrite as two equations.

$2w = 8$ or $2w = -2$ — Solve each equation.

$w = 4$ or $w = -1$

Answers

1. {7, −7}
2. {4, −4}
3. {0} **4.** { }

Check: $w = 4$	Check: $w = -1$	Check the solutions in the original equation.
$\|2w - 3\| = 5$	$\|2w - 3\| = 5$	
$\|2(4) - 3\| \stackrel{?}{=} 5$	$\|2(-1) - 3\| \stackrel{?}{=} 5$	
$\|8 - 3\| \stackrel{?}{=} 5$	$\|-2 - 3\| \stackrel{?}{=} 5$	
$\|5\| \stackrel{?}{=} 5$ ✓	$\|-5\| \stackrel{?}{=} 5$ ✓	

The solution set is $\{4, -1\}$.

Skill Practice Solve the equation.

5. $|4x + 1| = 9$

Example 3 Solving an Absolute Value Equation

Solve the equation. $|2c - 5| + 6 = 2$

Solution:

$|2c - 5| + 6 = 2$

$|2c - 5| = -4$ Isolate the absolute value. The equation is in the form $|x| = a$, where $x = 2c - 5$ and $a = -4$. Because $a < 0$, there is no solution.

No solution, { } There are no numbers c that will make an absolute value equal to a negative number.

Avoiding Mistakes

Always isolate the absolute value first. Otherwise you will get answers that do not check.

Skill Practice Solve the equation.

6. $|3z + 10| + 3 = 1$

Example 4 Solving an Absolute Value Equation

Solve the equation. $-2\left|\frac{2}{5}p + 3\right| - 7 = -19$

Solution:

$$-2\left|\frac{2}{5}p + 3\right| - 7 = -19$$

$$-2\left|\frac{2}{5}p + 3\right| = -12 \qquad \text{Isolate the absolute value.}$$

$$\frac{\cancel{-2}\left|\frac{2}{5}p + 3\right|}{\cancel{-2}} = \frac{-12}{-2}$$

$$\left|\frac{2}{5}p + 3\right| = 6$$

$\frac{2}{5}p + 3 = 6$ or $\frac{2}{5}p + 3 = -6$ Rewrite as two equations.

$2p + 15 = 30$ or $2p + 15 = -30$ Multiply by 5 to clear fractions.

$2p = 15$ or $2p = -45$

$p = \frac{15}{2}$ or $p = -\frac{45}{2}$ Both values check in the original equation.

The solution set is $\left\{\frac{15}{2}, -\frac{45}{2}\right\}$.

Answers

5. $\left\{2, -\frac{5}{2}\right\}$ **6.** { }

Skill Practice Solve the equation.

7. $3\left|\frac{3}{2}a + 1\right| + 2 = 14$

Example 5 Solving an Absolute Value Equation

Solve the equation. $6.9 = |4.1 - p| + 6.9$

Solution:

$6.9 = |4.1 - p| + 6.9$

$|4.1 - p| + 6.9 = 6.9$ First write the absolute value on the left. Then subtract 6.9 from both sides to write the equation in the form $|x| = a$.

$|4.1 - p| = 0$ Isolate the absolute value.

$4.1 - p = 0$ or $4.1 - p = -0$ Rewrite as two equations. Notice that the equations are the same.

$-p = -4.1$ Subtract 4.1 from both sides.

$p = 4.1$

Check $p = 4.1$ in the original equation.

Check: $p = 4.1$

$|4.1 - p| + 6.9 = 6.9$

$|4.1 - 4.1| + 6.9 \stackrel{?}{=} 6.9$

$|0| + 6.9 \stackrel{?}{=} 6.9$

$6.9 \stackrel{?}{=} 6.9$ ✓

The solution set is $\{4.1\}$.

Skill Practice Solve the equation.

8. $-3.5 = |1.2 + x| - 3.5$

2. Solving Equations Containing Two Absolute Values

Some equations have two absolute values such as $|x| = |y|$. If two quantities have the same absolute value, then the quantities are equal or the quantities are opposites.

Equality of Absolute Values

$|x| = |y|$ implies that $x = y$ or $x = -y$.

Answers

7. $\left\{2, -\frac{10}{3}\right\}$ **8.** $\{-1.2\}$

Example 6 Solving an Equation Having Two Absolute Values

Solve the equation. $|2w - 3| = |5w + 1|$

Solution:

$$|2w - 3| = |5w + 1|$$

$2w - 3 = 5w + 1$ or $2w - 3 = -(5w + 1)$ Rewrite as two equations, $x = y$ or $x = -y$.

Avoiding Mistakes

To take the opposite of the quantity $5w + 1$, use parentheses and apply the distributive property.

$$\begin{aligned} 2w - 3 &= 5w + 1 &\quad\text{or}\quad& 2w - 3 = -5w - 1 &\quad& \text{Solve for } w. \\ -3w - 3 &= 1 &\quad\text{or}\quad& 7w - 3 = -1 \\ -3w &= 4 &\quad\text{or}\quad& 7w = 2 \\ w &= -\frac{4}{3} &\quad\text{or}\quad& w = \frac{2}{7} && \text{Both values check in the original equation.} \end{aligned}$$

The solution set is $\left\{-\frac{4}{3}, \frac{2}{7}\right\}$.

Skill Practice Solve the equation.

9. $|3 - 2x| = |3x - 1|$

Example 7 Solving an Equation Having Two Absolute Values

Solve the equation. $|x - 4| = |x + 8|$

Solution:

$$|x - 4| = |x + 8|$$

$$\begin{aligned} x - 4 &= x + 8 &\quad\text{or}\quad& x - 4 = -(x + 8) &\quad& \text{Rewrite as two equations, } x = y \text{ or } x = -y. \\ -4 &= 8 &\quad\text{or}\quad& x - 4 = -x - 8 && \text{Solve for } x. \\ &\uparrow\ \text{contradiction} && 2x - 4 = -8 \\ &&& 2x = -4 \\ &&& x = -2 && x = -2 \text{ checks in the original equation.} \end{aligned}$$

The solution set is $\{-2\}$.

FOR REVIEW

Recall that if an equation reduces to a contradiction such as $-4 = 8$, the equation has no solution. If an equation reduces to an identity such as $0 = 0$, the solution set is the set of real numbers.

Skill Practice Solve the equation.

10. $|4t + 3| = |4t - 5|$

Answers

9. $\left\{\frac{4}{5}, -2\right\}$ **10.** $\left\{\frac{1}{4}\right\}$

Section 1.6 Activity

A.1. **a.** The expression $|x|$ represents the distance between the real number x and ________ on the number line.
b. Using this definition, solve the equation $|x| = 3$.
c. Solve the equation $|x| = 11$.
d. Solve the equation $|x| = 0$.
e. Explain why the equation $|x| = -2$ has no solution.

A.2. **a.** The equation $|u| = 5$ is equivalent to $u = 5$ or $u = -5$. Write the equation $|x - 4| = 5$ as two equations joined by "or."
b. Solve $|x - 4| = 5$.

A.3. **a.** What is the first step to solve the equation $4 = -6 + |3x + 1|$?
b. Solve the equation $4 = -6 + |3x + 1|$.

A.4. **a.** Solve the equation $|5x - 6| + 8 = 6$.
b. Solve the equation $|5x - 6| + 8 = 8$.

A.5. Consider the expression $|x| = |y|$. If two quantities have the same absolute value, then the quantities are equal, or the quantities are opposites. For example: $|5| = |5|$ and $|5| = |-5|$. With this in mind, write the equation $|x| = |y|$ as two equations joined by "or."

A.6. Given the equation $|x - 5| = |2x - 1|$,
a. Write the equation as two equations joined by "or."
b. Solve the equation.

A.7. Solve the equation $|4x + 3| = |4x - 1|$.

Practice Exercises Section 1.6

Prerequisite Review

For Exercises R.1–R.10, solve the equations.

R.1. **a.** $5x + 7 = 32$
b. $5x + 7 = -32$

R.2. **a.** $-4x - 3 = 17$
b. $-4x - 3 = -17$

R.3. **a.** $3w + 7 = 2w - 6$
b. $3w + 7 = -(2w - 6)$

R.4. **a.** $-4x + 5 = 2x - 1$
b. $-4x + 5 = -(2x - 1)$

R.5. $2(t - 3) + 4 = 2t - 3$

R.6. $3 - 4(p + 1) = -4p + 5$

R.7. $5x - 4 - x = 4(x - 1)$

R.8. $3(y + 1) - y = 2y + 3$

R.9. $\frac{1}{3}t - \frac{3}{4} = \frac{1}{6}t + \frac{1}{12}$

R.10. $\frac{2}{5}w + \frac{1}{3} = -\frac{1}{15}w - \frac{1}{5}$

Vocabulary and Key Concepts

1. **a.** An __________ value equation is an equation of the form $|x| = a$. If a is a positive real number, then the solution set is __________.

   b. What is the first step to solve the absolute value equation $|x| + 5 = 7$?

   c. The absolute value equation $|x| = |y|$ implies that $x =$ __________ or $x =$ __________.

   d. The solution set to the equation $|x + 4| = -2$ is __________. The solution set to the equation $|x + 4| = 0$ is __________.

Concept 1: Solving Absolute Value Equations

For Exercises 2–6, solve the equation.

2. a. $|x| = 4$
b. $|x| = -4$
c. $|x| = 0$

3. a. $|t| = 5$
b. $|t| = -5$
c. $|t| = 0$

4. a. $|x + 3| = 2$
b. $|x + 3| = -2$
c. $|x + 3| = 0$

5. a. $|y - 4| = 3$
b. $|y - 4| = -3$
c. $|y - 4| = 0$

6. a. $|2m| = 12$
b. $|2m| = -12$
c. $|2m| = 0$

For Exercises 7–38, solve the equations. **(See Examples 1–5.)**

7. $|p| = 7$
8. $|q| = 10$
9. $|x| + 5 = 11$
10. $|x| - 3 = 20$

11. $|y| + 8 = 5$
12. $|x| + 12 = 6$
13. $|w| - 3 = -1$
14. $|z| - 14 = -10$

15. $|3q| = 0$
16. $|4p| = 0$
17. $|3x - 4| = 8$
18. $|4x + 1| = 6$

19. $5 = |2x - 4|$
20. $10 = |3x + 7|$
21. $\left|\frac{7z}{3} - \frac{1}{3}\right| + 3 = 6$
22. $\left|\frac{w}{2} + \frac{3}{2}\right| - 2 = 7$

23. $|0.2x - 3.5| = -5.6$
24. $|1.81 + 2x| = -2.2$
25. $1 = -4 + \left|2 - \frac{1}{4}w\right|$
26. $-12 = -6 - |6 - 2x|$

27. $10 = 4 + |2y + 1|$
28. $-1 = -|5x + 7|$
29. $-2|3b - 7| - 9 = -9$
30. $-3|5x + 1| + 4 = 4$

31. $-2|x + 3| = 5$
32. $-3|x - 5| = 7$
33. $0 = |6x - 9|$
34. $7 = |4k - 6| + 7$

35. $\left|-\frac{1}{5} - \frac{1}{2}k\right| = \frac{9}{5}$
36. $\left|-\frac{1}{6} - \frac{2}{9}h\right| = \frac{1}{2}$
37. $-3|2 - 6x| + 5 = -10$
38. $5|1 - 2x| - 7 = 3$

Concept 2: Solving Equations Containing Two Absolute Values

For Exercises 39–56, solve the absolute value equations. **(See Examples 6–7.)**

39. $|4x - 2| = |-8|$
40. $|3x + 5| = |-5|$
41. $|4w + 3| = |2w - 5|$

42. $|3y + 1| = |2y - 7|$
43. $|2y + 5| = |7 - 2y|$
44. $|9a + 5| = |9a - 1|$

45. $\left|\frac{4w - 1}{6}\right| = \left|\frac{2w}{3} + \frac{1}{4}\right|$
46. $\left|\frac{6p + 3}{8}\right| = \left|\frac{3}{4}p - 2\right|$
47. $|2h - 6| = |2h + 5|$

48. $|6n - 7| = |4 - 6n|$

49. $|3.5m - 1.2| = |8.5m + 6|$

50. $|11.2n + 9| = |7.2n - 2.1|$

51. $|4x - 3| = -|2x - 1|$

52. $-|3 - 6y| = |8 - 2y|$

53. $|8 - 7w| = |7w - 8|$

54. $|4 - 3z| = |3z - 4|$

55. $|x + 2| + |x - 4| = 0$

56. $|t + 6| + |t - 1| = 0$

Technology Connections

For Exercises 57–62, enter the left side of the equation as Y_1 and enter the right side of the equation as Y_2. Then use the *Intersect* feature to approximate the x values where the two graphs intersect (if they intersect).

57. $|4x - 3| = 5$

58. $|x - 4| = 3$

59. $|8x + 1| + 8 = 1$

60. $|3x - 2| + 4 = 2$

61. $|x - 3| = |x + 2|$

62. $|x + 4| = |x - 2|$

Expanding Your Skills

63. Write an absolute value equation whose solution is the set of real numbers 6 units from zero on the number line.

64. Write an absolute value equation whose solution is the set of real numbers $\frac{7}{2}$ units from zero on the number line.

65. Write an absolute value equation whose solution is the set of real numbers $\frac{4}{3}$ units from zero on the number line.

66. Write an absolute value equation whose solution is the set of real numbers 9 units from zero on the number line.

Absolute Value Inequalities

Section 1.7

Concepts

1. Solving Absolute Value Inequalities by Definition
2. Solving Absolute Value Inequalities by the Test Point Method
3. Translating to an Absolute Value Expression

1. Solving Absolute Value Inequalities by Definition

In this section, we will solve absolute value *inequalities*. An inequality in any of the forms $|x| < a$, $|x| \le a$, $|x| > a$, or $|x| \ge a$ is called an **absolute value inequality**.

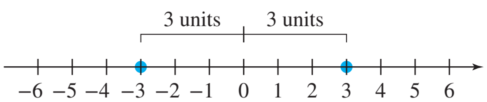
Recall that an absolute value represents distance from zero on the real number line. Consider the following absolute value equation and inequalities.

1. $|x| = 3$

$x = 3$ or $x = -3$

Solution:

The set of all points 3 units from zero on the number line

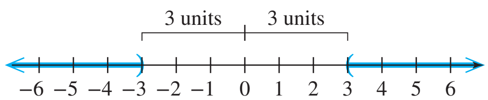


2. $|x| > 3$

$x < -3$ or $x > 3$

Solution:

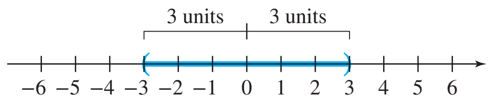
The set of all points more than 3 units from zero



3. $|x| < 3$
$-3 < x < 3$

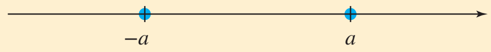
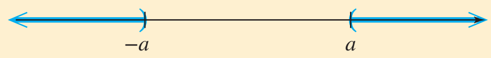
Solution:

The set of all points less than 3 units from zero



Solving Absolute Value Equations and Inequalities

Let a be a real number such that $a > 0$. Then

Equation/ Inequality	Solution (Equivalent Form)	Graph
$\|x\| = a$	$x = -a$ or $x = a$	
$\|x\| > a$	$x < -a$ or $x > a$	
$\|x\| < a$	$-a < x < a$	

To solve an absolute value inequality, first isolate the absolute value and then rewrite the absolute value inequality in its equivalent form.

Example 1 Solving an Absolute Value Inequality

Solve the inequality. $|3w + 1| - 4 < 7$

Solution:

$|3w + 1| - 4 < 7$

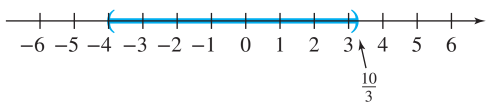
$|3w + 1| < 11$ ← Isolate the absolute value first.

The inequality is in the form $|x| < a$, where $x = 3w + 1$.

$-11 < 3w + 1 < 11$ Rewrite in the equivalent form $-a < x < a$.

$-12 < 3w < 10$ Solve for w.

$-4 < w < \frac{10}{3}$



The solution is $\{w \mid -4 < w < \frac{10}{3}\}$, or equivalently in interval notation, $\left(-4, \frac{10}{3}\right)$.

Skill Practice Solve the inequality. Write the solution in interval notation.

1. $|2t + 5| + 2 \le 11$

Answer

1. $[-7, 2]$

Example 2 Solving an Absolute Value Inequality

Solve the inequality. $3 \le 1 + \left|\frac{1}{2}t - 5\right|$

Solution:

$$3 \le 1 + \left|\frac{1}{2}t - 5\right|$$

$1 + \left|\frac{1}{2}t - 5\right| \ge 3$ Write the inequality with the absolute value on the left.

$\left|\frac{1}{2}t - 5\right| \ge 2$ Isolate the absolute value.

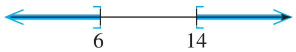
The inequality is in the form $|x| \ge a$, where $x = \frac{1}{2}t - 5$.

$\frac{1}{2}t - 5 \le -2$ or $\frac{1}{2}t - 5 \ge 2$ Rewrite in the equivalent form $x \le -a$ or $x \ge a$.

$\frac{1}{2}t \le 3$ or $\frac{1}{2}t \ge 7$ Solve the compound inequality.

$2\left(\frac{1}{2}t\right) \le 2(3)$ or $2\left(\frac{1}{2}t\right) \ge 2(7)$ Clear fractions.

$t \le 6$ or $t \ge 14$



The solution is $\{t \mid t \le 6 \text{ or } t \ge 14\}$, or equivalently in interval notation, $(-\infty, 6] \cup [14, \infty)$.

Skill Practice Solve the inequality. Write the solution in interval notation.

2. $5 < 1 + \left|\frac{1}{3}c - 1\right|$

TIP: It is generally easier to solve an absolute value inequality if the absolute value appears on the left-hand side of the inequality.

By definition, the absolute value of a real number will always be nonnegative. Therefore, the absolute value of any expression will always be greater than a negative number. Similarly, an absolute value can never be less than a negative number. If a represents a positive real number, then

- The solution to the inequality $|x| > -a$ is all real numbers, $(-\infty, \infty)$.
- There is no solution to the inequality $|x| < -a$.

Example 3 Solving Absolute Value Inequalities

Solve the inequalities.

a. $|3d - 5| + 7 < 4$ **b.** $|3d - 5| + 7 > 4$

Solution:

a. $|3d - 5| + 7 < 4$

$|3d - 5| < -3$

Isolate the absolute value. An absolute value expression cannot be less than a negative number. Therefore, there is no solution.

No solution, { }

Answer

2. $(-\infty, -9) \cup (15, \infty)$

b. $|3d - 5| + 7 > 4$

$|3d - 5| > -3$

All real numbers, $(-\infty, \infty)$

Isolate the absolute value. The inequality is in the form $|x| > a$, where a is negative. An absolute value of any real number is greater than a negative number. Therefore, the solution is all real numbers.

Skill Practice Solve the inequalities.

3. $|4p + 2| + 6 < 2$ **4.** $|4p + 2| + 6 > 2$

Example 4 Solving Absolute Value Inequalities

Solve the inequalities.

a. $|4x + 2| \geq 0$ **b.** $|4x + 2| > 0$ **c.** $|4x + 2| \leq 0$

Solution:

a. $|4x + 2| \geq 0 \longleftarrow$ The absolute value is already isolated.

The absolute value of any real number is nonnegative. Therefore, the solution is all real numbers, $(-\infty, \infty)$.

b. $|4x + 2| > 0$

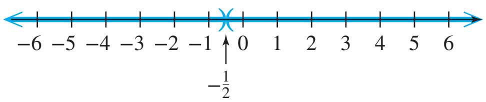
An absolute value will be greater than zero at all points *except where it is equal to zero*. That is, the value(s) of x for which $|4x + 2| = 0$ must be excluded from the solution set.

$|4x + 2| = 0$

$4x + 2 = 0$ or $4x + 2 = -0$ The second equation is the same as the first.

$4x = -2$

$x = -\frac{1}{2}$ Therefore, exclude $x = -\frac{1}{2}$ from the solution.



The solution is $\{x \mid x \neq -\frac{1}{2}\}$, or equivalently in interval notation, $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$.

c. $|4x + 2| \leq 0$

An absolute value of a number cannot be less than zero. However, it can be *equal* to zero. Therefore, the only solutions to this inequality are the solutions to the related equation:

$|4x + 2| = 0$ From part (b), we see that the solution set is $\left\{-\frac{1}{2}\right\}$.

Skill Practice Solve the inequalities.

5. $|3x - 1| \geq 0$ **6.** $|3x - 1| > 0$ **7.** $|3x - 1| \leq 0$

Answers

3. { }

4. All real numbers; $(-\infty, \infty)$

5. $(-\infty, \infty)$

6. $\left\{x \middle| x \neq \frac{1}{3}\right\}$ or $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

7. $\left\{\frac{1}{3}\right\}$

2. Solving Absolute Value Inequalities by the Test Point Method

In Examples 1 and 2, each absolute value inequality was converted to an equivalent compound inequality. However, sometimes students have difficulty setting up the appropriate compound inequality. To avoid this problem, you may want to use the test point method to solve absolute value inequalities.

Solving Inequalities by Using the Test Point Method

Step 1 Find the boundary points of the inequality. (Boundary points are the real solutions to the related equation and points where the inequality is undefined.)

Step 2 Plot the boundary points on the number line. This divides the number line into intervals.

Step 3 Select a test point from each interval and substitute it into the original inequality.

- If a test point makes the original inequality true, then that interval is part of the solution set.

Step 4 Test the boundary points in the original inequality.

- If the original inequality is strict ($<$ or $>$), do not include the boundary points in the solution set.
- If the original inequality is defined using $\le$ or $\ge$, then include the boundary points that are defined within the inequality.

Note: Any boundary point that makes an expression within the inequality undefined must *always* be excluded from the solution set.

To demonstrate the use of the test point method, we will repeat the absolute value inequalities from Examples 1 and 2. Notice that regardless of the method used, the absolute value is always isolated *first* before any further action is taken.

Example 5 Solving an Absolute Value Inequality by the Test Point Method

Solve the inequality by using the test point method. $|3w + 1| - 4 < 7$

Solution:

$|3w + 1| - 4 < 7$

$|3w + 1| < 11$ ← Isolate the absolute value.

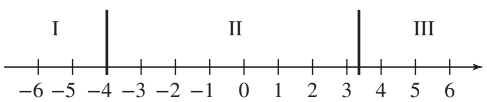
$|3w + 1| = 11$ **Step 1:** Solve the related equation.

$3w + 1 = 11$ or $3w + 1 = -11$ Write as an equivalent system of two equations.

$3w = 10$ or $3w = -12$

$w = \dfrac{10}{3}$ or $w = -4$ These are the only boundary points.

Step 2: Plot the boundary points.



Step 3: Select a test point from each interval.

Test $w = -5$:

$|3(-5) + 1| - 4 \overset{?}{<} 7$

$|-14| - 4 \overset{?}{<} 7$

$14 - 4 \overset{?}{<} 7$

$10 \overset{?}{<} 7$ False

Test $w = 0$:

$|3(0) + 1| - 4 \overset{?}{<} 7$

$|1| - 4 \overset{?}{<} 7$

$-3 \overset{?}{<} 7$ True

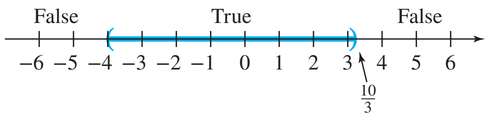
Test $w = 4$:

$|3(4) + 1| - 4 \overset{?}{<} 7$

$|13| - 4 \overset{?}{<} 7$

$13 - 4 \overset{?}{<} 7$

$9 \overset{?}{<} 7$ False



Step 4: Because the original inequality is a strict inequality, the boundary points (where equality occurs) are not included.

The solution is $\{w \mid -4 < w < \frac{10}{3}\}$, or equivalently in interval notation, $\left(-4, \frac{10}{3}\right)$.

Skill Practice Solve the inequality.

8. $6 + |3t - 4| \le 10$

Example 6 Solving an Absolute Value Inequality by the Test Point Method

Solve the inequality by using the test point method. $3 \le 1 + \left|\frac{1}{2}t - 5\right|$

Solution:

$3 \le 1 + \left|\frac{1}{2}t - 5\right|$

$1 + \left|\frac{1}{2}t - 5\right| \ge 3$ — Write the inequality with the absolute value on the left.

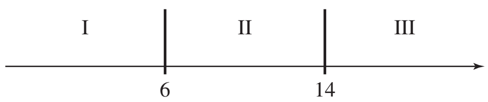
$\left|\frac{1}{2}t - 5\right| \ge 2$ ← Isolate the absolute value.

$\left|\frac{1}{2}t - 5\right| = 2$ — **Step 1:** Solve the related equation.

$\frac{1}{2}t - 5 = 2$ or $\frac{1}{2}t - 5 = -2$ — Write as an equivalent system of two equations.

$\frac{1}{2}t = 7$ or $\frac{1}{2}t = 3$

$t = 14$ or $t = 6$ — These are the boundary points.



Step 2: Plot the boundary points.

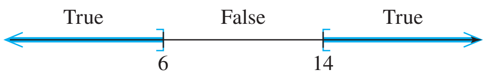
Step 3: Select a test point from each interval.

Answer

8. $\left[0, \frac{8}{3}\right]$

Test $t = 0$:	**Test $t = 10$:**	**Test $t = 16$:**
$3 \overset{?}{\leq} 1 + \left\lvert\frac{1}{2}(0) - 5\right\rvert$	$3 \overset{?}{\leq} 1 + \left\lvert\frac{1}{2}(10) - 5\right\rvert$	$3 \overset{?}{\leq} 1 + \left\lvert\frac{1}{2}(16) - 5\right\rvert$
$3 \overset{?}{\leq} 1 + \lvert 0 - 5\rvert$	$3 \overset{?}{\leq} 1 + \lvert 5 - 5\rvert$	$3 \overset{?}{\leq} 1 + \lvert 8 - 5\rvert$
$3 \overset{?}{\leq} 1 + \lvert -5\rvert$	$3 \overset{?}{\leq} 1 + \lvert 0\rvert$	$3 \overset{?}{\leq} 1 + \lvert 3\rvert$
$3 \overset{?}{\leq} 6$ True	$3 \overset{?}{\leq} 1$ False	$3 \overset{?}{\leq} 4$ True

Step 4: The original inequality uses the sign $\geq$. Therefore, the boundary points (where equality occurs) must be part of the solution set.



The solution is $\{t \mid t \leq 6 \text{ or } t \geq 14\}$, or equivalently in interval notation, $(-\infty, 6] \cup [14, \infty)$.

Skill Practice Solve the inequality.

9. $\left\lvert\frac{1}{2}c + 4\right\rvert + 1 > 6$

3. Translating to an Absolute Value Expression

Absolute value expressions can be used to describe distances. The distance between c and d is given by $\lvert c - d\rvert$. For example, the distance between -2 and 3 on the number line is $\lvert(-2) - 3\rvert = 5$ as expected.

Example 7 Expressing Distances With Absolute Value

Write an absolute value inequality to represent the following phrases.

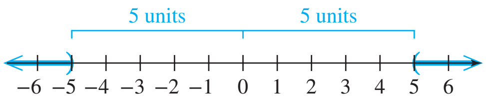
a. All real numbers x, whose distance from zero is greater than 5 units

b. All real numbers x, whose distance from -7 is less than 3 units

Solution:

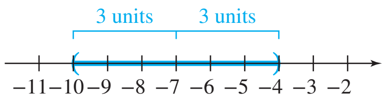
a. All real numbers x, whose distance from zero is greater than 5 units

$\lvert x - 0\rvert > 5$ or simply $\lvert x\rvert > 5$



b. All real numbers x, whose distance from -7 is less than 3 units

$\lvert x - (-7)\rvert < 3$ or simply $\lvert x + 7\rvert < 3$



Skill Practice Write an absolute value inequality to represent the following phrases.

10. All real numbers whose distance from zero is greater than 10 units

11. All real numbers whose distance from 4 is less than 6 units

Answers

9. $(-\infty, -18) \cup (2, \infty)$

10. $\lvert x\rvert > 10$

11. $\lvert x - 4\rvert < 6$

Absolute value expressions can also be used to describe boundaries for measurement error.

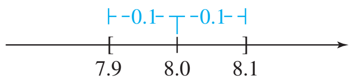
Example 8 Expressing Measurement Error With Absolute Value

Latoya measured a certain compound on a scale in the chemistry lab at school. She measured 8 g of the compound, but the scale is only accurate to ± 0.1 g. Write an absolute value inequality to express an interval for the true mass, x, of the compound she measured.

Solution:

Because the scale is only accurate to ± 0.1 g, the true mass, x, of the compound may deviate by as much as 0.1 g above or below 8 g. This may be expressed as an absolute value inequality:

$|x - 8.0| \leq 0.1$ or equivalently $7.9 \leq x \leq 8.1$



Skill Practice

12. Vonzell molded a piece of metal in her machine shop. She measured the thickness at 12 mm. Her machine is accurate to ± 0.05 mm. Write an absolute value inequality to express an interval for the true measurement of the thickness, t, of the metal.

Answer

12. $|t - 12| \leq 0.05$

Section 1.7 Activity

A.1. a. The inequality $|x| \geq 3$ is the set of real numbers that are 3 or more units from zero on the number line. Shade the region(s) on the number line that satisfy the inequality $|x| \geq 3$.

b. Write the solution set to the inequality $|x| \geq 3$ in interval notation.

c. Shade the region(s) on the number line that satisfy the inequality $|x| \leq 3$.

d. Write the solution set to the inequality $|x| \leq 3$ in interval notation.

A.2. Given the inequality $|2x + 5| - 1 < 7$,

a. What is the first step to solve the inequality?

b. Write the inequality as an equivalent compound inequality without absolute value bars.

c. Solve the inequality and write the solution set in interval notation.

A.3. Given the inequality $3 - |x - 9| < -7$,

a. What is the first step to solve the inequality?

b. Write the inequality as an equivalent compound inequality without absolute value bars.

c. Solve the inequality and write the solution set in interval notation.

A.4. Solve each inequality by inspection and explain your answer.

a. $|3x + 4| < -5$

b. $|3x + 4| \leq -5$

c. $|3x + 4| > -5$

d. $|3x + 4| \geq -5$

A.5. Solve each inequality by inspection and explain your answer.

a. $|2x - 6| < 0$

b. $|2x - 6| \leq 0$

c. $|2x - 6| \geq 0$

d. $|2x - 6| > 0$

Practice Exercises

Section 1.7

Study Skills Exercise

As a college student, you need to balance the demands of your job, your family, your personal life, and your education. The first step in achieving this balance is to efficiently manage your time. To do so, analyze your schedule and plan out your week. Keep a calendar to maintain your school, home, and work responsibilities. Insert due dates into the calendar, and schedule your study time throughout the week in 20- to 30-min sessions.

Prerequisite Review

For Exercises R.1–R.4, solve the compound inequality. Write the solution set in interval notation.

R.1. $2x + 3 < -5$ or $2x + 3 > 5$

R.2. $5 - x \leq -1$ or $5 - x \geq 1$

R.3. $3 + y \geq -2$ and $3 + y \leq 2$

R.4. $4x + 1 > -9$ and $4x + 1 < 9$

For Exercises R.5–R.10, solve the equation.

R.5. $|3x - 4| + 1 = 9$

R.6. $|3 + 2x| - 8 = 11$

R.7. $|6x + 7| + 9 = 4$

R.8. $|-4x + 9| + 5 = 3$

R.9. $|-5x + 3| + 2 = 2$

R.10. $|-2x - 7| + 3 = 3$

Vocabulary and Key Concepts

1. a. If a is a positive real number, then the inequality $|x| < a$ is equivalent to ________ $< x <$ ________.

b. If a is a positive real number, then the inequality $|x| > a$ is equivalent to $x <$ ________ or x ________ a.

c. The solution set to the inquality $|x + 2| < -6$ is ________, whereas the solution set to the inequality $|x + 2| > -6$ is ________.

2. The solution set to the inequality $|x + 4| \leq 0$ (includes/excludes) -4, whereas the solution set to the inequality $|x + 4| < 0$ (includes/excludes) -4.

Concepts 1 and 2: Solving Absolute Value Inequalities

For Exercises 3–14, solve the equations and inequalities. For each inequality, graph the solution set and express the solution in interval notation. **(See Examples 1–6.)**

3. **a.** $|x| = 5$
b. $|x| > 5$
c. $|x| < 5$

4. **a.** $|a| = 4$
b. $|a| > 4$
c. $|a| < 4$

5. **a.** $|x - 3| = 7$
b. $|x - 3| > 7$
c. $|x - 3| < 7$

6. **a.** $|w + 2| = 6$
b. $|w + 2| > 6$
c. $|w + 2| < 6$

7. **a.** $|p| = -2$
b. $|p| > -2$
c. $|p| < -2$

8. **a.** $|x| = -14$
b. $|x| > -14$
c. $|x| < -14$

9. **a.** $|y + 1| = -6$
b. $|y + 1| > -6$
c. $|y + 1| < -6$

10. **a.** $|z - 4| = -3$
b. $|z - 4| > -3$
c. $|z - 4| < -3$

11. **a.** $|x| = 0$
b. $|x| > 0$
c. $|x| < 0$

12. **a.** $|p + 3| = 0$
b. $|p + 3| > 0$
c. $|p + 3| < 0$

13. **a.** $|k - 7| = 0$
b. $|k - 7| > 0$
c. $|k - 7| < 0$

14. **a.** $|2x + 4| + 3 = 2$
b. $|2x + 4| + 3 > 2$
c. $|2x + 4| + 3 < 2$

For Exercises 15–44, solve the absolute value inequality. Graph the solution set and write the solution in interval notation. **(See Examples 1–6.)**

15. $|x| > 6$

16. $|x| \le 6$

17. $|t| \le 3$

18. $|p| > 3$

19. $|y + 2| \ge 0$

20. $0 \le |7n + 2|$

21. $5 \le |2x - 1|$

22. $|x - 2| \ge 7$

23. $|k - 7| < -3$

24. $|h + 2| < -9$

25. $\left|\dfrac{w - 2}{3}\right| - 3 \le 1$

26. $\left|\dfrac{x + 3}{2}\right| - 2 \ge 4$

27. $12 \le |9 - 4y| - 2$

28. $5 > |2m - 7| + 4$

29. $4 > -1 + \left|\dfrac{2x + 1}{4}\right|$

30. $9 \geq 2 + \left|\frac{x-4}{5}\right|$

31. $8 < |4 - 3x| + 12$

32. $-16 < |5x - 1| - 1$

33. $5 - |2m + 1| > 5$

34. $3 - |5x + 3| > 3$

35. $|p + 5| \leq 0$

36. $|y + 1| - 4 \leq -4$

37. $|z - 6| + 5 > 5$

38. $|2c - 1| - 4 > -4$

39. $5|2y - 6| + 3 \geq 13$

40. $7|y + 1| - 3 \geq 11$

41. $-3|6 - t| + 1 > -5$

42. $-4|8 - x| + 2 > -14$

43. $|0.02x + 0.06| - 0.1 < 0.05$

44. $|0.05x - 0.04| - 0.01 < 0.11$

Concept 3: Translating to an Absolute Value Expression

For Exercises 45–48, write an absolute value inequality equivalent to the expression given. **(See Example 7.)**

45. All real numbers whose distance from 0 is greater than 7

46. All real numbers whose distance from -3 is less than 4

47. All real numbers whose distance from 2 is at most 13

48. All real numbers whose distance from 0 is at least 6

49. A 32-oz jug of orange juice may not contain exactly 32 oz of juice. The possibility of measurement error exists when the jug is filled in the factory. If the maximum measurement error is ± 0.05 oz, write an absolute value inequality representing the range of volumes, x, in which the orange juice jug may be filled. **(See Example 8.)**

50. The length of a board is measured to be 32.3 in. The maximum measurement error is ± 0.2 in. Write an absolute value inequality that represents the range for the length of the board, x.

51. A bag of potato chips states that its weight is $6\frac{3}{4}$ oz. The maximum measurement error is $\pm\frac{1}{8}$ oz. Write an absolute value inequality that represents the range for the weight, x, of the bag of chips.

Andrew Bret Wallis/BananaStock/Getty Images

52. A $\frac{7}{8}$-in. bolt varies in length by at most $\pm\frac{1}{16}$ in. Write an absolute value inequality that represents the range for the length, x, of the bolt.

53. The width, w, of a bolt is supposed to be 2 cm but may have a 0.01-cm margin of error. Solve $|w - 2| \leq 0.01$, and interpret the solution to the inequality in the context of this problem.

54. In a political poll, the front-runner was projected to receive 53% of the votes with a margin of error of 3%. Solve $|p - 0.53| \leq 0.03$ and interpret the solution in the context of this problem.

Technology Connections

To solve an absolute value inequality by using a graphing calculator, let Y_1 equal the left side of the inequality and let Y_2 equal the right side of the inequality. Graph both Y_1 and Y_2 on a standard viewing window and use an *Intersect* feature to approximate the intersection of the graphs. To solve $Y_1 > Y_2$, determine all x values where the graph of Y_1 is above the graph of Y_2. To solve $Y_1 < Y_2$, determine all x values where the graph of Y_1 is below the graph of Y_2.

For Exercises 55–64, solve the inequalities using a graphing calculator.

55. $|x + 2| > 4$

56. $|3 - x| > 6$

57. $\left|\dfrac{x + 1}{3}\right| < 2$

58. $\left|\dfrac{x - 1}{4}\right| < 1$

59. $|x - 5| < -3$

60. $|x + 2| < -2$

61. $|2x + 5| > -4$

62. $|1 - 2x| > -4$

63. $|6x + 1| \leq 0$

64. $|3x - 4| \leq 0$

Problem Recognition Exercises

Identifying Equations and Inequalities

For Exercises 1–8, solve each equation or inequality. Express the solution in interval notation where appropriate.

1. **a.** $3x - 9 = 18$
b. $|3x - 9| = 18$
c. $|3x - 9| < 18$
d. $|3x - 9| \geq 18$

2. **a.** $5y + 2 = -20$
b. $|5y + 2| = -20$
c. $|5y + 2| \leq -20$
d. $|5y + 2| > -20$

3. **a.** $-2t - 14 = 0$
b. $-2t - 14 > 0$
c. $-2t - 14 \leq 0$

4. **a.** $\dfrac{x - 2}{3} = 9$
b. $\dfrac{x - 2}{3} \geq 9$
c. $\dfrac{x - 2}{3} < 9$

5. **a.** $|8t - 2| = |-2t + 3|$
b. $8t - 2 = -2t + 3$

6. **a.** $-5 < x + 2$ and $x + 2 \leq 8$
b. $-5 < x + 2 \leq 8$

7. **a.** $-4x - 9 < 11$ or $2 \leq x + 1$
b. $-4x - 9 < 11$ and $2 \leq x + 1$

8. **a.** $4 < 2y$ or $-3(y + 2) > -2y + 1$
b. $4 < 2y$ and $-3(y + 2) > -2y + 1$

For Exercises 9–28,

a. Identify the type of equation or inequality. Choose from:
- linear equation
- absolute value equation
- linear inequality
- compound inequality
- absolute value inequality

b. Solve the equation or inequality. Express the solution set in interval notation where appropriate.

9. $-0.5y + 0.7 = 3.7$

10. $3m - 9 = 18$

11. $|2t + 8| \leq 4$

12. $|1 - 3x| < -1$

13. $-11 < 2t + 1 < 19$

14. $2z - 3 \geq 11$ or $3z + 3 < 9$

15. $\left|\frac{1}{2}y + 3\right| = 5$

16. $|4x + 3| = |9 - 2x|$

17. $-\frac{3}{4}p \geq -9$

18. $8w + 4 \geq 5w + 1$

19. $\left|\frac{2x - 9}{3}\right| \geq 5$

20. $\left|\frac{10 - x}{5}\right| < 3$

21. $|2 - c| + 5 = 3$

22. $|10n + 2| + 7 = 7$

23. $\frac{w - 4}{5} - \frac{w + 1}{3} = 1$

24. $\frac{1}{3}y - \frac{5}{6} = \frac{1}{2}y + 1$

25. $2x - 7 > 9$ and $3x \leq 36$

26. $-3 + x > 2x$ and $2 \geq -\frac{1}{3}x$

27. $5(x - 2) + 7 = 2x + 3(x - 1)$

28. $7y - 4 = 3(y + 1) + 4y$

Chapter 1 Summary

Section 1.1 Linear Equations in One Variable

Key Concepts

A **linear equation in one variable** can be written in the form $ax + b = c$ $(a \neq 0)$.

Steps to Solve a Linear Equation in One Variable

1. Simplify both sides of the equation.
   - Clear parentheses.
   - Consider clearing fractions or decimals (if any are present) by multiplying both sides of the equation by a common denominator of all terms.
   - Combine *like* terms.
2. Use the addition or subtraction property of equality to collect the variable terms on one side of the equation.
3. Use the addition or subtraction property of equality to collect the constant terms on the other side.
4. Use the multiplication or division property of equality to make the coefficient on the variable term equal to 1.
5. Check your answer and write the solution set.

An equation that has no solution is called a **contradiction**.

An equation that has all real numbers as its solutions is called an **identity**.

Examples

Example 1

$$\frac{1}{2}(x-4) - \frac{3}{4}(x+2) = \frac{1}{4}$$

$$\frac{1}{2}x - 2 - \frac{3}{4}x - \frac{3}{2} = \frac{1}{4}$$

$$4\left(\frac{1}{2}x - 2 - \frac{3}{4}x - \frac{3}{2}\right) = 4\left(\frac{1}{4}\right)$$

$$2x - 8 - 3x - 6 = 1$$

$$-x - 14 = 1$$

$$-x = 15$$

$$x = -15$$

The solution -15 checks in the original equation.

The solution set is $\{-15\}$.

Example 2

$$3x + 6 = 3(x - 5)$$

$$3x + 6 = 3x - 15$$

$$6 = -15 \qquad \text{Contradiction}$$

There is no solution, $\{\ \}$.

Example 3

$$-(5x + 12) - 3 = 5(-x - 3)$$

$$-5x - 12 - 3 = -5x - 15$$

$$-5x - 15 = -5x - 15$$

$$-15 = -15 \qquad \text{Identity}$$

All real numbers are solutions.

The solution set is $\{x \mid x \text{ is a real number}\}$.

Section 1.2 Applications of Linear Equations in One Variable

Key Concepts

Problem-Solving Steps for Word Problems

1. Read the problem carefully.
2. Assign labels to unknown quantities.
3. Write a verbal model.
4. Write a mathematical equation.
5. Solve the equation.
6. Interpret the results and write the final answer in words.

Sales tax: (cost of merchandise)(tax rate)

Commission: (dollars in sales)(commission rate)

Simple interest: (Principal)(interest rate)(time in years)
$I = Prt$

Distance = (rate)(time) $\quad d = rt$

Examples

Example 1

1. Estella needs to borrow \$8500. She borrows part of the money from a friend and agrees to pay the friend 6% simple interest. She borrows the rest of the money from a bank that charges 10% simple interest. If she pays back the money at the end of 1 yr and also pays \$750 in interest, find the amount that Estella borrowed from each source.

2. Let x represent the amount borrowed at 6%. Then $8500 - x$ is the amount borrowed at 10%.

	6% Account	10% Account	Total
Principal	x	$8500 - x$	8500
Interest	$0.06x$	$0.10(8500 - x)$	750

3. $$\begin{pmatrix}\text{Interest}\\\text{owed at 6\%}\end{pmatrix} + \begin{pmatrix}\text{interest}\\\text{owed at 10\%}\end{pmatrix} = \begin{pmatrix}\text{total}\\\text{interest}\end{pmatrix}$$

4. $0.06x + 0.10(8500 - x) = 750$

5. $$\begin{aligned} 6x + 10(8500 - x) &= 75{,}000 \\ 6x + 85{,}000 - 10x &= 75{,}000 \\ -4x &= -10{,}000 \\ x &= 2500 \end{aligned}$$

6. $x = 2500$
$8500 - 2500 = 6000$

\$2500 was borrowed at 6% and \$6000 was borrowed at 10%.

Section 1.3 Applications to Geometry and Literal Equations

Key Concepts

Some useful formulas for word problems:

Perimeter

Rectangle: $P = 2l + 2w$

Area

Rectangle: $A = lw$

Square: $A = s^2$

Triangle: $A = \frac{1}{2}bh$

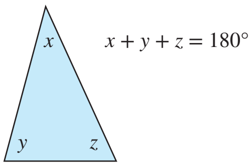
Trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$

Angles

Two angles whose measures total 90° are complementary angles.

Two angles whose measures total 180° are supplementary angles.

The sum of the measures of the angles of a triangle is 180°.

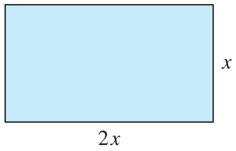


Literal equations (or formulas) are equations with several variables. To solve for a specific variable, follow the steps to solve a linear equation.

Examples

Example 1

A border of marigolds is to enclose a rectangular flower garden. If the length is twice the width and the perimeter is 25.5 ft, what are the dimensions of the garden?



$$
\begin{aligned}
P &= 2l + 2w \\
25.5 &= 2(2x) + 2(x) \\
25.5 &= 4x + 2x \\
25.5 &= 6x \\
4.25 &= x
\end{aligned}
$$

The width is 4.25 ft, and the length is 2(4.25) ft or 8.5 ft.

Example 2

Solve for y.

$$
\begin{aligned}
4x - 5y &= 20 \\
-5y &= -4x + 20 \\
\frac{-5y}{-5} &= \frac{-4x + 20}{-5} \\
y &= \frac{-4x + 20}{-5} \quad \text{or} \quad y = \frac{4}{5}x - 4
\end{aligned}
$$

Section 1.4 Linear Inequalities in One Variable

Key Concepts

A **linear inequality** is an inequality that can be written in one of the following forms, provided that $a \neq 0$.

$ax + b < c, \quad ax + b > c, \quad ax + b \leq c, \quad \text{or} \quad ax + b \geq c$

Properties of Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$, then $a - c < b - c$.
3. If c is positive and $a < b$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
4. If c is negative and $a < b$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

Properties 3 and 4 indicate that if we multiply or divide an inequality by a negative value, the direction of the inequality sign must be reversed.

Examples

Example 1

Solve.

$$\frac{14 - x}{-2} < -3x$$ Multiply both sides by -2 to clear fractions.

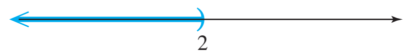
$$-\cancel{2}\left(\frac{14 - x}{-\cancel{2}}\right) > -2(-3x)$$ (Reverse the inequality sign.)

$$14 - x > 6x$$

$$-7x > -14$$

$$\frac{-7x}{-7} < \frac{-14}{-7}$$ (Reverse the inequality sign.)

$$x < 2$$



Set-builder notation: $\{x \mid x < 2\}$
Interval notation: $(-\infty, 2)$

Section 1.5 Compound Inequalities

Key Concepts

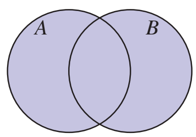
$A \cup B$ is the **union** of A and B. This is the set of elements that belong to set A or set B or both sets A and B.

$A \cap B$ is the **intersection** of A and B. This is the set of elements common to both A and B.

Examples

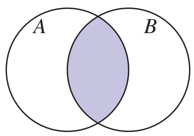
Example 1

Union



$A \cup B$

Intersection

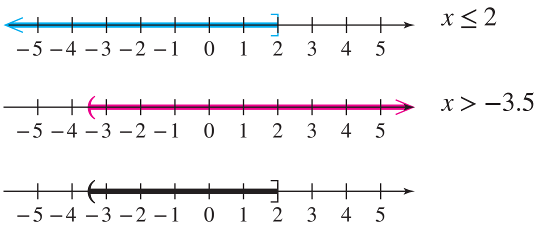


$A \cap B$

- Solve two or more inequalities joined by *and* by finding the intersection of their solution sets.

Example 2

$$-7x + 3 \geq -11 \quad \text{and} \quad 1 - x < 4.5$$
$$-7x \geq -14 \quad \text{and} \quad -x < 3.5$$
$$x \leq 2 \quad \text{and} \quad x > -3.5$$



The solution is $\{x \mid -3.5 < x \leq 2\}$, or equivalently, $(-3.5, 2]$.

Inequalities of the form $a < x < b$:

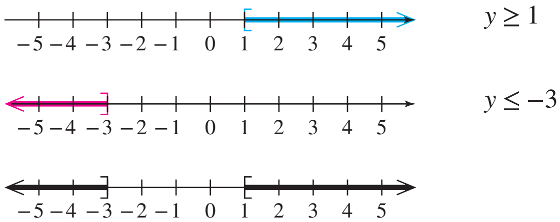
The inequality $a < x < b$ is equivalent to the compound inequality $a < x$ and $x < b$.

or, in interval notation, (a, b).

- Solve two or more inequalities joined by *or* by finding the union of their solution sets.

Example 3

$$5y + 1 \geq 6 \quad \text{or} \quad 2y - 5 \leq -11$$
$$5y \geq 5 \quad \text{or} \quad 2y \leq -6$$
$$y \geq 1 \quad \text{or} \quad y \leq -3$$



The solution is $\{y \mid y \leq -3 \text{ or } y \geq 1\}$, or equivalently, $(-\infty, -3] \cup [1, \infty)$.

Example 4

Solve.

$$-13 \leq 3x - 1 < 5$$
$$-13 + 1 \leq 3x - 1 + 1 < 5 + 1$$
$$-12 \leq 3x < 6$$
$$\frac{-12}{3} \leq \frac{3x}{3} < \frac{6}{3}$$
$$-4 \leq x < 2$$

Interval notation: $[-4, 2)$

Section 1.6 Absolute Value Equations

Key Concepts

The equation $|x| = a$ is an **absolute value equation**. For $a \geq 0$, the solution to the equation $|x| = a$ is $x = a$ or $x = -a$.

Steps to Solve an Absolute Value Equation

1. Isolate the absolute value to write the equation in the form $|x| = a$.
2. If $a < 0$, there is no solution.
3. Otherwise, if $a \geq 0$, rewrite the equation $|x| = a$ as $x = a$ or $x = -a$.
4. Solve the equations from step 3.
5. Check answers in the original equation.

The equation $|x| = |y|$ implies $x = y$ or $x = -y$.

Examples

Example 1

$|2x - 3| + 5 = 10$

$|2x - 3| = 5$ Isolate the absolute value.

$2x - 3 = 5$ or $2x - 3 = -5$

$2x = 8$ or $2x = -2$

$x = 4$ or $x = -1$

The solution set is $\{4, -1\}$.

Example 2

$|x + 2| + 5 = 1$

$|x + 2| = -4$ No solution, { }

Example 3

$|2x - 1| = |x + 4|$

$2x - 1 = x + 4$ or $2x - 1 = -(x + 4)$

$x = 5$ or $2x - 1 = -x - 4$

or $3x = -3$

or $x = -1$

The solution set is $\{5, -1\}$.

Section 1.7 Absolute Value Inequalities

Key Concepts

Solutions to Absolute Value Inequalities

For $a > 0$, we have:

$|x| > a \Rightarrow x < -a$ or $x > a$

$|x| < a \Rightarrow -a < x < a$

Examples

Example 1

$|5x - 2| < 12$

$-12 < 5x - 2 < 12$

$-10 < 5x < 14$

$-2 < x < \frac{14}{5}$

The solution is $\left(-2, \frac{14}{5}\right)$.

Test Point Method to Solve Inequalities

1. Find the boundary points of the inequality. (Boundary points are the real solutions to the related equation and points where the inequality is undefined.)
2. Plot the boundary points on the number line. This divides the number line into intervals.
3. Select a test point from each interval and substitute it into the original inequality.
   - If a test point makes the original inequality true, then that interval is part of the solution set.
4. Test the boundary points in the original inequality.
   - If the original inequality is strict ($<$ or $>$), do not include the boundary in the solution set.
   - If the original inequality is defined using $\le$ or $\ge$, then include the boundary points that are defined within the inequality.

Note: Any boundary point that makes an expression within the inequality undefined must *always* be excluded from the solution set.

Example 2

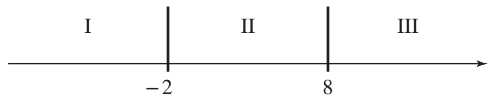
$|x-3|+2 \ge 7$

$|x-3| \ge 5$ Isolate the absolute value.

$|x-3| = 5$ Solve the related equation.

$x-3=5$ or $x-3=-5$

$x=8$ or $x=-2$ Boundary points



Interval I:

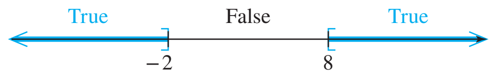
Test $x=-3$: $|(-3)-3|+2 \overset{?}{\ge} 7$ True

Interval II:

Test $x=0$: $|(0)-3|+2 \overset{?}{\ge} 7$ False

Interval III:

Test $x=9$: $|(9)-3|+2 \overset{?}{\ge} 7$ True



The solution is $(-\infty, -2] \cup [8, \infty)$.

If *a* is *negative* ($a < 0$), then

1. $|x| < a$ has no solution.
2. $|x| > a$ is true for all real numbers.

Example 3

$|x+5| > -2$

The solution is all real numbers because an absolute value will always be greater than a negative number.

$(-\infty, \infty)$

Example 4

$|x+5| < -2$

There is no solution because an absolute value cannot be less than a negative number.

The solution set is $\{ \ \}$.

Chapter 1 Review Exercises

Section 1.1

1. Describe the solution set for a contradiction.

2. Describe the solution set for an identity.

For Exercises 3–12, solve the equations and identify each as a conditional equation, a contradiction, or an identity.

3. $x - 27 = -32$

4. $y + \dfrac{7}{8} = 1$

5. $7.23 + 0.6x = 0.2x$

6. $0.1y + 1.122 = 5.2y$

7. $-(4 + 3m) = 9(3 - m)$

8. $-2(5n - 6) = 3(-n - 3)$

9. $\dfrac{x - 3}{5} - \dfrac{2x + 1}{2} = 1$

10. $3(x + 3) - 2 = 3x + 2$

11. $\dfrac{10}{8}m + 18 - \dfrac{7}{8}m = \dfrac{3}{8}m + 25$

12. $\dfrac{2}{3}m + \dfrac{1}{3}(m - 1) = -\dfrac{1}{3}m + \dfrac{1}{3}(4m - 1)$

Section 1.2

13. How would you label three consecutive integers?

14. How would you label two consecutive odd integers?

15. Explain what the formula $d = rt$ means.

16. Explain what the formula $I = Prt$ means.

17. **a.** Cory makes \$85,200 in taxable income. If he pays an average of 28% in taxes on his income, determine the amount of tax he must pay.

    b. What is his net income (after taxes)?

18. For a recent year, approximately 7.2 million men were in college in the United States. This represents an 8% increase over the number of men in college in the year 2000. Approximately how many men were in college in 2000? (Round to the nearest tenth of a million.)

19. For a recent year, there were 17,430 deaths due to alcohol-related accidents in the United States. This was a 5% increase over the number of alcohol-related deaths in 1999. How many such deaths were there in 1999?

20. Of three consecutive even integers, the sum of the smallest two integers is equal to 6 less than the largest. Find the integers.

21. To do a rope trick, a magician needs to cut a piece of rope so that one piece is one-third the length of the other piece. If she begins with a $2\frac{2}{3}$-ft rope, what will be the lengths of the two pieces of rope?

22. Sharyn invests \$2000 more in an account that earns 9% simple interest than she invests in an account that earns 6% simple interest. How much did she invest in each account if her total interest is \$405 after 1 yr?

23. How much 10% acid solution should be mixed with 1 L of 25% acid solution to produce a solution that is 15% acid?

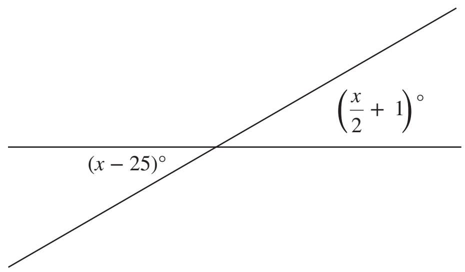
24. Two friends plan to meet at a restaurant for lunch. They both leave their homes at 11:30 A.M. and between the two of them, they drive a total of 37.5 mi. Lynn drives in from a neighboring town and averages 15 mph faster than her friend Linda. If they meet at noon, find the average driving speed for each.

Section 1.3

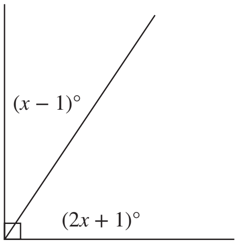
25. The length of a rectangle is 2 ft more than the width. Find the dimensions if the perimeter is 40 ft.

For Exercises 26–27, solve for x, and then find the measure of each angle.

26.



27.



For Exercises 28–31, solve for the indicated variable.

28. $3x - 2y = 4$ for y

29. $-6x + y = 12$ for y

30. $S = 2\pi r + \pi r^2 h$ for h

31. $A = \frac{1}{2}bh$ for b

32. **a.** The circumference of a circle is given by $C = 2\pi r$. Solve this equation for π.

b. Tom measures the radius of a circle to be 6 cm and the circumference to be 37.7 cm. Use these values to approximate π. (Round to 2 decimal places.)

Section 1.4

For Exercises 33–38, solve the inequality and graph the solution set. Write the solution set in (a) set-builder notation and (b) interval notation.

33. $-6x - 2 > 6$

34. $-10x \leq 15$

35. $5 - 7(x + 3) > 19x$

36. $4 - 3x \geq 10(-x + 5)$

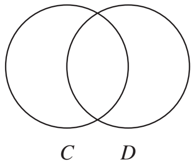
37. $\frac{5 - 4x}{8} \geq 9$

38. $\frac{3 + 2x}{4} \leq 8$

39. Dave earned the following test scores in his biology class: 82, 88, 92, and 93. How high does he have to score on the fifth test to have an average of 90 or more?

Section 1.5

40. Explain the difference between the union and intersection of two sets. You may use the sets C and D in the following diagram to provide an example.



Let $X = \{x \mid x \geq -10\}$, $Y = \{x \mid x < 1\}$, $Z = \{x \mid x > -1\}$, and $W = \{x \mid x \leq -3\}$. For Exercises 41–46, find the intersection or union of the sets X, Y, Z, and W. Write the answers in interval notation.

41. $X \cap Y$

42. $X \cup Y$

43. $Y \cup Z$

44. $Y \cap Z$

45. $Z \cup W$

46. $Z \cap W$

For Exercises 47–56, solve the compound inequalities. Write the solutions in interval notation.

47. $4m > -11$ and $4m - 3 \leq 13$

48. $4n - 7 < 1$ and $7 + 3n \geq -8$

49. $-3y + 1 \geq 10$ and $-2y - 5 \leq -15$

50. $\frac{1}{2} - \frac{h}{12} \leq -\frac{7}{12}$ and $\frac{1}{2} - \frac{h}{10} > -\frac{1}{5}$

51. $\frac{2}{3}t - 3 \leq 1$ or $\frac{3}{4}t - 2 > 7$

52. $2(3x + 1) < -10$ or $3(2x - 4) \geq 0$

53. $-7 < -7(2w + 3)$ or $-2 < -4(3w - 1)$

54. $5(p + 3) + 4 > p - 1$ or $4(p - 1) + 2 > p + 8$

55. $2 \geq -(b - 2) - 5b \geq -6$

56. $-4 \leq \frac{1}{2}(x - 1) < -\frac{3}{2}$

57. The product of $\frac{1}{3}$ and the sum of a number and 3 is between -1 and 5. Find all such numbers.

58. Normal levels of total cholesterol vary according to age. For adults between 25 and 40 yr old, the normal range is generally accepted to be between 140 and 225 mg/dL (milligrams per deciliter), inclusive. Let x represent cholesterol level.

a. Write an inequality representing the normal range for total cholesterol for adults between 25 and 40 yr old.

b. Write a compound inequality representing abnormal ranges for total cholesterol for adults between 25 and 40 yr old.

59. Normal levels of total cholesterol vary according to age. For adults younger than 25 yr old, the normal range is generally accepted to be between 125 and 200 mg/dL, inclusive. Let x represent cholesterol level.

a. Write an inequality representing the normal range for total cholesterol for adults younger than 25 yr old.

b. Write a compound inequality representing abnormal ranges for total cholesterol for adults younger than 25 yr old.

60. One method to approximate your maximum heart rate is to subtract your age from 220. To maintain an aerobic workout, it is recommended that you sustain a heart rate of between 60% and 75% of your maximum heart rate.

a. If the maximum heart rate h is given by the formula $h = 220 - A$, where A is a person's age, find your own maximum heart rate. (Answers will vary.)

b. Find the interval for your own heart rate that will sustain an aerobic workout. (Answers will vary.)

Section 1.6

For Exercises 61–74, solve the absolute value equations.

61. $|x| = 10$

62. $|x| = 17$

63. $|8.7 - 2x| = 6.1$

64. $|5.25 - 5x| = 7.45$

65. $16 = |x + 2| + 9$

66. $5 = |x - 2| + 4$

67. $|4x - 1| + 6 = 4$

68. $|3x - 1| + 7 = 3$

69. $\left|\frac{7x - 3}{5}\right| + 4 = 4$

70. $\left|\frac{4x + 5}{-2}\right| - 3 = -3$

71. $|3x - 5| = |2x + 1|$

72. $|8x + 9| = |8x - 1|$

73. $|2 + 7d| = |-7d - 2|$

74. $-|4y + 6| = |2y - 3|$

75. Which absolute value expression represents the distance between 3 and -2 on the number line?

$|3 - (-2)|$ $\quad$ $|-2 - 3|$

Section 1.7

76. Write the compound inequality $x < -5$ or $x > 5$ as an absolute value inequality.

77. Write the compound inequality $-4 < x < 4$ as an absolute value inequality.

For Exercises 78–79, write an absolute value inequality that represents the solution set graphed here.

78.

79.

For Exercises 80–93, solve the absolute value inequalities. Graph each solution set and write the solution in interval notation.

80. $|x + 6| \geq 8$

81. $|x + 8| \leq 3$

82. $2|7x - 1| + 4 > 4$

83. $4|5x + 1| - 3 > -3$

84. $|3x + 4| - 6 \leq -4$

85. $|5x - 3| + 3 \leq 6$

86. $\left|\frac{x}{2} - 6\right| < 5$

87. $\left|\frac{x}{3} + 2\right| < 2$

88. $|4 - 2x| + 8 \geq 8$

89. $|9 + 3x| + 1 \geq 1$

90. $-2|5.2x - 7.8| < 13$

91. $-|2.5x + 15| < 7$

92. $|3x - 8| < -1$

93. $|x + 5| < -4$

94. State one possible situation in which an absolute value inequality will have no solution.

95. State one possible situation in which an absolute value inequality will have a solution of all real numbers.

96. The Nielsen ratings estimated that the percent, p, of the television viewing audience watching a popular music show was 20% with a 3% margin of error. Solve the inequality $|p - 0.20| \leq 0.03$ and interpret the answer in the context of this problem.

97. The length, L, of a screw is supposed to be $3\frac{3}{8}$ in. Due to variation in the production equipment, there is a $\frac{1}{4}$-in. margin of error. Solve the inequality $|L - 3\frac{3}{8}| \leq \frac{1}{4}$ and interpret the answer in the context of this problem.

Chapter 1 Test

Study Skills Exercise

Studying for a mathematics test is very different than studying for other subjects. There are a number of test-taking strategies that you can practice to be more prepared mentally and physically.

- Space out your study time and practice time so that you give your subconscious mind time to process the information. This is better than cramming information in one sitting.
- Prepare a one-page summary sheet with the most important information that you need for the test. (Consider referring to the Chapter Summary for examples and key concepts.) Also, include a positive statement to yourself on your summary sheet to build a growth mindset.
- On the day of the test, look at this sheet several times to refresh your memory instead of trying to memorize new information.
- When you sit down to take your test, consider writing important formulas on the test paper or scrap paper first, before starting work on the problems.

For Exercises 1–9, solve the equations.

1. $\dfrac{x}{7} + 1 = 20$

2. $8 - 5(4 - 3z) = 2(4 - z) - 8z$

3. $0.12(x) + 0.08(60{,}000 - x) = 10{,}500$

4. $\dfrac{5 - x}{6} - \dfrac{2x - 3}{2} = \dfrac{x}{3}$

5. $\left|\dfrac{1}{2}x + 3\right| - 4 = 4$

6. $|3x + 4| = |x - 12|$

7. $-5 = -8 + |2y - 3|$

8. $|3.7x - 5| + 7 = 6.2$

9. $|8x + 11| = |8x + 5|$

10. Label each equation as a conditional equation, an identity, or a contradiction.

a. $(5x - 9) + 19 = 5(x + 2)$

b. $2a - 2(1 + a) = 5$

c. $(4w - 3) + 4 = 3(5 - w)$

11. The difference between two numbers is 72. If the larger is 5 times the smaller, find the two numbers.

12. Joëlle is determined to get some exercise and walks to the store at a brisk rate of 4.5 mph. She meets her friend Yun Ling at the store, and together they walk back at a slower rate of 3 mph. Joëlle's total walking time was 1 hr.

a. How long did it take her to walk to the store?

b. What is the distance to the store?

13. Shawnna has money distributed between two accounts: an account that earns 5% simple interest and an account that earns 3.5% simple interest. She has \$100 less invested at 3.5% than at 5%. If after 1 yr her total interest is \$81.50, how much did she invest at 5%?

14. A yield sign is in the shape of an equilateral triangle (all sides have equal length). Its perimeter is 81 in. Find the length of the sides.

15. The sum of three consecutive odd integers is 41 less than four times the largest. Find the numbers.

16. How many gallons of a 20% acid solution must be mixed with 6 gal of a 30% acid solution to make a 22% solution?

For Exercises 17–18, solve the equations for the indicated variable.

17. $4x + 2y = 6$ for y

18. $x = \mu + z\sigma$ for z

For Exercises 19–21, solve the inequalities. Graph the solution and write the solution set in interval notation.

19. $x + 8 > 42$

20. $-\frac{3}{2}x + 6 \geq x - 3$

21. $-2 < 3x - 1 \leq 5$

For Exercises 22–32, solve the compound and absolute value inequalities. Write the answers in interval notation.

22. $-2 \leq 3x - 1 \leq 5$

23. $-4 \leq \dfrac{6 - 2x}{5} < 2$

24. $-\frac{3}{5}x - 1 \leq 8$ or $-\frac{2}{3}x \geq 16$

25. $-2x - 3 > -3$ and $x + 3 \geq 0$

26. $5x + 1 \leq 6$ or $2x + 4 > -6$

27. $2x - 3 > 1$ and $x + 4 < -1$

28. $|3 - 2x| + 6 < 2$

29. $|3x - 8| \geq 9$

30. $|0.4x + 0.3| - 0.2 < 7$

31. $|7 - 3x| + 1 > -3$

32. $6 \geq |2x - 5| - 5$

33. An elevator can accommodate a maximum weight of 2000 lb. If four passengers on the elevator have an average weight of 180 lb each, how many additional passengers of the same average weight can the elevator carry before the maximum weight capacity is exceeded?

Keith Brofsky/Photodisc/Getty Images

34. The normal range in humans of the enzyme adenosine deaminase (ADA), is between 9 and 33 IU (international units), inclusive. Let x represent the ADA level in international units.

a. Write an inequality representing the normal range for ADA.

b. Write a compound inequality representing abnormal ranges for ADA.

35. The mass of a small piece of metal is measured to be 15.41 g. If the measurement error is at most ± 0.01 g, write an absolute value inequality that represents the possible mass, x, of the piece of metal.

Linear Equations in Two Variables and Functions

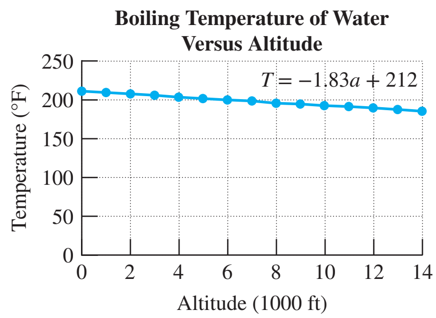
2

CHAPTER OUTLINE

Mathematics in Exploration

If you have ever been hiking or skiing in the mountains, you are probably familiar with localized weather changes at higher altitudes. The temperature drops with increasing altitude, and the air becomes "thinner." The "thin" air makes physical exertion more difficult because it is less dense and there is not as much oxygen in a breath of fixed volume. Furthermore, the decrease in density of the air also causes water to boil at a lower temperature.

In the mid-nineteenth century, explorers knew that the boiling temperature of water T (in °F) was related to altitude a (in thousands of feet) by the relationship $T = -1.83a + 212$. This equation is called a **linear equation in two variables**. One characteristic of a linear equation in two variables is that one variable changes with respect to the other at a fixed rate. For example, given an altitude a, this equation tells us that the boiling point of water *drops* by 1.83°F for every 1000 ft of altitude. This equation can also be manipulated to solve for a in terms of T.



$$a = -\frac{1}{1.83}T + 115.8$$

The equation in this form was used by explorers to measure their altitude a based on the temperature of a boiling cup of water T.

Section 2.1 Linear Equations in Two Variables

Concepts

1. The Rectangular Coordinate System
2. Linear Equations in Two Variables
3. Graphing Linear Equations in Two Variables
4. *x*- and *y*-Intercepts
5. Horizontal and Vertical Lines

1. The Rectangular Coordinate System

One application of algebra is the graphical representation of numerical information (or data). For example, Table 2-1 shows the percentage of individuals who participate in leisure sports activities according to the age of the individual.

Table 2-1

Age (years)	Percentage of Individuals Participating in Leisure Sports Activities
20	59%
30	52%
40	44%
50	34%
60	21%
70	18%

Source: U.S. National Endowment for the Arts

Information in table form is difficult to picture and interpret. However, when the data are presented in a graph, there appears to be a downward trend in the participation in leisure sports activities as age increases (Figure 2-1).

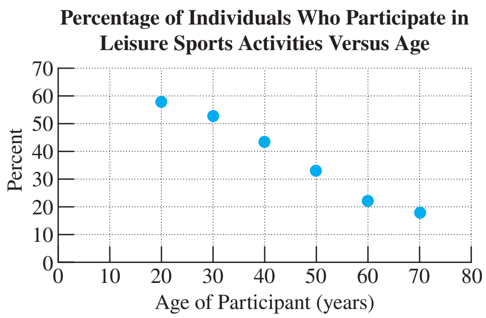


Figure 2-1

In this example, two variables are related: age and the percentage of individuals who participate in leisure sports activities.

To picture two variables simultaneously, we use a graph with two number lines drawn at right angles to each other (Figure 2-2). This forms a **rectangular coordinate system**. The horizontal line is called the ***x*-axis**, and the vertical line is called the ***y*-axis**. The point where the lines intersect is called the **origin**. On the *x*-axis, the numbers to the right of the origin are positive, and the numbers to the left are negative. On the *y*-axis, the numbers above the origin are positive, and the numbers below are negative. The *x*- and *y*-axes divide the graphing area into four regions called **quadrants**.

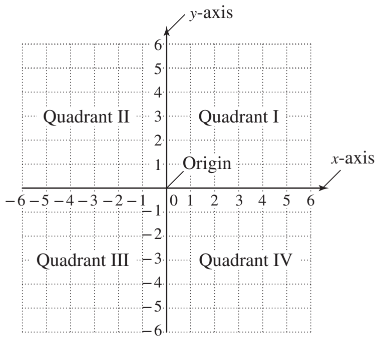


Figure 2-2

Points graphed in a rectangular coordinate system are defined by two numbers as an **ordered pair** (x, y). The first number (called the ***x*-coordinate** or abscissa) is the horizontal position from the origin. The second number (called the ***y*-coordinate** or ordinate) is the vertical position from the origin. Example 1 shows how points are plotted in a rectangular coordinate system.

Example 1 **Plotting Points**

Plot each point and state the quadrant or axis where it is located.

a. $(4, 1)$ **b.** $(-3, 4)$ **c.** $(4, -3)$

d. $(-\frac{5}{2}, -2)$ **e.** $(0, 3)$ **f.** $(-4, 0)$

Solution:

a. The point $(4, 1)$ is in quadrant I.

b. The point $(-3, 4)$ is in quadrant II.

c. The point $(4, -3)$ is in quadrant IV.

d. The point $(-\frac{5}{2}, -2)$ can also be written as $(-2.5, -2)$. This point is in quadrant III.

e. The point $(0, 3)$ is on the y-axis.

f. The point $(-4, 0)$ is on the x-axis.

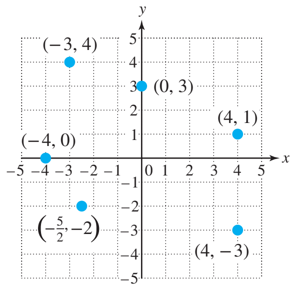


Figure 2-3

TIP: Notice that the points $(-3, 4)$ and $(4, -3)$ are in different quadrants. Changing the order of the coordinates changes the location of the point. That is why points are represented by *ordered* pairs (Figure 2-3).

Skill Practice Plot the point and state the quadrant or axis where it is located.

1. a. $(3, 5)$ **b.** $(-2, 0)$ **c.** $(2, -1)$

d. $(0, 4)$ **e.** $(-2, -2)$ **f.** $(-5, 2)$

2. Linear Equations in Two Variables

Recall that an equation in the form $ax + b = c$ is called a linear equation in one variable. In this section, we will study linear equations in *two* variables.

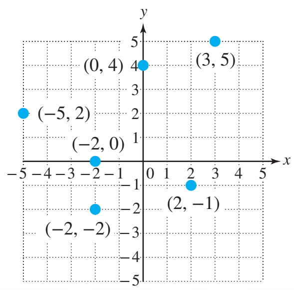
Definition of a Linear Equation in Two Variables

Let A, B, and C be real numbers such that A and B are not both zero. A **linear equation in two variables** is an equation that can be written in the form

$Ax + By = C$ This form is called *standard form*.

Answers

1. a. $(3, 5)$; quadrant I
b. $(-2, 0)$; x-axis
c. $(2, -1)$; quadrant IV
d. $(0, 4)$; y-axis
e. $(-2, -2)$; quadrant III
f. $(-5, 2)$; quadrant II



A *solution* to a linear equation in two variables is an ordered pair (x, y) that makes the equation a true statement.

Example 2 Determining Solutions to a Linear Equation

Determine whether the ordered pair is a solution to the equation $-2x + 3y = 8$.

a. $(-4, 0)$ **b.** $(2, -4)$ **c.** $\left(1, \frac{10}{3}\right)$

FOR REVIEW

Remember to use parentheses when substituting numerical values for variables within an expression or an equation.

$$-2x + 3y = 8$$
$$-2(\ \ ) + 3(\ \ ) = 8$$
$$-2(-4) + 3(0) = 8$$
$$8 - 0 = 8 \quad \checkmark$$

Solution:

a. $-2x + 3y = 8$ The ordered pair $(-4, 0)$ indicates that $x = -4$ and $y = 0$.

$-2(-4) + 3(0) \stackrel{?}{=} 8$ Substitute $x = -4$ and $y = 0$ into the equation.

$8 + 0 \stackrel{?}{=} 8 \checkmark$ (true) The ordered pair $(-4, 0)$ makes the equation a true statement. The ordered pair is a solution to the equation.

b. $-2x + 3y = 8$ Test the ordered pair $(2, -4)$.

$-2(2) + 3(-4) \stackrel{?}{=} 8$ Substitute $x = 2$ and $y = -4$ into the equation.

$-4 + (-12) \stackrel{?}{=} 8$

$-16 \stackrel{?}{=} 8$ (false) The ordered pair $(2, -4)$ does not make the equation a true statement. The ordered pair is *not* a solution to the equation.

c. $-2x + 3y = 8$ Test the ordered pair $(1, \frac{10}{3})$.

$-2(1) + 3\left(\frac{10}{3}\right) \stackrel{?}{=} 8$ Substitute $x = 1$ and $y = \frac{10}{3}$.

$-2 + 10 \stackrel{?}{=} 8 \checkmark$ (true) The ordered pair $(1, \frac{10}{3})$ is a solution to the equation.

Skill Practice Determine whether the ordered pair is a solution to the equation $x + 4y = -8$.

2. $(-2, -1)$ **3.** $(4, -3)$ **4.** $\left(-14, \frac{3}{2}\right)$

3. Graphing Linear Equations in Two Variables

Consider the linear equation $x - y = 3$. The solutions to the equation are ordered pairs such that the difference of x and y is 3. Several solutions are given in the following list:

Answers

2. Not a solution
3. Solution
4. Solution

Solution	Check
(x, y)	$x - y = 3$
$(3, 0)$	$(3) - (0) = 3$ ✓
$(4, 1)$	$(4) - (1) = 3$ ✓
$(0, -3)$	$(0) - (-3) = 3$ ✓
$(-1, -4)$	$(-1) - (-4) = 3$ ✓
$(2, -1)$	$(2) - (-1) = 3$ ✓

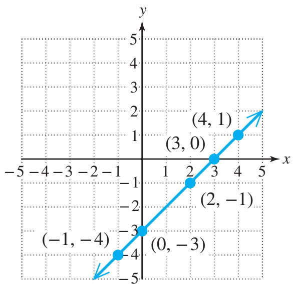


Figure 2-4

By graphing these ordered pairs, we see that the solution points line up (see Figure 2-4). There are actually an infinite number of solutions to the equation $x - y = 3$. The graph of all solutions to a linear equation forms a line in the xy-plane. Conversely, each ordered pair on the line is a solution to the equation.

The Graph of an Equation in Two Variables

To graph an equation in two variables means that we will graph all ordered pair solutions to the equation.

To graph a linear equation, it is sufficient to find two solution points and draw the line through them. We will find three solution points and use the third point as a check point. This is demonstrated in Example 3.

Example 3 Graphing a Linear Equation in Two Variables

Graph the equation $3x + 5y = 15$.

Solution:

We will find three ordered pairs that are solutions to the equation. In the table, we have selected arbitrary values for x or y and must complete the ordered pairs.

x	y	
0		⟶ (0,)
	2	⟶ (, 2)
5		⟶ (5,)

From the first row, substitute $x = 0$.

$$3x + 5y = 15$$
$$3(0) + 5y = 15$$
$$5y = 15$$
$$y = 3$$

From the second row, substitute $y = 2$.

$$3x + 5y = 15$$
$$3x + 5(2) = 15$$
$$3x + 10 = 15$$
$$3x = 5$$
$$x = \frac{5}{3}$$

From the third row, substitute $x = 5$.

$$3x + 5y = 15$$
$$3(5) + 5y = 15$$
$$15 + 5y = 15$$
$$5y = 0$$
$$y = 0$$

The completed list of ordered pairs is shown. To graph the equation, plot the three solutions and draw the line through the points (Figure 2-5). Arrows on the ends of the line indicate that points on the line extend infinitely in both directions.

x	y	
0	3	$\longrightarrow (0, 3)$
$\frac{5}{3}$	2	$\longrightarrow (\frac{5}{3}, 2)$
5	0	$\longrightarrow (5, 0)$

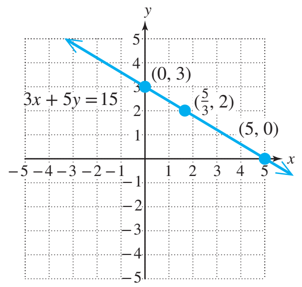


Figure 2-5

Skill Practice Given $2x - y = 1$, complete the table and graph the line through the points.

5.

x	y
0	
	5
1	

Example 4 Graphing a Linear Equation in Two Variables

Graph the equation $y = \frac{1}{2}x - 2$.

Solution:

Because the y-variable is isolated in the equation, it is easy to substitute a value for x and simplify the right-hand side to find y. Since any number can be used for x, choose numbers that are multiples of 2 that will simplify easily when multiplied by $\frac{1}{2}$.

x	y
0	
2	
4	

Substitute $x = 0$.

$y = \frac{1}{2}(0) - 2$

$y = 0 - 2$

$y = -2$

Substitute $x = 2$.

$y = \frac{1}{2}(2) - 2$

$y = 1 - 2$

$y = -1$

Substitute $x = 4$.

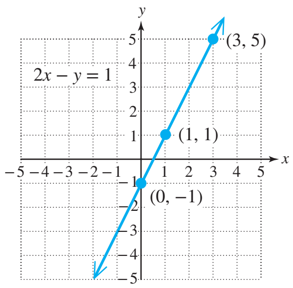
$y = \frac{1}{2}(4) - 2$

$y = 2 - 2$

$y = 0$

Answer

5.



The completed list of ordered pairs is as follows. To graph the equation, plot the three solutions and draw the line through the points (Figure 2-6).

x	y	
0	−2	⟶ (0, −2)
2	−1	⟶ (2, −1)
4	0	⟶ (4, 0)

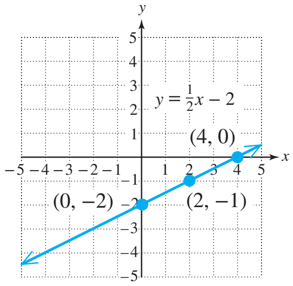


Figure 2-6

Skill Practice

6. Graph the equation $y = -\frac{1}{3}x + 1$.

Hint: Select values of x that are multiples of 3.

4. *x*- and *y*-Intercepts

For many applications of graphing, it is advantageous to know the points where a graph intersects the x- or y-axis. These points are called the x- and y-intercepts.

In Figure 2-5, the x-intercept is (5, 0). In Figure 2-6, the x-intercept is (4, 0). In general, a point on the x-axis must have a y-coordinate of zero. In Figure 2-5, the y-intercept is (0, 3). In Figure 2-6, the y-intercept is (0, −2). In general, a point on the y-axis must have an x-coordinate of zero.

Definition of *x*- and *y*-Intercepts

An ***x*-intercept*** is a point $(a, 0)$ where a graph intersects the x-axis. (See Figure 2-7.)

A ***y*-intercept** is a point $(0, b)$ where a graph intersects the y-axis. (See Figure 2-7.)

*In some applications, an x-intercept is defined as the x-*coordinate* of a point of intersection that a graph makes with the x-axis. For example, if an x-intercept is at the point (3, 0), it is sometimes stated simply as 3 (the y-coordinate is understood to be zero). Similarly, a y-intercept is sometimes defined as the y-*coordinate* of a point of intersection that a graph makes with the y-axis. For example, if a y-intercept is at the point (0, 7), it may be stated simply as 7 (the x-coordinate is understood to be zero).

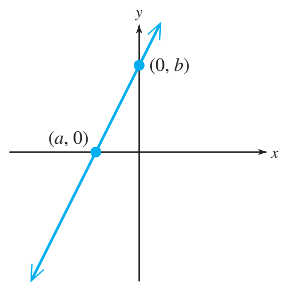


Figure 2-7

To find the x- and y-intercepts from an equation in x and y, follow these steps:

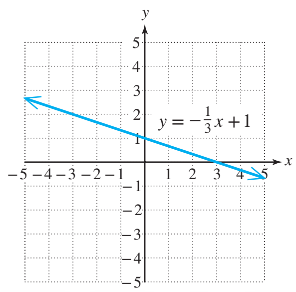
Determining the *x*- and *y*-Intercepts from an Equation

Given an equation in x and y,

- Find the x-intercept(s) by substituting $y = 0$ into the equation and solving for x.
- Find the y-intercept(s) by substituting $x = 0$ into the equation and solving for y.

Answer

6.



Example 5 Finding the x- and y-Intercepts of a Line

Given $2x + 4y = 8$, find the x- and y-intercepts. Then graph the equation.

Solution:

To find the x-intercept, substitute $y = 0$.

$$2x + 4y = 8$$
$$2x + 4(0) = 8$$
$$2x = 8$$
$$x = 4$$

The x-intercept is $(4, 0)$.

To find the y-intercept, substitute $x = 0$.

$$2x + 4y = 8$$
$$2(0) + 4y = 8$$
$$4y = 8$$
$$y = 2$$

The y-intercept is $(0, 2)$.

In this case, the intercepts are two distinct points and may be used to graph the line. A third point can be found to verify that the points all fall on the same line (points that lie on the same line are said to be *collinear*). Choose a different value for either x or y, such as $y = 4$.

$2x + 4y = 8$

$2x + 4(4) = 8$ Substitute $y = 4$.

$2x + 16 = 8$ Solve for x.

$2x = -8$

$x = -4$ The point $(-4, 4)$ lines up with the other two points (Figure 2-8).

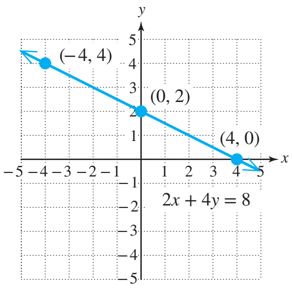


Figure 2-8

Skill Practice

7. Given $-2x + y = -4$, find the x- and y-intercepts. Then graph the equation.

Example 6 Finding the x- and y-Intercepts of a Line

Given $y = \frac{1}{4}x$, find the x- and y-intercepts. Then graph the equation.

Solution:

To find the x-intercept, substitute $y = 0$.

$$y = \frac{1}{4}x$$
$$(0) = \frac{1}{4}x$$
$$0 = x$$

The x-intercept is $(0, 0)$.

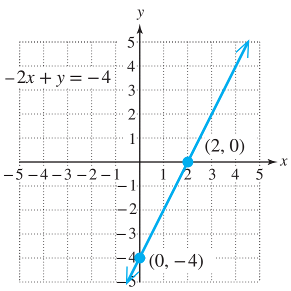
To find the y-intercept, substitute $x = 0$.

$$y = \frac{1}{4}x$$
$$y = \frac{1}{4}(0)$$
$$y = 0$$

The y-intercept is $(0, 0)$.

Answer

7.



Notice that the x- and y-intercepts are both located at the origin (0, 0). In this case, the intercepts do not yield two distinct points. Therefore, another point is necessary to draw the line. We may pick any value for either x or y. However, for this equation, it would be particularly convenient to pick a value for x that is a multiple of 4 such as $x = 4$.

$$y = \frac{1}{4}x$$

$$y = \frac{1}{4}(4) \qquad \text{Substitute } x = 4.$$

$$y = 1$$

The point (4, 1) is a solution to the equation (Figure 2-9).

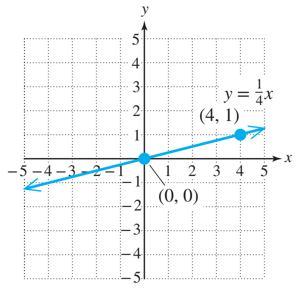


Figure 2-9

Avoiding Mistakes

You can always find a third point on a line to check the accuracy of your graph. For example, the point (−4, −1) satisfies the equation $y = \frac{1}{4}x$ and lines up with the other points from Example 6.

Skill Practice

8. Given $y = -5x$, find the x- and y-intercepts. Then graph the equation.

Example 7 Interpreting the *x*- and *y*-Intercepts of a Line

Companies and corporations are permitted to depreciate assets that have a known useful life span. This accounting practice is called *straight-line depreciation*. In this procedure the useful life span of the asset is determined, and then the asset is depreciated by an equal amount each year until the taxable value of the asset is equal to zero.

A trucking company purchases a new truck for \$65,000. The truck will be depreciated at \$13,000 per year. The equation that describes the depreciation line is

$$y = 65{,}000 - 13{,}000x$$

where y represents the value of the truck in dollars and x is the age of the truck in years.

Corbis

a. Find the x- and y-intercepts. Plot the intercepts on a rectangular coordinate system, and draw the line that represents the straight-line depreciation.

b. What does the x-intercept represent in the context of this problem?

c. What does the y-intercept represent in the context of this problem?

Solution:

a. To find the x-intercept, substitute $y = 0$.

$$0 = 65{,}000 - 13{,}000x$$

$$13{,}000x = 65{,}000$$

$$x = 5$$

The x-intercept is (5, 0).

To find the y-intercept, substitute $x = 0$.

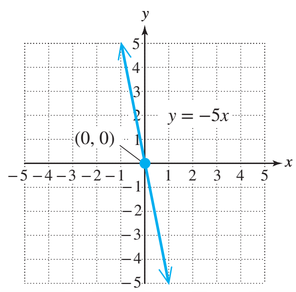
$$y = 65{,}000 - 13{,}000(0)$$

$$y = 65{,}000$$

The y-intercept is (0, 65,000).

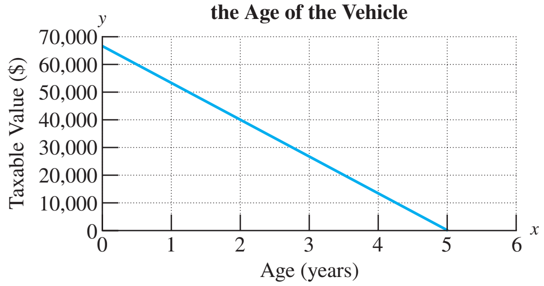
Answer

8. x-intercept: (0, 0); y-intercept: (0, 0)



TIP: In Example 7 we graphed the line only in the first quadrant where both the *x*- and *y*-coordinates are positive. (A negative *x*-coordinate would imply a negative age, and a negative *y*-coordinate would imply a negative value of a car, neither of which makes sense.)

Taxable Value of a Truck Versus the Age of the Vehicle



b. The x-intercept (5, 0) indicates that when the truck is 5 yr old, the taxable value of the truck will be \$0.

c. The y-intercept (0, 65,000) indicates that when the truck was new (0 yr old), its taxable value was \$65,000.

Skill Practice

9. A truck company tests the engines of its trucks by running the engines in a laboratory. The engines burn 4 gal of fuel per hour. The engines begin the test with 30 gal of fuel. The equation $y = 30 - 4x$ represents the amount of fuel y left in the engine after x hours.

a. Find the x- and y-intercepts.

b. Interpret the x-intercept in the context of this problem.

c. Interpret the y-intercept in the context of this problem.

5. Horizontal and Vertical Lines

Recall that a linear equation can be written in the form $Ax + By = C$, where A and B are not both zero. If either A or B is 0, then the resulting line is horizontal or vertical, respectively.

Vertical and Horizontal Lines

1. A **vertical line** is a line whose equation can be written in the form $x = k$, where k is a constant.
2. A **horizontal line** is a line whose equation can be written in the form $y = k$, where k is a constant.

Example 8 Graphing a Vertical Line

Graph the equation $x = 6$.

Solution:

Because this equation is in the form $x = k$, the line is vertical and must cross the x-axis at $x = 6$. We can also construct a table of solutions to the equation $x = 6$. The choice for the x-coordinate must be 6, but y can be any real number (Figure 2-10).

Answer

9. a. x-intercept: (7.5, 0); y-intercept: (0, 30)

b. The x-intercept (7.5, 0) represents the amount of fuel in the truck after 7.5 hr. After 7.5 hr the tank contains 0 gal. It is empty.

c. The y-intercept (0, 30) represents the amount of fuel in the truck initially (after 0 hr). After 0 hr, the tank contains 30 gal of fuel.

x	y	
6	−4	⟶ (6, −4)
6	1	⟶ (6, 1)
6	5	⟶ (6, 5)

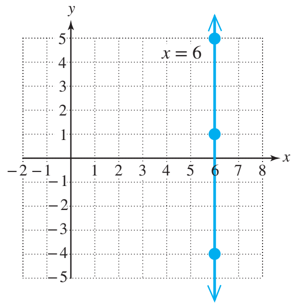


Figure 2-10

Skill Practice

10. Graph the equation $x = -4$.

Example 9 Graphing a Horizontal Line

Graph the equation $4y = -7$.

Solution:

The equation $4y = -7$ is equivalent to $y = -\frac{7}{4}$. Because the equation is in the form $y = k$, the line must be horizontal and must pass through the y-axis at $y = -\frac{7}{4}$ (Figure 2-11).

We can also construct a table of solutions to the equation $4y = -7$. The choice for the y-coordinate must be $-\frac{7}{4}$, but x can be any real number.

x	y	
0	$-\frac{7}{4}$	⟶ $(0, -\frac{7}{4})$
−3	$-\frac{7}{4}$	⟶ $(-3, -\frac{7}{4})$
2	$-\frac{7}{4}$	⟶ $(2, -\frac{7}{4})$

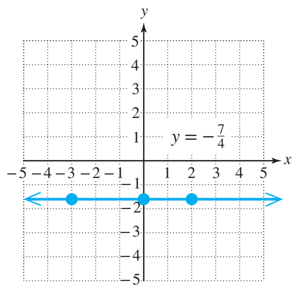


Figure 2-11

Skill Practice

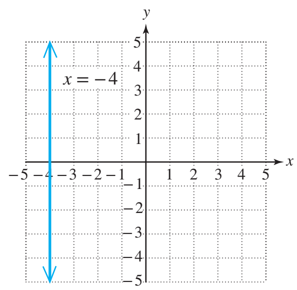
11. Graph the equation $-2y = 9$.

TIP: Notice that horizontal and vertical lines that do not pass through the origin have only one intercept. For instance,

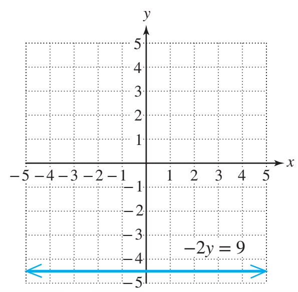
- The vertical line in Figure 2-10 has an x-intercept but no y-intercept.
- The horizontal line in Figure 2-11 has a y-intercept but no x-intercept.

Answers

10.



11.

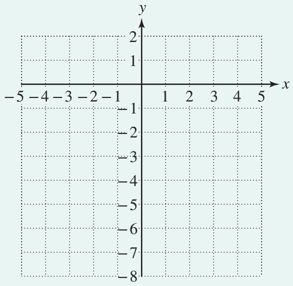


Section 2.1 Activity

A.1. **a.** Complete the table for the equation $2x - y = 3$.
b. Graph the ordered pairs from part (a).
c. Draw a line through the points to represent all solutions to the equation $2x - y = 3$.
d. The equation $2x - y = 3$ is a linear equation in two variables. Based on the graph in part (c), why is the equation categorized as linear?

e. Define a linear equation in two variables, and show that $2x - y = 3$ is linear.

x	y
0	
	0
1	
	1
−2	
	−5

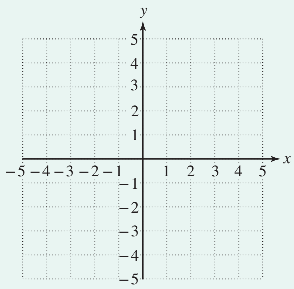


A.2. What are the x- and y-intercepts of the graph of an equation?

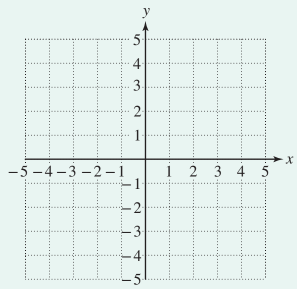
A.3. Given an equation such as $-2x + 3y = 6$, how do you find the x- and y-intercepts?

For Exercises A.4–A.7,
a. Find the x-intercept (if it exists).
b. Find the y-intercept (if it exists).
c. Find another point on the line.
d. Plot the points and sketch the line.

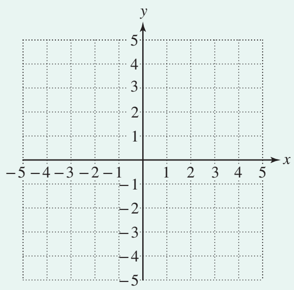
A.4. $-2x + 3y = 6$



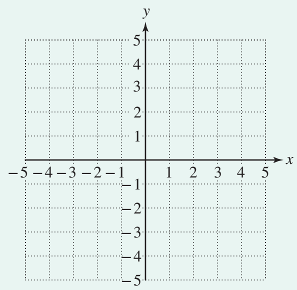
A.5. $x - 2y = 0$



A.6. $4y - 8 = 0$



A.7. $3x + 1 = -8$



A.8. By looking at an equation of a line, how can you tell if the line is horizontal?

A.9. By looking at an equation of a line, how can you tell if the line is vertical?

A.10. By looking at an equation of a line, how can you tell if the line is a slanted line passing through the origin?

Practice Exercises

Section 2.1

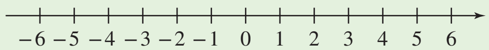
Study Skills Exercise

Math anxiety is a feeling of nervousness, tension, fear, or worry that interferes with a student's ability to concentrate and solve math problems. Math anxiety causes students *to believe* that they are not capable of learning math or performing computations, even if they know how. If you can relate to these feelings, the good news is that there are strategies to overcome math anxiety. Create the following lists, and reflect on your responses during times of angst.

- List the people who support you and your educational goals. Rely on these individuals for encouragement.
- List one or more successes that you have had in this course. Reframe your thoughts to focus on these positive moments and times of achievement.
- List your study habits that are beneficial to your success in this course. Are there other study skills or habits that you practice that would help you understand the content? Give these a try!

Prerequisite Review

For Exercises R.1–R.6, plot the points on the number line.



R.1. 5 **R.2.** -2 **R.3.** $-\frac{13}{4}$ **R.4.** $\frac{5}{3}$ **R.5.** $2.\overline{6}$ **R.6.** $-0.\overline{4}$

For Exercises R.7–R.10, solve for x or y.

R.7. $4x + 3 = 23$

R.8. $-7x - 5 = 65$

R.9. $9 - 3y = -24$

R.10. $11 + 3y = -10$

For Exercises R.11–R.14, evaluate the expression for the given values of x and y.

R.11. $-5x + 2y$ for $x = -3$ and $y = 6$

R.12. $10x - 5y$ for $x = 3$ and $y = -5$

R.13. $4x - y$ for $x = 0$ and $y = -7$

R.14. $3x + 5y$ for $x = -5$ and $y = 0$

Vocabulary and Key Concepts

1. **a.** In a rectangular coordinate system, two number lines are drawn at right angles to each other. The horizontal line is called the ______________-axis, and the vertical line is called the ______________.

b. A point in a rectangular coordinate system is defined by an ______________ pair, (x, y).

c. In a rectangular coordinate system, the point where the x- and y-axes intersect is called the ______________ and is represented by the ordered pair ______________.

d. The x- and y-axes divide the coordinate plane into four regions called ______________.

e. A point with a positive x-coordinate and a ______________ y-coordinate is located in quadrant IV.

f. In quadrant ______________, both the x- and y-coordinates are negative.

g. A linear equation in two variables is an equation that can be written in the form ______________ where A and B are not both zero.

h. A point where a graph intersects the x-axis is called a(n) ____________.

i. A point where a graph intersects the y-axis is called a(n) ____________.

j. A ____________ line can be represented by an equation of the form $x = k$, where k is a constant.

k. A ____________ line can be represented by an equation of the form $y = k$, where k is a constant.

Concept 1: The Rectangular Coordinate System

2. Let a and b represent nonzero real numbers.

a. An ordered pair of the form $(0, b)$ represents a point on which axis?

b. An ordered pair of the form $(a, 0)$ represents a point on which axis?

3. Given the coordinates of a point, explain how to determine in which quadrant the point lies.

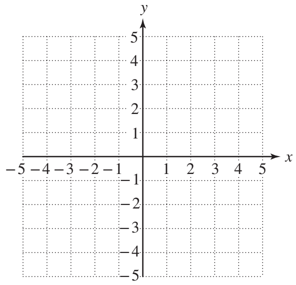
4. What is meant by the word *ordered* in the term *ordered pair?*

5. Plot the points on a rectangular coordinate system. **(See Example 1.)**

a. $(-2, 1)$ **b.** $(0, 4)$

c. $(0, 0)$ **d.** $(-3, 0)$

e. $\left(\frac{3}{2}, -\frac{7}{3}\right)$ **f.** $(-4.1, -2.7)$

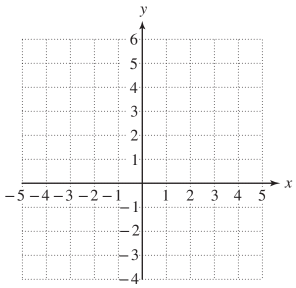


6. Plot the points on a rectangular coordinate system.

a. $(-2, 5)$ **b.** $\left(\frac{5}{2}, 0\right)$

c. $(4, -3)$ **d.** $(0, -2)$

e. $(2, 2)$ **f.** $(-3, -3)$

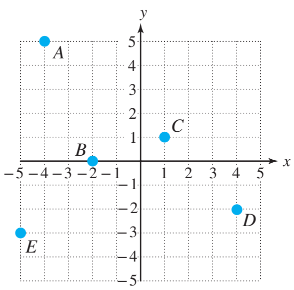


7. A point on the x-axis will have what y-coordinate?

8. A point on the y-axis will have what x-coordinate?

For Exercises 9–10, give the coordinates of the labeled points, and state the quadrant or axis where the point is located.

9.



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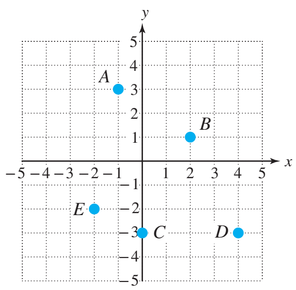


Concept 2: Linear Equations in Two Variables

For Exercises 11–14, determine if the ordered pair is a solution to the linear equation. **(See Example 2.)**

11. $2x - 3y = 9$

- **a.** $(0, -3)$
- **b.** $(-6, 1)$
- **c.** $\left(1, -\frac{7}{3}\right)$

12. $-5x - 2y = 6$

- **a.** $(0, 3)$
- **b.** $\left(-\frac{6}{5}, 0\right)$
- **c.** $(-2, 2)$

13. $x = \frac{1}{3}y + 1$

- **a.** $(-1, 0)$
- **b.** $(2, 3)$
- **c.** $(-6, 1)$

14. $y = -\frac{3}{2}x - 4$

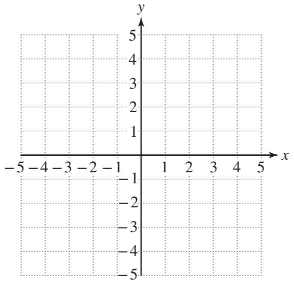
- **a.** $(0, -4)$
- **b.** $(2, -7)$
- **c.** $(-4, -2)$

Concept 3: Graphing Linear Equations in Two Variables

For Exercises 15–18, complete the table. Then graph the line defined by the points. **(See Examples 3–4.)**

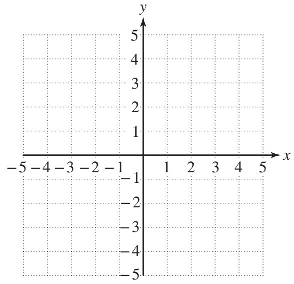
15. $3x - 2y = 4$

x	y
0	
	4
-1	



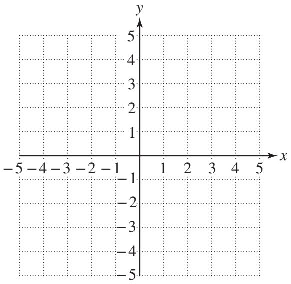
16. $4x + 3y = 6$

x	y
	2
3	
	-1



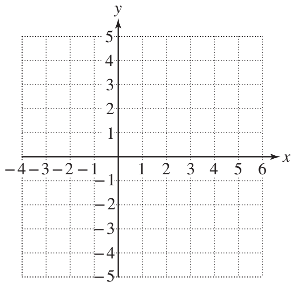
17. $y = -\frac{1}{5}x$

x	y
0	
5	
-5	



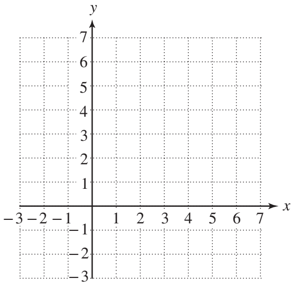
18. $y = \frac{1}{3}x$

x	y
0	
3	
6	

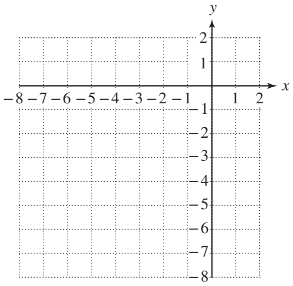


In Exercises 19–30, graph the linear equation by using a table of solutions. **(See Examples 3–4.)**

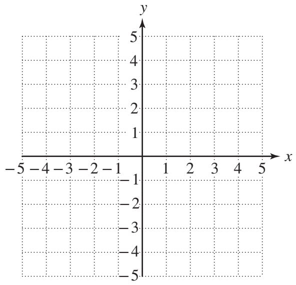
19. $x + y = 5$



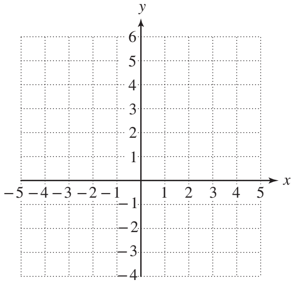
20. $x + y = -8$



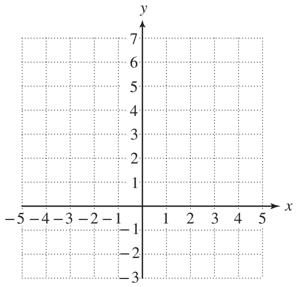
21. $3x - 4y = 12$



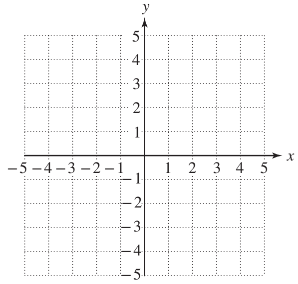
22. $5x + 3y = 15$



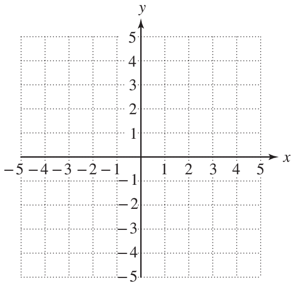
23. $y = -3x + 5$



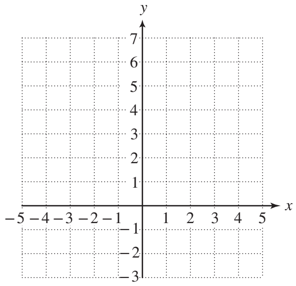
24. $y = -2x + 2$



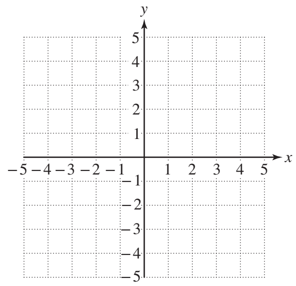
25. $y = \dfrac{2}{5}x - 1$



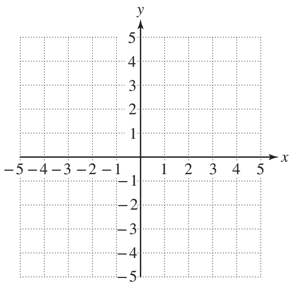
26. $y = \dfrac{5}{3}x + 1$



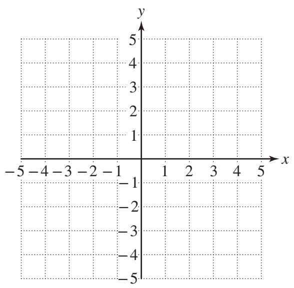
27. $x = -5y - 5$



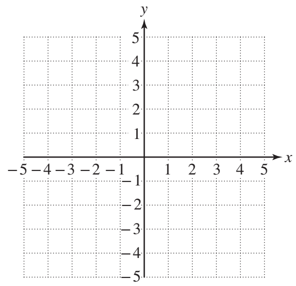
28. $x = 4y + 2$



29. $x = 2y$



30. $x = -3y$



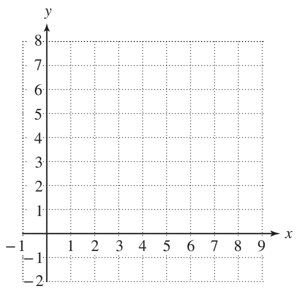
Concept 4: *x*- and *y*-Intercepts

31. Given a linear equation, how do you find an x-intercept? How do you find a y-intercept?

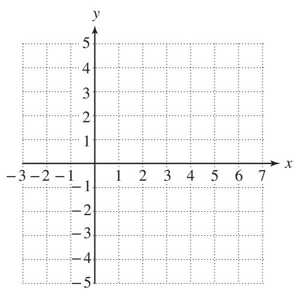
32. Can the point $(4, -1)$ be an x- or y-intercept? Why or why not?

For Exercises 33–44, **a.** find the x-intercept, **b.** find the y-intercept, and **c.** graph the equation. **(See Examples 5–6.)**

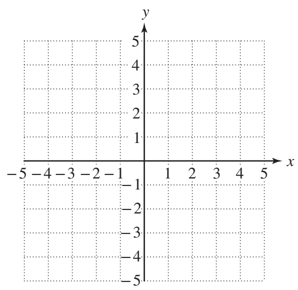
33. $2x + 3y = 18$



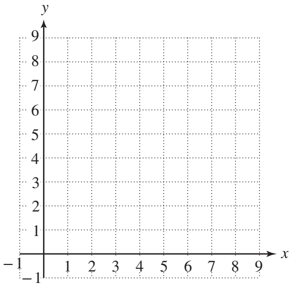
34. $2x - 5y = 10$



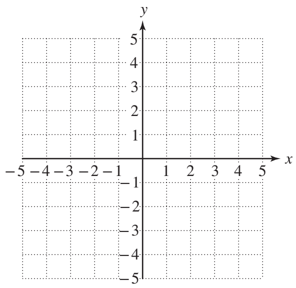
35. $x - 2y = 4$



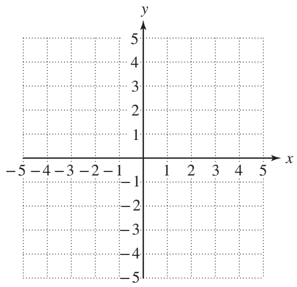
36. $x + y = 8$



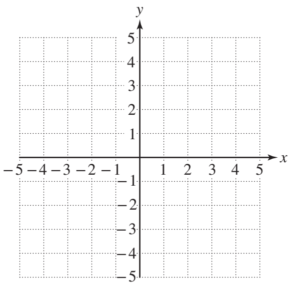
37. $5x = 3y$



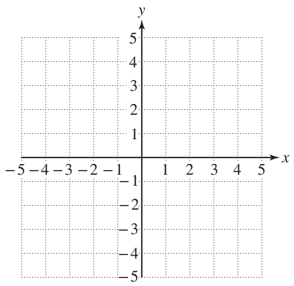
38. $3y = -5x$



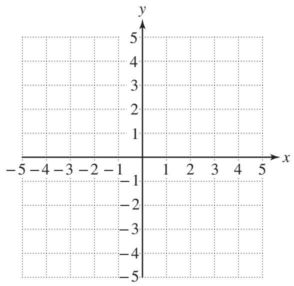
39. $y = 2x + 4$



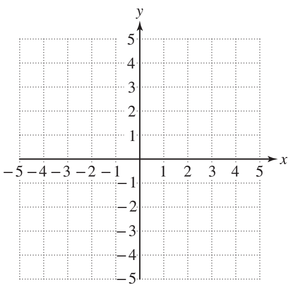
40. $y = -3x - 1$



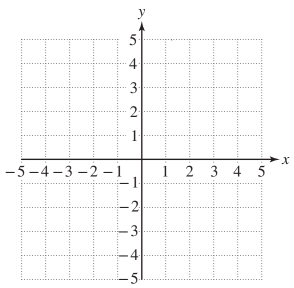
41. $y = -\frac{4}{3}x + 2$



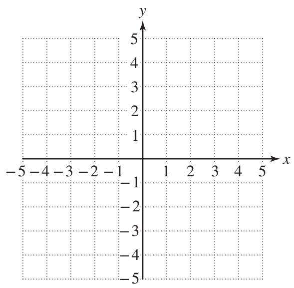
42. $y = -\frac{2}{5}x - 1$



43. $x = \frac{1}{4}y$



44. $x = \frac{2}{3}y$



45. A salesperson makes a base salary of \$15,000 a year plus an 8% commission on sales for the year. The total yearly salary can be expressed as a linear equation. **(See Example 7.)**

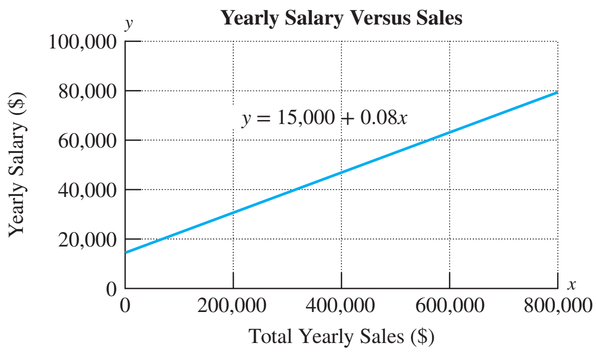
$$y = 15{,}000 + 0.08x$$

a. What is the salesperson's salary for a year in which his sales total \$500,000?

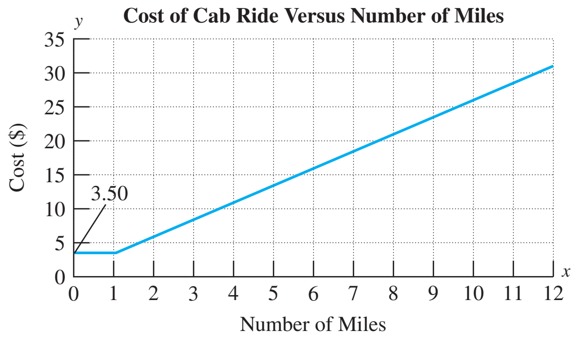
b. What is the salary for a year in which sales total \$300,000?

c. What does the y-intercept mean in the context of this problem?

d. Why is it unreasonable to use negative values for x in this equation?



46. A taxi company in Portland charges \$3.50 for any distance up to the first mile and \$2.50 for every mile thereafter. The cost of a cab ride can be modeled graphically.



a. Explain why the first part of the model is represented by a horizontal line.

b. What does the y-intercept mean in the context of this problem?

c. Explain why the line representing the cost of traveling more than 1 mi is not horizontal.

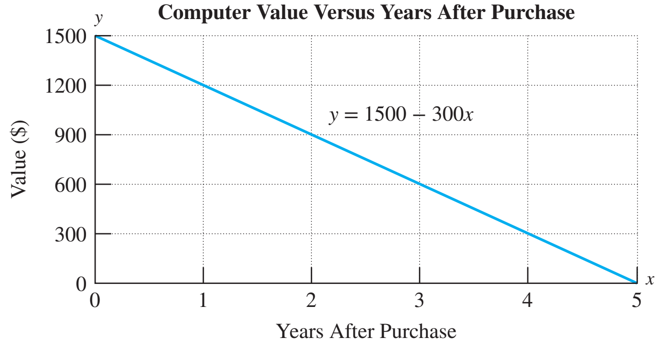
47. A business owner buys several new computers for the office for \$1500 each. The accounting office depreciates each computer by \$300 per year. The value y (in \$) for each computer can be represented by $y = 1500 - 300x$, where x is the number of years after purchase.

a. How much will a computer be worth 1 yr after purchase?

b. After how many years will the computer be worth only \$300?

c. Determine the y-intercept and interpret its meaning in the context of this problem.

d. Determine the x-intercept and interpret its meaning in the context of this problem.



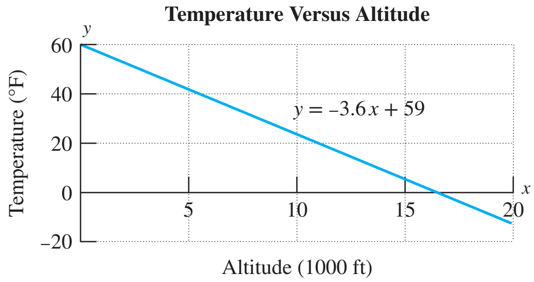
48. The equation $y = -3.6x + 59$ can be used to approximate the air temperature y (in °F) at an altitude x (in 1000 ft).

a. Determine the air temperature at 10,000 ft.

b. At what altitude is the air temperature −5.8°F?

c. Determine the y-intercept and interpret its meaning in the context of this problem.

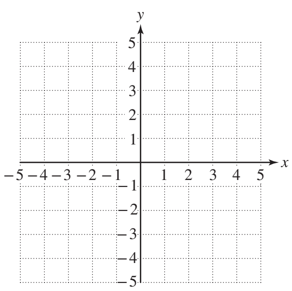
d. Determine the x-intercept and interpret its meaning in the context of this problem.



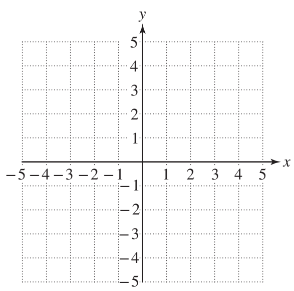
Concept 5: Horizontal and Vertical Lines

For Exercises 49–56, determine if the equation represents a horizontal line or a vertical line. Then graph the line and identify the x- and y-intercepts. **(See Examples 8–9.)**

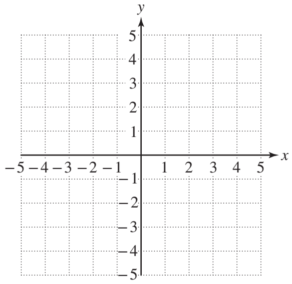
49. $y = -1$



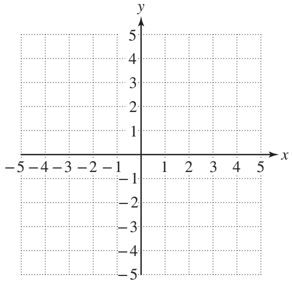
50. $y = 3$



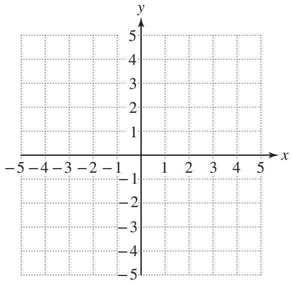
51. $x = 2$



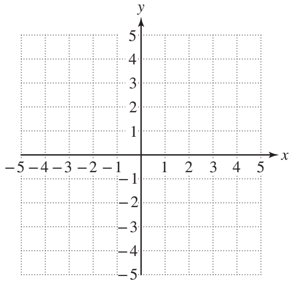
52. $x = -5$



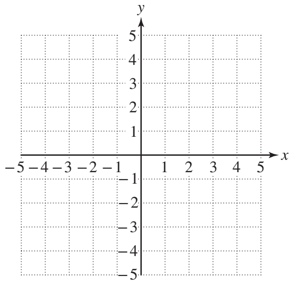
53. $2x + 6 = 5$



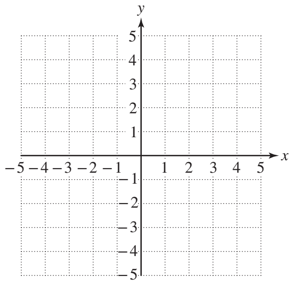
54. $-3x = 12$



55. $-2y + 1 = 9$



56. $-5y = -10$



57. Explain why not every line has both an x-intercept and a y-intercept.

58. Is it possible for a line to have no x-intercept and no y-intercept?

59. Which of the lines defined here has only one unique intercept?

a. $2x - 3y = 8$ **b.** $x = 7$ **c.** $3y = 9$ **d.** $-x + y = 0$

60. Which of the lines defined here has only one unique intercept?

a. $y = -5$ **b.** $x + 2y = 0$ **c.** $3x - 4 = 2$ **d.** $x + 3y = 6$

Technology Connections

For Exercises 61–64, solve the equation for y. Use a graphing calculator to graph the equation on the standard viewing window.

61. $2x - 3y = 7$ **62.** $4x + 2y = -2$ **63.** $3y = 9$ **64.** $2y + 10 = 0$

For Exercises 65–68, use a graphing calculator to graph the lines on the suggested viewing window.

65. $y = -\frac{1}{2}x - 10$

$-30 \le x \le 10$
$-15 \le y \le 5$

66. $y = -\frac{1}{3}x + 12$

$-10 \le x \le 40$
$-10 \le y \le 20$

67. $-2x + 4y = 1$

$-1 \le x \le 1$
$-1 \le y \le 1$

68. $5y = 4x - 1$

$-0.5 \le x \le 0.5$
$-0.5 \le y \le 0.5$

For Exercises 69–70, graph the lines in parts (a)–(c) on the same viewing window. Compare the graphs. Are the lines exactly the same?

69. **a.** $y = x + 3$

b. $y = x + 3.1$

c. $y = x + 2.9$

70. **a.** $y = 2x + 1$

b. $y = 1.9x + 1$

c. $y = 2.1x + 1$

Expanding Your Skills

For Exercises 71–74, find the x- and y-intercepts.

71. $\frac{x}{2} + \frac{y}{3} = 1$

72. $\frac{x}{7} + \frac{y}{4} = 1$

73. $\frac{x}{a} + \frac{y}{b} = 1$

74. $Ax + By = C$

Section 2.2 Slope of a Line and Rate of Change

Concepts

1. Introduction to the Slope of a Line
2. The Slope Formula
3. Parallel and Perpendicular Lines
4. Applications and Interpretation of Slope

1. Introduction to the Slope of a Line

We have already learned how to graph a linear equation and to identify its x- and y-intercepts. In this section, we learn about another important feature of a line called the *slope* of a line. Geometrically, slope measures the "steepness" of a line.

Figure 2-12 shows a set of stairs with a wheelchair ramp to the side. Notice that, even though the stairs and ramp both rise the same vertical distance, the stairs are steeper than the ramp. This is because the stairs rise 3 ft over a shorter horizontal distance than the ramp.

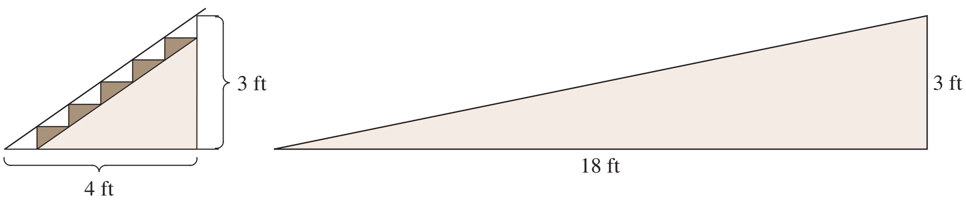
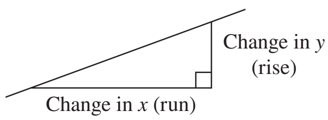


Figure 2-12

To measure the slope of a line quantitatively, consider two points on the line. The slope is the ratio of the vertical change between the two points to the horizontal change. That is, the **slope** is the ratio of the change in y to the change in x. As a memory device, we might think of the slope of a line as "rise over run."

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$$



To move from point A to point B on the stairs, rise 3 ft and move to the right 4 ft (Figure 2-13).

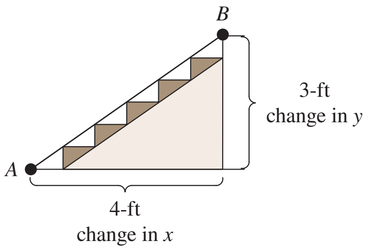


Figure 2-13

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{3\text{ ft}}{4\text{ ft}} = \frac{3}{4}$$

To move from point A to point B on the wheelchair ramp, rise 3 ft and move to the right 18 ft (Figure 2-14).

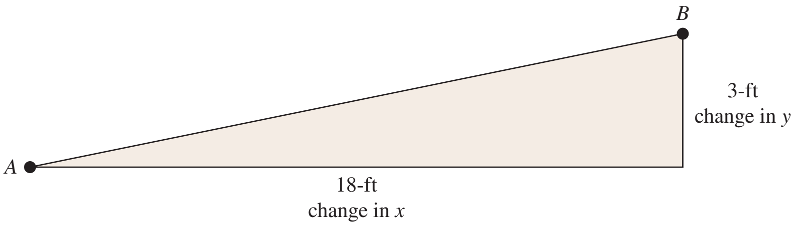


Figure 2-14

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{3\text{ ft}}{18\text{ ft}} = \frac{1}{6}$$

The slope of the stairs is $\frac{3}{4}$, which is greater than the slope of the ramp, which is $\frac{1}{6}$.

Example 1 Finding the Slope in an Application

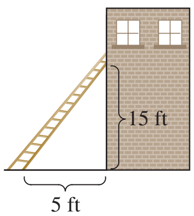
Find the slope of the ladder against the wall.

Solution:

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{15\text{ ft}}{5\text{ ft}}$$

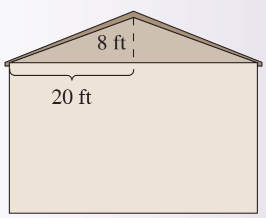
$$= \frac{3}{1} \text{ or } 3$$



The slope is $\frac{3}{1}$, which indicates that a person climbs 3 ft vertically for every 1 ft traveled horizontally.

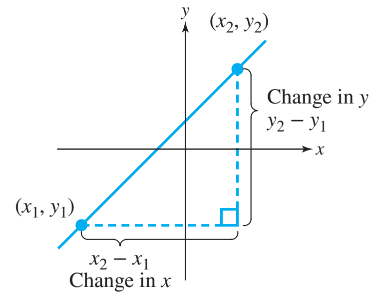
Skill Practice

1. Find the slope of the roof.



Answer

1. $\frac{2}{5}$



2. The Slope Formula

The slope of a line may be found by using *any* two points on the line—call these points (x_1, y_1) and (x_2, y_2). The change in y between the points can be found by taking the difference of the y values: $y_2 - y_1$. The change in x can be found by taking the difference of the x values in the same order: $x_2 - x_1$.

The slope of a line is often symbolized by the letter m and is given by the following formula.

Formula for the Slope of a Line Given Two Distinct Points

The **slope** of a line passing through the distinct points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{provided} \quad x_2 - x_1 \neq 0$$

Example 2 Finding the Slope of a Line Through Two Points

Find the slope of the line passing through the points $(1, -1)$ and $(7, 2)$.

Solution:

To use the slope formula, first label the coordinates of each point, and then substitute their values into the slope formula.

$(1, -1)$ and $(7, 2)$
(x_1, y_1) (x_2, y_2) Label the points.

$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2 - (-1)}{7 - 1}$ Apply the slope formula.

$= \dfrac{3}{6}$ Simplify.

$= \dfrac{1}{2}$

Avoiding Mistakes

When applying the slope formula, y_2 and x_2 are taken from the same ordered pair. Likewise y_1 and x_1 are taken from the same ordered pair.

The slope of the line can be verified from the graph (Figure 2-15).

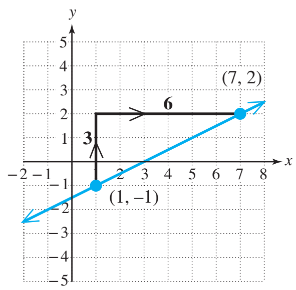


Figure 2-15

TIP: The slope formula does not depend on which point is labeled (x_1, y_1) and which point is labeled (x_2, y_2). For example, reversing the order in which the points are labeled in Example 2 results in the same slope:

$(1, -1)$ and $(7, 2)$ then $m = \dfrac{-1 - 2}{1 - 7} = \dfrac{-3}{-6} = \dfrac{1}{2}$
(x_2, y_2) (x_1, y_1)

Skill Practice

2. Find the slope of the line that passes through the points $(-4, 5)$ and $(6, 8)$.

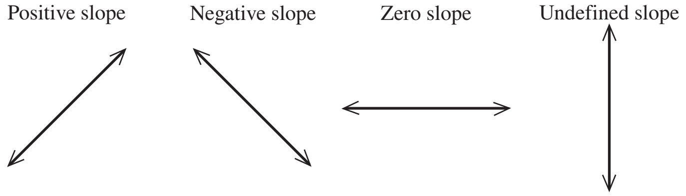
When you apply the slope formula, you will see that the slope of a line may be positive, negative, zero, or undefined.

- Lines that "increase," or "rise," from left to right have a *positive slope*.
- Lines that "decrease," or "fall," from left to right have a *negative slope*.

Answer

2. $\dfrac{3}{10}$

- Horizontal lines have a *zero slope*.
- Vertical lines have an *undefined slope*.



Example 3 Finding the Slope of a Line Through Two Points

Find the slope of the line passing through the points (3, −4) and (−5, −1).

Solution:

$(3, -4)$ and $(-5, -1)$

(x_1, y_1) (x_2, y_2) Label points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-(-4)}{-5-3}$$ Apply the slope formula.

$$= \frac{3}{-8} = -\frac{3}{8}$$ Simplify.

The two points can be graphed to verify that $-\frac{3}{8}$ is the correct slope (Figure 2-16).

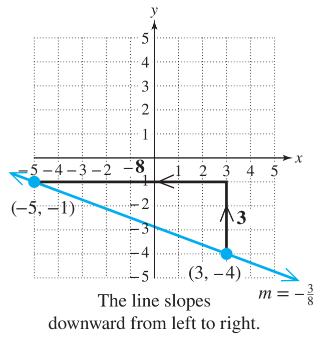


Figure 2-16

Skill Practice Find the slope of the line passing through the given points.

3. (1, −8) and (−5, −4)

Example 4 Finding the Slope of a Line Through Two Points

Find the slope of the line passing through the points (−3, 4) and (−3, −2).

Solution:

$(-3, 4)$ and $(-3, -2)$

(x_1, y_1) (x_2, y_2) Label the points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-4}{-3-(-3)}$$ Apply the slope formula.

$$= \frac{-6}{-3+3}$$

$$= \frac{-6}{0}$$ Undefined

The slope is undefined. The points define a vertical line (Figure 2-17).

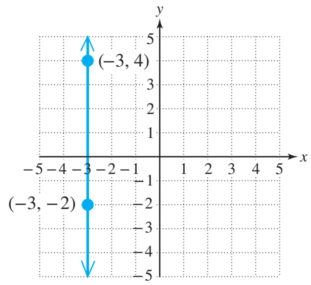


Figure 2-17

Skill Practice Find the slope of the line passing through the given points.

4. (5, −2) and (5, 5)

FOR REVIEW

Recall that division by zero is undefined. $\frac{-6}{0}$ is undefined because no number that when multiplied by 0 will equal −6.

Answers

3. $-\frac{2}{3}$ **4.** Undefined

Example 5 **Finding the Slope of a Line Through Two Points**

Find the slope of the line passing through the points (0, 2) and (−4, 2).

Solution:

(0, 2) and (−4, 2)
(x_1, y_1) (x_2, y_2) Label the points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-2}{-4-0}$$ Apply the slope formula.

$$= \frac{0}{-4}$$

$$= 0$$ Simplify.

The slope is zero. The line through the two points is a horizontal line (Figure 2-18).

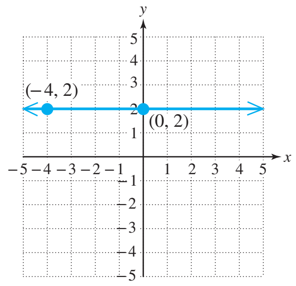


Figure 2-18

Skill Practice Find the slope of the line passing through the points.

5. (1, 6) and (−7, 6)

3. Parallel and Perpendicular Lines

Lines in the same plane that do not intersect are *parallel*. Nonvertical parallel lines have the same slope and different *y*-intercepts (Figure 2-19).

Lines that intersect at a right angle are *perpendicular*. If two lines are perpendicular, then the slope of one line is the opposite of the reciprocal of the slope of the other (provided neither line is vertical) (Figure 2-20).

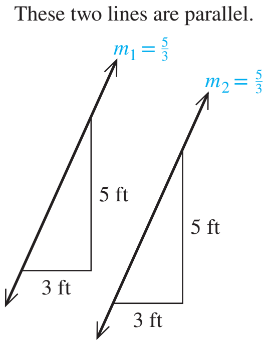


Figure 2-19

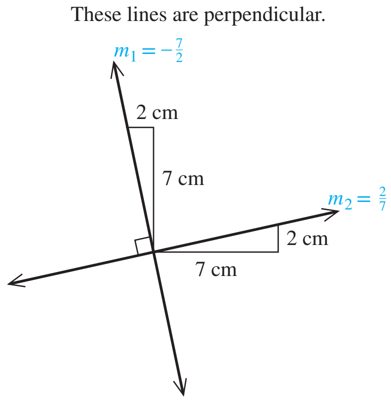


Figure 2-20

Slopes of Parallel Lines

If m_1 and m_2 represent the slopes of two parallel (nonvertical) lines, then

$$m_1 = m_2$$

See Figure 2-19.

Answer
5. 0

Slopes of Perpendicular Lines

If $m_1 \neq 0$ and $m_2 \neq 0$ represent the slopes of two perpendicular lines, then

$$m_1 = -\frac{1}{m_2} \text{ or, equivalently, } m_1 \cdot m_2 = -1$$

See Figure 2-20.

Example 6 Determining the Slope of Parallel and Perpendicular Lines

Suppose a given line has a slope of -5.

a. Find the slope of a line parallel to the given line.

b. Find the slope of a line perpendicular to the given line.

Solution:

a. The slope of a line parallel to the given line is $m = -5$ (same slope).

b. The slope of a line perpendicular to the given line is $m = \frac{1}{5}$ (the opposite of the reciprocal of -5).

FOR REVIEW

Recall that the reciprocal of a number carries the same sign as the number. Thus, the reciprocal of -5 is $-\frac{1}{5}$. The *opposite* of the reciprocal of -5 is $\frac{1}{5}$.

Skill Practice The slope of line L_1 is $-\frac{4}{3}$.

6. Find the slope of a line parallel to L_1.

7. Find the slope of a line perpendicular to L_1.

Example 7 Determining Whether Two Lines Are Parallel, Perpendicular, or Neither

Two points are given from each of two lines: L_1 and L_2. Without graphing the points, determine if the lines are parallel, perpendicular, or neither.

L_1: $(2, -3)$ and $(4, 1)$

L_2: $(5, -6)$ and $(-3, -2)$

Solution:

First determine the slope of each line. Then compare the values of the slopes to determine if the lines are parallel or perpendicular.

For line 1:

L_1: $(2, -3)$ and $(4, 1)$
(x_1, y_1) (x_2, y_2)

$$m = \frac{1-(-3)}{4-2} = \frac{4}{2} = 2$$

For line 2:

L_2: $(5, -6)$ and $(-3, -2)$
(x_1, y_1) (x_2, y_2)

$$m = \frac{-2-(-6)}{-3-5} = \frac{4}{-8} = -\frac{1}{2}$$

Label the points

Apply the slope formula.

TIP: You can also verify that the lines in Example 7 are perpendicular by noting that the product of their slopes is -1.

$$2 \cdot \left(-\frac{1}{2}\right) = -1$$

The slope of L_1 is 2. The slope of L_2 is $-\frac{1}{2}$. The slope of L_1 is the opposite of the reciprocal of the slope of L_2. By comparing the slopes, the lines must be perpendicular.

Answers

6. $-\frac{4}{3}$ **7.** $\frac{3}{4}$

Skill Practice Two points are given for lines L_1 and L_2. Determine if the lines are parallel, perpendicular, or neither.

8. L_1: $(4, -1)$ and $(-3, 6)$
L_2: $(-1, 3)$ and $(2, 0)$

4. Applications and Interpretation of Slope

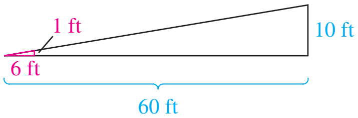


Figure 2-21

In applications, the slope of a line represents a rate of change between the y variable and the x variable. For example, a hiker walking up a hill with a slope of $\frac{1}{6}$ means that 1 ft of elevation is gained for every 6 ft traveled horizontally. Using this rate of change, we can also say that a hiker gains 10 ft of elevation for 60 ft traveled horizontally. See Figure 2-21.

Example 8 Interpreting the Slope of a Line in an Application

The number of males 20 yr old or older who were employed full-time in the United States grew linearly between 1970 and 2010. Approximately 43.0 million males 20 yr old or older were employed full-time in 1970. By 2005, this number had grown to 65.4 million (Figure 2-22).

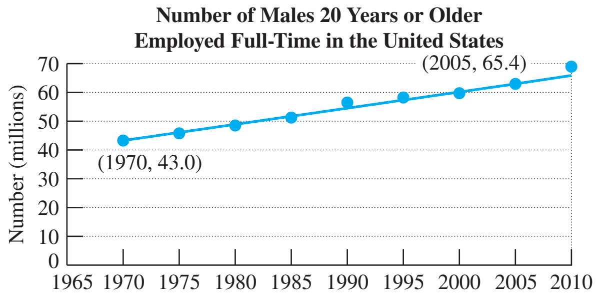


Figure 2-22

Source: U.S. Census Bureau

a. Find the slope of the line, using the points (1970, 43.0) and (2005, 65.4).

b. Interpret the meaning of the slope in the context of this problem.

Solution:

a. (1970, 43.0) and (2005, 65.4) — (x_1, y_1) and (x_2, y_2) — Label the points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{65.4 - 43.0}{2005 - 1970}$$ Apply the slope formula.

$$m = \frac{22.4}{35} \quad \text{or} \quad m = 0.64$$

b. The slope is approximately 0.64, meaning that the full-time workforce has increased at a rate of 0.64 million men (or 640,000 men) per year between 1970 and 2005.

TIP: The slope, $m = \frac{22.4}{35}$, means that the workforce increased by 22.4 million men over 35 yr. This is the same rate as $\frac{0.64}{1}$, meaning that the workforce increased by 0.64 million men per year.

Skill Practice The number of people per square mile (called population density) in Alaska has increased linearly since 1990. In 1990, there were 0.96 people per square mile. By 2010, this increased to 1.32 people per square mile. If x represents the year and y represents population density,

9. Find the slope of the line passing through the points (1990, 0.96) and (2010, 1.32).

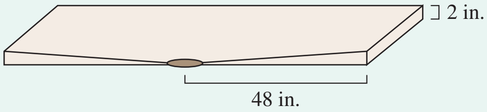
10. Interpret the meaning of the slope in the context of this problem.

Answers

8. Parallel
9. 0.018
10. The population density increased at a rate of 0.018 people per square mile per year.

Section 2.2 Activity

A.1. The recommended slope for a shower stall is approximately 4%. For example, this means that for a horizontal distance of 100 in., there is a drop of 4 in. Determine the slope of the shower stall shown here. Does the slope appear to be adequate for drainage?

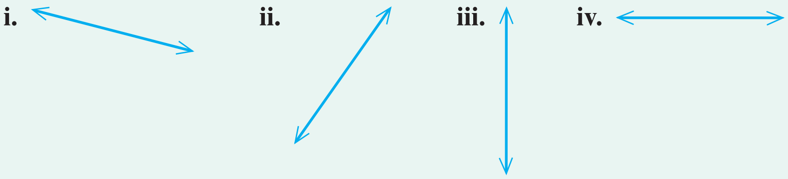


A.2. Write a formula for the slope m of a line passing through (x_1, y_1) and (x_2, y_2).

For Exercises A.3–A.6,

a. Find the slope m of the line between the two points.

b. Match the slope with the appropriate graph: i, ii, iii, or iv.



A.3. (4, −3) and (6, 5)

A.4. (−4, −1) and (−4, 5)

A.5. (2, −3) and (−6, −3)

A.6. (4, 2) and (−2, 4)

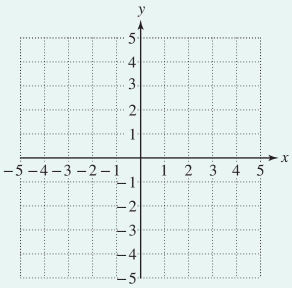
For Exercises A.7–A.8, two points are given for each of two lines l_1 and l_2.

a. Find the slope of each line.

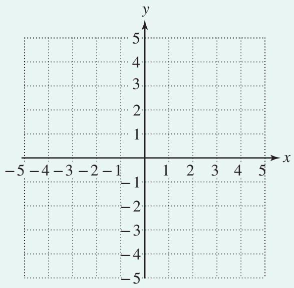
b. Based on the slopes, are the lines parallel, perpendicular, or neither?

c. Verify your results from part (b) by graphing l_1 and l_2.

A.7. l_1: (0, −3) and (2, 1)
l_2: (−2, −3) and (−1, −1)



A.8. l_1: (−3, −1) and (3, −3)
l_2: (0, 2) and (−2, −4)

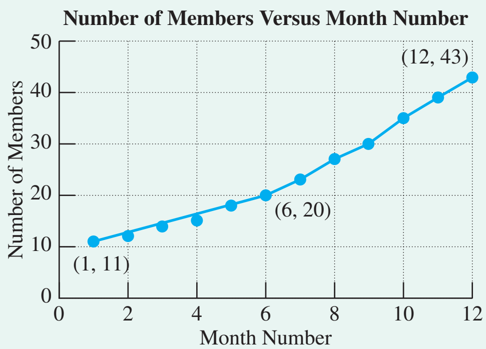


A.9. A bookstore starts a monthly book club. The graph shows the number of members by month number.

a. Find the slope of the line between the points (1, 11) and (6, 20). Interpret the meaning of the slope.

b. Find the slope of the line between the points (6, 20) and (12, 43). Interpret the meaning of the slope.

c. The slope from part (a) is greater than the slope from part (b). What does this mean in the context of this problem?



Section 2.2 Practice Exercises

Study Skills Exercise

Mnemonics are a great way of remembering things in mathematics. A mnemonic is a statement or sentence that uses the first letter of each word that you are trying to remember. A well-known mnemonic in mathematics is "Please Excuse My Dear Aunt Sally," which helps students remember the order of operations. A mnemonic can be created for almost any content with some creativity.

Create a mnemonic to help you remember whether the slope of a line is positive, negative, zero, or undefined.

Prerequisite Review

For Exercises R.1–R.6, simplify the expression, if possible.

R.1. $\dfrac{8-(-2)}{-2-4}$ **R.2.** $\dfrac{-1-5}{4-(-8)}$ **R.3.** $\dfrac{-5-(-5)}{3-10}$

R.4. $\dfrac{-6-8}{9-9}$ **R.5.** $\dfrac{\frac{2}{3}}{\frac{6}{5}}$ **R.6.** $\dfrac{\frac{-5}{4}}{\frac{5}{6}}$

R.7. **a.** What is the reciprocal of $-\frac{3}{7}$? **b.** What is the opposite of the reciprocal of $-\frac{3}{7}$?

R.8. **a.** What is the reciprocal of -8? **b.** What is the opposite of the reciprocal of -8?

R.9. A 52-week digital subscription to a newspaper is \$155.48. What is the weekly rate?

R.10. The bill for a 5-day hotel stay is \$775. What is the daily rate?

Vocabulary and Key Concepts

1. **a.** The ratio of the vertical change and the horizontal change between two distinct points (x_1, y_1) and (x_2, y_2) on a line is called the ________ of the line. The slope can be computed from the formula $m =$ ________.

   b. Lines in the same plane that do not intersect are called ________ lines. Such lines, if they are nonvertical, have the ________ slope and different y-intercepts.

   c. Two lines are perpendicular if they intersect at a ________ angle.

   d. If m_1 and m_2 represent the slopes of two nonvertical perpendicular lines, then $m_1 \cdot m_2 =$ ________.

For Exercises 2–6, determine if the graph of the indicated variable versus time would show a positive slope, a negative slope, a zero slope, or a nonlinear shape.

2. The value of a car steadily decreased over time.

3. Housing cost remained unchanged over time.

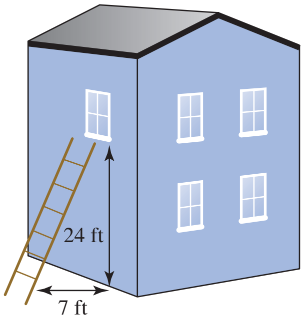
4. Gabriella's salary steadily increased over time.

5. The rate of increase in population increases at an increasing rate over time.

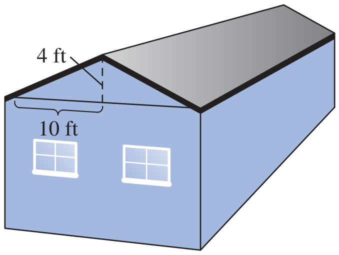
6. Brian's batting average in baseball remained the same over time.

Concept 1: Introduction to the Slope of a Line

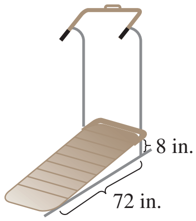
7. A 25-ft ladder is leaning against a house, as shown in the diagram. Find the slope of the ladder. **(See Example 1.)**



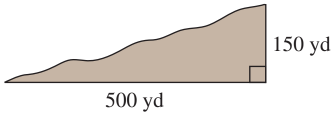
8. Find the slope of the roof shown in the figure.



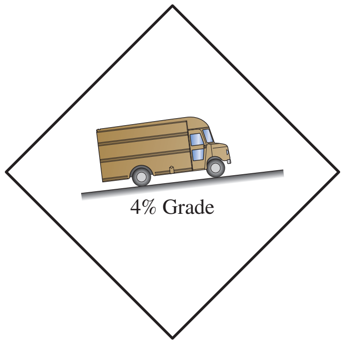
9. Find the slope of the treadmill.



10. Find the average slope of the hill.



11. The road sign shown in the figure indicates the percent grade of a hill. This gives the slope of the road as the change in elevation per 100 horizontal ft. Given a 4% grade, write this as a slope in fractional form.



12. If a plane gains 1000 ft in altitude over a distance of 12,000 horizontal ft, what is the slope? Explain what this value means in the context of the problem.

Concept 2: The Slope Formula

For Exercises 13–30, use the slope formula to determine the slope of the line containing the two points. **(See Examples 2–5.)**

13. $(6, 0)$ and $(0, -3)$

14. $(-5, 0)$ and $(0, 4)$

15. $(-2, 3)$ and $(4, -7)$

16. $(-5, -4)$ and $(1, -7)$

17. $(-2, 5)$ and $(2, -3)$

18. $(4, -2)$ and $(6, -8)$

19. $(0.3, -1.1)$ and $(-0.1, -0.8)$

20. $(0.4, -0.2)$ and $(0.3, -0.1)$

21. $(2, 3)$ and $(2, 7)$

22. $(-1, 5)$ and $(-1, 0)$

23. $(5, -1)$ and $(-3, -1)$

24. $(-8, 4)$ and $(1, 4)$

25. $(-4.6, 4.1)$ and $(0, 6.4)$

26. $(1.1, 4)$ and $(-3.2, -0.3)$

27. $\left(\frac{3}{2}, \frac{4}{3}\right)$ and $\left(\frac{7}{2}, 1\right)$

28. $\left(\frac{2}{3}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{6}, -\frac{3}{2}\right)$

29. $\left(\frac{3}{4}, \frac{7}{3}\right)$ and $\left(\frac{1}{2}, 2\frac{1}{3}\right)$

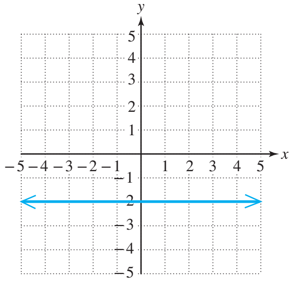
30. $\left(\frac{9}{4}, \frac{2}{5}\right)$ and $\left(2\frac{1}{4}, \frac{1}{10}\right)$

31. Explain how to use the graph of a line to determine whether the slope of a line is positive, negative, zero, or undefined.

32. If the slope of a line is $\frac{4}{3}$, how many units of change in y will be produced by 6 units of change in x?

For Exercises 33–38, estimate the slope of the line from its graph.

33.



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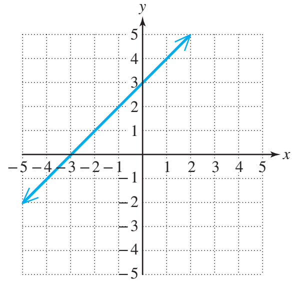

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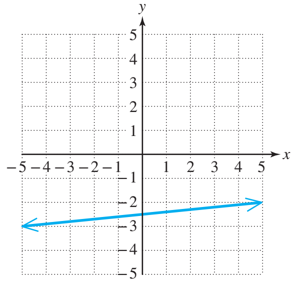
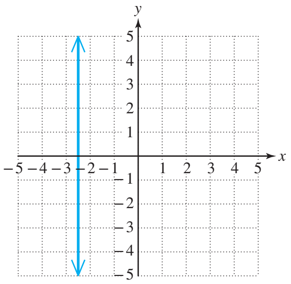
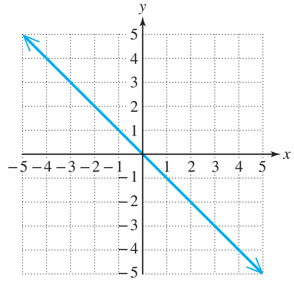


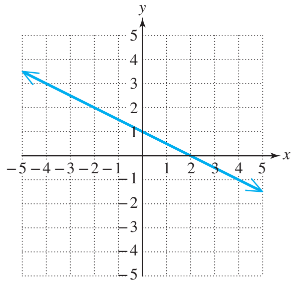
36.



37.



38.



Concept 3: Parallel and Perpendicular Lines

For Exercises 39–44, the slope of a line is given.

a. Find the slope of a line parallel to the given line.

b. Find the slope of a line perpendicular to the given line. **(See Example 6.)**

39. $m = 5$

40. $m = 3$

41. $m = -\dfrac{4}{7}$

42. $m = -\dfrac{2}{11}$

43. $m = 0$

44. m is undefined.

45. Can the slopes of two perpendicular lines both be positive? Explain your answer.

46. Suppose a line is defined by the equation $x = 2$. What is the slope of a line perpendicular to this line?

47. Suppose a line is defined by the equation $y = -5$. What is the slope of a line perpendicular to this line?

48. Suppose a line is defined by the equation $x = -3$. What is the slope of a line parallel to this line?

49. What is the slope of a line parallel to the x-axis?

50. What is the slope of a line perpendicular to the y-axis?

51. What is the slope of a line perpendicular to the x-axis?

52. What is the slope of a line parallel to the y-axis?

In Exercises 53–60, two points are given from each of two lines L_1 and L_2. Without graphing the points, determine if the lines are parallel, perpendicular, or neither. **(See Example 7.)**

53. L_1: (2, 5) and (4, 9)
L_2: (−1, 4) and (3, 2)

54. L_1: (−3, −5) and (−1, 2)
L_2: (0, 4) and (7, 2)

55. L_1: (4, −2) and (3, −1)
L_2: (−5, −1) and (−10, −16)

56. L_1: (0, 0) and (2, 3)
L_2: (−2, 5) and (0, −2)

57. L_1: (5, 3) and (5, 9)
L_2: (4, 2) and (0, 2)

58. L_1: (3, 5) and (2, 5)
L_2: (2, 4) and (0, 4)

59. L_1: (−3, −2) and (2, 3)
L_2: (−4, 1) and (0, 5)

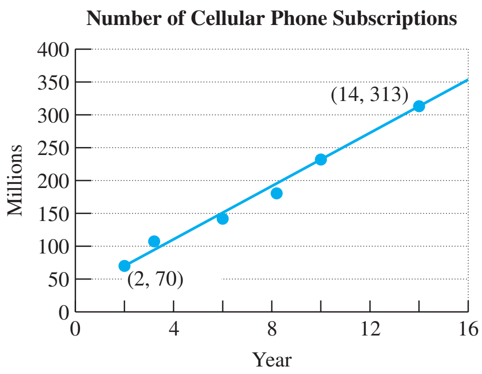
60. L_1: (7, 1) and (0, 0)
L_2: (−10, −8) and (4, −6)

Concept 4: Applications and Interpretation of Slope

61. The graph shows the number of cellular phone subscriptions (in millions) purchased in the United States over a period of 16 yr. **(See Example 8.)**

a. Use the coordinates of the given points to find the slope of the line, and express the answer in decimal form.

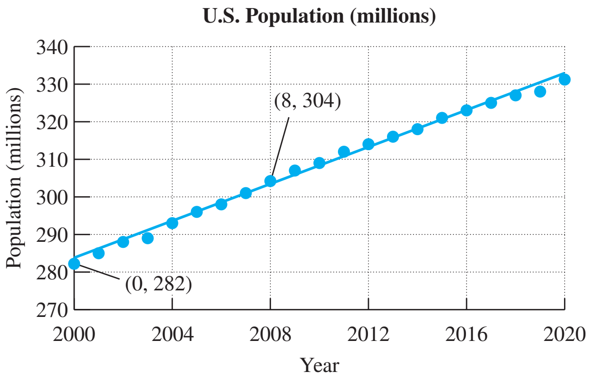
b. Interpret the meaning of the slope in the context of this problem.



62. The U.S. population (in millions) has grown approximately linearly since the year 2000.

a. Find the slope of the line defined by the two given points.

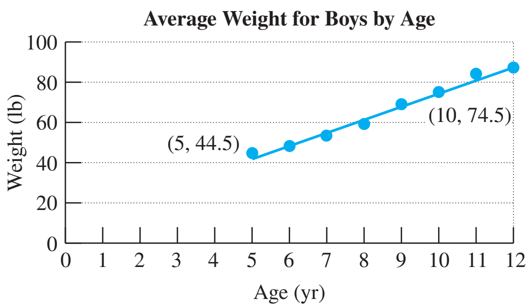
b. Interpret the meaning of the slope in the context of this problem.



63. The data in the graph show the average weight for boys based on age.

a. Use the coordinates of the given points to find the slope of the line.

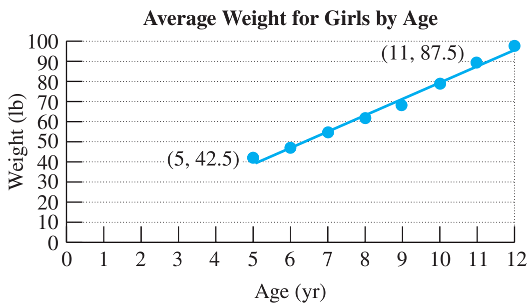
b. Interpret the meaning of the slope in the context of this problem.



64. The data in the graph show the average weight for girls based on age.

a. Use the coordinates of the given points to find the slope of the line, and write the answer in decimal form.

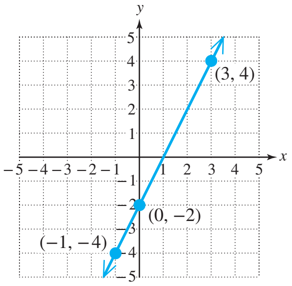
b. Interpret the meaning of the slope in the context of this problem.



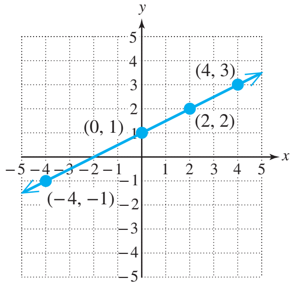
Mixed Exercises

The slope of a line is constant. This also means that the rate of change of y versus x is constant between any two points on the line. For Exercises 65–66, demonstrate this statement by computing the slope of the line using each pair of points on the line.

65. **a.** $(-1, -4)$ and $(0, -2)$
b. $(0, -2)$ and $(3, 4)$
c. $(-1, -4)$ and $(3, 4)$

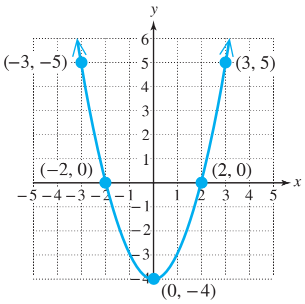


66. **a.** $(-4, -1)$ and $(0, 1)$
b. $(0, 1)$ and $(2, 2)$
c. $(0, 1)$ and $(4, 3)$

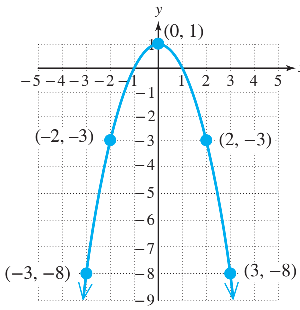


Given a nonlinear graph, the rate of change of y versus x may differ for different pairs of points on the graph. For Exercises 67–68, demonstrate this statement by computing the slope formula with each pair of points on the curve.

67. **a.** $(-2, 0)$ and $(0, -4)$
b. $(0, -4)$ and $(2, 0)$
c. $(2, 0)$ and $(3, 5)$



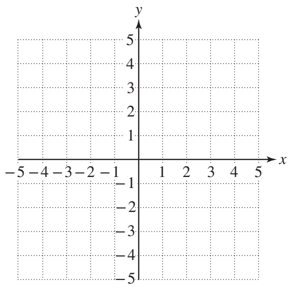
68. **a.** $(-3, -8)$ and $(-2, -3)$
b. $(-2, -3)$ and $(0, 1)$
c. $(0, 1)$ and $(2, -3)$



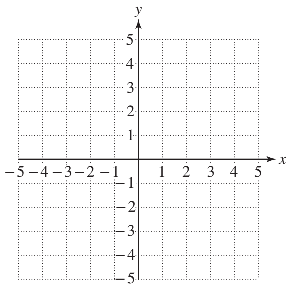
Expanding Your Skills

For Exercises 69–74, given a point P on a line and the slope m of the line, find a second point on the line (answers may vary). *Hint:* Graph the line to help you find the second point.

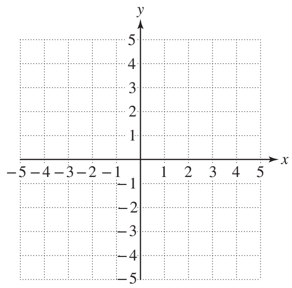
69. $P(0, 0)$ and $m = 2$



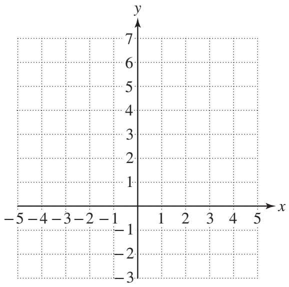
70. $P(-2, 1)$ and $m = -\frac{1}{3}$



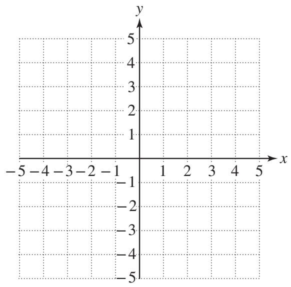
71. $P(2, -3)$ and m is undefined



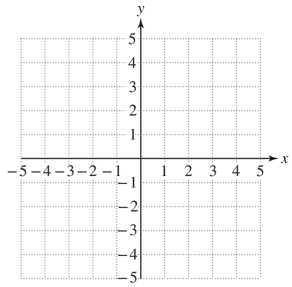
72. $P(-2, 4)$ and $m = 0$



73. $P(-1, 2)$ and $m = -\dfrac{2}{3}$



74. $P(-1, -4)$ and $m = \dfrac{4}{5}$



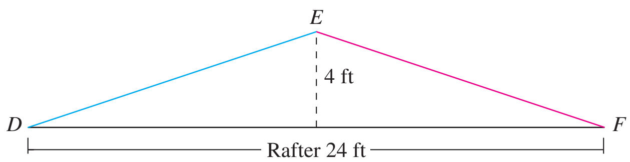
75. Given that $(-2, y)$ and $(4, 6)$ are points on a line whose slope is $-\frac{3}{2}$, find y.

76. Given that $(x, -4)$ and $(3, 2)$ are points on a line whose slope is $\frac{6}{7}$, find x.

77. The pitch of a roof is defined as $\dfrac{\text{rise}}{\text{rafter}}$.

a. Determine the pitch of the roof shown.

b. Determine the slope of the line segment from point D to point E.



Equations of a Line

Section 2.3

Concepts

1. Slope-Intercept Form
2. The Point-Slope Formula
3. Equations of a Line: A Summary

1. Slope-Intercept Form

We have already learned that an equation of the form $Ax + By = C$ (where A and B are not both zero) represents a line in a rectangular coordinate system. An equation of a line written in this way is in **standard form**. In this section, we will learn a new form, called the **slope-intercept form**, which is useful in determining the slope and y-intercept of a line.

Let $(0, b)$ represent the y-intercept of a line. Let (x, y) represent any other point on the line where $x \neq 0$. Then, labeling the points and applying the slope formula, we have

$$(0, b) \qquad (x, y)$$
$$(x_1, y_1) \qquad (x_2, y_2)$$

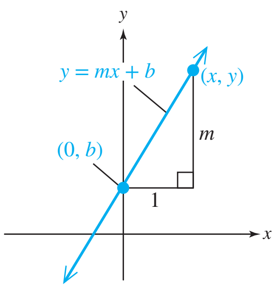
$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \rightarrow \quad m = \frac{y - b}{x - 0} \qquad \text{Apply the slope formula.}$$

$$m = \frac{y - b}{x} \qquad \text{Simplify.}$$

$$m \cdot x = \left(\frac{y - b}{\cancel{x}}\right) \cdot \cancel{x} \qquad \text{Clear fractions.}$$

$$mx = y - b \qquad \text{Simplify.}$$

$$mx + b = y \quad \text{or} \quad y = mx + b \qquad \text{Solve for } y \text{ to get slope-intercept form.}$$



Slope-Intercept Form of a Line

$y = mx + b$ is the slope-intercept form of a line.

m is the slope and the point $(0, b)$ is the y-intercept.

The equation $y = -4x + 7$ is written in slope-intercept form. By inspection, we can see that the slope of the line is -4 and the y-intercept is $(0, 7)$.

Example 1 Finding the Slope and y-Intercept of a Line

Given $3x + 4y = 4$, write the equation of the line in slope-intercept form. Then find the slope and y-intercept.

Solution:

Write the equation in slope-intercept form, $y = mx + b$, by solving for y.

$$3x + 4y = 4$$

$$4y = -3x + 4 \qquad \text{Subtract } 3x \text{ from both sides.}$$

$$\frac{4y}{4} = \frac{-3x}{4} + \frac{4}{4} \qquad \text{To isolate } y\text{, divide both sides by 4.}$$

$$y = -\frac{3}{4}x + 1 \qquad \text{The slope is } -\frac{3}{4} \text{ and the } y\text{-intercept is } (0, 1).$$

Skill Practice Write the equation in slope-intercept form. Determine the slope and the y-intercept.

1. $2x - 4y = 3$

The slope-intercept form is a useful tool to graph a line. The y-intercept is a known point on the line, and the slope indicates the "direction" of the line and can be used to find a second point. Using slope-intercept form to graph a line is demonstrated in Example 2.

Example 2 Graphing a Line Using the Slope and y-Intercept

Graph the equation $y = -\frac{3}{4}x + 1$ using the slope and y-intercept.

Solution:

First plot the y-intercept $(0, 1)$. The slope $m = -\frac{3}{4}$ can be written as

$$m = \frac{-3}{4} \qquad \begin{array}{l}\text{The change in } y \text{ is } -3.\\ \text{The change in } x \text{ is } 4.\end{array}$$

To find a second point on the line, start at the y-intercept and move *down* 3 *units* and to the *right* 4 *units*. Then draw the line through the two points (Figure 2-23).

Similarly, the slope can be written as

$$m = \frac{3}{-4} \qquad \begin{array}{l}\text{The change in } y \text{ is } 3.\\ \text{The change in } x \text{ is } -4.\end{array}$$

To find a second point on the line, start at the y-intercept and move *up* 3 *units* and to the *left* 4 *units*. Then draw the line through the two points (see Figure 2-23).

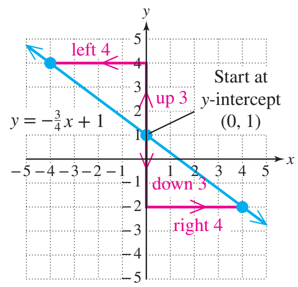


Figure 2-23

Skill Practice

2. Graph the equation $y = \frac{1}{5}x - 2$ using the slope and y-intercept.

FOR REVIEW

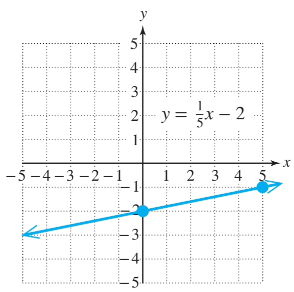
Recall that the negative sign in a fraction can be written in the numerator, denominator, or out in front.

$$\frac{-3}{4} = \frac{3}{-4} = -\frac{3}{4}$$

Answers

1. $y = \frac{1}{2}x - \frac{3}{4}$
Slope: $\frac{1}{2}$; y-intercept: $\left(0, -\frac{3}{4}\right)$

2.



As we have seen earlier, two lines are parallel if they have the same slope and different y-intercepts. Two lines are perpendicular if the slope of one line is the opposite of the reciprocal of the slope of the other line. Otherwise, the lines are neither parallel nor perpendicular.

Example 3 Determining if Two Lines Are Parallel, Perpendicular, or Neither

Given the equations for two lines, L_1 and L_2, determine if the lines are parallel, perpendicular, or neither.

a. L_1: $y = -2x + 7$
L_2: $y = -2x - 1$

b. L_1: $2y = -3x + 2$
L_2: $-4x + 6y = -12$

c. L_1: $x + y = 6$
L_2: $y = 6$

Solution:

a. The equations are written in slope-intercept form.

L_1: $y = -2x + 7$ The slope is -2 and the y-intercept is $(0, 7)$.

L_2: $y = -2x - 1$ The slope is -2 and the y-intercept is $(0, -1)$.

Because the slopes are the same and the y-intercepts are different, the lines are parallel.

b. Write each equation in slope-intercept form by solving for y.

L_1: $2y = -3x + 2$

$$\frac{2y}{2} = \frac{-3x}{2} + \frac{2}{2} \quad \text{Divide by 2.}$$

$$y = -\frac{3}{2}x + 1$$

L_2: $-4x + 6y = -12$

$$6y = 4x - 12 \quad \text{Add } 4x \text{ to both sides.}$$

$$\frac{6y}{6} = \frac{4}{6}x - \frac{12}{6} \quad \text{Divide by 6.}$$

$$y = \frac{2}{3}x - 2$$

The slope of L_1 is $-\frac{3}{2}$. The slope of L_2 is $\frac{2}{3}$.

The value $-\frac{3}{2}$ is the opposite of the reciprocal of $\frac{2}{3}$. Therefore, the lines are perpendicular.

c. L_1: $x + y = 6$ is equivalent to $y = -x + 6$. The slope is -1.

L_2: $y = 6$ is a horizontal line, and the slope is 0.

The slopes are not the same. Therefore, the lines are not parallel. The slope of one line is not the opposite of the reciprocal of the other slope. Therefore, the lines are not perpendicular. The lines are neither parallel nor perpendicular.

Skill Practice Given the pair of equations, determine if the lines are parallel, perpendicular, or neither.

3. $y = -\frac{3}{4}x + 1$
$y = \frac{4}{3}x + 3$

4. $3x + y = 4$
$6x = 6 - 2y$

5. $x - y = 7$
$x = 1$

Answers

3. Perpendicular **4.** Parallel
5. Neither

Example 4 **Using Slope-Intercept Form to Find an Equation of a Line**

Use slope-intercept form to find an equation of the line with slope -3 and passing through the point $(1, -4)$.

Solution:

To find an equation of a line in slope-intercept form, $y = mx + b$, it is necessary to find the slope, m, and the y-intercept, b. The slope is given in the problem as $m = -3$. Therefore, the slope-intercept form becomes

$$y = mx + b$$

$$y = -3x + b$$

Furthermore, because the point $(1, -4)$ is on the line, it is a solution to the equation. Therefore, if we substitute $(1, -4)$ for x and y in the equation, we can solve for b.

$$\begin{aligned} -4 &= -3(1) + b \\ -4 &= -3 + b \\ -1 &= b \end{aligned}$$

Thus, the slope-intercept form is $y = -3x - 1$.

Skill Practice

6. Use slope-intercept form to find an equation of the line with slope 2 and passing through $(-3, -5)$.

2. The Point-Slope Formula

In Example 4, we used the slope-intercept form of a line to construct an equation of a line given its slope and a known point on the line. Here we provide another tool called the *point-slope formula* that (as its name suggests) can accomplish the same result.

Suppose a nonvertical line passes through a given point (x_1, y_1) and has slope m. If (x, y) is any other point on the line, then

$$m = \frac{y - y_1}{x - x_1} \qquad \text{Slope formula}$$

$$m(x - x_1) = \frac{y - y_1}{x - x_1}(x - x_1) \qquad \text{Clear fractions.}$$

$$m(x - x_1) = y - y_1$$

or

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope formula}$$

The Point-Slope Formula

The **point-slope formula** is given by

$$y - y_1 = m(x - x_1)$$

where m is the slope of the line and (x_1, y_1) is a known point on the line.

Answer

6. $y = 2x + 1$

The point-slope formula is used specifically to find an equation of a line when a point on the line is known and the slope is known. To illustrate the point-slope formula, we will repeat the problem from Example 4.

Example 5 Using the Point-Slope Formula to Find an Equation of a Line

Use the point-slope formula to find an equation of the line having a slope of −3 and passing through the point (1, −4). Write the answer in slope-intercept form.

Solution:

$m = -3 \quad$ and $\quad (x_1, y_1) = (1, -4)$

$y - y_1 = m(x - x_1)$

$y - (-4) = -3(x - 1)$ Apply the point-slope formula.

$y + 4 = -3(x - 1)$ Simplify.

To write the answer in slope-intercept form, clear parentheses and solve for y.

$y + 4 = -3x + 3$ Clear parentheses.

$y = -3x - 1$ Solve for y. The answer is written in slope-intercept form. Notice that this is the same equation as in Example 4.

Skill Practice

7. Use the point-slope formula to write an equation for the line passing through the point (−2, −6) and having a slope of −5. Write the answer in slope-intercept form.

TIP: The solution to Example 5 can also be written in standard form, $Ax + By = C$.

$y = -3x - 1$ Slope-intercept form

$3x + y = -3x + 3x - 1$ Add $3x$ to both sides.

$3x + y = -1$ Standard form

In general, we will write the solution in slope-intercept form, because the slope and y-intercept can be easily identified.

Example 6 Finding an Equation of a Line Given Two Points

Find an equation of the line passing through the points (5, −1) and (3, 1). Write the answer in slope-intercept form.

Solution:

The slope formula can be used to compute the slope of the line between two points. Once the slope is known, the point-slope formula can be used to find an equation of the line.

First find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-1)}{3 - 5} = \frac{2}{-2} = -1$$

Answer

7. $y = -5x - 16$

TIP: In Example 6, once the slope is found from the two given points, we can alternatively use slope-intercept form to find an equation of the line. Substitute the coordinates from *either* known point for x and y. Using the point (3, 1), we have:

$$y = mx + b$$
$$1 = -1(3) + b$$
$$1 = -3 + b$$
$$4 = b$$

Therefore, $y = -x + 4$.

Next, apply the point-slope formula.

$y - y_1 = m(x - x_1)$

$y - 1 = -1(x - 3)$ Substitute $m = -1$ and use *either* point for (x_1, y_1). We will use (3, 1) for (x_1, y_1).

$y - 1 = -x + 3$ Clear parentheses.

$y = -x + 4$ Solve for y. The final answer is in slope-intercept form.

Skill Practice

8. Use the point-slope formula to write an equation of the line that passes through the points $(-5, 2)$ and $(-1, -1)$. Write the answer in slope-intercept form.

TIP: In Example 6, the point (3, 1) was used for (x_1, y_1) in the point-slope formula. However, either point could have been used. Using the point (5, –1) for (x_1, y_1) produces the same final equation:

$$y - (-1) = -1(x - 5)$$
$$y + 1 = -x + 5$$
$$y = -x + 4$$

Example 7 Finding an Equation of a Line Parallel to Another Line

Find an equation of the line passing through the point $(-2, -3)$ and parallel to the line $4x + y = 8$. Write the answer in slope-intercept form.

Solution:

To find an equation of a line, we must know a point on the line and the slope. The known point is $(-2, -3)$. Because the line is parallel to $4x + y = 8$, the two lines must have the same slope. Writing the equation $4x + y = 8$ in slope-intercept form, we have $y = -4x + 8$. Therefore, the slope of both lines must be -4.

Now find an equation of the line passing through $(-2, -3)$ having a slope of -4.

$y - y_1 = m(x - x_1)$ Apply the point-slope formula.

$y - (-3) = -4[x - (-2)]$ Substitute $m = -4$ and $(-2, -3)$ for (x_1, y_1).

$y + 3 = -4(x + 2)$

$y + 3 = -4x - 8$ Clear parentheses.

$y = -4x - 11$ Write the answer in slope-intercept form.

Skill Practice

9. Find an equation of a line containing $(4, -1)$ and parallel to $2x = y - 7$. Write the answer in slope-intercept form.

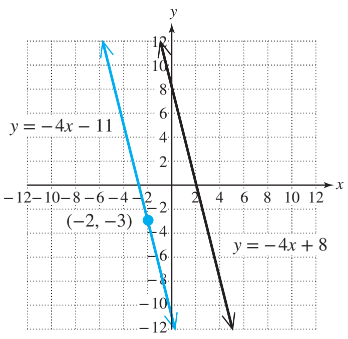


Figure 2-24

We can verify the answer to Example 7 by graphing both equations. We see that the line defined by $y = -4x - 11$ passes through the point $(-2, -3)$ and is parallel to the line $y = -4x + 8$. See Figure 2-24.

Answers

8. $y = -\frac{3}{4}x - \frac{7}{4}$ **9.** $y = 2x - 9$

Example 8 Finding an Equation of a Line Perpendicular to Another Line

Find an equation of the line passing through the point (4, 3) and perpendicular to the line $2x + 3y = 3$. Write the answer in slope-intercept form.

Solution:

The slope of the given line can be found from its slope-intercept form.

$$2x + 3y = 3$$

$$3y = -2x + 3 \qquad \text{Solve for } y.$$

$$\frac{3y}{3} = \frac{-2x}{3} + \frac{3}{3}$$

$$y = -\frac{2}{3}x + 1 \qquad \text{The slope is } -\frac{2}{3}.$$

The slope of a line *perpendicular* to this line must be the opposite of the reciprocal of $-\frac{2}{3}$; hence, $m = \frac{3}{2}$. Using $m = \frac{3}{2}$ and the known point (4, 3), we can apply the point-slope formula to find an equation of the line.

$$y - y_1 = m(x - x_1) \qquad \text{Apply the point-slope formula.}$$

$$y - 3 = \frac{3}{2}(x - 4) \qquad \text{Substitute } m = \tfrac{3}{2} \text{ and } (4, 3) \text{ for } (x_1, y_1).$$

$$y - 3 = \frac{3}{2}x - 6 \qquad \text{Clear parentheses.}$$

$$y = \frac{3}{2}x - 3 \qquad \text{Solve for } y.$$

Skill Practice

10. Find an equation of the line passing through the point $(1, -6)$ and perpendicular to the line $x + 2y = 8$. Write the answer in slope-intercept form.

Avoiding Mistakes

To check the result of Example 8, note that the product of slopes is -1, indicating that the lines are perpendicular.

$$-\frac{2}{3} \cdot \frac{3}{2} = -1$$

Furthermore, we can show that the line $y = \frac{3}{2}x - 3$ passes through the point (4, 3) by using substitution.

$$y = \frac{3}{2}x - 3$$

$$3 = \frac{3}{2}(4) - 3$$

$$3 = 6 - 3 \checkmark$$

3. Equations of a Line: A Summary

A linear equation can be written in several different forms, as summarized in Table 2-2.

Table 2-2

Form	Example	Comments
Standard Form $Ax + By = C$	$2x + 3y = 6$	A and B must not *both* be zero.
Horizontal Line $y = k$ (k is constant)	$y = 3$	The slope is zero, and the y-intercept is $(0, k)$.
Vertical Line $x = k$ (k is constant)	$x = -2$	The slope is undefined and the x-intercept is $(k, 0)$.
Slope-Intercept Form $y = mx + b$ Slope is m. y-Intercept is $(0, b)$.	$y = -2x + 5$ Slope $= -2$ y-Intercept is $(0, 5)$.	Solving a linear equation for y results in slope-intercept form. The coefficient of the x-term is the slope, and the constant defines the location of the y-intercept.
Point-Slope Formula $y - y_1 = m(x - x_1)$ Slope is m and (x_1, y_1) is a point on the line.	$m = -2$ $(x_1, y_1) = (3, 1)$ $y - 1 = -2(x - 3)$	This formula is typically used to build an equation of a line when a point on the line is known and the slope is known.

Answer

10. $y = 2x - 8$

Although it is important to understand and apply slope-intercept form and the point-slope formula, they are not necessarily applicable to all problems. Example 9 illustrates how a little ingenuity may lead to a simple solution.

Example 9 Finding an Equation of a Line

Find an equation of the line passing through the point $(-4, 1)$ and perpendicular to the x-axis.

Solution:

Any line perpendicular to the x-axis must be *vertical*. Recall that all vertical lines can be written in the form $x = k$, where k is constant. A quick sketch can help find the value of the constant (Figure 2-25).

Because the line must pass through a point whose x-coordinate is -4, the equation of the line is $x = -4$.

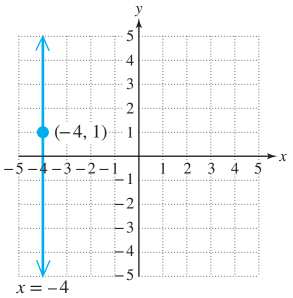


Figure 2-25

Skill Practice

11. Write an equation of the line through the point (20, 50) and having a slope of 0.

Answer

11. $y = 50$

Section 2.3 Activity

A.1. Given a line with slope m and y-intercept $(0, b)$, write the slope-intercept form of the line.

A.2. Consider two lines l_1 and l_2 represented by the following equations.

l_1: $2x - y = 4$ and l_2: $x + 2y = 4$

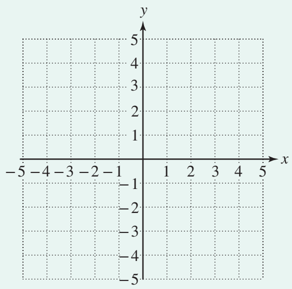
a. Write the slope-intercept form of l_1. Identify the slope and y-intercept.

b. Write the slope-intercept form of l_2. Identify the slope and y-intercept.

c. Based on the results of parts (a) and (b), are the lines parallel, perpendicular, or neither? Why?

d. Graph l_1 by graphing the y-intercept and using the slope to find a second point on the line.

e. On the same coordinate system, graph l_2 by using the slope and y-intercept. Does the graph support your answer to part (c)?

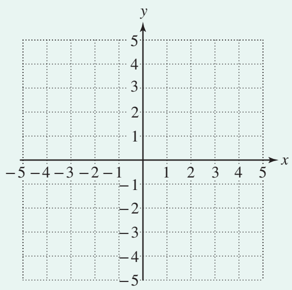


A.3. A line passes through the point $(3, -2)$ and has a slope of $-\frac{4}{3}$.

a. Use the slope-intercept form $y = mx + b$ and substitute $-\frac{4}{3}$ for m, 3 for x and -2 for y. Then solve the equation for b.

b. Using the values of m and b, write an equation of the line in slope-intercept form.

c. Graph the equation from part (b) to confirm that the line passes through $(3, -2)$ with slope $-\frac{4}{3}$.

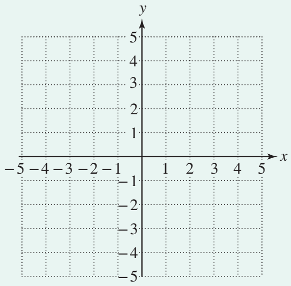


A.4. Write the point-slope formula for an equation of a line with slope m and known point on the line (x_1, y_1).

A.5. A line passes through the point $(3, -4)$ and has a slope of -2.

a. Label the point $(3, -4)$ as (x_1, y_1) and the slope as $m = -2$. Use the point-slope formula to write an equation of the line. Write the final answer in slope-intercept form.

b. Use the equation from part (a) to graph the line. Verify that the line passes through $(3, -4)$ and has a slope of -2.



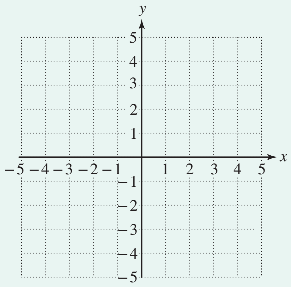
A.6. Write an equation of the line passing through $(-4, -3)$ and perpendicular to the line $4x + y = 1$ by following these steps.

a. Write the equation $4x + y = 1$ in slope-intercept form.

b. What is the slope of a line perpendicular to $4x + y = 1$?

c. Use the point-slope formula with the point $(-4, -3)$ and the slope from part (b) to write an equation of the line. Write the answer in slope-intercept form.

d. Graph the line $4x + y = 1$ and the line from part (c). Does the line from part (c) pass through the point $(-4, -3)$ as expected? Do the two lines appear to be perpendicular?



A.7. How would the procedure have changed in Exercise A.6 if the two lines were parallel rather than perpendicular?

A.8. Write an equation of the line passing through the points $(-2, -4)$ and $(2, 6)$ by following these steps.

a. Find the slope of the line between the two given points.

b. Apply the point-slope formula with the slope found in part (a) and the point $(-2, -4)$ to find an equation of the line through the points. Write the answer in slope-intercept form.

c. Apply the point-slope formula with the slope found in part (a) and the point $(2, 6)$ to find an equation of the line through the points. Write the answer in slope-intercept form.

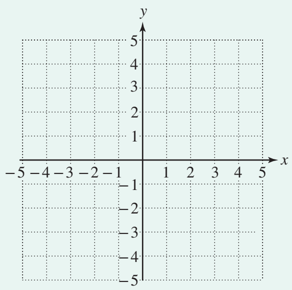
d. Are the equations in parts (b) and (c) the same? If not, revisit your work and look for errors.

A.9. **a.** Does an equation of the form $x = k$ represent a horizontal, vertical, or slanted line? What is the slope of the line?

b. Does an equation of the form $y = k$ represent a horizontal, vertical, or slanted line? What is the slope of the line?

A.10. **a.** Graph the line $x = -1$.

b. On the same coordinate system, graph the line perpendicular to $x = -1$ that passes through the point $(2, -3)$.



c. Write an equation of the line from part (b).

Section 2.3 Practice Exercises

Study Skills Exercise

Research shows that writing mathematically has significant advantages to learning. When you can communicate a mathematical concept in words, you are demonstrating a full understanding of that concept. You are also communicating your understanding and reasoning to yourself and others. This skill will help you move mathematical concepts into long-term memory.

- Suppose you are given two linear equations in slope-intercept form. Explain how you know if the equations of the lines are parallel, perpendicular, or neither. Describe how you might check that your answers are correct.

Prerequisite Review

For Exercises R.1–R.2, find the slope of the line containing the given points.

R.1. $(3, -4)$ and $(-1, 6)$

R.2. $(-1, -8)$ and $(4, 2)$

For Exercises R.3–R.6, the slope of a line is given.

a. Find the slope of a line parallel to the line with the given slope.
b. Find the slope of a line perpendicular to the line with the given slope, if possible.

R.3. $m = -\frac{5}{4}$

R.4. $m = \frac{4}{7}$

R.5. $m = 0$

R.6. $m = -1$

For Exercises R.7–R.8, solve the equation.

R.7. $-8 = -6 + b$

R.8. $11 = -3 + b$

For Exercises R.9–R.10, solve for the indicated variable.

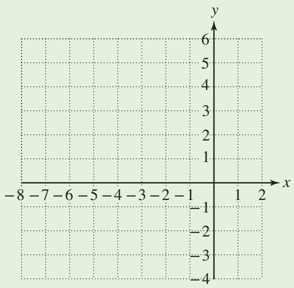
R.9. $3x - 7y = 14$ for y

R.10. $5x + 2y = 6$ for y

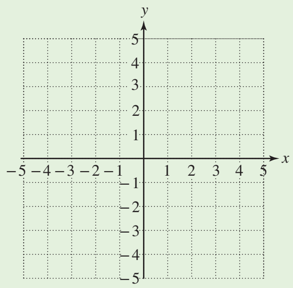
For Exercises R.11–R.12,

a. Determine the x- and y-intercepts.
b. Determine the slope.
c. Graph the equation.

R.11. $-2x + 5y = 15$



R.12. $4x + 3y = 6$



Vocabulary and Key Concepts

1. **a.** Consider a line with slope m and y-intercept $(0, b)$. The slope-intercept form of an equation of the line is ____________.

b. An equation of a line written in the form $Ax + By = C$, where A and B are not both zero, is said to be in ____________ form.

c. A line defined by an equation $y = k$, where k is a constant is a (horizontal/vertical) line.

d. A line defined by an equation $x = k$, where k is a constant is a (horizontal/vertical) line.

e. Given the slope-intercept form of an equation of a line, $y = mx + b$, the value of m is the __________ and b is the __________.

f. Given a point (x_1, y_1) on a line with slope m, the point-slope formula is given by __________.

2. Using slopes, how do you determine whether two lines are parallel?

3. Using slopes, how do you determine whether two lines are perpendicular?

4. Determine the slope and y-intercept of the line defined by the given equation.

a. $y = -4x + 3$ **b.** $y = 5 + 2x$

Concept 1: Slope-Intercept Form

For Exercises 5–18, determine the slope and the y-intercept of the line. **(See Example 1.)**

5. $y = 3x + 1$ **6.** $y = -2x - 7$ **7.** $y = -\frac{2}{3}x - 4$ **8.** $y = \frac{3}{7}x - 1$

9. $y = 2 + 3x$ **10.** $y = 5 - 7x$ **11.** $17x + y = 0$ **12.** $x + y = 0$

13. $18 = 2y$ **14.** $-7 = \frac{1}{2}y$ **15.** $8x + 12y = 9$ **16.** $-9x + 10y = -4$

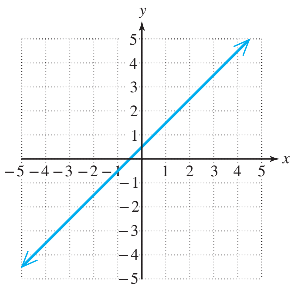
17. $y = 0.625x - 1.2$ **18.** $y = -2.5x + 1.8$

In Exercises 19–24, match the equation with the correct graph.

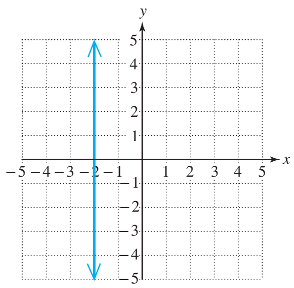
19. $y = \frac{3}{2}x - 2$ **20.** $y = -x + 3$ **21.** $y = \frac{13}{4}$

22. $y = x + \frac{1}{2}$ **23.** $x = -2$ **24.** $y = -\frac{1}{2}x + 2$

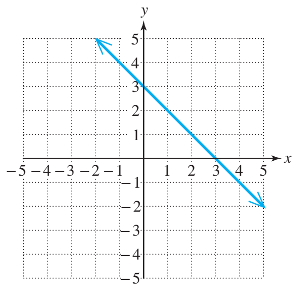
a.



b.



c.



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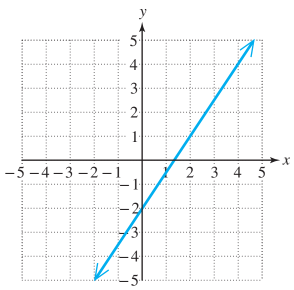

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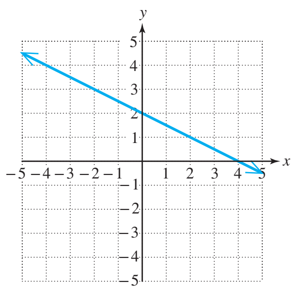

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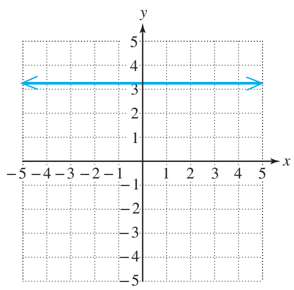
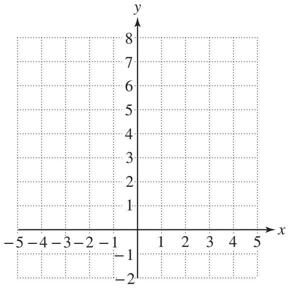
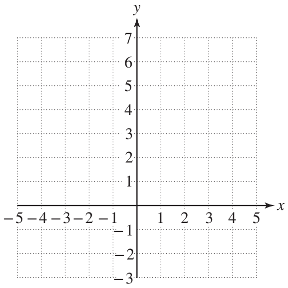


For Exercises 25–30, write the equations in slope-intercept form (if possible). Then graph each line, using the slope and y-intercept. **(See Example 2.)**

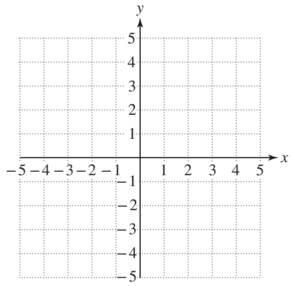
25. $y - 2 = 4x$



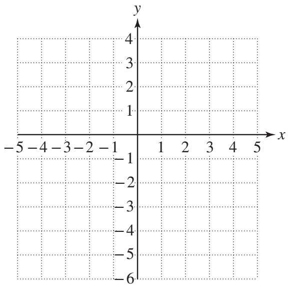
26. $3x = 5 - y$



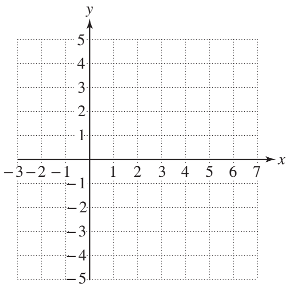
27. $3x + 2y = 6$



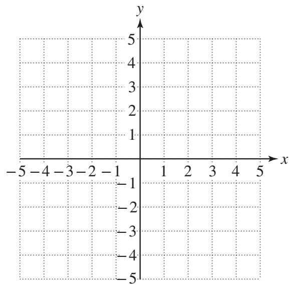
28. $x - 2y = 8$



29. $2x - 5y = 0$



30. $3x - y = 0$



31. Given the standard form of a linear equation $Ax + By = C$, $B \neq 0$, solve for y and write the equation in slope-intercept form. What is the slope of the line? What is the y-intercept?

32. Use the result of Exercise 31 to determine the slope and y-intercept of the line $3x + 7y = 9$.

For Exercises 33–38, the equations of two lines are given. Determine if the lines are parallel, perpendicular, or neither. **(See Example 3.)**

33. $-3y = 5x - 1$
$6x = 10y - 12$

34. $x = 6y - 3$
$3x + \frac{1}{2}y = 0$

35. $3x - 4y = 12$
$\frac{1}{2}x - \frac{2}{3}y = 1$

36. $4.8x = 1.2y + 3.6$
$y - 1 = 4x$

37. $3y = 5x + 6$
$5x + 3y = 9$

38. $-y = 3x - 2$
$-6x + 2y = 6$

For Exercises 39–44, use the slope-intercept form of a line to find an equation of the line having the given slope and passing through the given point. **(See Example 4.)**

39. $m = 3$, $(0, 5)$

40. $m = -4$, $(0, 3)$

41. $m = 2$, $(4, -3)$

42. $m = 3$, $(-1, 5)$

43. $m = -\frac{4}{5}$, $(10, 0)$

44. $m = -\frac{2}{7}$, $(-3, 1)$

Concept 2: The Point-Slope Formula

For Exercises 45–74, write an equation of the line satisfying the given conditions. Write the answer in slope-intercept form or standard form.

45. The line passes through the point $(0, -2)$ and has a slope of 3.

46. The line passes through the point $(0, 5)$ and has a slope of $-\frac{1}{2}$.

47. The line passes through the point $(2, 7)$ and has a slope of 2. **(See Example 5.)**

48. The line passes through the point $(3, 10)$ and has a slope of -2.

49. The line passes through the point $(-2, -5)$ and has a slope of -3.

50. The line passes through the point $(-1, -6)$ and has a slope of 4.

51. The line passes through the point $(6, -3)$ and has a slope of $-\frac{4}{5}$.

52. The line passes through the point $(7, -2)$ and has a slope of $\frac{7}{2}$.

53. The line passes through $(0, 4)$ and $(3, 0)$. **(See Example 6.)**

54. The line passes through $(1, 1)$ and $(3, 7)$.

55. The line passes through $(6, 12)$ and $(4, 10)$.

56. The line passes through $(-2, -1)$ and $(3, -4)$.

57. The line passes through $(-5, 2)$ and $(-1, 2)$.

58. The line passes through $(-4, -1)$ and $(2, -1)$.

59. The line contains the point $(3, 2)$ and is parallel to a line with a slope of $-\frac{3}{4}$. **(See Example 7.)**

60. The line contains the point $(-1, 4)$ and is parallel to a line with a slope of $\frac{1}{2}$.

61. The line contains the point $(3, 2)$ and is perpendicular to a line with a slope of $-\frac{3}{4}$. **(See Example 8.)**

62. The line contains the point $(-2, 5)$ and is perpendicular to a line with a slope of $\frac{1}{2}$.

63. The line contains the point $(2, -5)$ and is parallel to $3x - 4y = -7$.

64. The line contains the point $(-6, -1)$ and is parallel to $2x + 3y = -12$.

65. The line contains the point $(-8, -1)$ and is perpendicular to $-15x + 3y = 9$.

66. The line contains the point $(4, -2)$ and is perpendicular to $4x + 3y = -6$.

67. The line contains the point $(4, 0)$ and is parallel to the line defined by $3x = 2y$.

68. The line contains the point $(-3, 0)$ and is parallel to the line defined by $-5x = 6y$.

69. The line is perpendicular to the line defined by $3y + 2x = 21$ and passes through the point $(2, 4)$.

70. The line is perpendicular to $7y - x = -21$ and passes through the point $(-14, 8)$.

71. The line is perpendicular to $\frac{1}{2}y = x$ and passes through $(-3, 5)$.

72. The line is perpendicular to $-\frac{1}{4}y = x$ and passes through $(-1, -5)$.

73. The line is parallel to the line $3x + y = 7$ and passes through the origin.

74. The line is parallel to the line $-2x + y = -5$ and passes through the origin.

Concept 3: Equations of a Line: A Summary

For Exercises 75–82, write an equation of the line satisfying the given conditions.

75. The line passes through $(2, -3)$ and has a zero slope.

76. The line contains the point $(\frac{5}{2}, 0)$ and has an undefined slope.

77. The line contains the point $(2, -3)$ and has an undefined slope. **(See Example 9.)**

78. The line contains the point $(\frac{5}{2}, 0)$ and has a zero slope.

79. The line is parallel to the x-axis and passes through (4, 5).

80. The line is perpendicular to the x-axis and passes through (4, 5).

81. The line is parallel to the line $x = 4$ and passes through (5, 1).

82. The line is parallel to the line $y = -2$ and passes through $(-3, 4)$.

Technology Connections

For Exercises 83–86, use a graphing calculator to graph the lines on the same viewing window. Then explain how the lines are related.

83. $y_1 = \frac{1}{2}x + 4$

$y_2 = \frac{1}{2}x - 2$

84. $y_1 = -\frac{1}{3}x + 5$

$y_2 = -\frac{1}{3}x - 3$

85. $y_1 = x - 2$

$y_2 = 2x - 2$

$y_3 = 3x - 2$

86. $y_1 = -2x + 1$

$y_2 = -3x + 1$

$y_3 = -4x + 1$

For Exercises 87–88, use a graphing calculator to graph the lines on a square viewing window. Then explain how the lines are related.

87. $y_1 = 4x - 1$

$y_2 = -\frac{1}{4}x - 1$

88. $y_1 = \frac{1}{2}x - 3$

$y_2 = -2x - 3$

Expanding Your Skills

89. Is the equation $x = -2$ in slope-intercept form? Identify the slope and y-intercept.

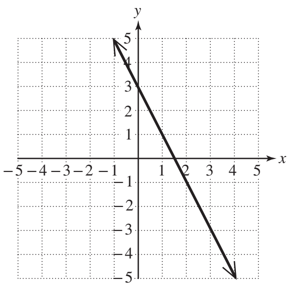
90. Is the equation $x = 1$ in slope-intercept form? Identify the slope and y-intercept.

91. Is the equation $y = 3$ in slope-intercept form? Identify the slope and y-intercept.

92. Is the equation $y = -5$ in slope-intercept form? Identify the slope and y-intercept.

For Exercises 93–96, write an equation of the given line. Write the answer in slope-intercept form, if possible.

93.



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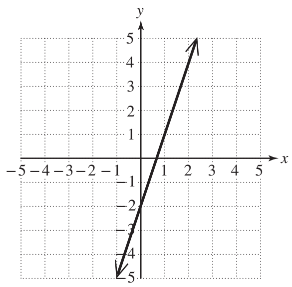

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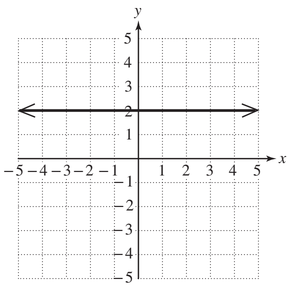

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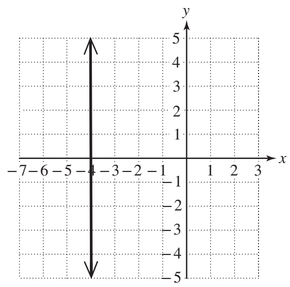


Problem Recognition Exercises

Characteristics of Linear Equations

For Exercises 1–20, choose the equation(s) from column B whose graph satisfies the condition described in column A. Give all possible answers.

Column A

1. Line whose slope is positive.

2. Line whose slope is negative.

3. Line that passes through the origin.

4. Line that contains the point (2, 0).

5. Line whose y-intercept is $(0, -3)$.

6. Line whose y-intercept is (0, 0).

7. Line whose slope is $-\frac{1}{3}$.

8. Line whose slope is $\frac{1}{2}$.

9. Line whose slope is 0.

10. Line whose slope is undefined.

11. Line that is parallel to the line with equation $x + 3y = 6$.

12. Line perpendicular to the line with equation $x - 4y = -4$.

13. Line that is vertical.

14. Line that is horizontal.

15. Line whose x-intercept is (12, 0).

16. Line whose x-intercept is $\left(\frac{1}{5}, 0\right)$.

17. Line that has no x-intercept.

18. Line that is perpendicular to the x-axis.

19. Line with a negative slope and positive y-intercept.

20. Line with a positive slope and negative y-intercept.

Column B

a. $y = -4x$

b. $2x - 4y = 4$

c. $y = -\frac{1}{3}x - 3$

d. $3x + 5y = 10$

e. $3y = -9$

f. $y = 5x - 1$

g. $4x + 1 = 9$

h. $x + 3y = 12$

Applications of Linear Equations and Modeling

Section 2.4

Concepts

1. Writing a Linear Model
2. Interpreting a Linear Model
3. Finding a Linear Model from Observed Data Points

1. Writing a Linear Model

A **mathematical model** is a formula or equation that represents a relationship between two or more variables in a real-world application. Algebra (or some other field of mathematics) can then be used to solve the problem. The use of mathematical models is found throughout the physical and biological sciences, sports, medicine, economics, business, and many other fields.

For an equation written in slope-intercept form, $y = mx + b$, the term mx is called the *variable term*. The value of this term changes with different values of x. The term b is called the *constant term* and it remains unchanged regardless of the value of x. The slope of the line, m, is called the *rate of change*. A linear equation can be created if the rate of change and the constant are known.

Example 1 Writing a Linear Model

Buffalo, New York, had 2 ft (24 in.) of snow on the ground before a snowstorm. During the storm, snow fell at an average rate of $\frac{5}{8}$ in./hr.

a. Write a linear equation to compute the total snow depth y after x hr of the storm.

b. Graph the equation.

c. Use the equation to compute the depth of snow after 8 hr.

d. If the snow depth was 31.5 in. at the end of the storm, determine how long the storm lasted.

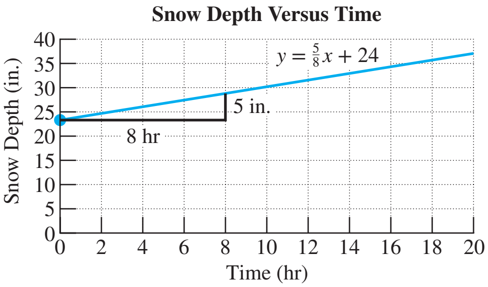
Solution:

a. The constant or base amount of snow before the storm began is 24 in. The rate of change is given by $\frac{5}{8}$ in. of snow per hour. If m is replaced by $\frac{5}{8}$ and b is replaced by 24, we have the linear equation

$$y = mx + b$$

$$y = \frac{5}{8}x + 24$$

b. The equation is in slope-intercept form, and the corresponding graph is shown in the figure.



c. $y = \frac{5}{8}x + 24$

$y = \frac{5}{8}(8) + 24$ Substitute $x = 8$.

$y = 5 + 24$ Solve for y.

$y = 29$ in.

The snow depth was 29 in. after 8 hr. The corresponding ordered pair is (8, 29) and can be confirmed from the graph.

d. $y = \frac{5}{8}x + 24$

$31.5 = \frac{5}{8}x + 24$ Substitute $y = 31.5$.

$8(31.5) = 8\left(\frac{5}{8}x + 24\right)$ Multiply by 8 to clear fractions.

$252 = 5x + 192$ Clear parentheses.

$60 = 5x$ Solve for x.

$12 = x$

The storm lasted 12 hr. The corresponding ordered pair is (12, 31.5) and can be confirmed from the graph.

Avoiding Mistakes

A mathematical model *may be reasonable only* for specific values of the input variable. In Example 1, the model $y = \frac{5}{8}x + 24$ is reasonable for $0 \le x \le 12$. For x values less than 0, the snow storm hadn't started. For x values greater than 12, the storm had ended. We say that "model breakdown" occurs for values of x outside the reasonable range.

Skill Practice When Joe graduated from college, he had \$1000 in his savings account. When he began working, he decided he would add \$120 per month to his savings account.

1. Write a linear equation to compute the amount of money y in Joe's account after x months of saving.
2. Use the equation to compute the amount of money in Joe's account after 6 months.
3. Joe needs \$3160 for a down payment for a car. How long will it take for Joe's account to reach this amount?

2. Interpreting a Linear Model

Example 2 Interpreting a Linear Model

In 1938, President Franklin D. Roosevelt signed a bill enacting the Fair Labor Standards Act of 1938 (FLSA). In its final form, the act banned oppressive child labor and set the minimum hourly wage at 25 cents and the maximum workweek at 44 hr. Over the years, the minimum hourly wage has been increased by the government to meet the rising cost of living.

The minimum hourly wage y (in dollars per hour) in the United States between 1970 and 2010 can be approximated by the equation

$$y = 0.14x + 1.60 \qquad x \geq 0$$

where x represents the number of years since 1970 ($x = 0$ corresponds to 1970, $x = 1$ corresponds to 1971, and so on) (Figure 2-26).

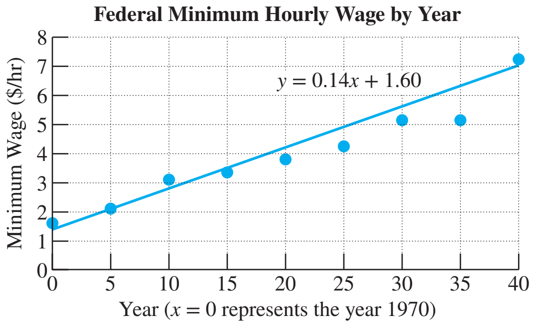


Figure 2-26

a. Determine the slope of the line and interpret the meaning of the slope as a rate of change.

b. Find the y-intercept of the line and interpret the meaning of the y-intercept in the context of this problem.

c. Use the linear equation to approximate the minimum wage in 1985.

Answers

1. $y = 120x + 1000$
2. \$1720 **3.** 18 months

Solution:

a. The equation $y = 0.14x + 1.60$ is written in slope-intercept form. The slope is 0.14 and indicates that minimum hourly wage rose an average of \$0.14 per year since 1970.

b. The y-intercept is (0, 1.60). The y-intercept indicates that the minimum wage in the year 1970 ($x = 0$) was approximately \$1.60 per hour.

c. The year 1985 is 15 years after the year 1970. Substitute $x = 15$ into the linear equation.

$$y = 0.14x + 1.60$$

$$y = 0.14(15) + 1.60 \qquad \text{Substitute } x = 15.$$

$$y = 2.1 + 1.60$$

$$y = 3.70$$

According to the linear model, the minimum wage in 1985 was approximately \$3.70 per hour. (The actual minimum wage in 1985 was \$3.35 per hour.)

Skill Practice One cell-phone plan charges a monthly fee plus a charge per minute for the number of minutes used beyond 400 min. If a customer exceeds the 400-min cap, then the monthly fee, y (in dollars), is given by $y = 0.40x + 49.99$, where x is the number of minutes used beyond 400 min.

4. What is the slope? Interpret its meaning in the context of the problem.

5. What is the y-intercept?

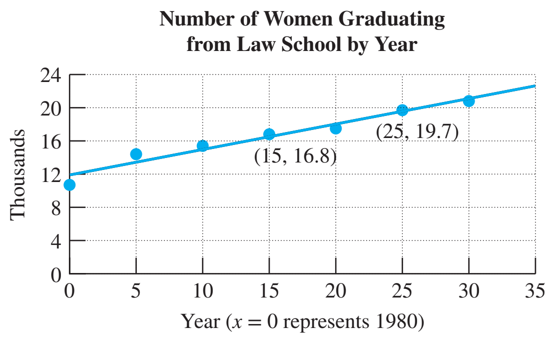
6. Use the equation to determine the cost of using 445 min in this plan.

3. Finding a Linear Model from Observed Data Points

Graphing a set of data points offers a visual method to determine whether the points follow a linear pattern. When two variables are related, it is often desirable to find a mathematical equation (or *model*) to describe the relationship.

Example 3 Writing a Linear Model from Observed Data

Figure 2-27 represents the number of women, y (in thousands), who graduated from law school in the United States by year. Let x represent the number of years since 1980.



Source: U.S. National Center for Education Statistics

Figure 2-27

Answers

4. The slope is 0.40. The customer is charged \$0.40 for each minute used beyond 400 min.

5. The y-intercept is (0, 49.99). This means that if 0 min is used beyond 400 min, the customer is charged \$49.99.

6. \$67.99

a. Use the given ordered pairs to find a linear equation to model the number of women graduating from law school by year.

b. Determine the slope of the line and interpret the meaning of the slope as a rate of change.

c. Use the linear equation to estimate the number of women who graduated from law school in the year 2015.

d. Would it be practical to use the linear model to predict the number of women who would graduate from law school for the year 2050?

Solution:

a. The slope formula can be used to compute the slope of the line between the points.

$\underset{(x_1, y_1)}{(15, 16.8)}$ and $\underset{(x_2, y_2)}{(25, 19.7)}$ — Label the points.

$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{19.7 - 16.8}{25 - 15} = \dfrac{2.9}{10} = 0.29$ — Apply the slope formula.

$y - y_1 = m(x - x_1)$ — Apply the point-slope formula.

$y - 16.8 = 0.29(x - 15)$ — Use the point (15, 16.8) and $m = 0.29$.

$y - 16.8 = 0.29x - 4.35$ — Clear parentheses.

$y = 0.29x + 12.45$ — Solve for y. The equation is in slope-intercept form.

b. The slope is 0.29. This means that the number of women who graduate from law school increased at a rate of 0.29 thousand (290) per year during this time period.

c. The year 2015 is 35 years after the year 1980. Therefore, substitute $x = 35$.

$y = 0.29(35) + 12.45$

$= 22.6$

For the year 2015, we estimate that approximately 22,600 women graduated from law school.

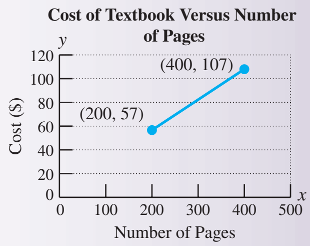
d. It would not be practical to use this equation to predict the number of women who will graduate from law school in the year 2050. It is unreasonable to assume that the linear trend will continue this far beyond the observed data.

Skill Practice The figure shows data relating the cost of college textbooks (in dollars) to the number of pages in the book. Let y represent the cost of the book, and let x represent the number of pages.

7. Use the ordered pairs given in the figure to write a linear equation to model the cost of textbooks (in dollars) versus the number of pages.

8. What is the slope of this line and what does it mean in the context of this problem?

9. Use the equation to predict the cost of a textbook that has 360 pages.



Answers

7. $y = 0.25x + 7$

8. $m = 0.25$; The cost of the textbook goes up by \$0.25 per page.

9. \$97

Section 2.4 Activity

A.1. A credit card offers travel points and gives 5000 points initially for signing up for the card. Thereafter, the cardholder is awarded 2 points for every dollar charged for the first month.

a. Write an equation that expresses the total number of points y during the first month if x dollars is charged to the card.

b. How many points would accumulate in the first month if \$104 is charged?

c. How much money must be charged to get 8000 points by the end of the first month?

A.2. The equation $y = \frac{1}{20}x + 4$ represents the expected number of teachers y in an elementary school based on the number of students x attending the school.

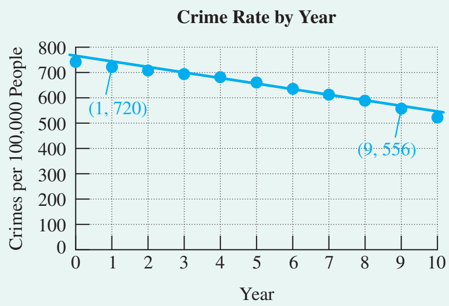
a. If 400 students attend a school, how many teachers would be expected?

b. If 420 students attend a school, how many teachers would be expected?

c. How many additional teachers would be expected for a school with 420 students versus a school with 400 students?

d. Identify the slope of the line and interpret its meaning in context.

A.3. The graph shows the crime rate (per 100,000 people) for reported crimes in a large city versus the time in years. For example, a crime rate of 556 means that there were approximately 556 crimes per 100,000 people in the city, or approximately 0.556%. From the graph, crime rate appears to follow a downward linear trend. The city believes that this is due, in part, to economic growth in the city, after-school programs for kids, and community forums in which police hear public concerns.



a. Interpret the ordered pair (1, 720).

b. Use the ordered pairs (1, 720) and (9, 556) to find the slope of the line shown. What does the slope mean in the context of this problem?

c. Use the slope from part (b) and one of the given points to find an equation of the line shown. Write the answer in slope-intercept form.

d. What is the y-intercept, and what does it mean in the context of this problem?

e. Use the equation from part (d) to approximate the crime rate in year 7.

Section 2.4 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, find the slope and y-intercept.

R.1. $2x - 9y = 18$

R.2. $-5x + y = 8$

For Exercises R.3–R.4, find the slope of the line containing the given points.

R.3. $(-3, 6)$ and $(-4, -2)$

R.4. $(-1, 8)$ and $(-5, 3)$

For Exercises R.5–R.8, find an equation of the line given the following information. Write the answer in slope-intercept form.

R.5. The line passes through the point $(3, -5)$ with slope $-\frac{1}{2}$.

R.6. The line passes through the point $(4, 1)$ with slope $\frac{3}{4}$.

R.7. The line passes through the points $(-4, 1)$ and $(-3, -6)$.

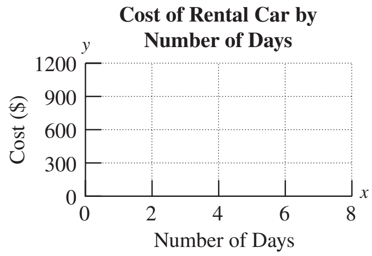
R.8. The line passes through the points $(2, -8)$ and $(-4, -3)$.

Concept 1: Writing a Linear Model

1. A luxury car rental company charges \$120 per day in addition to a flat fee of \$65 for insurance. **(See Example 1.)**

a. Write an equation that represents the cost y (in dollars) to rent the car for x days.

b. Graph the equation.



c. What is the y-intercept and what does it mean in the context of this problem?

d. Use the model from part (a) to determine the cost of driving the rental car for 2 days, 5 days, and 7 days.

e. If the company offers a weekly rate of \$799, is this a better deal than paying by the day for 7 days?

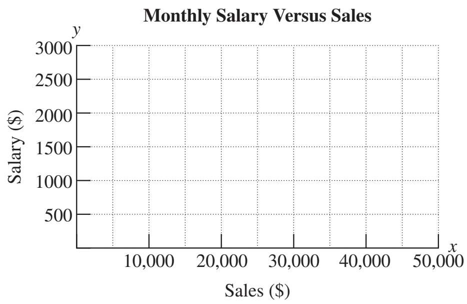
f. Find the total cost of driving the car for 4 days if the sales tax is 6%.

g. Is it reasonable to use negative values of x in the equation from part (a)? Why or why not?

2. Alex is a sales representative and earns a base salary of \$1000 per month plus a 4% commission on his sales for the month.

a. Write a linear equation that expresses Alex's monthly salary y in terms of his sales x.

b. Graph the equation.



c. What is the y-intercept and what does it represent in the context of this problem?

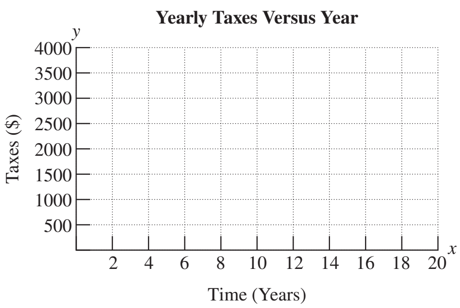
d. What is the slope of the line and what does it represent in the context of this problem?

e. How much will Alex make if his sales for a given month are \$30,000?

3. Ava recently purchased a home in Crescent Beach, Florida. Her annual property tax was initially \$2742, and Ava estimates that her taxes will increase at a rate of \$52 per year.

a. Write an equation to compute Ava's yearly property taxes. Let y be the amount she pays in taxes, and let x be the time in years after purchasing the house.

b. Graph the line.




Amanda Clement/Getty Images

c. What is the slope of this line? What does the slope of the line represent in the context of this problem?

d. What is the y-intercept? What does the y-intercept represent in the context of this problem?

e. What will Ava's yearly property tax be in 10 years? In 15 years?

4. Millage rate is the amount per \$1000 that is often used to calculate property tax. For example, a home with an \$80,000 taxable value in a municipality with a 19 mil tax rate would require $(0.019)(\$80{,}000) = \$1520$ in property taxes.

a. In one county, homeowners pay a flat tax of \$156 plus a rate of 19 mil on the taxable value of a home. Write a linear equation that represents the total property tax y (in \$) for a home with a taxable value of x dollars.

b. Determine the amount of property tax on a home with a taxable value of \$90,000.

c. If the property tax on a home is \$2436, determine the taxable value of the home.

Concept 2: Interpreting a Linear Model

5. Sound travels at approximately one-fifth of a mile per second. Therefore, the difference in time, x (in seconds), between seeing lightning and hearing thunder can be used to estimate the distance y (in miles) between a storm and an observer. The distance of the storm can be approximated by the equation $y = 0.2x$, where $x \geq 0$. **(See Example 2.)**


View Stock/age fotostock

a. Use the linear model to determine the distance between a storm and an observer for the following times between seeing lightning and hearing thunder: 4 sec, 12 sec, and 16 sec.

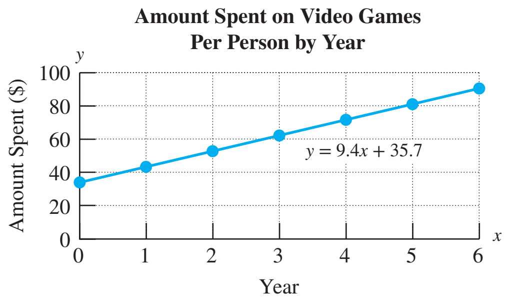
b. If a storm is 4.2 mi away, how many seconds will pass between seeing lightning and hearing thunder?

6. The force y (in pounds) required to stretch a particular spring x in. beyond its rest (or "equilibrium") position is given by the equation $y = 2.5x$, where $0 \leq x \leq 20$.

a. Use the equation to determine the amount of force necessary to stretch the spring 6 in. from its rest position. How much force is necessary to stretch the spring twice as far?

b. If 45 lb of force is exerted on the spring, how far will the spring be stretched?

7. The equation $y = 9.4x + 35.7$ represents the average amount y (in \$) spent on video games per person in the United States where x represents the number of years since the study began.



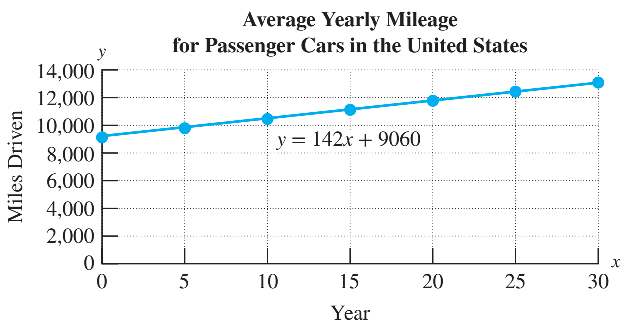
a. Use the equation to approximate the average amount spent per person in year 4.

b. Use the equation to approximate the average amount spent per person in year 2 and compare it with the actual amount spent of \$55.80.

c. What is the slope of the line and what does it mean in the context of this problem?

d. What is the y-intercept and what does it mean in the context of this problem?

8. Let y represent the average number of miles driven per year for passenger cars in the United States and let x represent the number of years since the study began. The average yearly mileage for passenger cars can be approximated by the equation $y = 142x + 9060$, where $x \geq 0$.



a. Use the linear equation to approximate the average yearly mileage for passenger cars in the United States in year 25.

b. Use the linear equation to approximate the average mileage for year 5, and compare it to the actual value of 9700 mi.

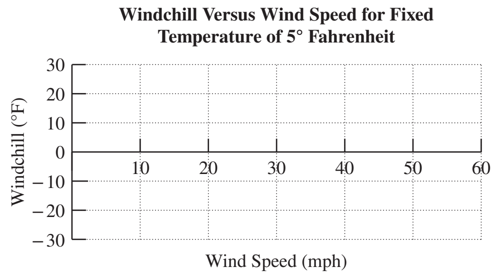
c. What is the slope of the line and what does it mean in the context of this problem?

d. What is the y-intercept and what does it mean in the context of this problem?

Concept 3: Finding a Linear Model from Observed Data Points

9. Windchill is the apparent temperature felt on exposed skin. This is a relationship between air temperature and wind speed. At a fixed air temperature of 5°F, windchill temperature is approximately linear for speeds between 10 mph and 110 mph. At 20 mph the windchill is −15°F and at 50 mph it is −24°F. **(See Example 3.)**

a. Make a graph with wind speed on the x-axis and windchill on the y-axis. Plot the points $(20, -15)$ and $(50, -24)$, and draw the line through the points.



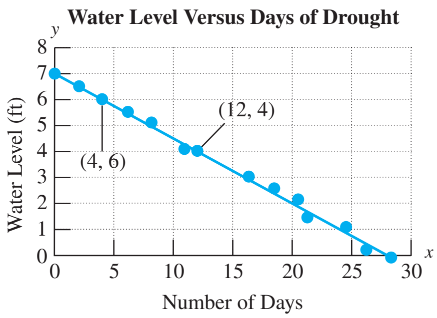
b. Write an equation of the line through the given points. Write the equation in slope-intercept form.

c. Use the equation from part (b) to estimate the windchill for a wind speed of 40 mph.

d. Use the equation from part (b) to estimate the windchill for a wind speed of 46 mph.

e. What is the slope of the line and what does it mean in the context of this problem?

10. During a drought, the average water level in a retention pond decreased linearly with time.



a. Let y represent the water level (in feet) and x represent the number of days since the beginning of the drought. Use the ordered pairs given to write a linear equation to estimate the water level based on the number of days since the drought began.

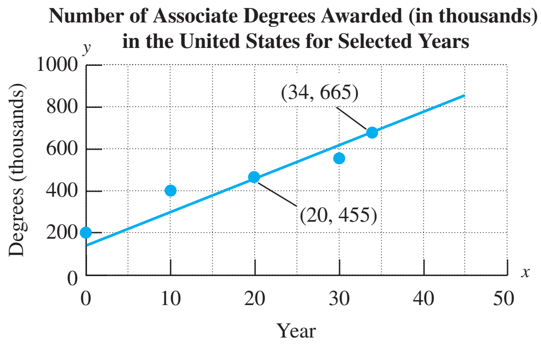
b. Use the linear equation to approximate the water level after 15 days.

c. Use the linear equation to approximate the water level after 25 days.

d. What is the slope of the line and what does it mean in the context of this problem?

e. Interpret the meaning of the x-intercept. Do you think that this linear trend will continue indefinitely? Explain your answer.

11. The graph displays the number of associate degrees conferred in the United States at the end of selected academic years. The variable x represents the number of years since the study began, and the variable y represents the number of associate degrees in thousands.

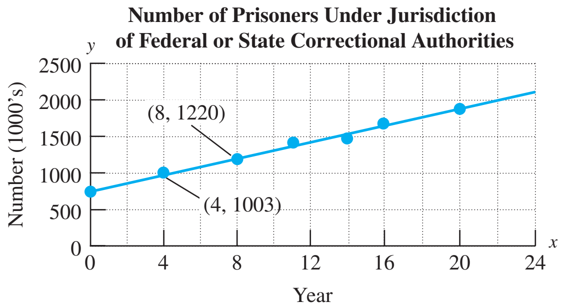


a. Use the points (20, 455) and (34, 665) to create a linear model of the data.

b. What does the slope mean in the context of this problem?

c. If this linear trend continues, estimate the number of associate degrees that will be conferred in the United States for year 45.

12. The number of prisoners y (in 1000's) in federal or state correctional facilities is shown in the graph for consecutive years x. (*Source:* U.S. Department of Justice)



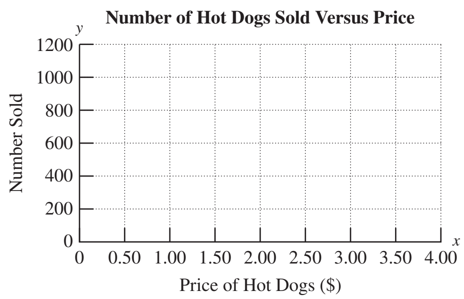
a. Use the given points to create a linear model of the data.

b. What does the slope mean in the context of this problem?

c. Use the equation in part (a) to estimate the number of prisoners in federal or state correctional facilities for year 25.

13. At a concession stand at a high school football game, the owner notices that the relationship between the price of a hot dog and the number of hot dogs sold is linear. If the price is \$2.50 per hot dog, then approximately 650 are sold each night. If the price is raised to \$3.50, then the number sold drops to 475 per night.

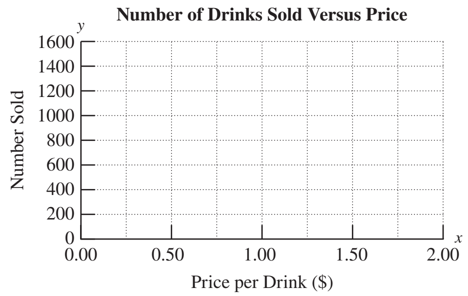
a. Make a graph with the price of hot dogs on the x-axis and the number of hot dogs sold on the y-axis. Use the points (2.50, 650) and (3.50, 475). Then graph the line through the points with $x \geq 0$.



b. Find an equation of the line through the points. Write the equation in slope-intercept form.

c. Use the equation from part (b) to predict the number of hot dogs that would sell if the price were raised to \$4.00. Round to the nearest whole unit.

14. Sales at a concession stand indicate that the relationship between the price of a drink and the number of drinks sold is linear. If drinks are sold at \$1.00 each, then approximately 1020 are sold each night. If the price is raised to \$1.50, then the number sold drops to 820 per night.



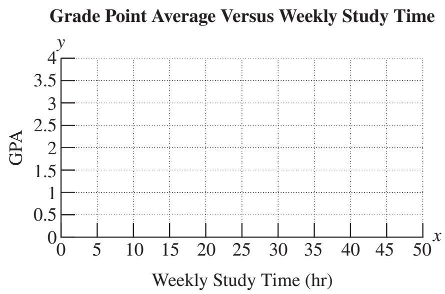
a. Make a graph with the price of drinks on the x-axis and the number of drinks sold on the y-axis. Graph the points (1.00, 1020) and (1.50, 820). Then graph the line through the points with $x \geq 0$.

b. Find an equation of the line through the points. Write the equation in slope-intercept form.

c. Use the equation from part (b) to predict the number of drinks that would sell if the price were \$2.00 per drink.

15. In order to advise students properly, a college advisor is interested in the relationship between the number of hours a student studies in an average week and the student's GPA. The data are shown in the table.

a. Let x represent study time and let y represent GPA. Graph the points.



Student	Study Time (in hours)	GPA
1	15	2.5
2	38	3.9
3	10	2.1
4	24	2.8
5	35	3.3
6	15	2.7
7	45	4.0
8	28	3.1
9	35	3.4
10	10	2.2
11	6	1.8

b. Does there appear to be a linear trend?

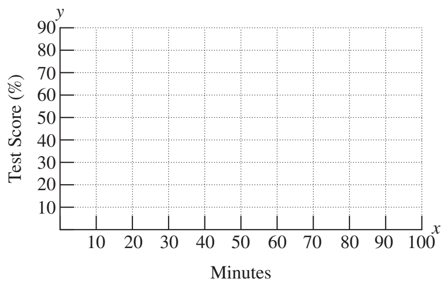
c. Use the data points (28, 3.1) and (10, 2.2) to find an equation of the line through these points.

d. Use the equation in part (c) to estimate the GPA of a student who studies for 30 hr a week.

e. Why would the linear model found in part (c) not be realistic for a student who studies more than 46 hr per week?

16. Loraine is enrolled in an algebra class that meets 5 days per week. Her instructor gives a test every Friday. Loraine has a study plan and keeps a portfolio with notes, homework, test corrections, and vocabulary. She also records the amount of time per day that she studies and does homework. The following data represent the amount of time she studied per day and her weekly test grades.

a. Graph the points on a rectangular coordinate system. Do the data points appear to follow a linear trend?



Time Studied per Day (min) x	Weekly Test Grade (percent) y
60	69
70	74
80	79
90	84
100	89

b. Find a linear equation that relates Loraine's weekly test score y to the amount of time she studied per day x. (*Hint:* Pick two ordered pairs from the observed data, and find an equation of the line through the points.)

c. How many minutes should Loraine study per day in order to score at least 90% on her weekly examination? Would the equation used to determine the time Loraine needs to study to get 90% work for other students? Why or why not?

d. If Loraine is only able to spend $\frac{1}{2}$ hr/day studying her math, predict her test score for that week.

Technology Connections

17. Use a *Table* feature to confirm your answers to Exercise 5(a).

18. Use a *Table* feature to confirm your answers to Exercise 6(a).

19. Graph the equation $y = -175x + 1087.5$ on the viewing window $0 \le x \le 5$ and $0 \le y \le 1200$. Use the *Value* feature to support your answer to Exercise 13 by showing that the line passes through the points (2.5, 650) and (3.5, 475).

20. Graph the equation $y = -400x + 1420$ on the viewing window $0 \le x \le 3$ and $0 \le y \le 1600$. Use the *Value* feature to support your answer to Exercise 14 by showing that the line passes through the points (1, 1020) and (1.5, 820).

Expanding Your Skills

Points are *collinear* if they lie on the same line. For Exercises 21–24, use the slope formula to determine if the points are collinear. (*Hint:* Three points are collinear if the slope calculated using one pair of points is equal to the slope calculated using a different pair of points.)

21. (3, −4), (0, −5), (9, −2)

22. (4, 3), (−4, −1), (2, 2)

23. (0, 2), (−2, 12), (−1, 6)

24. (−2, −2), (0, −3), (−4, −1)

Introduction to Relations

Section 2.5

Concepts

1. Definition of a Relation
2. Domain and Range of a Relation
3. Applications Involving Relations

1. Definition of a Relation

In many naturally occurring phenomena, two variables may be linked by some other type of relationship. Table 2-3 shows a correspondence between the length of a woman's femur and her height. (The femur is the large bone in the thigh attached to the knee and hip.)

Table 2-3

Length of Femur (cm) x	Height (in.) y	Ordered Pair
45.5	65.5	→ (45.5, 65.5)
48.2	68.0	→ (48.2, 68.0)
41.8	62.2	→ (41.8, 62.2)
46.0	66.0	→ (46.0, 66.0)
50.4	70.0	→ (50.4, 70.0)

Each data point from Table 2-3 may be represented as an ordered pair. In this case, the first value represents the length of a woman's femur and the second, the woman's height. The set of ordered pairs {(45.5, 65.5), (48.2, 68.0), (41.8, 62.2), (46.0, 66.0), (50.4, 70.0)} defines a relation between femur length and height.

2. Domain and Range of a Relation

Definition of Relation in *x* and *y*

A set of ordered pairs (x, y) is called a **relation in *x* and *y***. Furthermore,

- The set of first components in the ordered pairs is called the **domain of the relation**.
- The set of second components in the ordered pairs is called the **range of the relation**.

Example 1 Finding the Domain and Range of a Relation

Find the domain and range of the relation linking the length of a woman's femur to her height {(45.5, 65.5), (48.2, 68.0), (41.8, 62.2), (46.0, 66.0), (50.4, 70.0)}.

Solution:

Domain:	{45.5, 48.2, 41.8, 46.0, 50.4}	Set of first components
Range:	{65.5, 68.0, 62.2, 66.0, 70.0}	Set of second components

Skill Practice Find the domain and range of the relation.

1. $\left\{(0, 0), (-8, 4), \left(\frac{1}{2}, 1\right), (-3, 4), (-8, 0)\right\}$

The x and y components that constitute the ordered pairs in a relation do not need to be numerical. This is demonstrated in Example 2.

Answer

1. Domain: $\left\{0, -8, \frac{1}{2}, -3\right\}$; range: {0, 4, 1}

Example 2 **Writing a Relation and Finding Its Domain and Range**

Table 2-4 gives five states in the United States and the corresponding number of representatives in the House of Representatives for a recent year.

a. The data in the table define a relation. Write a list of ordered pairs for this relation.

b. Write the domain and range.

Table 2-4

State x	Number of Representatives y
Alabama	7
California	53
Colorado	7
Florida	27
Kansas	4

Solution:

a. {(Alabama, 7), (California, 53), (Colorado, 7), (Florida, 27), (Kansas, 4)}

b. Domain: {Alabama, California, Colorado, Florida, Kansas}

Range: {7, 53, 27, 4} (*Note:* The element 7 is not listed twice.)

Skill Practice The table depicts six types of animals and their average longevity.

2. Write the ordered pairs indicated by the relation in the table.

3. Find the domain and range of the relation.

Animal x	Longevity (years) y
Bear	22.5
Cat	11
Cow	20.5
Deer	12.5
Dog	11
Elephant	35

A relation may consist of a finite number of ordered pairs or an infinite number of ordered pairs. Furthermore, a relation may be defined by several different methods.

- A relation may be defined as a set of ordered pairs.

$$\{(1, 2), (-3, 4), (1, -4), (3, 4)\}$$

- A relation may be defined by a correspondence (Figure 2-28). The corresponding ordered pairs are $\{(1, 2), (1, -4), (-3, 4), (3, 4)\}$.

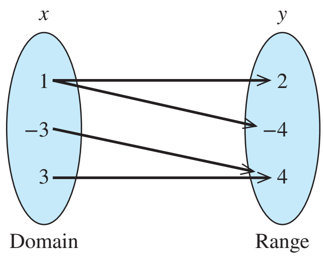


Figure 2-28

- A relation may be defined by a graph (Figure 2-29). The corresponding ordered pairs are $\{(1, 2), (-3, 4), (1, -4), (3, 4)\}$.

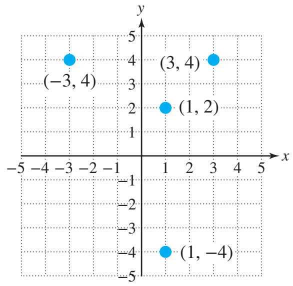


Figure 2-29

Answers

2. {(Bear, 22.5), (Cat, 11), (Cow, 20.5), (Deer, 12.5), (Dog, 11), (Elephant, 35)}

3. Domain: {Bear, Cat, Cow, Deer, Dog, Elephant}; range: {22.5, 11, 20.5, 12.5, 35}

- A relation may be expressed by an equation such as $x = y^2$. The solutions to this equation define an infinite set of ordered pairs of the form $\{(x, y) \mid x = y^2\}$. The solutions can also be represented by a graph in a rectangular coordinate system (Figure 2-30).

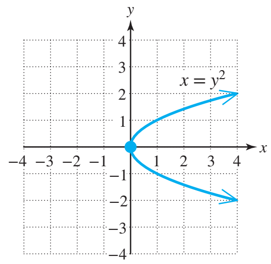
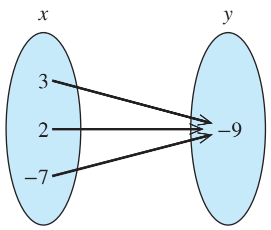


Figure 2-30

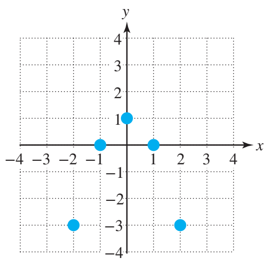
Example 3 Finding the Domain and Range of a Relation

Write the relation as a set of ordered pairs. Then find the domain and range.

a.



b.



Solution:

a. From the figure, the relation defines the set of ordered pairs:
$\{(3, -9), (2, -9), (-7, -9)\}$

Domain: $\{3, 2, -7\}$

Range: $\{-9\}$

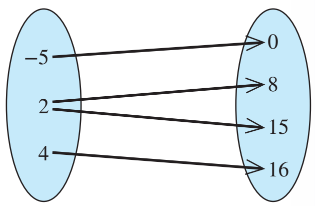
b. The points in the graph make up the set of ordered pairs:
$\{(-2, -3), (-1, 0), (0, 1), (1, 0), (2, -3)\}$

Domain: $\{-2, -1, 0, 1, 2\}$

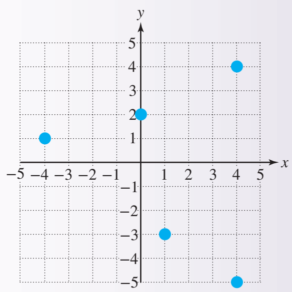
Range: $\{-3, 0, 1\}$

Skill Practice Find the domain and range of the relations.

4.



5.



Answers

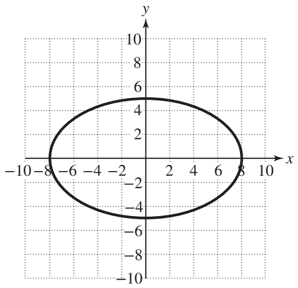
4. Domain: $\{-5, 2, 4\}$; range: $\{0, 8, 15, 16\}$

5. Domain: $\{-4, 0, 1, 4\}$; range: $\{-5, -3, 1, 2, 4\}$

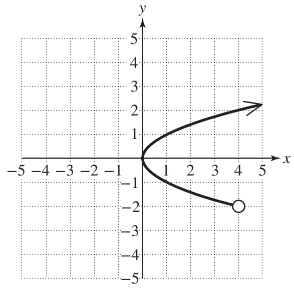
Example 4 Finding the Domain and Range of a Relation

Use interval notation to express the domain and range of the relation.

a.

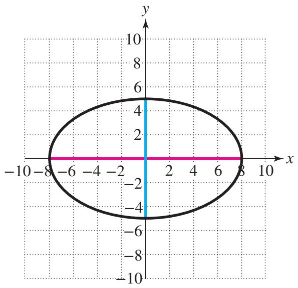


b.



Solution:

a.

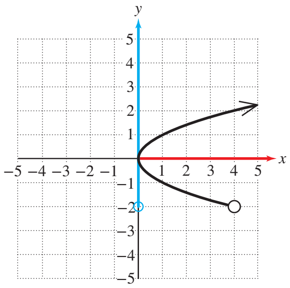


The domain consists of an infinite number of x values extending from -8 to 8 (shown in red). The range consists of all y values from -5 to 5 (shown in blue). Thus, the domain and range must be expressed in set-builder notation or in interval notation.

Domain: $[-8, 8]$

Range: $[-5, 5]$

b.



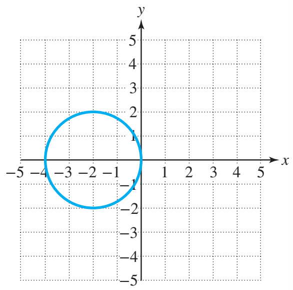
The arrow on the curve indicates that the graph extends infinitely far up and to the right. The open circle means that the graph will end at the point $(4, -2)$, but not include that point.

Domain: $[0, \infty)$

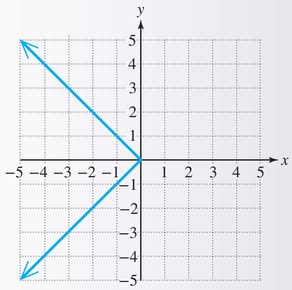
Range: $(-2, \infty)$

Skill Practice Use interval notation to express the domain and range of the relations.

6.



7.



Answers

6. Domain: $[-4, 0]$
Range: $[-2, 2]$

7. Domain: $(-\infty, 0]$
Range: $(-\infty, \infty)$

3. Applications Involving Relations

Example 5 Analyzing a Relation

The data in Table 2-5 depict the length of a woman's femur and her corresponding height. Based on these data, a forensics specialist can find a linear relationship between height y (in inches) and femur length x (in centimeters):

$$y = 0.91x + 24 \qquad 40 \leq x \leq 55$$

From this type of relationship, the height of a woman can be inferred based on skeletal remains.

a. Find the height of a woman whose femur is 46.0 cm.

b. Find the height of a woman whose femur is 51.0 cm.

c. Why is the domain restricted to $40 \leq x \leq 55$?

Solution:

a. $y = 0.91x + 24$

$= 0.91(46.0) + 24$ Substitute $x = 46.0$ cm.

$= 65.86$ The woman is approximately 65.9 in. tall.

b. $y = 0.91x + 24$

$= 0.91(51.0) + 24$ Substitute $x = 51.0$ cm.

$= 70.41$ The woman is approximately 70.4 in. tall.

c. The domain restricts femur length to values between 40 cm and 55 cm inclusive. These values are within the normal lengths for an adult female and are in the proximity of the observed data (Figure 2-31).

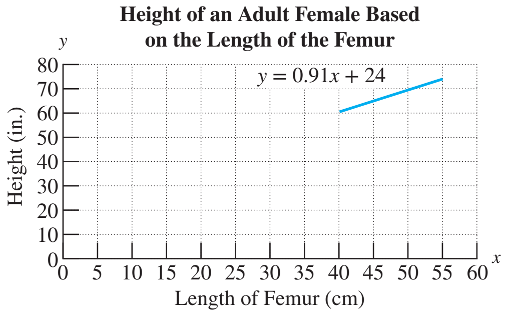


Figure 2-31

Skill Practice The linear equation, $y = -0.014x + 64.5$, for $1500 \leq x \leq 4000$, relates the weight of a car, x (in pounds), to its gas mileage, y (in mpg).

8. Find the gas mileage in miles per gallon for a car weighing 2550 lb.

9. Find the gas mileage for a car weighing 2850 lb.

10. Why is the domain restricted to $1500 \leq x \leq 4000$?

Table 2-5

Length of Femur (cm) x	Height (in.) y
45.5	65.5
48.2	68.0
41.8	62.2
46.0	66.0
50.4	70.0

Answers

8. 28.8 mpg **9.** 24.6 mpg

10. The relation is valid only for cars weighing between 1500 lb and 4000 lb, inclusive.

Section 2.5 Activity

A.1. A set of ordered pairs of the form (x, y) is called a relation in x and y. Consider the relation $\{(-3, 1), (0, 4), (2, -5), (-1, -4)\}$.

a. The domain of a relation is the set of _____ (choose one: x values, y values).

b. Write the domain of the given relation.

c. The set of y values from a relation is called the ______ of the relation.

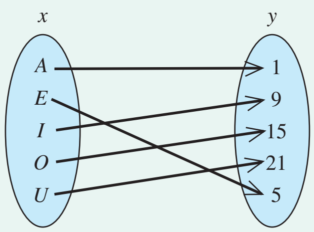
d. Write the range of the given relation.

A relation may be expressed as a figure, a table, or a graph. For Exercises A.2–A.3,

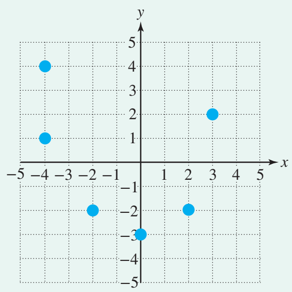
a. Write the domain of the relation.

b. Write the range of the relation.

A.2.



A.3.



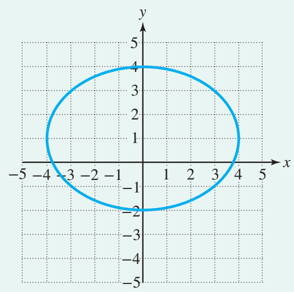
A.4. Consider the relation shown in the graph.

a. What is the leftmost x value in the graph? What is the rightmost x value?

b. Because the graph has no "gaps" or "breaks," every x value between the leftmost and rightmost value is accounted for. Write the domain in interval notation.

c. What is the lowermost y value in the graph? What is the uppermost y value in the graph?

d. Write the range of the relation in interval notation.



A.5. Consider the relation shown in the graph.

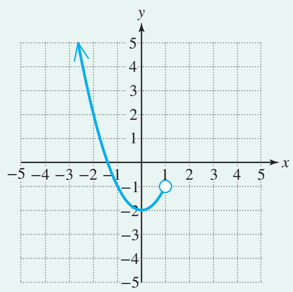
a. What does the open dot at the point $(1, -1)$ represent?

b. How far to the left does the graph extend? How far to the right does the graph extend?

c. Write the domain in interval notation.

d. What is the lowermost y value in the graph? Does the graph have an uppermost y value?

e. Write the range of the relation in interval notation.



Practice Exercises

Study Skills Exercise

It is always helpful to read the material in a section and make notes before it is presented in class. This will help reduce anxiety about new material. Refer to your class syllabus and identify the section that will be covered in your next class. Read the section in the text before the class meets to familiarize yourself with the material and terminology. As you preview the material, respond to the following in your own words:

- What is the section about?
- What concepts do you find challenging or confusing?
- Write down at least one question to ask your instructor.

Prerequisite Review

For Exercises R.1–R.6, write the set in interval notation.

R.1. $\{x\,|\,x > -6\}$

R.2. $\{x\,|\,x < 2\}$

R.3. $\left\{x \,\middle|\, \frac{3}{4} \geq x\right\}$

R.4. $\left\{x \,\middle|\, -\frac{10}{3} \leq x\right\}$

R.5. $\{x\,|\,0.5 < x \leq 1.2\}$

R.6. $\{x\,|\,0 \leq x < 4.6\}$

For Exercises R.7–R.12, estimate the coordinates of the given point.

R.7. *A*

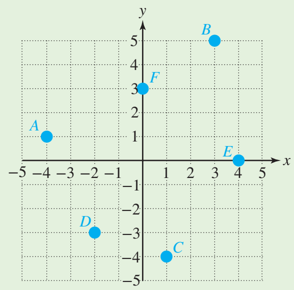
R.8. *B*

R.9. *C*

R.10. *D*

R.11. *E*

R.12. *F*



Vocabulary and Key Concepts

1. **a.** A set of ordered pairs (x, y) is called a ________ in x and y.

b. The ________ of a relation is the set of first components in the ordered pairs.

c. The ________ of a relation is the set of second components in the ordered pairs.

Concept 2: Domain and Range of a Relation

2. Explain how to determine the domain and range of a relation represented by a set of ordered pairs.

For Exercises 3–14,

a. Write the relation as a set of ordered pairs.

b. Determine the domain and range. **(See Examples 1–3.)**

3.

Region	Number Living in Poverty (thousands)
Northeast	54.1
Midwest	65.6
South	110.7
West	70.7

4.

x	y
0	3
−2	$\frac{1}{2}$
−7	1
−2	8
5	1

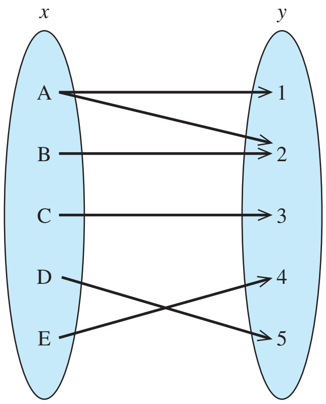
5.

Country	Year of First Man or Woman in Space
USSR	1961
USA	1962
Poland	1978
Vietnam	1980
Cuba	1980

6.

State, x	Year of Statehood, y
Maine	1820
Nebraska	1867
Utah	1896
Hawaii	1959
Alaska	1959

7.



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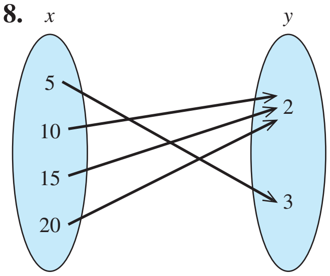

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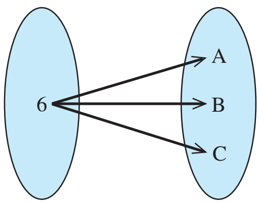

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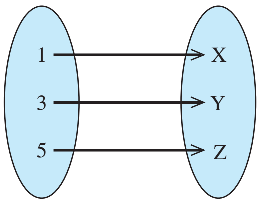

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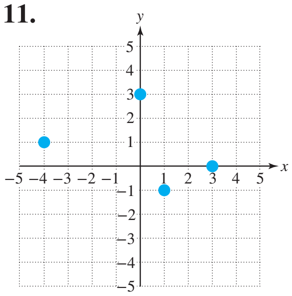

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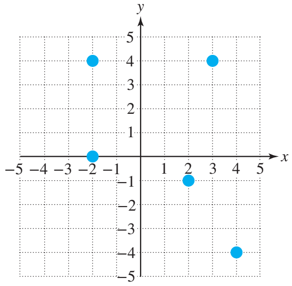

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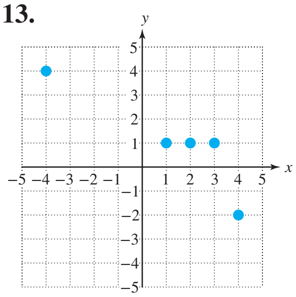

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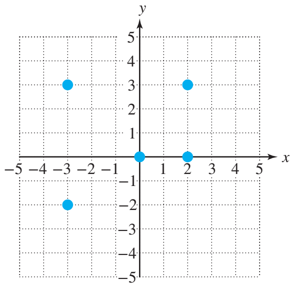
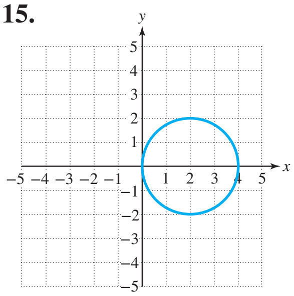


For Exercises 15–30, find the domain and range of the relations. Use interval notation where appropriate. **(See Example 4.)**

15.



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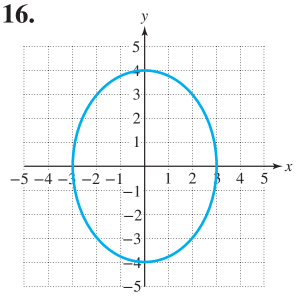

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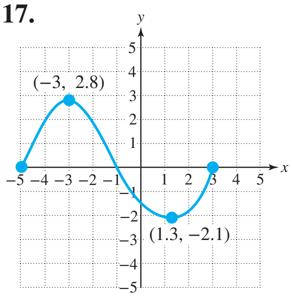

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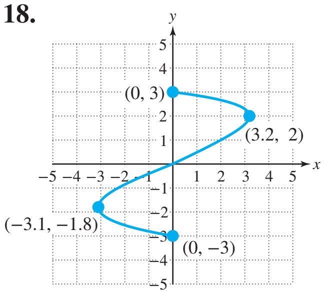

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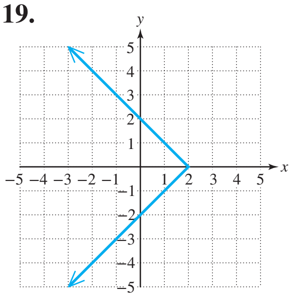

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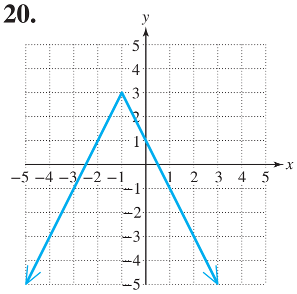

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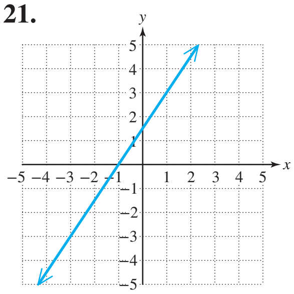

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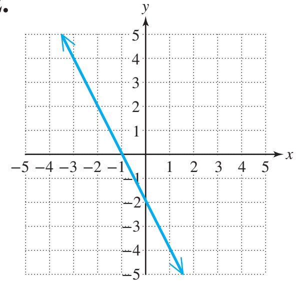

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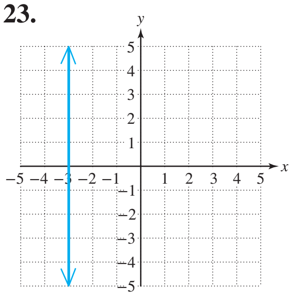

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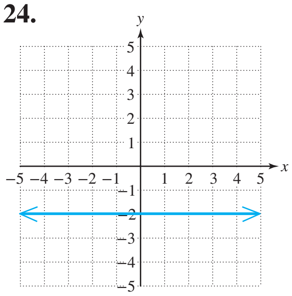

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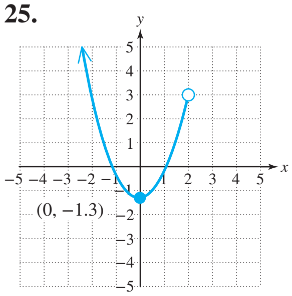

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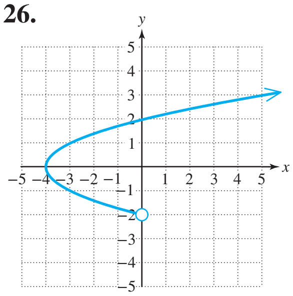

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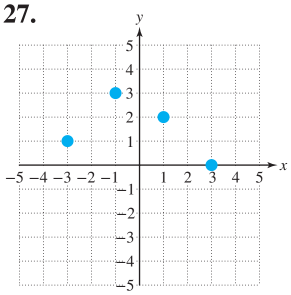

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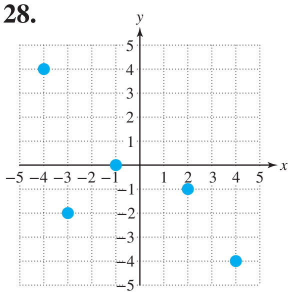

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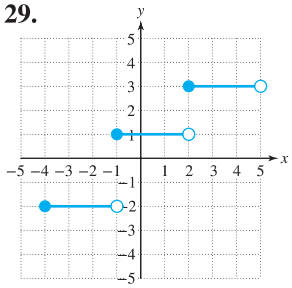

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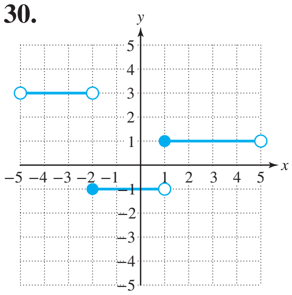


Concept 3: Applications Involving Relations

31. The table gives a relation between the month of the year and the average precipitation for that month for Miami, Florida. **(See Example 5.)**

Month x	Precipitation (in.) y	Month x	Precipitation (in.) y
Jan.	2.01	July	5.70
Feb.	2.08	Aug.	7.58
Mar.	2.39	Sept.	7.63
Apr.	2.85	Oct.	5.64
May	6.21	Nov.	2.66
June	9.33	Dec.	1.83

Source: U.S. National Oceanic and Atmospheric Administration

a. What is the range element corresponding to April?

b. What is the range element corresponding to June?

c. Which element in the domain corresponds to the least value in the range?

d. Complete the ordered pair: (, 2.66)

e. Complete the ordered pair: (Sept.,   )

f. What is the domain of this relation?

32. The table gives a relation between a person's age and the person's maximum recommended heart rate.

Age (years) x	Maximum Recommended Heart Rate (Beats per Minute) y
20	200
30	190
40	180
50	170
60	160

a. What is the domain?

b. What is the range?

c. The range element 200 corresponds to what element in the domain?

d. Complete the ordered pair: (50,   )

e. Complete the ordered pair: (, 190)

33. The population of Canada y (in millions) can be approximated by the relation $y = 0.35x + 30.7$, where x represents the number of years since 2000.

a. Approximate the population of Canada in the year 2010.

b. In what year did the population of Canada reach approximately 37,350,000?

34. The world record times for women's track and field events are shown in the table. The women's world record time y (in seconds) required to run x meters can be approximated by the relation $y = 0.159x - 10.79$.


Digital Vision/Getty Images

Distance (m)	Time (sec)	Winner's Name and Country
100	10.49	Florence Griffith Joyner (United States)
200	21.34	Florence Griffith Joyner (United States)
400	47.60	Marita Koch (East Germany)
800	113.28	Jarmila Kratochvilova (Czechoslovakia)
1000	148.98	Svetlana Masterkova (Russia)
1500	230.07	Genzebe Dibaba (Ethiopia)

a. Predict the time required for a 500-m race.

b. Use this model to predict the time for a 1000-m race. Is this value exactly the same as the data value given in the table? Explain.

Expanding Your Skills

35. **a.** Define a relation with four ordered pairs such that the first element of the ordered pair is the name of a friend and the second element is your friend's place of birth.

b. State the domain and range of this relation.

36. **a.** Define a relation with four ordered pairs such that the first element is a state and the second element is its capital.

b. State the domain and range of this relation.

37. Use a mathematical equation to define a relation whose second component y is 1 less than 2 times the first component x.

38. Use a mathematical equation to define a relation whose second component y is 3 more than the first component x.

39. Use a mathematical equation to define a relation whose second component y is the square of the first component x.

40. Use a mathematical equation to define a relation whose second component y is one-fourth the first component x.

Introduction to Functions

Section 2.6

Concepts

1. Definition of a Function
2. Vertical Line Test
3. Function Notation
4. Finding Function Values From a Graph
5. Domain of a Function

1. Definition of a Function

In this section, we introduce a special type of relation called a function.

Definition of a Function

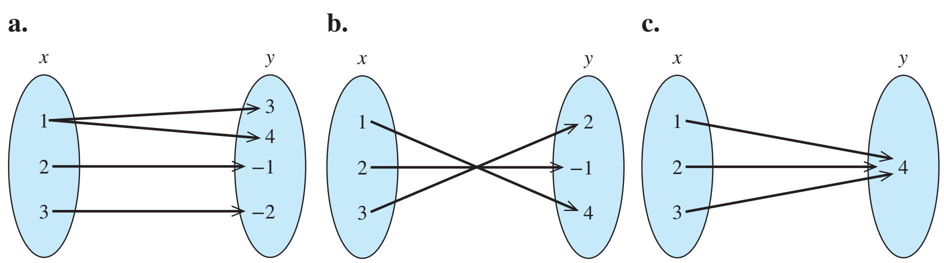
Given a relation in x and y, we say "y is a **function** of x" if, for each element x in the domain, there is exactly one value of y in the range.

Note: This means that no two ordered pairs may have the same first coordinate and different second coordinates.

To understand the difference between a relation that is a function and a relation that is not a function, consider Example 1.

Example 1 Determining Whether a Relation Is a Function

Determine which of the relations define y as a function of x.



Solution:

a. This relation is defined by the set of ordered pairs

same x

$$\{(1, 3), (1, 4), (2, -1), (3, -2)\}$$

different y values

When $x = 1$, there are *two* possible range elements: $y = 3$ and $y = 4$. Therefore, this relation is *not* a function.

b. This relation is defined by the set of ordered pairs $\{(1, 4), (2, -1), (3, 2)\}$.

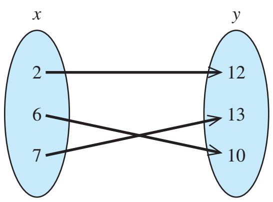
Notice that no two ordered pairs have the same value of x but different values of y. Therefore, this relation *is* a function.

c. This relation is defined by the set of ordered pairs $\{(1, 4), (2, 4), (3, 4)\}$.

Notice that no two ordered pairs have the same value of x but different values of y. Therefore, this relation *is* a function.

Skill Practice Determine if the relation defines y as a function of x.

1.



2. $\{(4, 2), (-5, 4), (0, 0), (8, 4)\}$

3. $\{(-1, 6), (8, 9), (-1, 4), (-3, 10)\}$

2. Vertical Line Test

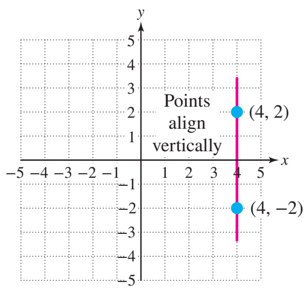


Figure 2-32

A relation that is not a function has at least one domain element x paired with more than one range value y. For example, the set $\{(4, 2), (4, -2)\}$ does not define a function because two different y values correspond to the same x. These two points are aligned vertically in the xy-plane, and a vertical line drawn through one point also intersects the other point (see Figure 2-32). If a vertical line drawn through a graph of a relation intersects the graph in more than one point, the relation cannot be a function. This idea is stated formally as the **vertical line test**.

The Vertical Line Test

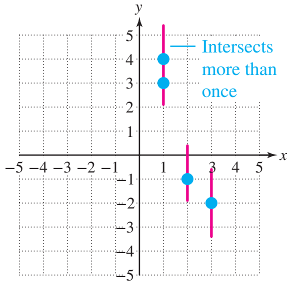
Consider a relation defined by a set of points (x, y) in a rectangular coordinate system. The graph defines y as a function of x if no vertical line intersects the graph in more than one point.

Answers

1. Yes **2.** Yes **3.** No

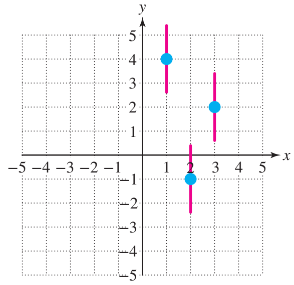
The vertical line test can be demonstrated by graphing the ordered pairs from the relations in Example 1.

a. $\{(1, 3), (1, 4), (2, -1), (3, -2)\}$



Not a Function
A vertical line intersects in more than one point.

b. $\{(1, 4), (2, -1), (3, 2)\}$

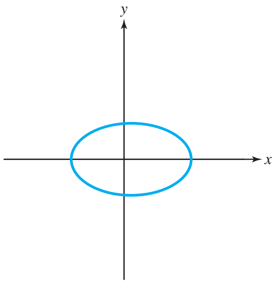


Function
No vertical line intersects more than once.

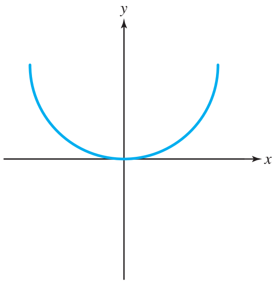
Example 2 Using the Vertical Line Test

Use the vertical line test to determine whether the relations define y as a function of x.

a.

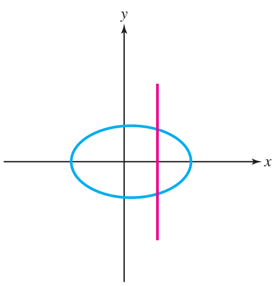


b.



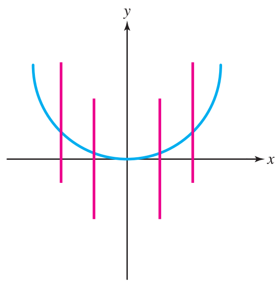
Solution:

a.



Not a Function
A vertical line intersects in more than one point.

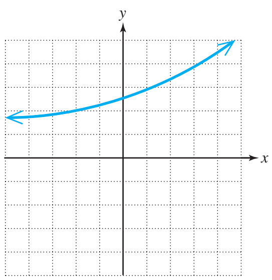
b.



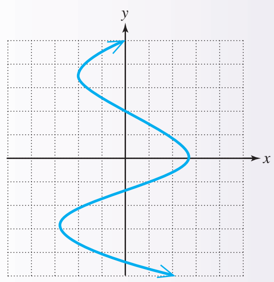
Function
No vertical line intersects in more than one point.

Skill Practice Use the vertical line test to determine whether the relations define y as a function of x.

4.



5.



Answers
4. Yes **5.** No

3. Function Notation

A function is defined as a relation with the added restriction that each value in the domain must have only one corresponding y value in the range. In mathematics, functions are often given by rules or equations to define the relationship between two or more variables. For example, the equation $y = 2x$ defines the set of ordered pairs such that the y value is twice the x value.

When a function is defined by an equation, we often use **function notation**. For example, the equation $y = 2x$ may be written in function notation as

$f(x) = 2x$
- f is the name of the function.
- x is an input value from the domain of the function.
- $f(x)$ is the function value (y value) corresponding to x.

Avoiding Mistakes
Be sure to note that $f(x)$ is *not* $f \cdot x$.

The notation $f(x)$ is read as "f of x" or "the value of the function f at x."

A function may be evaluated at different values of x by substituting x values from the domain into the function. For example, to evaluate the function defined by $f(x) = 2x$ at $x = 5$, substitute $x = 5$ into the function.

$$f(x) = 2x$$
$$f(5) = 2(5)$$
$$f(5) = 10$$

TIP: $f(5) = 10$ can be interpreted as the ordered pair (5, 10).

Thus, when $x = 5$, the corresponding function value is 10. We say:

- f of 5 is 10.
- f at 5 is 10.
- f evaluated at 5 is 10.

The names of functions are often given by either lowercase or uppercase letters, such as f, g, h, p, K, and M. The input variable may also be a letter other than x. For example, $y = P(t)$ might represent population as a function of time.

Example 3 Evaluating a Function

Given the function defined by $g(x) = \frac{1}{2}x - 1$, find the function values.

a. $g(0)$ **b.** $g(2)$ **c.** $g(4)$ **d.** $g(-2)$

Solution:

a. $g(x) = \frac{1}{2}x - 1$

$$g(0) = \frac{1}{2}(0) - 1$$
$$= 0 - 1$$
$$= -1$$

We say that "g of 0 is -1." This is equivalent to the ordered pair $(0, -1)$.

b. $g(x) = \frac{1}{2}x - 1$

$$g(2) = \frac{1}{2}(2) - 1$$
$$= 1 - 1$$
$$= 0$$

We say that "g of 2 is 0." This is equivalent to the ordered pair (2, 0).

c. $g(x) = \frac{1}{2}x - 1$

$g(4) = \frac{1}{2}(4) - 1$

$= 2 - 1$

$= 1$

We say that "g of 4 is 1." This is equivalent to the ordered pair (4, 1).

d. $g(x) = \frac{1}{2}x - 1$

$g(-2) = \frac{1}{2}(-2) - 1$

$= -1 - 1$

$= -2$

We say that "g of -2 is -2." This is equivalent to the ordered pair $(-2, -2)$.

Skill Practice Given the function defined by $f(x) = -2x - 3$, find the function values.

6. $f(1)$ **7.** $f(0)$ **8.** $f(-3)$ **9.** $f\left(\frac{1}{2}\right)$

In Example 3, notice that $g(0)$, $g(2)$, $g(4)$, and $g(-2)$ correspond to the ordered pairs $(0, -1)$, $(2, 0)$, $(4, 1)$, and $(-2, -2)$. In the graph, these points "line up." The graph of *all* ordered pairs defined by this function is a line with a slope of $\frac{1}{2}$ and y-intercept of $(0, -1)$ (Figure 2-33). This should not be surprising because the function defined by $g(x) = \frac{1}{2}x - 1$ is equivalent to $y = \frac{1}{2}x - 1$.

A function may be evaluated at numerical values or at algebraic expressions, as shown in Example 4.

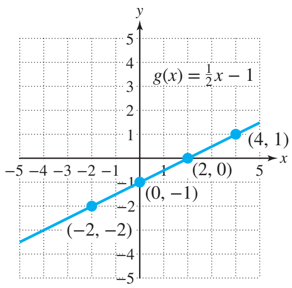


Figure 2-33

Example 4 Evaluating Functions

Given the functions defined by $f(x) = x^2 - 2x$ and $g(x) = 3x + 5$, find the function values.

a. $f(t)$ **b.** $g(w + 4)$ **c.** $f(-t)$

Solution:

a. $f(x) = x^2 - 2x$

$f(t) = (t)^2 - 2(t)$ Substitute $x = t$ for all values of x in the function.

$= t^2 - 2t$ Simplify.

b. $g(x) = 3x + 5$

$g(w + 4) = 3(w + 4) + 5$ Substitute $x = w + 4$ for all values of x in the function.

$= 3w + 12 + 5$

$= 3w + 17$ Simplify.

c. $f(x) = x^2 - 2x$ Substitute $-t$ for x.

$f(-t) = (-t)^2 - 2(-t)$

$= t^2 + 2t$ Simplify.

Skill Practice Given the function defined by $g(x) = 4x - 3$, find the function values.

10. $g(a)$ **11.** $g(x + h)$ **12.** $g(-x)$

FOR REVIEW

Remember to use parentheses when substituting values into an expression. For example, the use of parentheses in the expression $3(w + 4) + 5$ reminds us to multiply 3 by both w and 4.

Answers

6. -5 **7.** -3 **8.** 3 **9.** -4
10. $4a - 3$ **11.** $4x + 4h - 3$
12. $-4x - 3$

4. Finding Function Values From a Graph

We can find function values by looking at the graph of a function. The value of $f(a)$ refers to the y-coordinate of a point with x-coordinate a.

Example 5 Finding Function Values From a Graph

Consider the function pictured in Figure 2-34.

a. Find $h(-1)$.

b. Find $h(2)$.

c. Find $h(5)$.

d. For what value of x is $h(x) = 3$?

e. For what values of x is $h(x) = 0$?

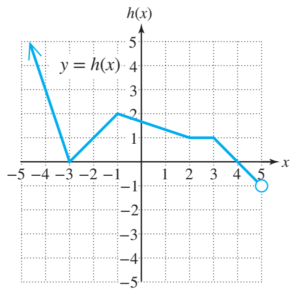


Figure 2-34

Solution:

a. $h(-1) = 2$ — From the graph, when $x = -1$, the function value (y value) is 2. This corresponds to the ordered pair $(-1, 2)$.

b. $h(2) = 1$ — From the graph, when $x = 2$, the function value (y value) is 1. This corresponds to the ordered pair $(2, 1)$.

c. $h(5)$ is not defined. — The open dot at $(5, -1)$ indicates that 5 is not in the domain.

d. $h(x) = 3$ for $x = -4$ — We want to find the x value(s) that correspond to a function value (y value) of 3. This corresponds to the ordered pair $(-4, 3)$.

e. $h(x) = 0$ for $x = -3$ and $x = 4$ — We want to find the x value(s) that correspond to a function value (y value) of 0. These correspond to the ordered pairs $(-3, 0)$ and $(4, 0)$.

Skill Practice Refer to the function graphed here.

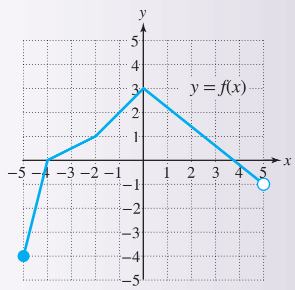
13. Find $f(0)$.

14. Find $f(-2)$.

15. Find $f(5)$.

16. For what value(s) of x is $f(x) = 0$?

17. For what value(s) of x is $f(x) = -4$?



5. Domain of a Function

A function is a relation, and it is often necessary to determine its domain and range. Consider a function defined by the equation $y = f(x)$. The **domain** of f is the set of all x values that when substituted into the function produce a real number. The **range** of f is the set of all y values corresponding to the values of x in the domain.

To find the domain of a function defined by $y = f(x)$, keep these guidelines in mind.

- Exclude values of x that make the denominator of a fraction zero.
- Exclude values of x that make the expression within a square root negative.

Answers

13. 3 **14.** 1
15. not defined
16. $x = -4$ and $x = 4$
17. $x = -5$

Example 6 Finding the Domain of a Function

Write the domain in interval notation.

a. $f(x) = \dfrac{x+7}{2x-1}$ **b.** $h(x) = \dfrac{x-4}{x^2+9}$

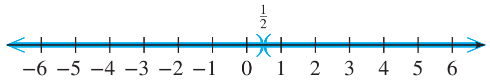
c. $k(t) = \sqrt{t+4}$ **d.** $g(t) = t^2 - 3t$

Solution:

a. $f(x) = \dfrac{x+7}{2x-1}$ will not be a real number when the denominator is zero, that is, when

$$2x - 1 = 0$$
$$2x = 1$$
$$x = \frac{1}{2}$$

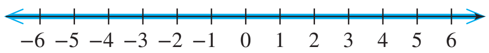
The value $x = \frac{1}{2}$ must be *excluded* from the domain.



The number line indicates that the domain consists of two intervals, $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$. We use the notation $\cup$ to combine the two intervals. The symbol $\cup$ stands for union. The union of two intervals consists of all the numbers in the first interval, along with all the numbers in the second interval.

Interval notation: $\left(-\infty, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$

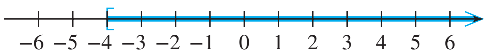
b. For $h(x) = \dfrac{x-4}{x^2+9}$ the quantity x^2 is greater than or equal to 0 for all real numbers x, and the number 9 is positive. The sum $x^2 + 9$ must be *positive* for all real numbers x. The denominator will never be zero; therefore, the domain is the set of all real numbers.



Interval notation: $(-\infty, \infty)$

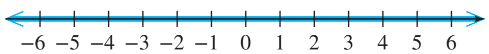
c. The value of the function defined by $k(t) = \sqrt{t+4}$ will not be a real number when the expression within the square root is negative. Therefore, the domain is the set of all t values that make $t + 4$ greater than or equal to zero.

$$t + 4 \geq 0$$
$$t \geq -4$$



Interval notation: $[-4, \infty)$

d. The function defined by $g(t) = t^2 - 3t$ has no restrictions on its domain because any real number substituted for t will produce a real number. The domain is the set of all real numbers.



Interval notation: $(-\infty, \infty)$

Skill Practice Write the domain in interval notation.

18. $f(x) = \dfrac{2x+1}{x-9}$ **19.** $k(x) = \dfrac{-5}{4x^2+1}$

20. $g(x) = \sqrt{x-2}$ **21.** $h(x) = x + 6$

FOR REVIEW

Recall that with interval notation, curved parentheses, (or), indicate that the endpoints *are not* included in the interval. On the other hand, square brackets, [or], indicate that the endpoints *are* included.

Answers

18. $(-\infty, 9) \cup (9, \infty)$ **19.** $(-\infty, \infty)$
20. $[2, \infty)$ **21.** $(-\infty, \infty)$

Section 2.6 Activity

A.1. Given a set of ordered pairs, how can you determine whether the relation defines y as a function of x?

For Exercises A.2–A.3, consider the given relation.

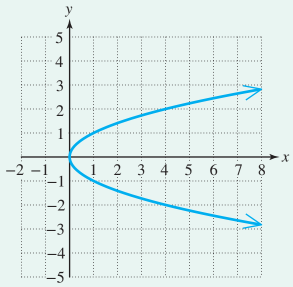
a. Do any two ordered pairs have the same x value but different y values?

b. Is the relation a function?

A.2. $\{(-5, 1), (3, 4), (-2, 6), (-5, 2), (0, -3)\}$

A.3. $\{(-1, 6), (2, 11), (8, 6), (-3, 1), (0.4, -0.5)\}$

A.4. **a.** For the graph given, draw a vertical line through the point (4, 2). Does the vertical line intersect the graph at any other point?



b. Does this graph define y as a function of x?

c. Using this example, explain how the vertical line test is used to determine if a graph defines y as a function of x.

A.5. Consider the equation $y = 2x + 1$.

a. If $x = 3$, what is the corresponding y value?

b. Write the result of part (a) as an ordered pair (x, y).

A.6. Consider the function defined by $f(x) = 2x + 1$.

a. Find $f(3)$. That is, evaluate the function for $x = 3$ by substituting 3 for x.

b. Write the result of part (a) as an ordered pair (x, y).

c. Refer to Exercise A.5 and compare the results.

A.7. Given $g(x) = \dfrac{4}{x-1}$, find the function values if possible.

a. $g(2)$ **b.** $g(-3)$

c. $g(0)$ **d.** $g(1)$

A.8. Refer to $g(x) = \dfrac{4}{x-1}$ from Exercise A.7.

a. What value(s) of x must be excluded from the domain of g? Why?

b. Write the domain of g in interval notation.

A.9. Given $h(x) = \sqrt{x+2}$, find the function values if possible.

a. $h(-1)$ **b.** $h(2)$

c. $h(7)$ **d.** $h(-6)$

A.10. Refer to $h(x) = \sqrt{x+2}$ from Exercise A.9.

a. The square root of a negative number is not a real number. Therefore, what restrictions must be imposed on x?

b. Write the domain of h in interval notation.

A.11. Consider the function defined by $f(x) = x^2 - 3x + 4$. Find the function values.

a. $f(-2)$ **b.** $f(t)$ **c.** $f(a + 2)$

Practice Exercises

Section 2.6

Prerequisite Review

For Exercises R.1–R.4, substitute the given value of x. Then solve for y.

R.1. $y = -2x - 4; x = 6$

R.2. $y = 5x - 10; x = -2$

R.3. $y = 2x^2 - 4x + 1; x = -2$

R.4. $y = 3x^2 + x - 4; x = -1$

For Exercises R.5–R.8, determine the values of x for which the expression is undefined.

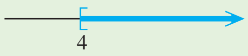
R.5. $\dfrac{3}{x - 5}$

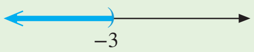
R.6. $\dfrac{-4}{x + 1}$

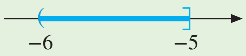
R.7. $\dfrac{x + 4}{x^2 - 9}$

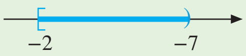
R.8. $\dfrac{x - 10}{x^2 - 25}$

For Exercises R.9–R.12, express the set in interval notation.

R.9. 

R.10. 

R.11. 

R.12. 

For Exercises R.13–R.16, solve the inequality. Write the solution set in interval notation.

R.13. $2x - 3 \geq 0$

R.14. $5x + 1 \geq 0$

R.15. $4 - x \geq 0$

R.16. $-3 - x \geq 0$

Vocabulary and Key Concepts

1. a. Given a relation in x and y, we say that y is a ______________ of x if for each element x in the domain, there is exactly one value of y in the range.

b. If a ______________ line intersects the graph of a relation in more than one point, the relation is not a function.

c. Function notation for the equation $y = 2x + 1$ is $f(x) =$ ______________.

d. Given a function defined by $y = f(x)$, the set of all ______________ that produce a real number when substituted into a function is called the domain of the function.

e. The set of all ______________ corresponding to x values in the domain is called the range of a function.

f. To find the domain of a function defined by $y = f(x)$, exclude any values of x that make the ______________ of a fraction equal to zero.

g. To find the domain of a function defined by $y = f(x)$, exclude any values of x that make the expression within a square root ______________.

For Exercises 2–4, consider the functions defined by $f(x) = x + 2$, $g(x) = 2x$, and $h(x) = x^2$. Fill in the blank with f, g, or h based on the description of the function.

2. Function ______________ doubles the values of x from its domain.

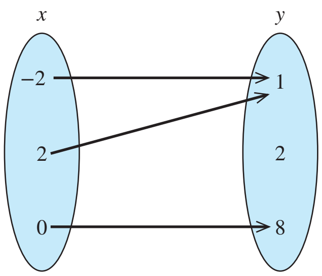
3. Function ______________ squares the values of x from its domain.

4. Function ______________ increases by two the values of x from its domain.

Concept 1: Definition of a Function

For Exercises 5–10, determine if the relation defines y as a function of x. **(See Example 1.)**

5.



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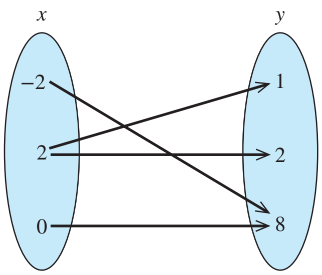

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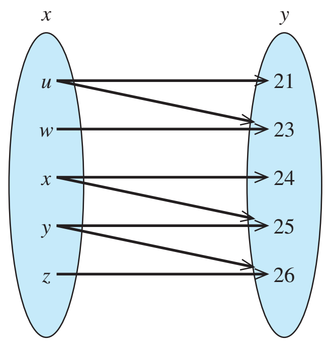

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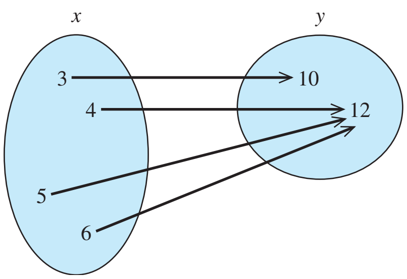


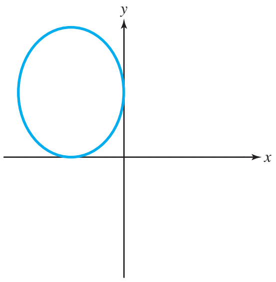
9. $\{(1, 2), (3, 4), (5, 4), (-9, 3)\}$

10. $\left\{(0, -1.1), \left(\frac{1}{2}, 8\right), (1.1, 8), \left(4, \frac{1}{2}\right)\right\}$

Concept 2: Vertical Line Test

For Exercises 11–16, use the vertical line test to determine whether the relation defines y as a function of x. **(See Example 2.)**

11.



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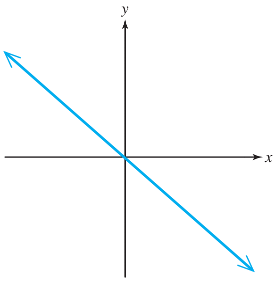

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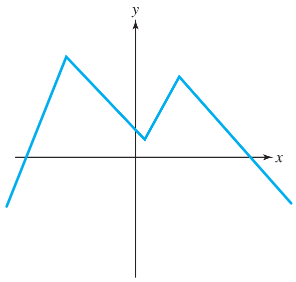

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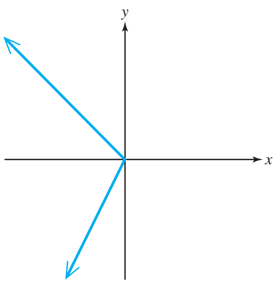

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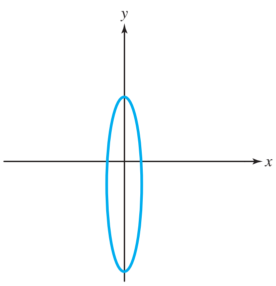

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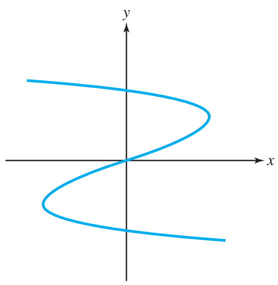


Concept 3: Function Notation

For Exercises 17–20, consider a function defined by $y = f(x)$.

17. Interpret the meaning of $f(2) = 5$.

18. Interpret the meaning of $f(-7) = 8$.

19. Write the ordered pair represented by $f(0) = -2$.

20. Write the ordered pair represented by $f(-10) = 11$.

Consider the functions defined by $f(x) = 6x - 2$, $g(x) = -x^2 - 4x + 1$, $h(x) = 7$, and $k(x) = |x - 2|$. For Exercises 21–52, find the following. **(See Examples 3–4.)**

21. $g(2)$ **22.** $k(2)$ **23.** $g(0)$ **24.** $h(0)$

25. $k(0)$ **26.** $f(0)$ **27.** $f(t)$ **28.** $g(a)$

29. $h(u)$ **30.** $k(v)$ **31.** $g(-3)$ **32.** $h(-5)$

33. $k(-2)$ **34.** $f(-6)$ **35.** $f(x + 1)$ **36.** $h(x + 1)$

37. $g(2x)$ **38.** $k(x - 3)$ **39.** $g(-\pi)$ **40.** $g(a^2)$

41. $h(a + b)$ **42.** $f(x + h)$ **43.** $f(-a)$ **44.** $g(-b)$

45. $k(-c)$ **46.** $h(-x)$ **47.** $f\left(\frac{1}{2}\right)$ **48.** $g\left(\frac{1}{4}\right)$

49. $h\left(\frac{1}{7}\right)$ **50.** $k\left(\frac{3}{2}\right)$ **51.** $f(-2.8)$ **52.** $k(-5.4)$

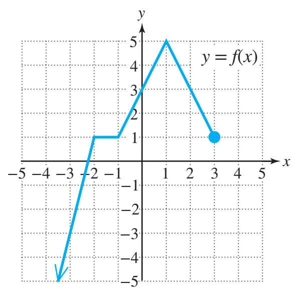
Consider the functions $p = \{(\frac{1}{2}, 6), (2, -7), (1, 0), (3, 2\pi)\}$ and $q = \{(6, 4), (2, -5), (\frac{3}{4}, \frac{1}{5}), (0, 9)\}$. For Exercises 53–60, find the function values.

53. $p(2)$ **54.** $p(1)$ **55.** $p(3)$ **56.** $p\left(\frac{1}{2}\right)$

57. $q(2)$ **58.** $q\left(\frac{3}{4}\right)$ **59.** $q(6)$ **60.** $q(0)$

Concept 4: Finding Function Values From a Graph

61. The graph of $y = f(x)$ is given. **(See Example 5.)**



a. Find $f(0)$.

b. Find $f(3)$.

c. Find $f(-2)$.

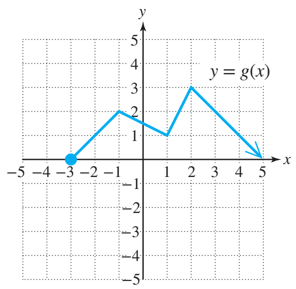
d. For what value(s) of x is $f(x) = -3$?

e. For what value(s) of x is $f(x) = 3$?

f. Write the domain of f.

g. Write the range of f.

62. The graph of $y = g(x)$ is given.



a. Find $g(-1)$.

b. Find $g(1)$.

c. Find $g(4)$.

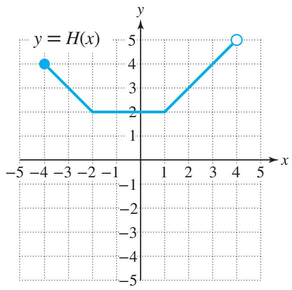
d. For what value(s) of x is $g(x) = 3$?

e. For what value(s) of x is $g(x) = 0$?

f. Write the domain of g.

g. Write the range of g.

63. The graph of $y = H(x)$ is given.



a. Find $H(-3)$.

b. Find $H(4)$.

c. Find $H(3)$.

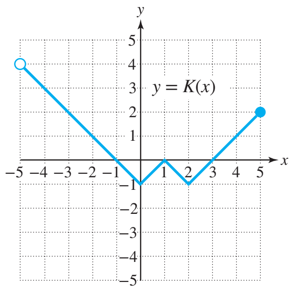
d. For what value(s) of x is $H(x) = 3$?

e. For what value(s) of x is $H(x) = 2$?

f. Write the domain of H.

g. Write the range of H.

64. The graph of $y = K(x)$ is given.



a. Find $K(0)$.

b. Find $K(-5)$.

c. Find $K(1)$.

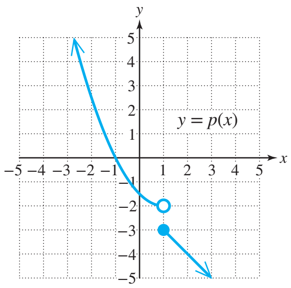
d. For what value(s) of x is $K(x) = 0$?

e. For what value(s) of x is $K(x) = 3$?

f. Write the domain of K.

g. Write the range of K.

65. The graph of $y = p(x)$ is given.



a. Find $p(2)$.

b. Find $p(-1)$.

c. Find $p(1)$.

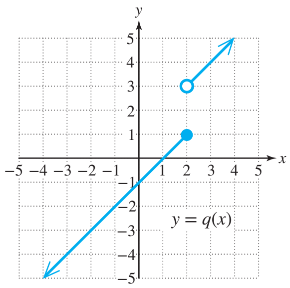
d. For what value(s) of x is $p(x) = 0$?

e. For what value(s) of x is $p(x) = -2$?

f. Write the domain of p.

g. Write the range of p.

66. The graph of $y = q(x)$ is given.



a. Find $q(3)$.

b. Find $q(-1)$.

c. Find $q(2)$.

d. For what value(s) of x is $q(x) = -4$?

e. For what value(s) of x is $q(x) = 3$?

f. Write the domain of q.

g. Write the range of q.

For Exercises 67–76, refer to the functions $y = f(x)$ and $y = g(x)$, defined as follows:

$$f = \{(-3, 5), (-7, -3), (-\tfrac{3}{2}, 4), (1.2, 5)\}$$
$$g = \{(0, 6), (2, 6), (6, 0), (1, 0)\}$$

67. Identify the domain of f.

68. Identify the range of f.

69. Identify the range of g.

70. Identify the domain of g.

71. For what value(s) of x is $f(x) = 5$?

72. For what value(s) of x is $f(x) = -3$?

73. For what value(s) of x is $g(x) = 0$?

74. For what value(s) of x is $g(x) = 6$?

75. Find $f(-7)$.

76. Find $g(0)$.

Concept 5: Domain of a Function

77. Explain how to determine the domain of the function defined by $f(x) = \dfrac{x+6}{x-2}$.

78. Explain how to determine the domain of the function defined by $g(x) = \sqrt{x-3}$.

For Exercises 79–94, find the domain. Write the answer in interval notation. **(See Example 6.)**

79. $k(x) = \dfrac{x-3}{x+6}$

80. $m(x) = \dfrac{x-1}{x-4}$

81. $f(t) = \dfrac{5}{t}$

82. $g(t) = \dfrac{t-7}{t}$

83. $h(p) = \dfrac{p-4}{p^2+1}$

84. $n(p) = \dfrac{p+8}{p^2+2}$

85. $h(t) = \sqrt{t+7}$

86. $k(t) = \sqrt{t-5}$

87. $f(a) = \sqrt{a-3}$

88. $g(a) = \sqrt{a+2}$

89. $m(x) = \sqrt{1-2x}$

90. $n(x) = \sqrt{12-6x}$

91. $p(t) = 2t^2 + t - 1$

92. $q(t) = t^3 + t - 1$

93. $f(x) = x + 6$

94. $g(x) = 8x - \pi$

Mixed Exercises

95. The height (in feet) of a ball that is dropped from an 80-ft building is given by $h(t) = -16t^2 + 80$, where t is the time in seconds after the ball is dropped.

a. Find $h(1)$ and $h(1.5)$.

b. Interpret the meaning of the function values found in part (a).

96. A ball is dropped from a 50-m building. The height (in meters) after t seconds is given by $h(t) = -4.9t^2 + 50$.

a. Find $h(1)$ and $h(1.5)$.

b. Interpret the meaning of the function values found in part (a).

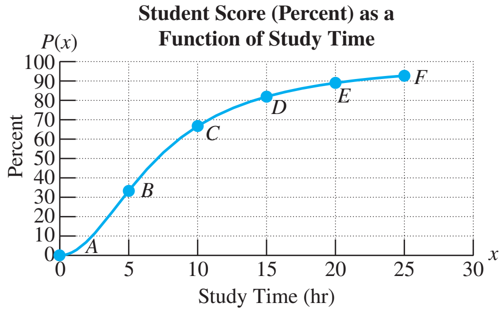
97. If Alicia rides a bike at an average speed of 11.5 mph, the distance that she rides can be represented by $d(t) = 11.5t$, where t is the time in hours.

a. Find $d(1)$ and $d(1.5)$.

b. Interpret the meaning of the function values found in part (a).

98. Brian's score on an exam is a function of the number of hours he spends studying. The function defined by $P(x) = \dfrac{100x^2}{50 + x^2}$ $(x \geq 0)$ indicates that he will achieve a score of $P\%$ if he studies for x hours.

Evaluate $P(0)$, $P(5)$, $P(10)$, $P(15)$, $P(20)$, and $P(25)$ and confirm the values on the graph. (Round to one decimal place.) Interpret $P(25)$ in the context of this problem.



For Exercises 99–102, write a function defined by $y = f(x)$ subject to the conditions given.

99. The value of $f(x)$ is three more than two times x.

100. The value of $f(x)$ is four less than the square of x.

101. The value of $f(x)$ is ten less than the absolute value of x.

102. The value of $f(x)$ is sixteen times the square root of x.

Technology Connections

103. Graph $k(t) = \sqrt{t - 5}$. Use the graph to support your answer to Exercise 86.

104. Graph $h(t) = \sqrt{t + 7}$. Use the graph to support your answer to Exercise 85.

Expanding Your Skills

For Exercises 105–106, write the domain in interval notation.

105. $q(x) = \dfrac{2}{\sqrt{x + 2}}$

106. $p(x) = \dfrac{8}{\sqrt{x - 4}}$

Section 2.7 Graphs of Functions

Concepts

1. Linear and Constant Functions
2. Graphs of Basic Functions
3. Definition of a Quadratic Function
4. Finding the x- and y-Intercepts of a Graph Defined by $y = f(x)$

1. Linear and Constant Functions

A function may be expressed as a mathematical equation that relates two or more variables. In this section, we will look at several elementary functions.

Recall that the graph of an equation $y = k$, where k is a constant, is a horizontal line. In function notation, this can be written as $f(x) = k$. For example, the graph of the function defined by $f(x) = 3$ is a horizontal line, as shown in Figure 2-35.

We say that a function defined by $f(x) = k$ is a constant function because for any value of x, the function value is constant.

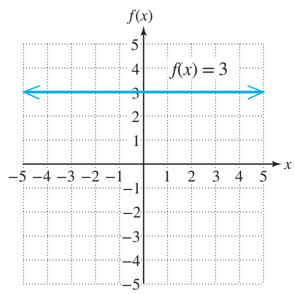


Figure 2-35

An equation of the form $y = mx + b$ is represented graphically by a line with slope m and y-intercept $(0, b)$. In function notation, this may be written as $f(x) = mx + b$. A function in this form is called a linear function. For example, the function defined by $f(x) = 2x - 3$ is a linear function with slope $m = 2$ and y-intercept $(0, -3)$ (Figure 2-36).

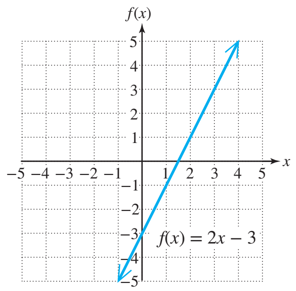


Figure 2-36

Linear Functions and Constant Functions

Let m and b represent real numbers such that $m \neq 0$. Then

A function that can be written in the form $f(x) = mx + b$ is a **linear function**.
A function that can be written in the form $f(x) = b$ is a **constant function**.

Note: The graphs of linear and constant functions are lines.

2. Graphs of Basic Functions

At this point, we are able to recognize the equations and graphs of linear and constant functions. In addition to linear and constant functions, the following equations define six basic functions that will be encountered in the study of algebra:

Equation		Function Notation	Type of Function
$y = x$		$f(x) = x$	Identity function
$y = x^2$		$f(x) = x^2$	Quadratic function
$y = x^3$		$f(x) = x^3$	Cubic function
$y = \lvert x\rvert$	equivalent function notation $\longrightarrow$	$f(x) = \lvert x\rvert$	Absolute value function
$y = \sqrt{x}$		$f(x) = \sqrt{x}$	Square root function
$y = \frac{1}{x}$		$f(x) = \frac{1}{x}$	Reciprocal function

The graph of the function defined by $f(x) = x$ is linear, with slope $m = 1$ and y-intercept $(0, 0)$ (Figure 2-37).

To determine the shapes of the other basic functions, we can plot several points to establish the pattern of the graph. Analyzing the equation itself may also provide insight into the domain, range, and shape of the graph. To demonstrate this, we will graph $f(x) = x^2$ and $g(x) = \frac{1}{x}$.

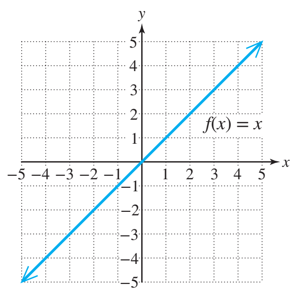


Figure 2-37

Example 1 Graphing Basic Functions

Graph the function defined by $f(x) = x^2$.

Solution:

The domain of the function given by $f(x) = x^2$ (or equivalently $y = x^2$) is all real numbers.

To graph the function, choose arbitrary values of x within the domain of the function. Be sure to choose values of x that are positive and values that are negative to determine the behavior of the function to the right and left of the origin (Table 2-6). The graph of $f(x) = x^2$ is shown in Figure 2-38.

The function values are equated to the square of x, so $f(x)$ will always be greater than or equal to zero. Hence, the y-coordinates on the graph will never be negative. The range of the function is $[0, \infty)$. The arrows on each branch of the graph imply that the pattern continues indefinitely.

Table 2-6

x	$f(x) = x^2$
0	0
1	1
2	4
3	9
−1	1
−2	4
−3	9

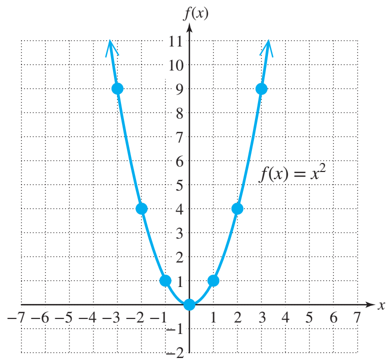


Figure 2-38

Skill Practice

1. Graph $f(x) = -x^2$ by first making a table of points.

Example 2 Graphing Basic Functions

Graph the function defined by $g(x) = \dfrac{1}{x}$.

Solution:

$g(x) = \dfrac{1}{x}$ Notice that $x = 0$ is not in the domain of the function. From the equation $y = \frac{1}{x}$, the y values will be the reciprocal of the x values. The graph defined by $g(x) = \frac{1}{x}$ is shown in Figure 2-39.

x	$g(x) = \dfrac{1}{x}$
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
−1	−1
−2	$-\frac{1}{2}$
−3	$-\frac{1}{3}$

x	$g(x) = \dfrac{1}{x}$
$\frac{1}{2}$	2
$\frac{1}{3}$	3
$\frac{1}{4}$	4
$-\frac{1}{2}$	−2
$-\frac{1}{3}$	−3
$-\frac{1}{4}$	−4

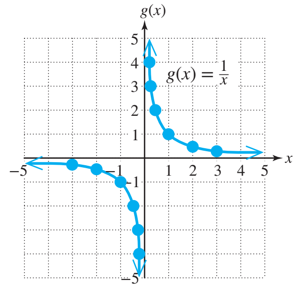


Figure 2-39

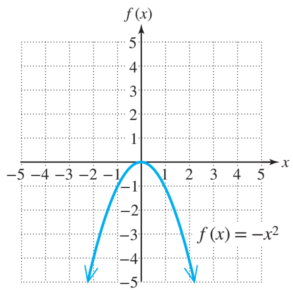
Notice that as x approaches ∞ and $-\infty$, the y values approach zero, and the graph approaches the x-axis. In this case, the x-axis is called a *horizontal asymptote*. Similarly, the graph of the function approaches the y-axis as x gets close to zero. In this case, the y-axis is called a *vertical asymptote*.

Skill Practice

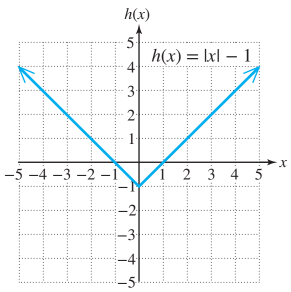
2. Graph $h(x) = |x| - 1$ by first making a table of points.

Answers

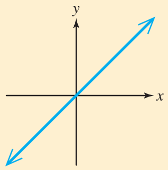
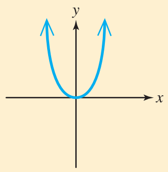
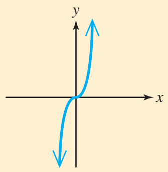
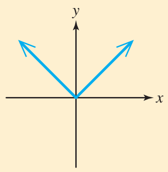
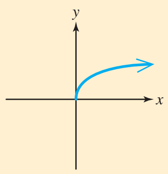
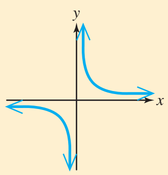
1.



2.



Summary of Six Basic Functions and Their Graphs

Function	**Graph**	**Domain and Range**
1. $f(x) = x$ Identity function		Domain $(-\infty, \infty)$ Range $(-\infty, \infty)$
2. $f(x) = x^2$ Quadratic function		Domain $(-\infty, \infty)$ Range $[0, \infty)$
3. $f(x) = x^3$ Cubic function		Domain $(-\infty, \infty)$ Range $(-\infty, \infty)$
4. $f(x) = \|x\|$ Absolute value function		Domain $(-\infty, \infty)$ Range $[0, \infty)$
5. $f(x) = \sqrt{x}$ Square root function		Domain $[0, \infty)$ Range $[0, \infty)$
6. $f(x) = \dfrac{1}{x}$ Reciprocal function		Domain $(-\infty, 0) \cup (0, \infty)$ Range $(-\infty, 0) \cup (0, \infty)$

TIP: Recall that the symbol $\cup$ means the combination (or union) of the two intervals.

The shapes of these six graphs will be developed in the homework exercises. These functions are used often in the study of algebra. Therefore, we recommend that you associate an equation with its graph and commit each to memory.

3. Definition of a Quadratic Function

In Example 1 we graphed the function defined by $f(x) = x^2$ by plotting points. This function belongs to a special category called quadratic functions.

Definition of a Quadratic Function

A **quadratic function** is a function defined by

$f(x) = ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$.

The graph of a quadratic function is in the shape of a **parabola**. The leading coefficient, a, determines the direction of the parabola.

- If $a > 0$, then the parabola opens upward, and the vertex is the minimum point on the parabola. For example, the graph of $f(x) = x^2$ is shown in Figure 2-40.
- If $a < 0$, then the parabola opens downward, and the vertex is the maximum point on the parabola. For example, the graph of $f(x) = -x^2$ is shown in Figure 2-41.

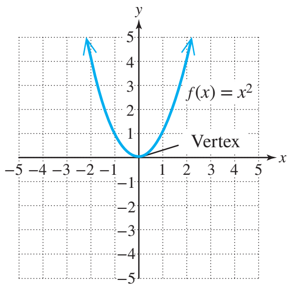


Figure 2-40

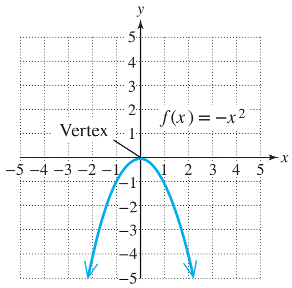


Figure 2-41

Example 3 Identifying Functions

Identify each function as linear, constant, quadratic, or none of these.

a. $f(x) = -4$ **b.** $f(x) = x^2 + 3x + 2$ **c.** $f(x) = 7 - 2x$

d. $f(x) = \dfrac{4x+8}{8}$ **e.** $f(x) = \dfrac{6}{x} + 2$

Solution:

a. $f(x) = -4$ is a constant function. It is in the form $f(x) = b$, where $b = -4$.

b. $f(x) = x^2 + 3x + 2$ is a quadratic function. It is in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

c. $f(x) = 7 - 2x$ is linear. Writing it in the form $f(x) = mx + b$, we get $f(x) = -2x + 7$, where $m = -2$ and $b = 7$.

d. $f(x) = \dfrac{4x+8}{8}$ is linear. Writing it in the form $f(x) = mx + b$, we get

$$f(x) = \frac{4x}{8} + \frac{8}{8}$$

$$= \frac{1}{2}x + 1, \text{ where } m = \frac{1}{2} \text{ and } b = 1.$$

e. $f(x) = \dfrac{6}{x} + 2$ fits none of these categories because the variable is in the denominator.

Skill Practice Identify each function as constant, linear, quadratic, or none of these.

3. $m(x) = -2x^2 - 3x + 7$ **4.** $n(x) = -6$

5. $W(x) = \dfrac{4}{3}x - \dfrac{1}{2}$ **6.** $R(x) = \dfrac{4}{3x} - \dfrac{1}{2}$

Answers

3. Quadratic
4. Constant
5. Linear
6. None of these

4. Finding the *x*- and *y*-Intercepts of a Graph Defined by $y = f(x)$

We have already learned that to find an x-intercept, we substitute $y = 0$ and solve the equation for x. Using function notation, this is equivalent to finding the real solutions of the equation $f(x) = 0$. To find the y-intercept, substitute $x = 0$ and solve the equation for y. In function notation, this is equivalent to finding $f(0)$.

Finding Intercepts Using Function Notation

Given a function defined by $y = f(x)$,

Step 1 The x-intercepts are the real solutions to the equation $f(x) = 0$.

Step 2 The y-intercept is given by $f(0)$.

Example 4 Finding *x*- and *y*-Intercepts

Given the function defined by $f(x) = 2x - 4$:

a. Find the x-intercept(s).

b. Find the y-intercept.

c. Graph the function.

Solution:

a. To find the x-intercept(s), find the real solutions to the equation $f(x) = 0$.

$f(x) = 2x - 4$

$0 = 2x - 4$ Substitute $f(x) = 0$.

$4 = 2x$

$2 = x$ The x-intercept is (2, 0).

b. To find the y-intercept, evaluate $f(0)$.

$f(0) = 2(0) - 4$ Substitute $x = 0$.

$f(0) = -4$ The y-intercept is (0, −4).

c. This function is linear, with a y-intercept of (0, −4), an x-intercept of (2, 0), and a slope of 2 (Figure 2-42).

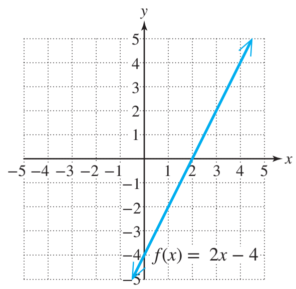


Figure 2-42

Skill Practice Consider $f(x) = -5x + 1$.

7. Find the x-intercept.

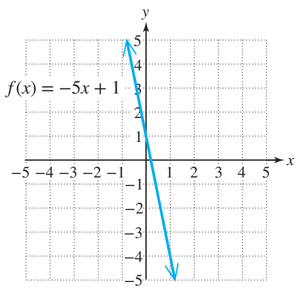
8. Find the y-intercept.

9. Graph the function.

Answers

7. $\left(\frac{1}{5}, 0\right)$ **8.** (0, 1)

9.



Example 5 **Finding x- and y-Intercepts**

For the function pictured in Figure 2-43, estimate

a. The real values of x for which $f(x) = 0$.

b. The value of $f(0)$.

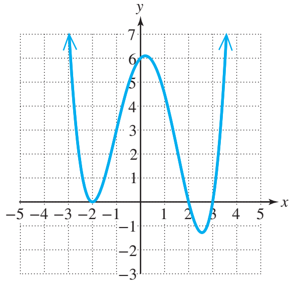


Figure 2-43

Solution:

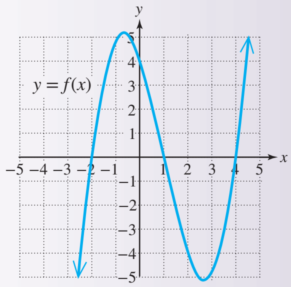
a. The real values of x for which $f(x) = 0$ are the x-intercepts of the function. For this graph, the x-intercepts are located at $x = -2$, $x = 2$, and $x = 3$.

b. The value of $f(0)$ is the value of y at $x = 0$. That is, $f(0)$ is the y-intercept, $f(0) = 6$.

Skill Practice Use the function pictured.

10. a. Estimate the real value(s) of x for which $f(x) = 0$.

b. Estimate the value of $f(0)$.



Answer

10. a. $x = -2$, $x = 1$, and $x = 4$
b. $f(0) = 4$

Section 2.7 Activity

For Exercises A.1–A.3,

a. Identify the graph of the equation as representing a slanted line, a horizontal line, a vertical line, or a parabola.

b. Write the equation using function notation $y = f(x)$.

c. Identify the function as being a constant function (characterized by a horizontal line), a linear function (slanted line), or a quadratic function (parabola).

A.1. $y = -3x - 2$

A.2. $y = 5$

A.3. $y = x^2 - 4x + 3$

A.4. Given the function defined by $f(x) = -3x - 2$,

a. Explain how to find the x-intercept.

b. Find the x-intercept.

c. Explain how to find the y-intercept.

d. Find the y-intercept.

For Exercises A.5–A.6, identify the x- and y-intercepts for the function defined by $y = f(x)$.

A.5. $f(x) = 5$

A.6. $f(x) = x^2 - 4x + 3$

Practice Exercises

Section 2.7

Prerequisite Review

For Exercises R.1–R.4, state whether the equation represents a slanted line, a horizontal line, or a vertical line.

R.1. $3x = 5$

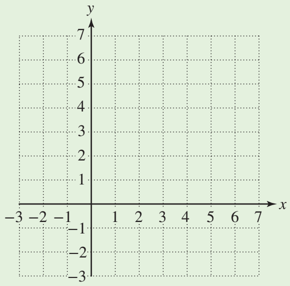
R.2. $y - 3 = 5$

R.3. $3x - 7y = 14$

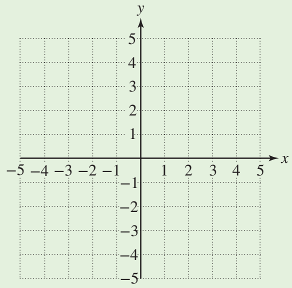
R.4. $y = -\dfrac{1}{2}x - 4$

For Exercises R.5–R.8, find the x- and y-intercepts and graph the equation.

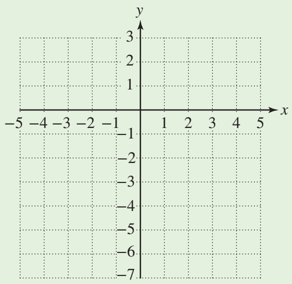
R.5. $y = -\dfrac{3}{5}x + 3$



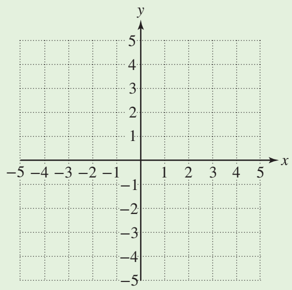
R.6. $y = -\dfrac{1}{4}x - 1$



R.7. $y = -4$



R.8. $x = -2$



Vocabulary and Key Concepts

1. **a.** A function that can be written in the form $f(x) = mx + b$, $m \neq 0$, is a ______________ function.

   b. A function that can be written in the form $f(x) = b$ is a ______________ function.

   c. A function that can be written in the form $f(x) = ax^2 + bx + c$, $a \neq 0$, is a ______________ function.

   d. The graph of a quadratic function is in the shape of a ______________.

   e. To find the x-intercept(s) of a graph defined by $y = f(x)$, find the real solutions to the equation $f(x) =$ ______________.

   f. To find the ______________-intercept of a graph defined by $y = f(x)$, evaluate $f(0)$.

Concept 1: Linear and Constant Functions

2. Given $f(x) = 4$, find the function values.

   a. $f(1)$ **b.** $f(2)$ **c.** $f(3)$

3. Given $g(x) = -3$, find the function values.

   a. $g(1)$ **b.** $g(2)$ **c.** $g(3)$

4. Given $h(x) = 4x$, find the function values.

a. $h(1)$ **b.** $h(2)$ **c.** $h(3)$

5. Given $k(x) = -3x$, find the function values.

a. $k(1)$ **b.** $k(2)$ **c.** $k(3)$

6. Refer to functions f, g, h, and k from Exercises 2–5.

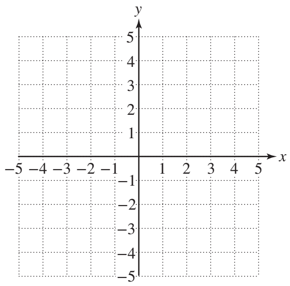
a. Which functions are linear?

b. Which functions are constant?

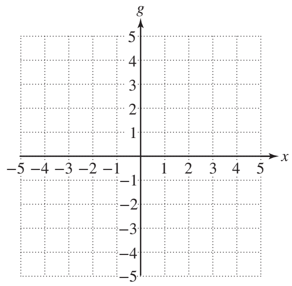
7. Fill in the blank with the word *vertical* or *horizontal*. The graph of a constant function is a _____ line.

8. For the linear function defined by $f(x) = mx + b$, identify the slope and y-intercept.

9. Graph the constant function $f(x) = 2$. Then use the graph to identify the domain and range of f.



10. Graph the linear function $g(x) = -2x + 1$. Then use the graph to identify the domain and range of g.



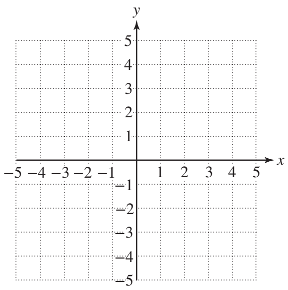
Concept 2: Graphs of Basic Functions

For Exercises 11–16, sketch a graph by completing the table and plotting the points. **(See Examples 1–2.)**

11. $f(x) = \dfrac{1}{x}$

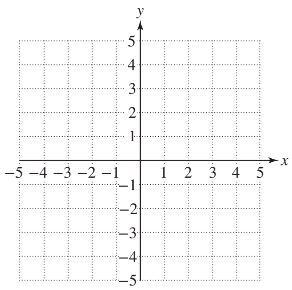
x	$f(x)$
-2	
-1	
$-\frac{1}{2}$	
$-\frac{1}{4}$	

x	$f(x)$
$\frac{1}{4}$	
$\frac{1}{2}$	
1	
2	



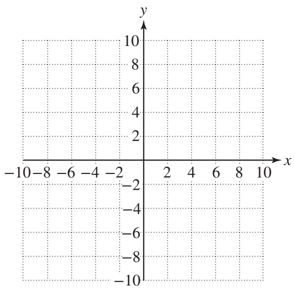
12. $g(x) = |x|$

x	$g(x)$
-2	
-1	
0	
1	
2	



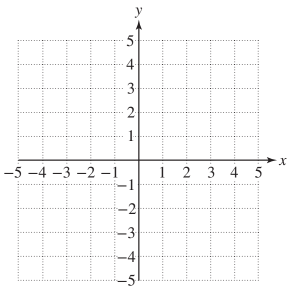
13. $h(x) = x^3$

x	$h(x)$
-2	
-1	
0	
1	
2	



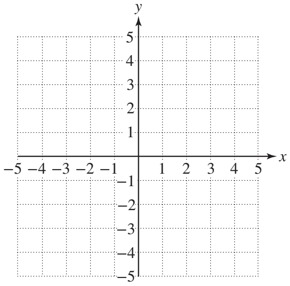
14. $k(x) = x$

x	$k(x)$
-2	
-1	
0	
1	
2	



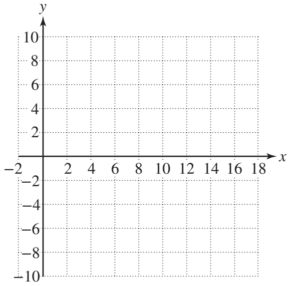
15. $q(x) = x^2$

x	$q(x)$
−2	
−1	
0	
1	
2	



16. $p(x) = \sqrt{x}$

x	$p(x)$
0	
1	
4	
9	
16	



Concept 3: Definition of a Quadratic Function

For Exercises 17–28, determine if the function is constant, linear, quadratic, or none of these. **(See Example 3.)**

17. $f(x) = 2x^2 + 3x + 1$

18. $g(x) = -x^2 + 4x + 12$

19. $k(x) = -3x - 7$

20. $h(x) = -x - 3$

21. $m(x) = \dfrac{4}{3}$

22. $n(x) = 0.8$

23. $p(x) = \dfrac{2}{3x} + \dfrac{1}{4}$

24. $Q(x) = \dfrac{1}{5x} - 3$

25. $t(x) = \dfrac{2}{3}x + \dfrac{1}{4}$

26. $r(x) = \dfrac{1}{5}x - 3$

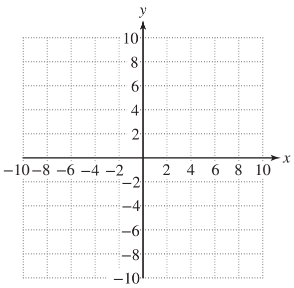
27. $w(x) = \sqrt{4 - x}$

28. $T(x) = -|x + 10|$

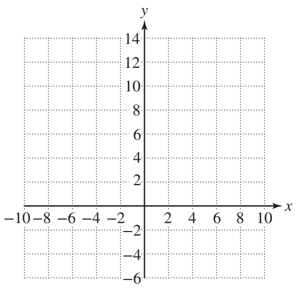
Concept 4: Finding the *x*- and *y*-Intercepts of a Graph Defined by *y* = *f*(*x*)

For Exercises 29–36, find the x- and y-intercepts, and graph the function. **(See Example 4.)**

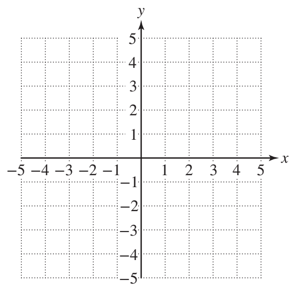
29. $f(x) = 5x - 10$



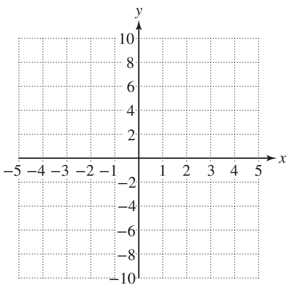
30. $f(x) = -3x + 12$



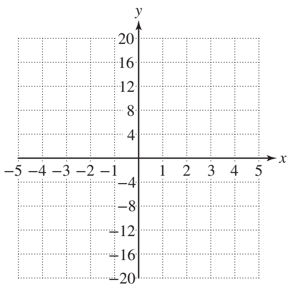
31. $g(x) = -6x + 5$



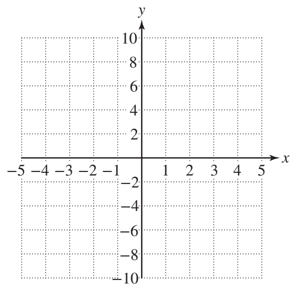
32. $h(x) = 2x + 9$



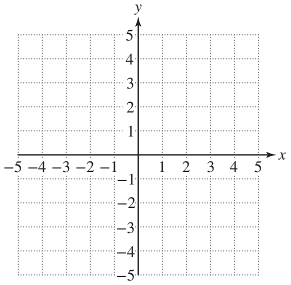
33. $f(x) = 18$



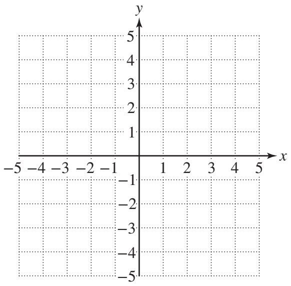
34. $g(x) = -7$



35. $g(x) = \frac{2}{3}x + 2$



36. $h(x) = -\frac{3}{5}x - 3$

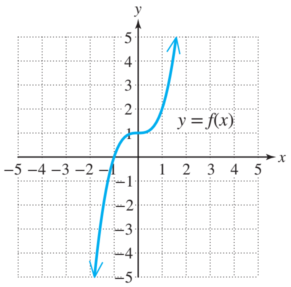


For Exercises 37–42, use the graph to estimate

a. The real values of x for which $f(x) = 0$.

b. The value of $f(0)$. **(See Example 5.)**

37.



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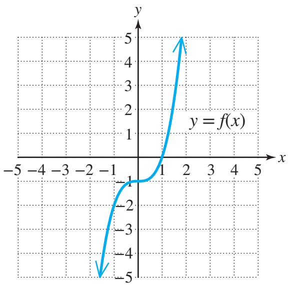

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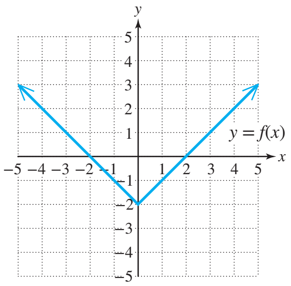

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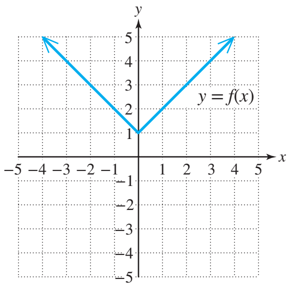

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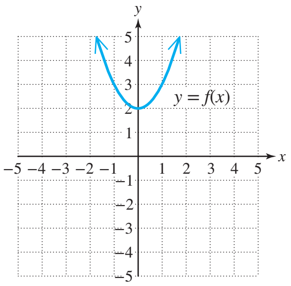

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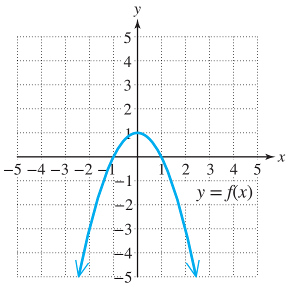


Mixed Exercises

For Exercises 43–52,

a. Identify the domain of the function.

b. Identify the y-intercept.

c. Match the function with its graph by recognizing the basic shape of the graph and using the results from parts (a) and (b). Plot additional points if necessary.

43. $q(x) = 2x^2$

44. $p(x) = -2x^2 + 1$

45. $h(x) = x^3 + 1$

46. $k(x) = x^3 - 2$

47. $r(x) = \sqrt{x + 1}$

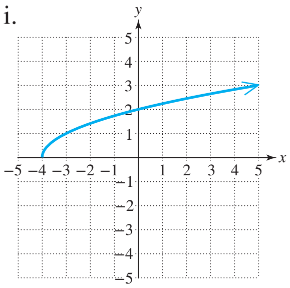
48. $s(x) = \sqrt{x + 4}$

49. $f(x) = \frac{1}{x - 3}$

50. $g(x) = \frac{1}{x + 1}$

51. $k(x) = |x + 2|$

52. $h(x) = |x - 1| + 2$

i.


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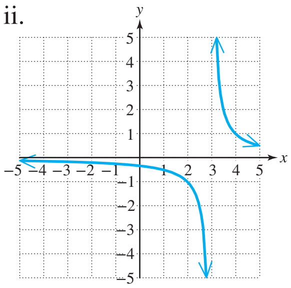

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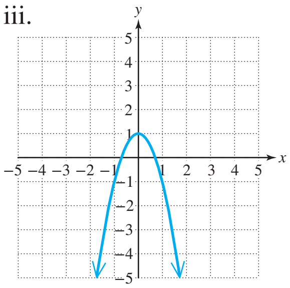

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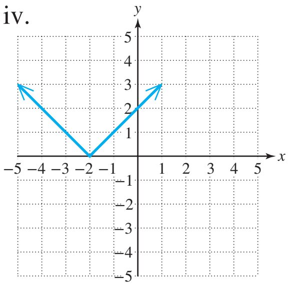
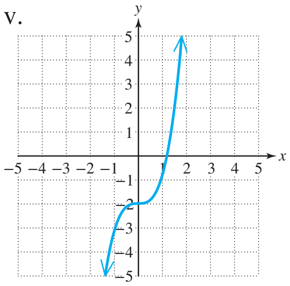
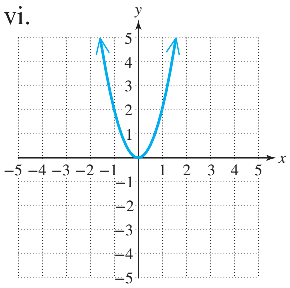
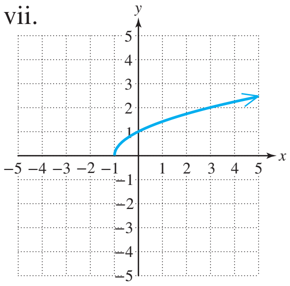
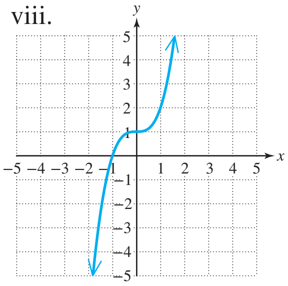
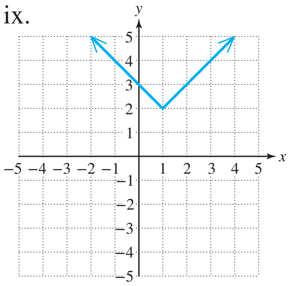


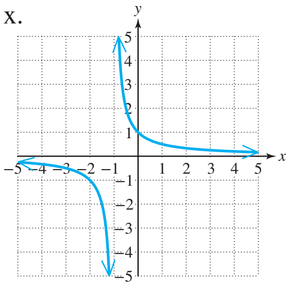
v.


vi.


vii.


viii.


ix.


x.


53. Suppose that a student has an 80% average on all of her chapter tests in her Intermediate Algebra class. This counts as $\frac{3}{4}$ of the student's overall grade. The final exam grade counts as the remaining $\frac{1}{4}$ of the student's overall grade. The student's overall course grade, $G(x)$, can be computed by

$$G(x) = \frac{3}{4}(80) + \frac{1}{4}x, \text{ where } x \text{ is the student's grade on the final exam.}$$

a. Is this function linear, quadratic, or neither?

b. Evaluate $G(90)$ and interpret its meaning in the context of the problem.

c. Evaluate $G(50)$ and interpret its meaning in the context of the problem.

54. The median weekly earnings, $E(x)$ in dollars, for women 16 years and older working full time can be approximated by $E(x) = 0.14x^2 + 7.8x + 540$. For this function, x represents the number of years since 2000. (*Source:* U.S. Department of Labor)

a. Is this function linear, quadratic, or neither?

b. Evaluate $E(10)$ and interpret its meaning in the context of this problem.

c. Evaluate $E(20)$ and interpret its meaning in the context of this problem.

For Exercises 55–60, write a function defined by $y = f(x)$ under the given conditions.

55. The value of $f(x)$ is two more than the square of x.

56. The value of $f(x)$ is the square of the sum of 2 and x.

57. The function f is a constant function passing through the point (4, 3).

58. The function f is a constant function with y-intercept (0, 5).

59. The function f is a linear function with slope $\frac{1}{2}$ and y-intercept $(0, -2)$.

60. The function f is a linear function with slope -4 and y-intercept $(0, \frac{1}{3})$.

Technology Connections

For Exercises 61–66, use a graphing calculator to graph the basic functions. Verify your answers from the Summary of Six Basic Functions and Their Graphs.

61. $f(x) = x$ **62.** $f(x) = x^2$ **63.** $f(x) = x^3$ **64.** $f(x) = |x|$ **65.** $f(x) = \sqrt{x}$ **66.** $f(x) = \dfrac{1}{x}$

For Exercises 67–70, find the x- and y-intercepts using a graphing calculator and the *Value* and *Zero* features.

67. $y = -\dfrac{1}{8}x + 1$ **68.** $y = -\dfrac{1}{2}x - 3$ **69.** $y = \dfrac{4}{5}x + 4$ **70.** $y = \dfrac{7}{2}x - 7$

Problem Recognition Exercises

Characteristics of Relations

Exercises 1–15 refer to the sets of points (x, y) described in a–h.

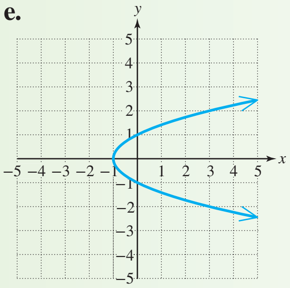
a. $\{(0, 8), (1, 4), (\frac{1}{2}, 4), (-3, 5), (2, 1)\}$

b. $\{(-6, 4), (2, 3), (-9, 6), (2, 1), (0, 10)\}$

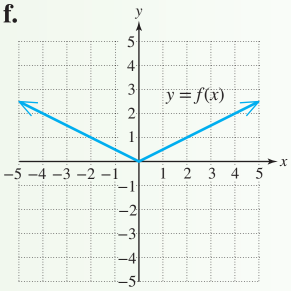
c. $c(x) = 3x^2 - 2x - 1$

d. $d(x) = 5x - 9$

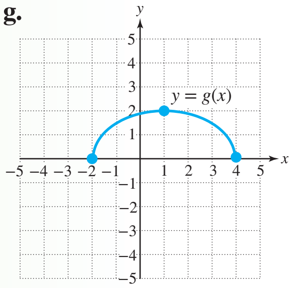
e.



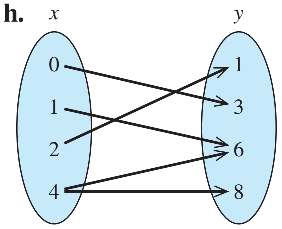
f.



g.



h.



1. Which relations define y as a function of x?

2. Which relations contain the point (2, 1)?

3. Use relation (c) to find $c(-1)$.

4. Use relation (f) to find $f(-4)$.

5. Find the domain of relation (a).

6. Find the range of relation (b).

7. Find the domain of relation (g).

8. Find the range of relation (f).

9. Use relation (h) to complete the ordered pair (__, 3).

10. Find the x-intercept(s) of the graph of relation (d).

11. Find the y-intercept(s) of the graph shown in relation (e).

12. Use relation (g) to determine the value of x such that $g(x) = 2$.

13. Which relation describes a quadratic function?

14. Which relation describes a linear function?

15. Use relation (d) to find the value of x such that $d(x) = 6$.

Chapter 2 Summary

Section 2.1 Linear Equations in Two Variables

Key Concepts

A **linear equation in two variables** can be written in the form $Ax + By = C$, where A, B, and C are real numbers and A and B are not both zero.

The graph of a linear equation in two variables is a line and can be represented in a rectangular coordinate system.

An ***x*-intercept** is a point $(a, 0)$ where a graph intersects the x-axis. Given an equation of a graph, to find an x-intercept, substitute 0 for y and solve for x.

A ***y*-intercept** is a point $(0, b)$ where a graph intersects the y-axis. Given an equation of a graph, to find a y-intercept, substitute 0 for x and solve for y.

An equation of a **vertical line** can be written in the form $x = k$, where k is a constant.

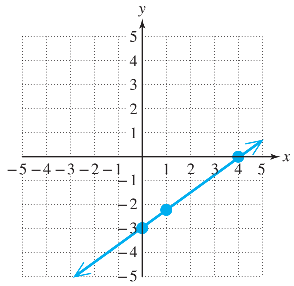
An equation of a **horizontal line** can be written in the form $y = k$, where k is a constant.

Examples

Example 1

To graph the equation $3x - 4y = 12$, we can construct a table of points.

x	y
0	-3
4	0
1	$-\frac{9}{4}$



Example 2

Given $2x + 3y = 8$, find the x- and y-intercepts.

x-intercept: $2x + 3(0) = 8$

$2x = 8$

$x = 4 \quad (4, 0)$

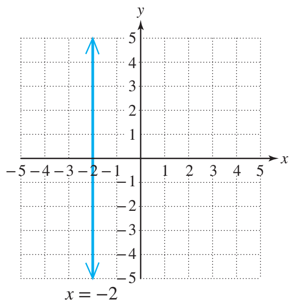
y-intercept: $2(0) + 3y = 8$

$3y = 8$

$y = \frac{8}{3} \quad \left(0, \frac{8}{3}\right)$

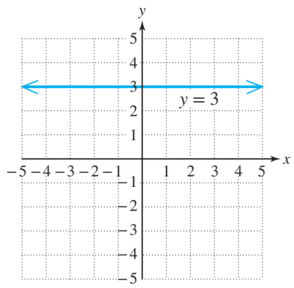
Example 3

Graph $x = -2$.



Example 4

Graph $y = 3$.



Section 2.2 Slope of a Line and Rate of Change

Key Concepts

The **slope** m of a line passing through two distinct points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_2 - x_1 \neq 0$$

The slope of a line may be positive, negative, zero, or undefined.

Two parallel (nonvertical) lines have the same slope: $m_1 = m_2$.

Two lines are perpendicular if the slope of one line is the opposite of the reciprocal of the slope of the other line:

$$m_1 = -\frac{1}{m_2} \text{ or equivalently, } m_1 m_2 = -1$$

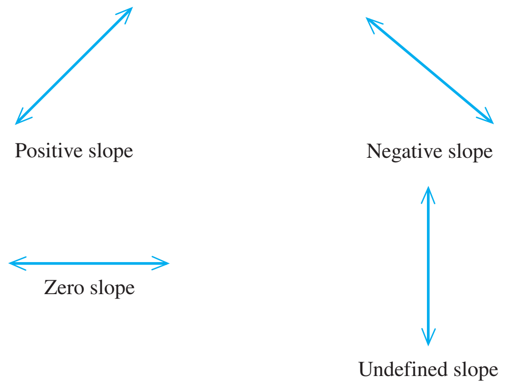
Examples

Example 1

The slope of the line passing through $(1, -3)$ and $(-3, 7)$ is

$$m = \frac{7 - (-3)}{-3 - 1} = \frac{10}{-4} = -\frac{5}{2}$$

Example 2



Example 3

The slopes of two lines are given. Determine whether the lines are parallel, perpendicular, or neither.

a.	$m_1 = -7$	and	$m_2 = -7$	Parallel
b.	$m_1 = -\frac{1}{5}$	and	$m_2 = 5$	Perpendicular
c.	$m_1 = -\frac{3}{2}$	and	$m_2 = -\frac{2}{3}$	Neither

Section 2.3 Equations of a Line

Key Concepts

Standard Form: $Ax + By = C$ (A and B are not both zero)

Horizontal line: $y = k$

Vertical line: $x = k$

Slope-intercept form: $y = mx + b$

Point-slope formula: $y - y_1 = m(x - x_1)$

Slope-intercept form is used to identify the slope and y-intercept of a line when the equation is given. Slope-intercept form can also be used to graph a line.

The point-slope formula can be used to construct an equation of a line, given a point and a slope.

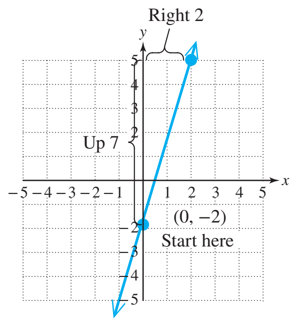
Examples

Example 1

Find the slope and y-intercept. Then graph the equation.

$$7x - 2y = 4 \qquad \text{Solve for } y.$$
$$-2y = -7x + 4$$
$$y = \frac{7}{2}x - 2 \qquad \text{Slope-intercept form}$$

The slope is $\frac{7}{2}$; the y-intercept is $(0, -2)$.



Example 2

Find an equation of the line passing through the point $(2, -3)$ and having slope $m = -4$.

Using the point-slope formula gives

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = -4(x - 2)$$
$$y + 3 = -4x + 8$$
$$y = -4x + 5$$

Section 2.4 Applications of Linear Equations and Modeling

Key Concepts

A linear model can be constructed to describe data for a given situation.

- Given two points from the data, use the point-slope formula to find an equation of the line.

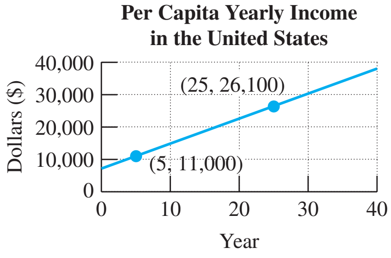
- Interpret the meaning of the slope and y-intercept in the context of the problem.

- Use the equation to predict values.

Examples

Example 1

The per capita income in the United States has been rising linearly over selected years. In the graph, x represents the number of years since the start of the study, and y represents average income in dollars.



Write an equation of the line, using the points (5, 11,000) and (25, 26,100).

Slope: $\dfrac{26{,}100 - 11{,}000}{25 - 5} = \dfrac{15{,}100}{20} = 755$

$y - 11{,}000 = 755(x - 5)$

$y - 11{,}000 = 755x - 3775$

$y = 755x + 7225$

- The slope 755 indicates that the average income has increased at a rate of \$755 per year.
- The y-intercept (0, 7225) means that the average income at the beginning of the study ($x = 0$) was \$7225.

By substituting different values of x, the equation can be used to approximate the average income for that year.
For year 30, we have:

$y = 755(30) + 7225$

$y = 29{,}875$

The average per capita income in year 30 is approximately \$29,875.

Section 2.5 Introduction to Relations

Key Concepts

A set of ordered pairs (x, y) is called a **relation in x and y**.

The **domain** of a relation is the set of first components in the ordered pairs in the relation. The **range** of a relation is the set of second components in the ordered pairs.

Examples

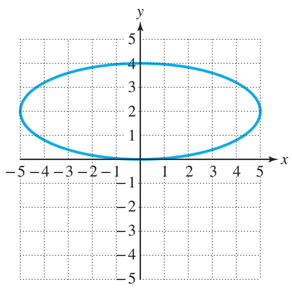
Example 1

Let $A = \{(0, 0), (1, 1), (2, 4), (3, 9), (-1, 1), (-2, 4)\}$.

Domain of A: $\{0, 1, 2, 3, -1, -2\}$

Range of A: $\{0, 1, 4, 9\}$

Example 2



Domain: $[-5, 5]$
Range: $[0, 4]$

Section 2.6 Introduction to Functions

Key Concepts

Given a relation in x and y, we say "y is a **function** of x" if, for each element x in the domain, there is exactly one value of y in the range.

Note: This means that no two ordered pairs may have the same first coordinate and different second coordinates.

The Vertical Line Test for Functions

Consider a relation defined by a set of points (x, y) in a rectangular coordinate system. The graph defines y as a function of x if no vertical line intersects the graph in more than one point.

Examples

Example 1

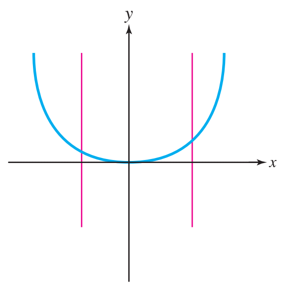
Function $\{(1, 3), (2, 5), (6, 3)\}$

Not a function $\{(1, 3), (2, 5), (1, 4)\}$

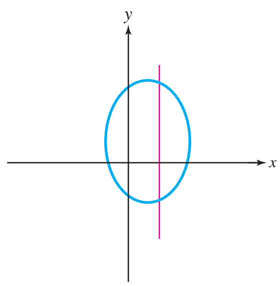
same x

different y

Example 2



Function



Not a function

Function Notation

$f(x)$ is the value of the function f at x.

The domain of a function defined by $y = f(x)$ is the set of x values that when substituted into the function produces a real number. In particular,

- Exclude values of x that make the denominator of a fraction zero.
- Exclude values of x that make the expression within a square root negative.

Example 3

Given $f(x) = -3x^2 + 5x$, find $f(-2)$.

$$\begin{aligned} f(-2) &= -3(-2)^2 + 5(-2) \\ &= -12 - 10 \\ &= -22 \end{aligned}$$

Example 4

Find the domain.

1. $f(x) = \dfrac{x+4}{x-5}$; Domain: $(-\infty, 5) \cup (5, \infty)$
2. $f(x) = \sqrt{x-3}$; Domain: $[3, \infty)$
3. $f(x) = 3x^2 - 5$; Domain: $(-\infty, \infty)$

Section 2.7 Graphs of Functions

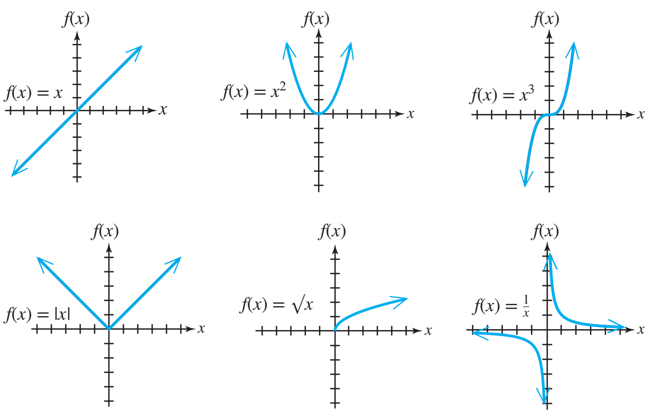
Key Concepts

A function defined by $f(x) = mx + b$ $(m \neq 0)$ is a **linear function**. Its graph is a line with slope m and y-intercept $(0, b)$.

A function defined by $f(x) = k$ is a **constant function**. Its graph is a horizontal line.

A function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is a **quadratic function**. Its graph is a **parabola**.

Graphs of basic functions:



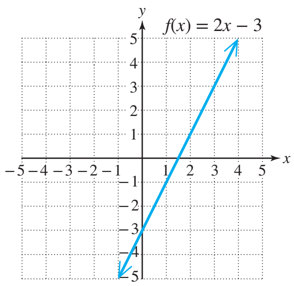
The x-intercepts of a function are determined by finding the real solutions to the equation $f(x) = 0$.

The y-intercept of a function is at $f(0)$.

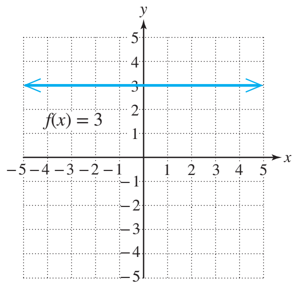
Examples

Example 1

$f(x) = 2x - 3$

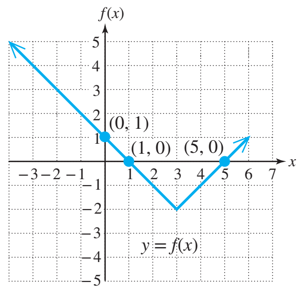


$f(x) = 3$



Example 2

Find the x- and y-intercepts for the function pictured.



$f(x) = 0$, when $x = 1$ and $x = 5$.

The x-intercepts are $(1, 0)$ and $(5, 0)$.

$f(0) = 1$. The y-intercept is $(0, 1)$.

Chapter 2 Review Exercises

Section 2.1

1. Label the following on the diagram:

a. Origin

b. x-Axis

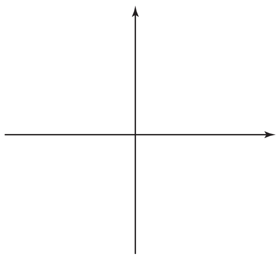
c. y-Axis

d. Quadrant I

e. Quadrant II

f. Quadrant III

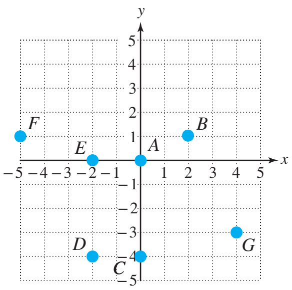
g. Quadrant IV



2. Determine if (2, 5) is a solution to $-2x + 4y = -16$.

3. Determine if $(-3, 4)$ is a solution to $5x = -15$.

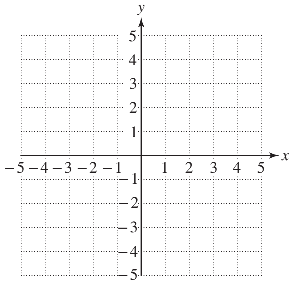
4. Determine the coordinates of the points labeled in the graph.



For Exercises 5–7, complete the table and graph the line defined by the points.

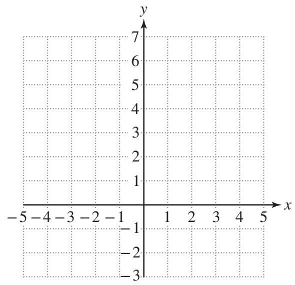
5. $3x - 2y = -6$

x	y
0	
	0
1	



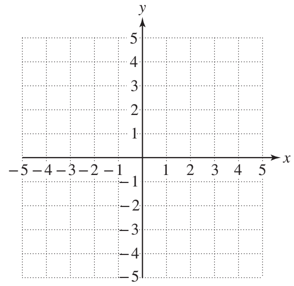
6. $2y - 3 = 10$

x	y
0	
5	
-4	



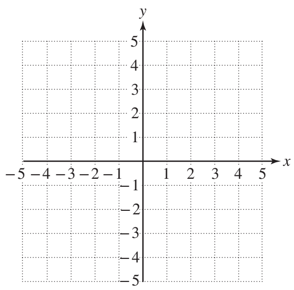
7. $6 - x = 2$

x	y
	0
	1
	-2

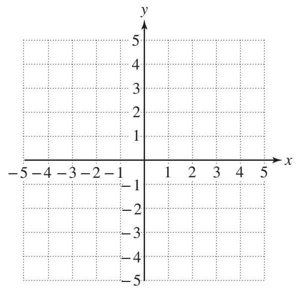


For Exercises 8–11, graph the lines. In each case find at least three points and identify the x- and y-intercepts (if possible).

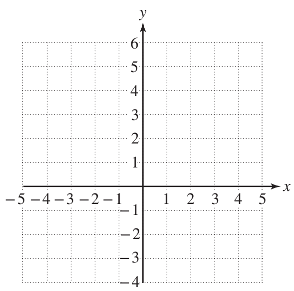
8. $2x = 3y - 6$



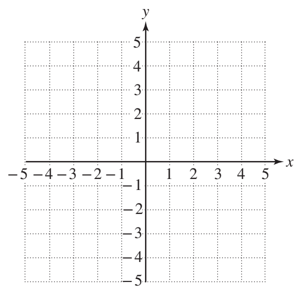
9. $5x - 2y = 0$



10. $2y = 6$



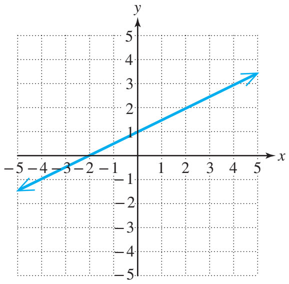
11. $-3x = 6$



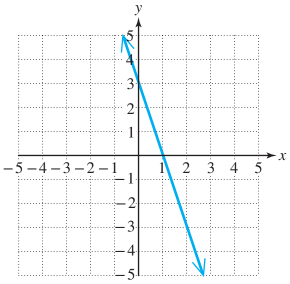
Section 2.2

12. Find the slope of the line.

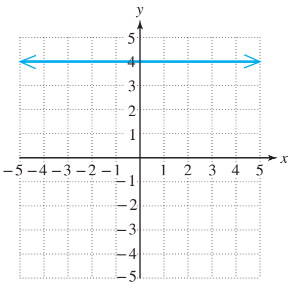
a.



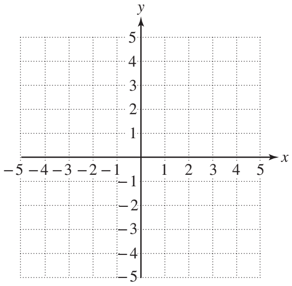
b.



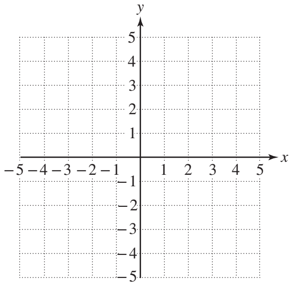
c.



13. Draw a line with slope 2 (answers may vary).



14. Draw a line with slope $-\frac{3}{4}$ (answers may vary).



For Exercises 15–18, find the slope of the line that passes through each pair of points.

15. $(2, 6), (-1, 0)$

16. $(7, 2), (-3, -5)$

17. $(8, 2), (3, 2)$

18. $\left(-4, \frac{1}{2}\right), (-4, 1)$

19. Two points for each of two lines are given. Determine if the lines are parallel, perpendicular, or neither.

L_1: $(4, -6)$ and $(3, -2)$

L_2: $(3, -1)$ and $(7, 0)$

For Exercises 20–22, the slopes of two lines are given. Based on the slopes, are the lines parallel, perpendicular, or neither?

20. $m_1 = -\frac{1}{3}, m_2 = 3$

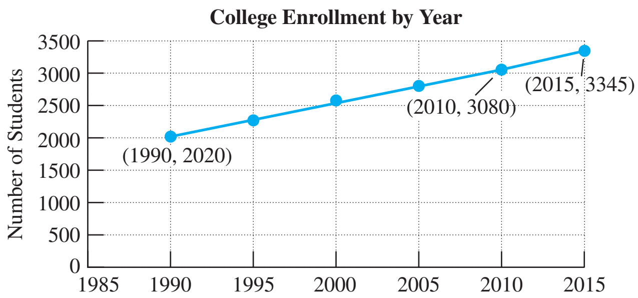
21. $m_1 = \frac{5}{4}, m_2 = \frac{4}{5}$

22. $m_1 = 7, m_2 = 7$

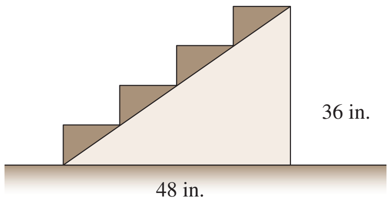
23. The graph indicates that the enrollment for a small college has been increasing linearly since 1990.

a. Use the two data points to find the slope of the line.

b. Interpret the meaning of the slope in the context of this problem.



24. Approximate the slope of the stairway pictured here.



Section 2.3

25. Write a formula.

a. Horizontal line

b. Point-slope formula

c. Standard form

d. Vertical line

e. Slope-intercept form

For Exercises 26–30, write your answer in slope-intercept form or in standard form.

26. Write an equation of the line that has slope $\frac{1}{9}$ and y-intercept $(0, 6)$.

27. Write an equation of the line that has slope $-\frac{2}{3}$ and x-intercept $(3, 0)$.

28. Write an equation of the line that passes through the points $(-8, -1)$ and $(-5, 9)$.

29. Write an equation of the line that passes through the point $(6, -2)$ and is perpendicular to the line $y = -\frac{1}{3}x + 2$.

30. Write an equation of the line that passes through the point $(0, -3)$ and is parallel to the line $4x + 3y = -1$.

31. For each of the given conditions, find an equation of the line

a. Passing through the point $(-3, -2)$ and parallel to the x-axis.

b. Passing through the point $(-3, -2)$ and parallel to the y-axis.

c. Passing through the point $(-3, -2)$ and having an undefined slope.

d. Passing through the point $(-3, -2)$ and having a zero slope.

32. Are any of the lines in Exercise 31 the same?

Section 2.4

33. Ally loves the beach and decides to spend the summer selling various ice cream products on the beach. From her accounting course, she knows that her total cost is calculated as

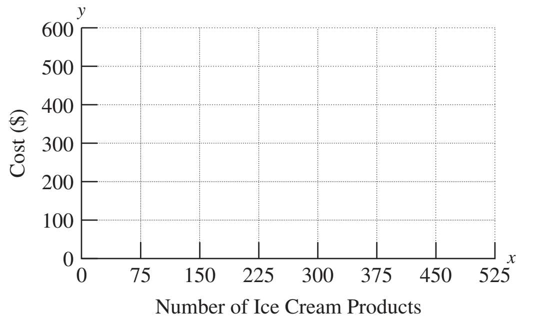
$$\text{Total cost} = \text{fixed cost} + \text{variable cost}$$

She estimates that her fixed cost for the summer season is \$50 per day. She also knows that each ice cream product costs her \$0.75 from her distributor.

Andersen Ross/Getty Images

a. Write a relationship for the daily cost y in terms of the number of ice cream products sold per day x.

b. Graph the equation from part (a) by letting the horizontal axis represent the number of ice cream products sold per day and letting the vertical axis represent the daily cost.



c. What does the y-intercept represent in the context of this problem?

d. What is her cost if she sells 450 ice cream products?

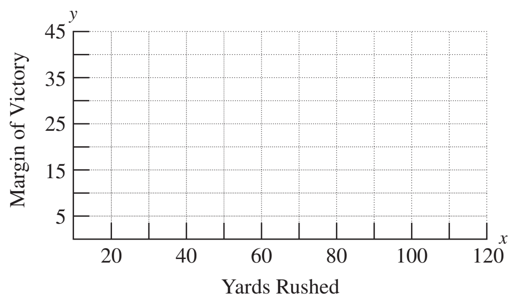
e. What is the slope of the line?

f. What does the slope of the line represent in the context of this problem?

34. The margin of victory for a certain college football team seems to be linearly related to the number of rushing yards gained by the star running back. The table shows the statistics.

Yards Rushed	Margin of Victory
100	20
60	10
120	24
50	7

a. Graph the data to determine if a linear trend exists. Let x represent the number of yards rushed by the star running back and y represent the points in the margin of victory.



b. Find an equation for the line through the points (50, 7) and (100, 20).

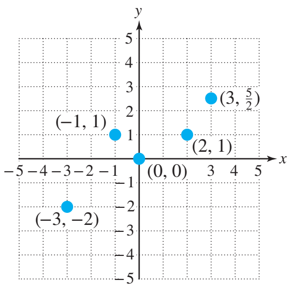
c. Based on the equation, what would be the result of the football game if the star running back did not play?

Section 2.5

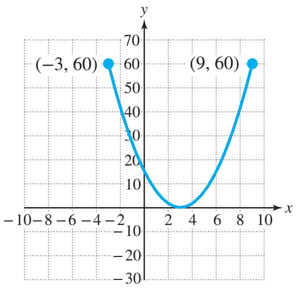
For Exercises 35–38, find the domain and range.

35. $\left\{\left(\frac{1}{3}, 10\right), \left(6, -\frac{1}{2}\right), \left(\frac{1}{4}, 4\right), \left(7, \frac{2}{5}\right)\right\}$

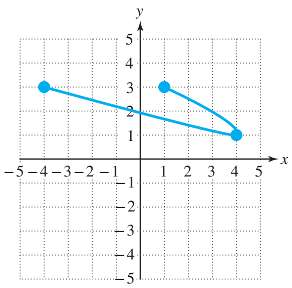
36.



37.

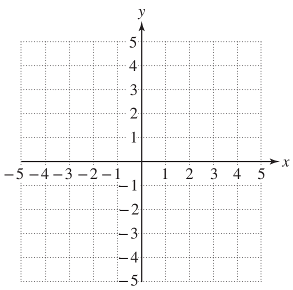


38.

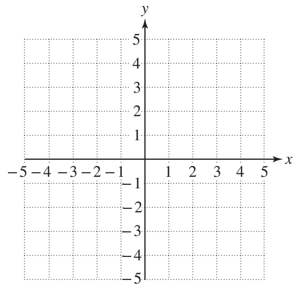


Section 2.6

39. Sketch a relation that is *not* a function. (Answers may vary.)



40. Sketch a relation that *is* a function. (Answers may vary.)



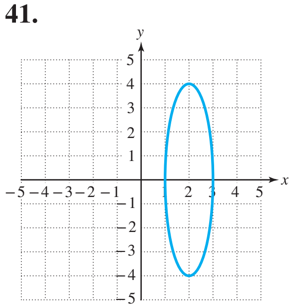
For Exercises 41–46:

a. Determine whether the relation defines y as a function of x.

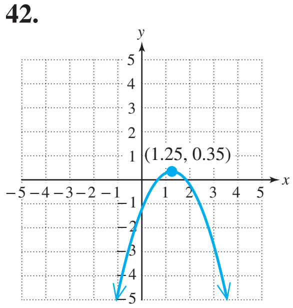
b. Find the domain.

c. Find the range.

41.



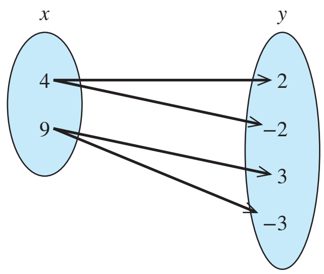
42.



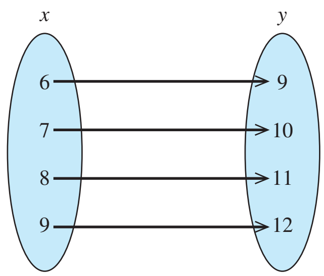
43. {(1, 3), (2, 3), (3, 3), (4, 3)}

44. {(0, 2), (0, 3), (4, 4), (0, 5)}

45.



46.



For Exercises 47–54, find the function values given $f(x) = 6x^2 - 4$.

47. $f(0)$

48. $f(1)$

49. $f(-1)$

50. $f(t)$

51. $f(b)$

52. $f(\pi)$

53. $f(a)$

54. $f(-2)$

For Exercises 55–58, write the domain of each function in interval notation.

55. $g(x) = 7x^3 + 1$

56. $h(x) = \dfrac{x + 10}{x - 11}$

57. $k(x) = \sqrt{x - 8}$

58. $w(x) = \sqrt{x + 2}$

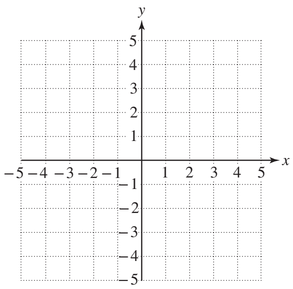
59. Anita is a waitress and makes \$6 per hour plus tips. Her tips average \$5 per table. In one 8-hr shift, Anita's pay can be described by $p(x) = 48 + 5x$, where x represents the number of tables she waits on. Determine how much Anita will earn if she waits on

a. 10 tables **b.** 15 tables **c.** 20 tables

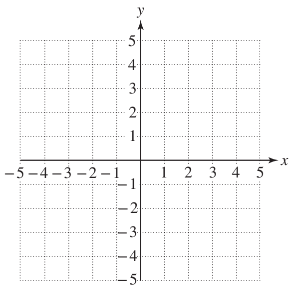
Section 2.7

For Exercises 60–65, sketch the functions from memory.

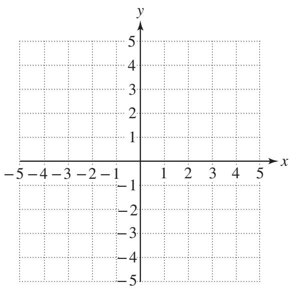
60. $h(x) = x$



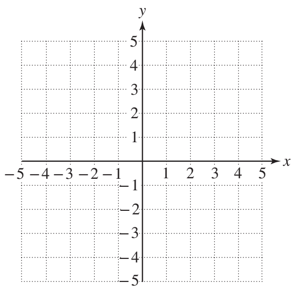
61. $f(x) = x^2$



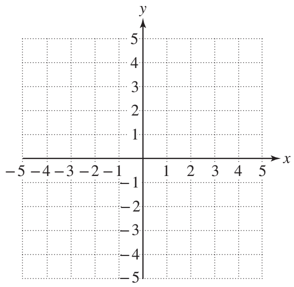
62. $g(x) = x^3$



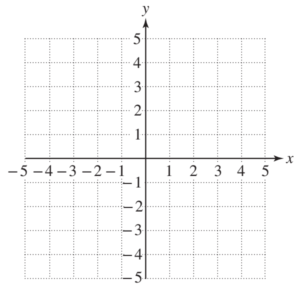
63. $w(x) = |x|$



64. $s(x) = \sqrt{x}$

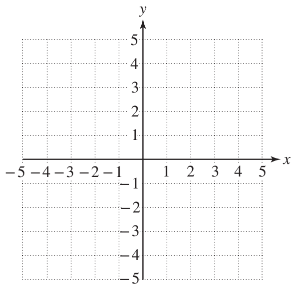


65. $r(x) = \dfrac{1}{x}$

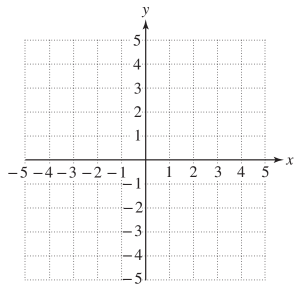


For Exercises 66–67, sketch the functions.

66. $q(x) = 3$



67. $k(x) = 2x + 1$



68. Given: $s(x) = (x - 2)^2$

a. Find $s(4)$, $s(-3)$, $s(2)$, $s(1)$, and $s(0)$.

b. What is the domain of s?

69. Given: $r(x) = 2\sqrt{x - 4}$

a. Find $r(2)$, $r(4)$, $r(5)$, and $r(8)$.

b. What is the domain of r?

70. Given: $h(x) = \dfrac{3}{x - 3}$

a. Find $h(-3)$, $h(0)$, $h(2)$, and $h(5)$.

b. What is the domain of h?

71. Given: $k(x) = -|x + 3|$

a. Find $k(-5)$, $k(-4)$, $k(-3)$, and $k(2)$.

b. What is the domain of k?

For Exercises 72–73, find the x- and y-intercepts.

72. $p(x) = 4x - 7$

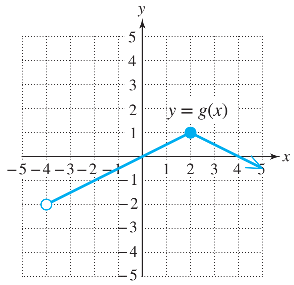
73. $q(x) = -2x + 9$

74. The function defined by $b(t) = 1.64t + 28.3$ represents the per capita consumption of bottled water in the United States since 2010. The values of $b(t)$ are measured in gallons, and $t = 0$ corresponds to the year 2010. (*Source:* U.S. Department of Agriculture)

a. Evaluate $b(0)$ and $b(5)$ and interpret the results in the context of this problem.

b. Determine the slope and interpret its meaning in the context of this problem.

For Exercises 75–80, refer to the graph.



75. Find $g(-2)$.

76. Find $g(4)$.

77. For what value(s) of x is $g(x) = 0$?

78. For what value(s) of x is $g(x) = 4$?

79. Write the domain of g.

80. Write the range of g.

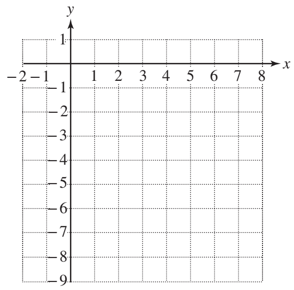
Chapter 2 Test

Study Skills Exercise

Take time to reflect on your test preparation for the first test and whether you achieved the results you desired. Did you feel sufficiently prepared for the test? Could you have done some things differently in studying for the test? Before the next test, examine the test preparation strategies listed below. Determine which ones you used in the past were helpful and consider whether there might be new strategies you can incorporate for the next test.

- Space test preparation over several days before the test.
- Complete and understand all homework problems.
- Review course notes and Chapter Summary.
- Work additional exercises in the Chapter Review Exercises and Chapter Test.
- Use other resources for additional support. These might include visiting your tutoring center, visiting your instructor during office hours, and seeking help from digital resources such as video instruction.

1. Given the equation $x - \frac{2}{3}y = 6$, complete the ordered pairs and graph the corresponding points. (0,) (, 0) (, −3)



2. Determine whether the following statements are true or false and explain your answer.

a. The product of the x- and y-coordinates is positive only for points in quadrant I.

b. The quotient of the x- and y-coordinates is negative only for points in quadrant IV.

c. The point $(-2, -3)$ is in quadrant III.

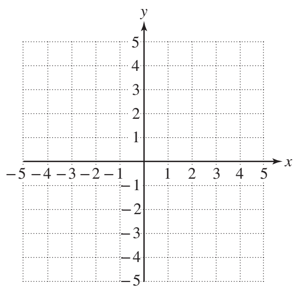
d. The point $(0, 0)$ lies on the x-axis.

3. Determine if $(-4, -1)$ is a solution to $y = -\frac{1}{2}x - 3$.

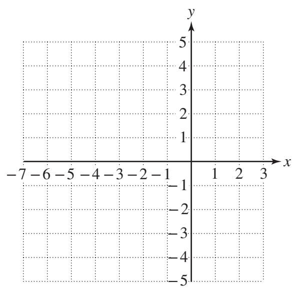
4. Explain the process for finding the x- and y-intercepts.

For Exercises 5–8, identify the x- and y-intercepts (if possible) and graph the line.

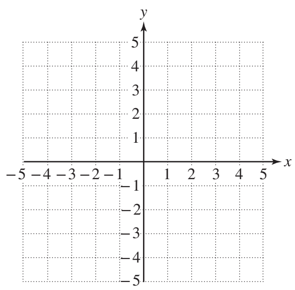
5. $6x - 8y = 24$



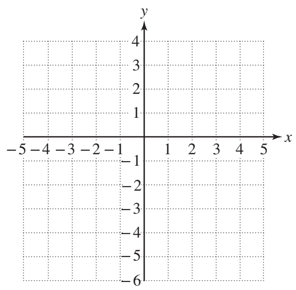
6. $x = -4$



7. $3x = 5y$



8. $2y = -6$



9. Find the slope of the line, given the following information:

   a. The line passes through the points $(7, -3)$ and $(-1, -8)$.

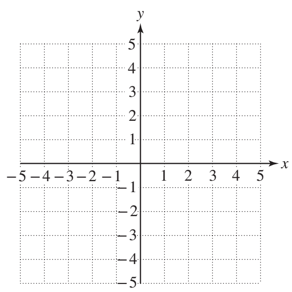
   b. The line is given by $6x - 5y = 1$.

10. Given the equation $-3x + 4y = 4$,

    a. Write the line in slope-intercept form.

    b. Determine the slope and y-intercept.

    c. Graph the line, using the slope and y-intercept.



11. The slope of a line is -7.

    a. Find the slope of a line parallel to the given line.

    b. Find the slope of a line perpendicular to the given line.

12. Describe the relationship of the slopes of

    a. Two parallel lines

    b. Two perpendicular lines

13. Two points are given for each of two lines. Determine if the lines are parallel, perpendicular, or neither.

    L_1: $(4, -4)$ and $(1, -6)$

    L_2: $(-2, 0)$ and $(0, 3)$

14. Determine if the lines are parallel, perpendicular, or neither.

    a. $y = -x + 4$
    $y = x - 3$

    b. $9x - 3y = 1$
    $15x - 5y = 10$

    c. $3y = 6$
    $x = 0.5$

    d. $5x - 3y = 9$
    $3x - 5y = 10$

15. Write an equation that represents a line subject to the following conditions. (Answers may vary.)

    a. A line that does not pass through the origin and has a positive slope

    b. A line with an undefined slope

    c. A line perpendicular to the y-axis. What is the slope of such a line?

    d. A slanted line that passes through the origin and has a negative slope

16. Write an equation of the line containing the points $(2, -3)$ and $(4, 0)$.

17. Write an equation of a line containing $(4, -3)$ and parallel to $6x - 3y = 1$.

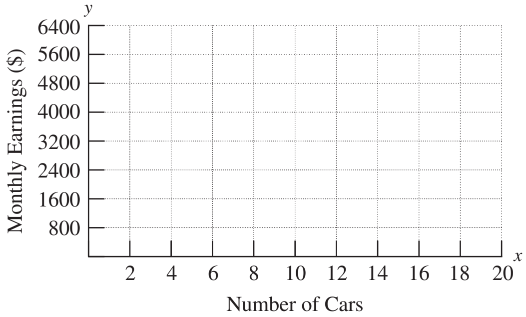
18. Write an equation of the line that passes through the point $\left(8, -\frac{1}{2}\right)$ with slope -2. Write the answer in slope-intercept form.

19. Write an equation of the line that passes through the point $(-10, -3)$ and is perpendicular to $3x + y = 7$. Write the answer in slope-intercept form.

20. Jack sells used cars. He is paid \$800 per month plus \$300 commission for each automobile he sells.

a. Write an equation that represents Jack's monthly earnings y in terms of the number of automobiles he sells x.

b. Graph the linear equation from part (a).

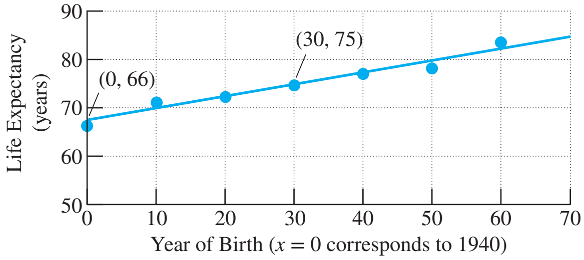


c. What does the y-intercept mean in the context of this problem?

d. How much will Jack earn in a month if he sells 17 automobiles?

21. The graph represents the life expectancy for females in the United States according to birth year. The value $x = 0$ represents a birth year of 1940.

Life Expectancy for Females in the United States According to Birth Year



Source: National Center for Health Statistics

a. Determine the y-intercept from the graph. What does the y-intercept represent in the context of this problem?

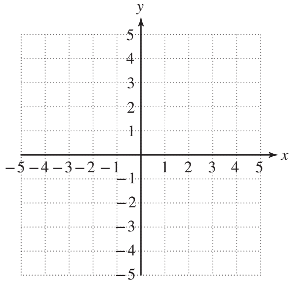
b. Using the two points (0, 66) and (30, 75), determine the slope of the line. What does the slope of the line represent in the context of this problem?

c. Use the y-intercept and the slope found in parts (a) and (b) to write an equation of the line by letting x represent the year of birth and y represent the corresponding life expectancy.

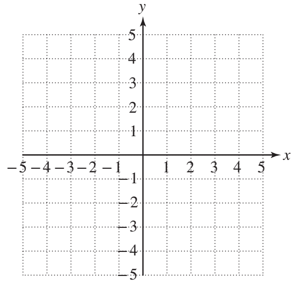
d. Using the linear equation from part (c), approximate the life expectancy for women born in the United States in 1994. How does your answer compare with the reported life expectancy of 79 yr?

For Exercises 22–25, graph the functions.

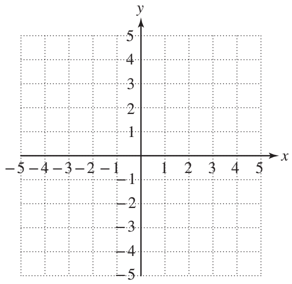
22. $f(x) = -3x - 1$



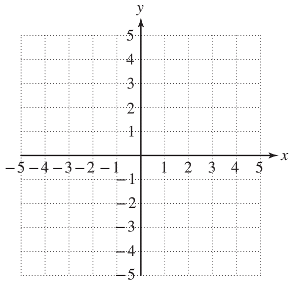
23. $k(x) = -2$



24. $p(x) = x^2$

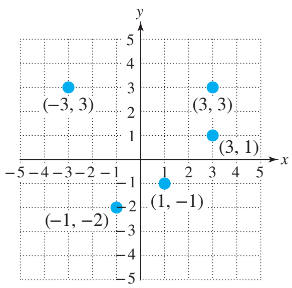


25. $w(x) = |x|$

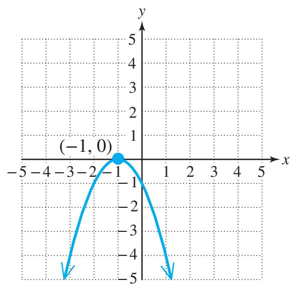


For Exercises 26–27, **a.** determine if the relation defines y as a function of x, **b.** identify the domain, and **c.** identify the range.

26.



27.



For Exercises 28–30, write the domain in interval notation.

28. $f(x) = \dfrac{x - 5}{x + 7}$

29. $f(x) = \sqrt{x + 7}$

30. $h(x) = (x + 7)(x - 5)$

31. Given: $r(x) = x^2 - 2x + 1$

a. Find $r(-2)$, $r(0)$, and $r(3)$.

b. What is the domain of r?

32. The function defined by $s(t) = -0.008t + 0.96$ approximates the per capita consumption of milk per day in the United States, t years after the study began. The values of $s(t)$ are measured in cups. (*Source:* U.S. Department of Agriculture)

a. Evaluate $s(0)$ and $s(20)$ and interpret the results in the context of this problem.

b. Determine the slope and interpret its meaning in the context of this problem.

For Exercises 33–36, determine if the function is constant, linear, quadratic, or none of these.

33. $f(x) = -3x^2$

34. $g(x) = -3x$

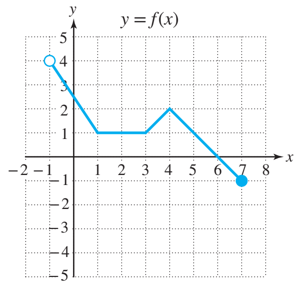
35. $h(x) = -3$

36. $k(x) = -\dfrac{3}{x}$

37. Explain how to find the x- and y-intercepts of the graph defined by $y = f(x)$.

38. Find the x- and y-intercepts for $f(x) = \dfrac{3}{4}x + 9$.

For Exercises 39–46, refer to the graph.



39. Find $f(1)$.

40. Find $f(4)$.

41. Write the domain of f.

42. Write the range of f.

43. Answer true or false. The value $y = 5$ is in the range of f.

44. Find the x-intercept.

45. For what value(s) of x is $f(x) = 0$?

46. For what value(s) of x is $f(x) = 1$?

Systems of Linear Equations and Inequalities

3

CHAPTER OUTLINE

Mathematics in Aeronautics

When a commercial airliner transports passengers between two cities, a flight plan is made outlining the route, cruising altitude, estimated time of the flight, and other important information. The flight time is based largely on the cruising speed of the plane once the plane is at the established flying altitude. The cruising speed (relative to the ground) is based on the plane's speed in still air, p, and the speed of the wind, w. If the plane is flying with the wind at its back (a tailwind), the net speed of the plane is $p + w$. If the plane is flying against the wind (a headwind), the net speed of the plane is $p - w$.

In the United States, prevailing winds aloft are typically from west to east. So suppose that a nonstop flight from New York to Los Angeles takes 5.5 hr, the return trip takes 4.5 hr, and the distance for each flight is 2475 mi. Using the familiar relationship $d = rt$ (distance equals the average rate multiplied by the time of travel), we can estimate the plane's still air speed and wind speed.

Bet_Noire/iStockphoto/Getty Images

	(rate)(time) = distance
Flight from New York to Los Angeles (against the wind):	$(p - w)(5.5) = 2475$
Flight from Los Angeles to New York (with the wind):	$(p + w)(4.5) = 2475$

These two equations make up a **system of linear equations in two variables**. A system of equations enables us to model a physical situation in which two variables are subject to two constraints. In this case, the variables p and w are subject to the constraints given by the flight times and distance of travel. In this chapter, you will learn several methods to solve a system of linear equations in two variables and will find that the plane's average speed in still air is 500 mph and the wind speed is 50 mph.

Section 3.1 Solving Systems of Linear Equations by the Graphing Method

Concepts

1. Solutions to Systems of Linear Equations
2. Solving Systems of Linear Equations by Graphing

1. Solutions to Systems of Linear Equations

A linear equation in two variables has an infinite number of solutions that form a line in a rectangular coordinate system. Two or more linear equations form a **system of linear equations**. For example:

$$x - 3y = -5$$
$$2x + 4y = 10$$

A **solution to a system of linear equations** is an ordered pair that is a solution to *both* individual linear equations.

Example 1 Determining Solutions to a System of Linear Equations

Determine whether the ordered pairs are solutions to the system.

$$x + y = -6$$
$$3x - y = -2$$

a. $(-2, -4)$ **b.** $(0, -6)$

Solution:

a. Substitute the ordered pair $(-2, -4)$ into both equations:

$$x + y = -6 \longrightarrow (-2) + (-4) \stackrel{?}{=} -6 \checkmark \text{ True}$$
$$3x - y = -2 \longrightarrow 3(-2) - (-4) \stackrel{?}{=} -2 \checkmark \text{ True}$$

Because the ordered pair $(-2, -4)$ is a solution to each equation, it is a solution to the *system* of equations.

b. Substitute the ordered pair $(0, -6)$ into both equations:

$$x + y = -6 \longrightarrow (0) + (-6) \stackrel{?}{=} -6 \checkmark \text{ True}$$
$$3x - y = -2 \longrightarrow 3(0) - (-6) \stackrel{?}{=} -2 \quad \text{False}$$

Because the ordered pair $(0, -6)$ is not a solution to the second equation, it is *not* a solution to the system of equations.

Skill Practice Determine whether the ordered pairs are solutions to the system.

$$3x + 2y = -8$$
$$y = 2x - 18$$

1. $(-2, -1)$ **2.** $(4, -10)$

A solution to a system of two linear equations can be interpreted graphically as a point of intersection between the two lines.

Answers
1. No **2.** Yes

Graphing the lines from Example 1 we see that the point of intersection is $(-2, -4)$. Therefore, we say that the solution set is $\{(-2, -4)\}$. See Figure 3-1.

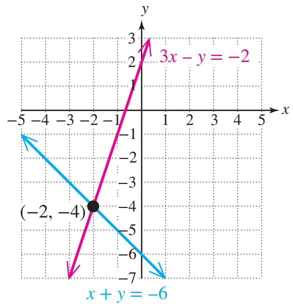
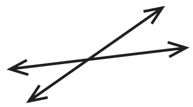


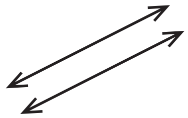
Figure 3-1

When two lines are drawn in a rectangular coordinate system, three geometric relationships are possible:

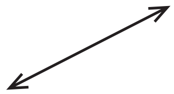
1. Two lines may intersect at *exactly one point.*



2. Two lines may intersect at *no point.* This occurs if the lines are parallel.



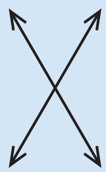
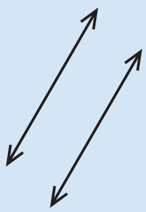
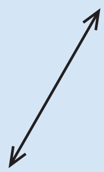
3. Two lines may intersect at *infinitely many points* along the line. This occurs if the equations represent the same line (the lines are coinciding).



If a system of linear equations has one or more solutions, the system is said to be a **consistent system**. If a linear system has no solution, it is said to be an **inconsistent system**.

If two equations represent the same line, then all points along the line are solutions to the system of equations. In such a case, the equations are said to be **dependent**. If two linear equations represent different lines, then the equations are said to be **independent**. The different possibilities for solutions to systems of linear equations are given in Table 3-1.

Table 3-1 **Solutions to Systems of Linear Equations in Two Variables**

One Unique Solution	No Solution	Infinitely Many Solutions
		
One point of intersection	Parallel lines	Coinciding lines
System is consistent.	System is inconsistent.	System is consistent.
Equations are independent.	Equations are independent.	Equations are dependent.

2. Solving Systems of Linear Equations by Graphing

Example 2 **Solving a System of Linear Equations by Graphing**

Solve the system by graphing both linear equations and finding the point(s) of intersection.

$$y = \frac{1}{2}x - 2$$

$$4x + 2y = 6$$

FOR REVIEW

Recall that a linear equation is graphed by finding two or more distinct points on the line. These points can also be found by using a table to find solutions to the equation.

Solution:

To graph each equation, write the equation in slope-intercept form $y = mx + b$.

First equation

$$y = \frac{1}{2}x - 2 \qquad \text{Slope: } \tfrac{1}{2}$$

Second equation

$$4x + 2y = 6$$
$$2y = -4x + 6$$
$$\frac{2y}{2} = \frac{-4x}{2} + \frac{6}{2}$$
$$y = -2x + 3 \qquad \text{Slope: } -2$$

From their slope-intercept forms, we see that the lines have different slopes, indicating that the lines must intersect at exactly one point. We can graph the lines using the slope and y-intercept to find the point of intersection (Figure 3-2).

Avoiding Mistakes

Using graph paper may help you be more accurate when graphing lines. There are many websites from which you can print graph paper.

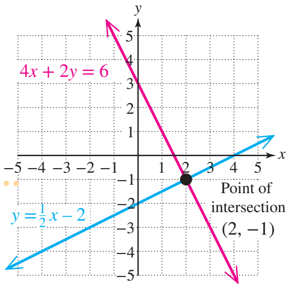


Figure 3-2

The point $(2, -1)$ appears to be the point of intersection. This can be confirmed by substituting $x = 2$ and $y = -1$ into both equations.

$$y = \frac{1}{2}x - 2 \qquad\qquad 4x + 2y = 6$$
$$-1 \stackrel{?}{=} \frac{1}{2}(2) - 2 \qquad\qquad 4(2) + 2(-1) \stackrel{?}{=} 6$$
$$-1 \stackrel{?}{=} 1 - 2 \qquad\qquad 8 - 2 \stackrel{?}{=} 6$$
$$-1 \stackrel{?}{=} -1 \checkmark \text{ True} \qquad\qquad 6 \stackrel{?}{=} 6 \checkmark \text{ True}$$

The solution set is $\{(2, -1)\}$.

Skill Practice Solve by using the graphing method.

3. $y = -3x - 5$
$x - 2y = -4$

TIP: In Example 2, the lines could also have been graphed by using the x- and y-intercepts or by using a table of points. However, the advantage of writing the equations in slope-intercept form is that we can compare the slope *and* y-intercept of each line.

1. If the slopes differ, the lines are different and nonparallel and must cross in exactly one point.
2. If the slopes are the same and the y-intercepts are different, the lines are parallel and do not intersect.
3. If the slopes are the same and the y-intercepts are the same, the two equations represent the same line.

Answer

3. $\{(-2, 1)\}$

Example 3 Solving a System of Linear Functions by Graphing

Solve the system.

$$f(x) = 3$$
$$g(x) = 2x + 1$$

Solution:

This first function can be written as $y = 3$. This is an equation of a horizontal line. Writing the second equation as $y = 2x + 1$, we have a slope of 2 and a y-intercept of $(0, 1)$.

The graphs of the functions are shown in Figure 3-3. The point of intersection is $(1, 3)$. Therefore, the solution set is $\{(1, 3)\}$.

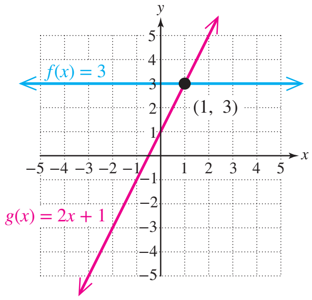


Figure 3-3

TIP: The equations in Example 3 are independent because they represent different lines. The system is consistent because it has a solution.

Skill Practice Solve the system by graphing.

4. $f(x) = 1$

$g(x) = -3x + 4$

Example 4 Solving a System of Linear Equations by Graphing

Solve the system by graphing.

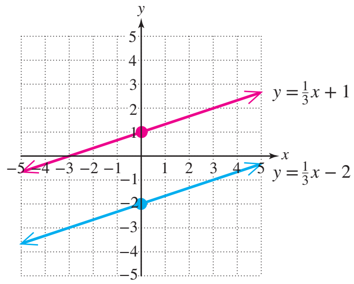
$$-x + 3y = -6$$
$$6y = 2x + 6$$

Solution:

To graph the lines, write each equation in slope-intercept form.

$$-x + 3y = -6 \qquad\qquad 6y = 2x + 6$$
$$3y = x - 6$$
$$\frac{3y}{3} = \frac{x}{3} - \frac{6}{3} \qquad\qquad \frac{6y}{6} = \frac{2x}{6} + \frac{6}{6}$$
$$y = \frac{1}{3}x - 2 \qquad\qquad y = \frac{1}{3}x + 1$$

Because the lines have the same slope but different y-intercepts, they are parallel (Figure 3-4). Two parallel lines do not intersect, which implies that the system has no solution. The system is inconsistent.



The solution set is the empty set, { }.

Figure 3-4

Skill Practice Solve the system by graphing.

5. $2y = 2x$

$-x + y = -3$

Answers

4. $\{(1, 1)\}$

5. The solution set is { }. The system is inconsistent.

Example 5 Solving a System of Linear Equations by Graphing

Solve the system by graphing.

$$x + 4y = 8$$
$$y = -\frac{1}{4}x + 2$$

Solution:

Write the first equation in slope-intercept form. The second equation is already in slope-intercept form.

First equation

$$x + 4y = 8$$
$$4y = -x + 8$$
$$\frac{4y}{4} = \frac{-x}{4} + \frac{8}{4}$$
$$y = -\frac{1}{4}x + 2$$

Second equation

$$y = -\frac{1}{4}x + 2$$

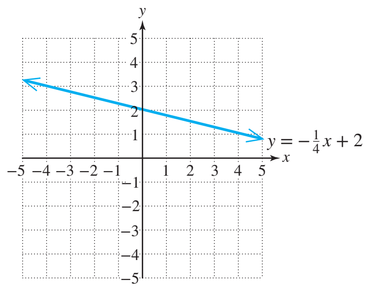


Figure 3-5

Notice that the slope-intercept forms of the two lines are identical. Therefore, the equations represent the same line (Figure 3-5). The equations are dependent, and the solution to the system of equations is the set of all points on the line.

Because the ordered pairs in the solution set cannot all be listed, we can write the solution in set-builder notation. Furthermore, the equations $x + 4y = 8$ and $y = -\frac{1}{4}x + 2$ represent the same line. Therefore, the solution set may be written as $\{(x, y) \mid y = -\frac{1}{4}x + 2\}$ or $\{(x, y) \mid x + 4y = 8\}$.

Skill Practice Solve the system by graphing.

6. $y = \frac{1}{2}x + 1$

$x - 2y = -2$

Answer

6. Infinitely many solutions; $\left\{(x, y) \,\middle|\, y = \frac{1}{2}x + 1\right\}$; dependent equations

Section 3.1 Activity

A.1. Consider the system. $3x + y = 1$
$x - 2y = 5$

a. Write each equation in slope-intercept form.

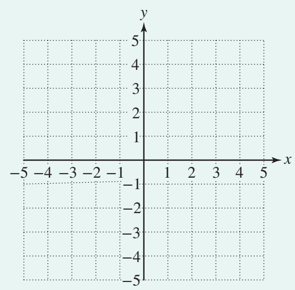
b. Identify the slope of each line.

c. Based on the slope-intercept form of each line, do the lines intersect in a single point, are the lines parallel, or do the equations represent the same line (coinciding lines)?

d. Graph the lines on the same coordinate system.

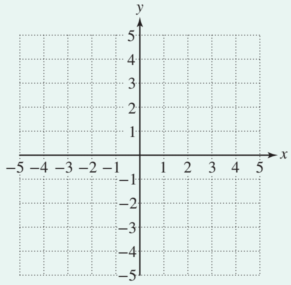
e. If the lines intersect, estimate the point of intersection.

f. The point of intersection is the solution to the system of equations. To verify that your answer is correct, check the ordered pair in both equations.



A.2. Consider the system defined by functions f and g.

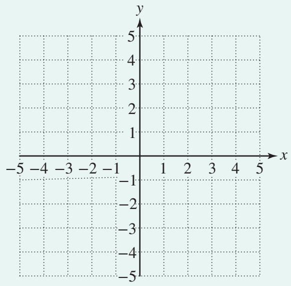
$$f(x) = 1$$
$$g(x) = -\frac{5}{3}x - 4$$



a. The graph of f is a (choose one: horizontal line, vertical line, slanted line). The graph of g is a (choose one: horizontal line, vertical line, slanted line).
b. Based on the answer to part (a), how many solutions does the system have?
c. Graph the functions on the same coordinate system.
d. If the graphs intersect, estimate the point of intersection, check the point in each function, and write the solution set.

A.3. Consider the system.

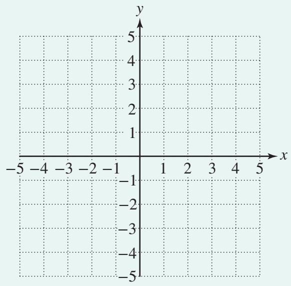
$$x + 2y = 4$$
$$2(x + y) = x - 2$$



a. Write each equation in slope-intercept form.
b. Identify the slope of each line.
c. Based on the slope-intercept form of each line, do the lines intersect in a single point, are the lines parallel, or do the equations represent the same line (coinciding lines)?
d. Graph the lines on the same coordinate system.
e. Is there a solution to this system? Write the solution set.
f. Is the system consistent or inconsistent?

A.4. Consider the system.

$$3x + y = 3$$
$$1 - x = \frac{1}{3}y$$



a. Write each equation in slope-intercept form.
b. Identify the slope of each line.
c. Based on the slope-intercept form of each line, do the lines intersect in a single point, are the lines parallel, or do the equations represent the same line (coinciding lines)?
d. Graph the lines on the same coordinate system.
e. How many solutions are there to this system?
f. Write the solution set.
g. Are the equations in this system dependent or independent?

Practice Exercises — Section 3.1

Study Skills Exercise

Note-taking is a skill that is more challenging than it may initially appear. There is a delicate balance between writing notes and listening to your instructor. But with practice, taking notes in class will help you stay engaged and will enable you to ask and answer meaningful questions.

The following questions are key concepts from this section. Using only your notes, complete the following sentences.

- An ordered pair that satisfies both equations in the system is a ________.
- Suppose that the graphs of the equations in a system represent two parallel lines. This means that the slopes are ________ and the ________ are different.
- Suppose that a system of equations when graphed results in lines that coincide. This indicates that the system has __________ solution(s).
- A system of equations is consistent if _________________.

Were you able to answer all of the questions using only your notes? Or did you find that key concepts are missing? If your notes are lacking important key concepts, refer to the Chapter Summary located at the end of the chapter.

Prerequisite Review

For Exercises R.1–R.4, identify the slope and *x*- and *y*-intercept, if possible.

R.1. $2x + 3y = 12$

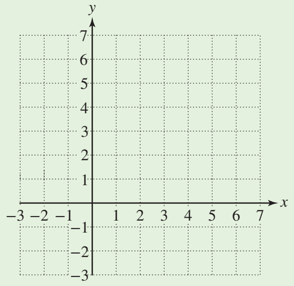
R.2. $-3x - y = 4$

R.3. $2y = 8$

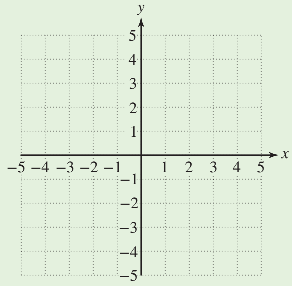
R.4. $3x + 2 = -1$

For Exercises R.5–R.8, graph the equation.

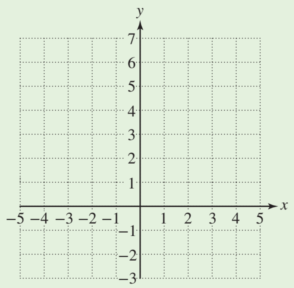
R.5. $2x + 3y = 12$



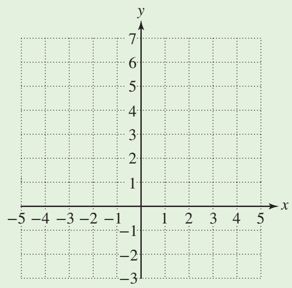
R.6. $-3x - y = 4$



R.7. $2y = 8$



R.8. $3x + 2 = -1$



Vocabulary and Key Concepts

1. **a.** A ______ of linear equations consists of two or more linear equations.

b. A ________ to a system of linear equations is an ordered pair that is a solution to both individual equations in the system.

c. Graphically, a solution to a system of linear equations in two variables is a point where the lines ________.

d. A system of equations that has one or more solutions is said to be ________.

e. The solution set to an inconsistent system of equations is _________.

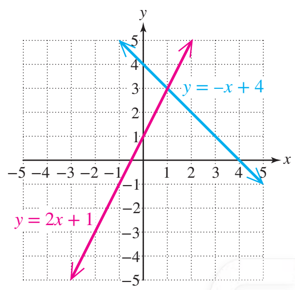
f. Two equations in a system of linear equations in two variables are said to be ________ if they represent the same line.

g. Two equations in a system of linear equations in two variables are said to be ________ if they represent different lines.

Concept 1: Solutions to Systems of Linear Equations

2. From the graph shown, determine the solution to the system.

$$x + y = 4$$
$$y = 2x + 1$$



For Exercises 3–8, determine which points are solutions to the given system. **(See Example 1.)**

3. $y = 8x - 5$
$y = 4x + 3$
$(-1, 13), (-1, 1), (2, 11)$

4. $y = -\frac{1}{2}x - 5$
$y = \frac{3}{4}x - 10$
$(4, -7), (0, -10), \left(3, -\frac{9}{2}\right)$

5. $2x - 7y = -30$
$y = 3x + 7$
$(0, -30), \left(\frac{3}{2}, 5\right), (-1, 4)$

6. $x + 2y = 4$
$y = -\frac{1}{2}x + 2$
$(-2, 3), (4, 0), \left(3, \frac{1}{2}\right)$

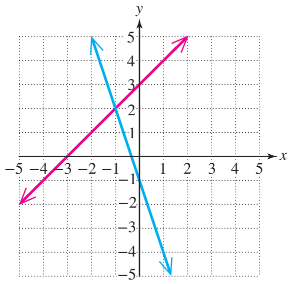
7. $x - y = 6$
$4x + 3y = -4$
$(4, -2), (6, 0), (2, 4)$

8. $x - 3y = 3$
$2x - 9y = 1$
$(0, 1), (4, -1), (9, 2)$

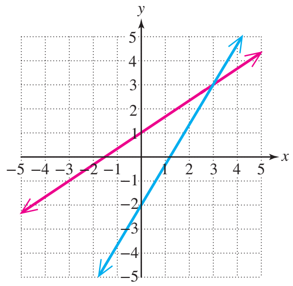
For Exercises 9–14, the graph of a system of linear equations is given.

a. Identify whether the system is consistent or inconsistent.
b. Identify the equations as dependent or independent.
c. Identify the number of solutions to the system.

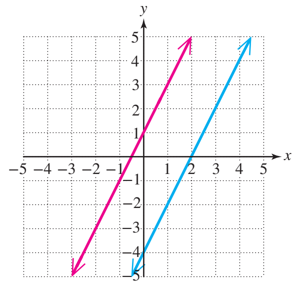
9. $y = x + 3$
$3x + y = -1$



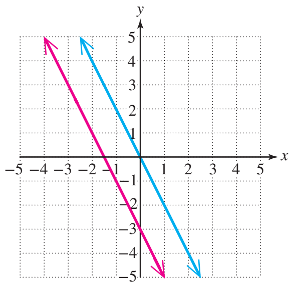
10. $5x - 3y = 6$
$3y = 2x + 3$



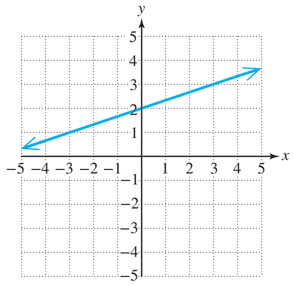
11. $2x = y + 4$
$-4x + 2y = 2$



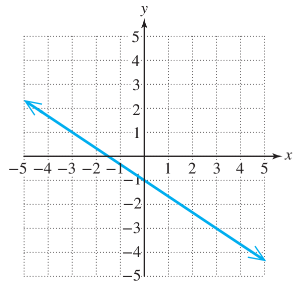
12. $y = -2x - 3$
$-4x - 2y = 0$



13. $y = \frac{1}{3}x + 2$
$-x + 3y = 6$



14. $y = -\frac{2}{3}x - 1$
$-4x - 6y = 6$

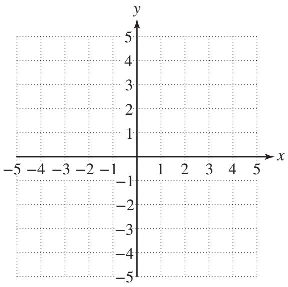


Concept 2: Solving Systems of Linear Equations by Graphing

For Exercises 15–32, solve the system by graphing. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent. **(See Examples 2–5.)**

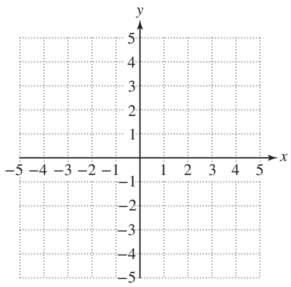
15. $2x + y = -3$

$-x + y = 3$



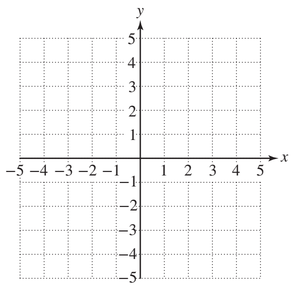
16. $4x - 3y = 12$

$3x + 4y = -16$



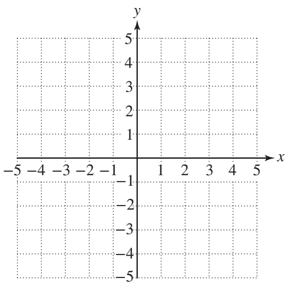
17. $f(x) = -2x + 3$

$g(x) = 5x - 4$



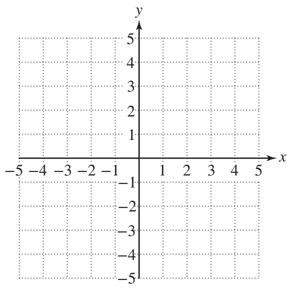
18. $h(x) = 2x + 5$

$g(x) = -x + 2$



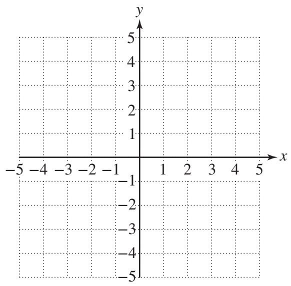
19. $k(x) = \frac{1}{3}x - 5$

$f(x) = -\frac{2}{3}x - 2$



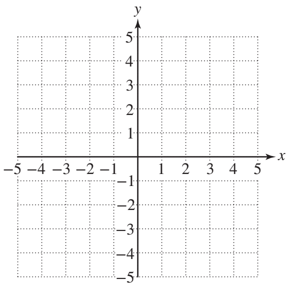
20. $f(x) = \frac{1}{2}x + 2$

$g(x) = \frac{5}{2}x - 2$



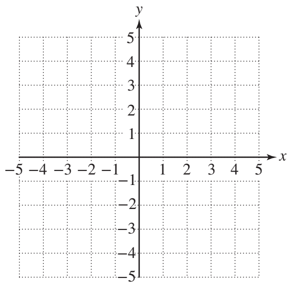
21. $x = 4$

$y = 2x - 3$



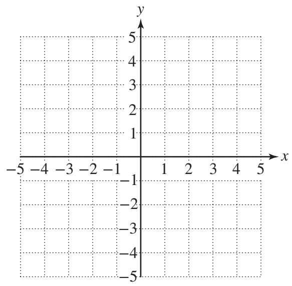
22. $3x + 2y = 6$

$y = -3$



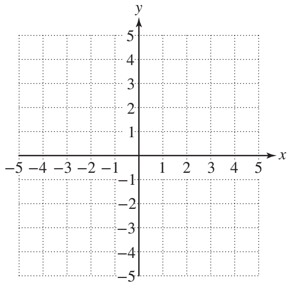
23. $y = -2x + 3$

$-2x = y + 1$



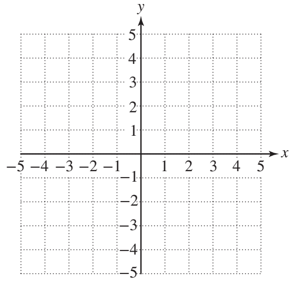
24. $y = \frac{1}{3}x - 2$

$x = 3y - 9$



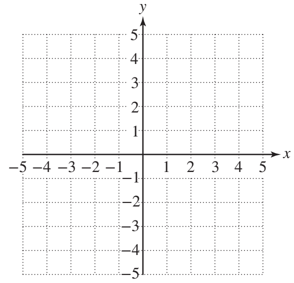
25. $y = \frac{2}{3}x - 1$

$2x = 3y + 3$



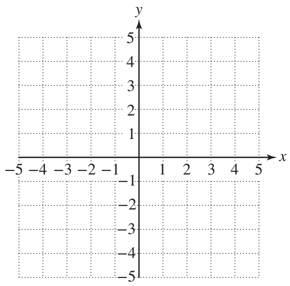
26. $4x = 16 - 8y$

$y = -\frac{1}{2}x + 2$



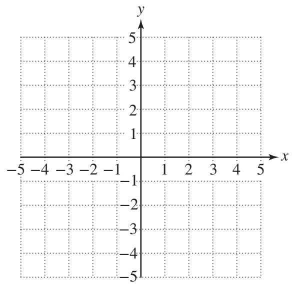
27. $2x = 4$

$\frac{1}{2}y = -1$



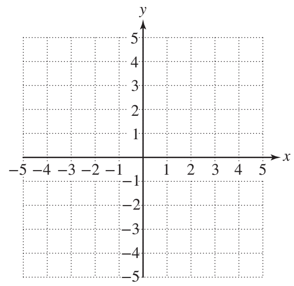
28. $y + 7 = 6$

$-5 = 2x$



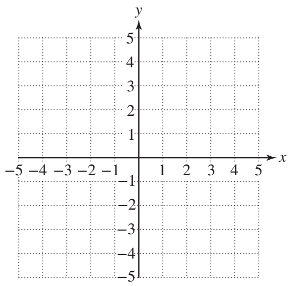
29. $-x + 3y = 6$

$6y = 2x + 12$



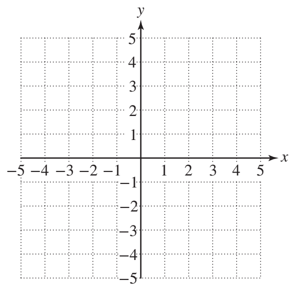
30. $3x = 2y - 4$

$-4y = -6x - 8$



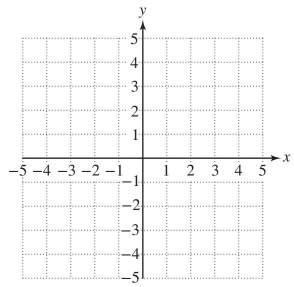
31. $2x - y = 4$

$4x + 2 = 2y$



32. $x = 4y + 4$

$-2x + 8y = -16$



For Exercises 33–36, identify each statement as true or false.

33. A consistent system is a system that always has a unique solution.

34. Dependent equations form a system that has no solution.

35. If two lines coincide, the equations are dependent.

36. If two lines are parallel, the equations are independent.

Technology Connections

For Exercises 37–40, use a graphing calculator to graph each linear equation on the same viewing window. Use a *Trace* or *Intersect* feature to find the point(s) of intersection.

37. $y = 5.62x + 15.46$
$y = -1.96x - 11.07$

38. $y = -2.3x - 5.48$
$y = 4.62x + 26.352$

39. $2.4x - 4.8y = -9.36$
$-1.8x + 5.4y = 12.456$

40. $36x - 90y = -36$
$-15.5x - 5y = -80.75$

Expanding Your Skills

41. Write a system of equations with solution set $\{(4, 5)\}$.

42. Write a system of equations with solution set $\{(-2, 6)\}$.

43. Find C and D such that the solution set to the system is $\{(1, 3)\}$.

$$Cx + 2y = 11$$
$$-3x + Dy = 9$$

44. Find M and N such that the solution set to the system is $\{(2, -4)\}$.

$$3x + My = -22$$
$$Nx + 4y = 6$$

Section 3.2 Solving Systems of Linear Equations by the Substitution Method

Concepts

1. The Substitution Method
2. Solving Inconsistent Systems and Systems of Dependent Equations

1. The Substitution Method

Graphing a system of equations is one method to find the solution of the system. However, sometimes it is difficult to determine the solution using this method because of limitations in the accuracy of the graph. This is particularly true when the coordinates of a solution are not integer values or when the solution is a point not sufficiently close to the origin. Identifying the coordinates of the point $(\frac{3}{17}, -\frac{23}{9})$ or $(-251, 8349)$, for example, might be difficult from a graph.

In this section we will present the first of two algebraic methods to solve a system of equations. This is called the *substitution method*. This technique is particularly important because it can be used to solve more advanced problems, including nonlinear systems of equations.

The first step in the substitution process is to isolate one of the variables from one of the equations. Consider the system

$$x + y = 16$$
$$x - y = 4$$

Solving the first equation for x yields $x = 16 - y$. Then, because x is equal to $16 - y$, the expression $16 - y$ may replace x in the second equation. This leaves the second equation in terms of y only.

First equation: $x + y = 16 \xrightarrow{\text{Solve for } x.} x = \underbrace{16 - y}$

Second equation: $(16 - y) - y = 4$ Substitute $x = 16 - y$.

$16 - 2y = 4$ Solve for y.

$-2y = -12$

$y = 6$

$x = 16 - y$ To find x, substitute $y = 6$ back into the equation $x = 16 - y$.

$x = 16 - (6)$

$x = 10$ Check the ordered pair (10, 6) in both original equations.

$x + y = 16$ $\qquad$ $x - y = 4$

$(10) + (6) \stackrel{?}{=} 16$ ✓ True $\qquad$ $(10) - (6) \stackrel{?}{=} 4$ ✓ True

The solution set is $\{(10, 6)\}$.

FOR REVIEW

Recall that when making a substitution for a variable, always use parentheses. For example, to substitute $16 - y$ for x in the following equation, we have:

$$x - y = 4$$
$$(\quad) - y = 4$$
$$(16 - y) - y = 4$$

Solving a System of Equations by the Substitution Method

Step 1 Isolate one of the variables from one equation.

Step 2 Substitute the quantity found in step 1 into the *other* equation.

Step 3 Solve the resulting equation.

Step 4 Substitute the value found in step 3 back into the equation in step 1 to find the value of the remaining variable.

Step 5 Check the solution in *both* equations, and write the answer as an ordered pair within set notation.

Example 1 Using the Substitution Method to Solve a System of Linear Equations

Solve the system by using the substitution method.

$$3x - 2y = -7$$
$$6x + \quad y = 6$$

Solution:

The y variable in the second equation is the easiest variable to isolate because its coefficient is 1.

$3x - 2y = -7$

$6x + \quad y = 6 \longrightarrow y = \underbrace{-6x + 6}$ **Step 1:** Solve the second equation for y.

$3x - 2(-6x + 6) = -7$ **Step 2:** Substitute the quantity $-6x + 6$ for y in the *other* equation.

$3x + 12x - 12 = -7$

$15x - 12 = -7$

$15x = 5$ **Step 3:** Solve for x.

$x = \dfrac{1}{3}$

Avoiding Mistakes

Do not substitute $y = -6x + 6$ into the same equation from which it came. This mistake will result in an identity:

$$6x + y = 6$$
$$6x + (-6x + 6) = 6$$
$$6x - 6x + 6 = 6$$
$$6 = 6$$

$$y = -6x + 6$$
$$y = -6\left(\frac{1}{3}\right) + 6$$
$$y = -2 + 6$$
$$y = 4$$

Step 4: Substitute $x = \frac{1}{3}$ into the equation $y = -6x + 6$.

$$3x - 2y = -7 \qquad 6x + y = 6$$
$$3\left(\frac{1}{3}\right) - 2(4) \stackrel{?}{=} -7 \qquad 6\left(\frac{1}{3}\right) + 4 \stackrel{?}{=} 6$$
$$1 - 8 \stackrel{?}{=} -7 \checkmark \qquad 2 + 4 \stackrel{?}{=} 6 \checkmark$$

Step 5: Check the ordered pair $(\frac{1}{3}, 4)$ in each original equation.

The solution set is $\{(\frac{1}{3}, 4)\}$.

Skill Practice Solve by using the substitution method.

1. $3x + y = 8$
$x - 2y = 12$

Example 2 Using the Substitution Method to Solve a System of Linear Equations

Solve the system by using the substitution method.

$$-3x + 4y = 9$$
$$x = -\frac{1}{3}y + 2$$

Solution:

$$-3x + 4y = 9$$
$$x = \underbrace{-\frac{1}{3}y + 2}$$

Step 1: In the second equation, x is already isolated.

$$-3\left(-\frac{1}{3}y + 2\right) + 4y = 9$$

Step 2: Substitute the quantity $-\frac{1}{3}y + 2$ for x in the *other* equation.

$$y - 6 + 4y = 9$$

Step 3: Solve for y.

$$5y = 15$$
$$y = 3$$

Now use the known value of y to solve for the remaining variable x.

$$x = -\frac{1}{3}y + 2$$
$$x = -\frac{1}{3}(3) + 2$$

Step 4: Substitute $y = 3$ into the equation $x = -\frac{1}{3}y + 2$.

$$x = -1 + 2$$
$$x = 1$$

Answer

1. $\{(4, -4)\}$

Step 5: Check the ordered pair (1, 3) in each original equation.

$$-3x + 4y = 9 \qquad\qquad x = -\frac{1}{3}y + 2$$

$$-3(1) + 4(3) \stackrel{?}{=} 9 \qquad\qquad 1 \stackrel{?}{=} -\frac{1}{3}(3) + 2$$

$$-3 + 12 \stackrel{?}{=} 9 \checkmark \text{ True} \qquad\qquad 1 \stackrel{?}{=} -1 + 2 \checkmark \text{ True}$$

The solution set is $\{(1, 3)\}$.

Skill Practice Solve by using the substitution method.

2. $x = 2y + 3$

$4x - 2y = 0$

2. Solving Inconsistent Systems and Systems of Dependent Equations

Example 3 Solving an Inconsistent System

Solve the system by using the substitution method.

$$x = 2y - 4$$
$$-2x + 4y = 6$$

Solution:

$x = 2y - 4$ **Step 1:** The x variable is already isolated.

$-2x + 4y = 6$

$-2(2y - 4) + 4y = 6$ **Step 2:** Substitute the quantity $x = 2y - 4$ into the *other* equation.

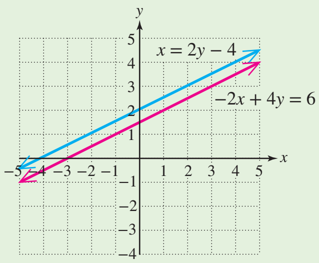
$-4y + 8 + 4y = 6$ **Step 3:** Solve for y.

$8 = 6$ The equation reduces to a contradiction, indicating that the system has no solution. The lines never intersect and must be parallel. The system is *inconsistent*.

There is no solution.
The system is inconsistent.
The solution set is { }.

TIP: The answer to Example 3 can be verified by writing each equation in slope-intercept form and graphing the equations.

Equation 1	**Equation 2**
$x = 2y - 4$	$-2x + 4y = 6$
$2y = x + 4$	$4y = 2x + 6$
$\frac{2y}{2} = \frac{x}{2} + \frac{4}{2}$	$\frac{4y}{4} = \frac{2x}{4} + \frac{6}{4}$
$y = \frac{1}{2}x + 2$	$y = \frac{1}{2}x + \frac{3}{2}$



Notice that the equations have the same slope, but different y-intercepts; therefore, the lines must be parallel. There is no solution to this system of equations.

FOR REVIEW

Recall that an equation that reduces to a contradiction such as $8 = 6$ has no solution. On the other hand, an equation that reduces to an identity such as $-6 = -6$ or $0 = 0$ has infinitely many solutions.

Answer

2. $\{(-1, -2)\}$

Skill Practice Solve by using the substitution method.

3. $8x - 16y = 3$

$y = \frac{1}{2}x + 1$

Example 4 Solving a System of Dependent Equations

Solve by using the substitution method.

$$4x - 2y = -6$$
$$y - 3 = 2x$$

Solution:

$4x - 2y = -6$

$y - 3 = 2x \longrightarrow y = \underbrace{2x + 3}$ **Step 1:** Solve for one of the variables.

$4x - 2(2x + 3) = -6$ **Step 2:** Substitute the quantity $2x + 3$ for y in the *other* equation.

$4x - 4x - 6 = -6$ **Step 3:** Solve for x. Apply the distributive property to clear the parentheses.

$-6 = -6$

The system reduces to the identity $-6 = -6$. Therefore, the two equations are equivalent. The solution set consists of all points on the common line, giving us an infinite number of solutions. The equations are dependent, and because the equations $4x - 2y = -6$ and $y - 3 = 2x$ represent the same line, the solution set can be written as

$$\{(x, y) \mid 4x - 2y = -6\} \qquad \text{or} \qquad \{(x, y) \mid y - 3 = 2x\}$$

Skill Practice Solve the system by using substitution.

4. $3x + 6y = 12$

$2y = -x + 4$

TIP: We can confirm the results of Example 4 by writing each equation in slope-intercept form. The slope-intercept forms are identical, indicating that the lines are the same.

slope-intercept form

$$4x - 2y = -6 \longrightarrow -2y = -4x - 6 \longrightarrow y = 2x + 3$$
$$y - 3 = 2x \longrightarrow y = 2x + 3$$

Answers

3. No solution; { }; inconsistent system

4. Infinitely many solutions; $\{(x, y) \mid 3x + 6y = 12\}$; dependent equations

Section 3.2 Activity

A.1. Consider the system.
$$y = 2x - 6$$
$$5x + 3y = -7$$

a. The first equation tells us that the variable y is equal to the quantity $2x - 6$. Therefore, the expression $2x - 6$ and the variable y are interchangeable. Write the second equation with y replaced by the quantity $2x - 6$.

b. After making the substitution in part (a), the equation should now have only one variable. Solve the equation for x.

c. Substitute the value of x you found in part (b) back into the first equation to solve for y. With the values of x and y now known, write the solution set.

d. Check the ordered pair in both equations.

A.2. Consider the system.
$$7x - 3y = -36$$
$$x + 4y = 17$$

a. Of the four variable terms, which is the easiest variable to isolate?

b. Solve the system by using the substitution method.

c. Check the ordered pair from part (b) in both equations.

A.3. Consider the system.
$$x - \frac{3}{7}y = 1$$
$$7x = 3y - 7$$

a. Of the four variable terms, which variable is the easiest to isolate?

b. Solve the system by using the substitution method.

c. How many solutions are there to this system?

d. Is the system consistent or inconsistent?

A.4. Consider the system.
$$\frac{4}{5}x - y = 1$$
$$4x = 5(y + 1)$$

a. Of the four variable terms, which variable is the easiest to isolate?

b. Solve the system by using the substitution method.

c. How many solutions are there to this system?

d. Are the equations in this system dependent or independent?

Practice Exercises Section 3.2

Study Skills Exercise

Spaced practice is an effective memorization technique in mathematics. In order to become proficient in mathematics, you should work on mathematics frequently and in spaced sessions. Repetition of similar problems should also occur. This will increase your understanding of the material and help you retain it for longer periods of time.

Evaluate your schedule to see how you are spending your time outside of the classroom. Ask yourself the following questions.

- How many hours are being spent outside of class studying math?
- Is my study time spread out over the course of a week, or do I cram right before the test?
- Am I retaining material, or am I forgetting the majority of the material after a day or two?

If you have not already done so, plan frequent, short intervals of study time in your weekly schedule.

Prerequisite Review

For Exercises R.1–R.6, write each pair of lines in slope-intercept form. Then identify whether the lines intersect in exactly one point or if the lines are parallel or coinciding.

R.1. $3x - y = 4$
$2y = 6x - 8$

R.2. $4x + 3y = 12$
$3(y - 4) = -4x$

R.3. $2x - 3y = 5$
$5x + 6y = 26$

R.4. $4x - y = 13$
$3x + 5y = -19$

R.5. $5y = 3x$
$6x - 10y = 8$

R.6. $8x = 2y - 4$
$4x - y = 0$

For Exercises R.7–R.10, an equation is given along with a numerical value for one of the variables. Solve for the other variable.

R.7. $-2x + 5y = 10$; Given that $y = 4$, solve for x.

R.8. $3x - 4y = -2$; Given that $y = 2$, solve for x.

R.9. $y = -\frac{3}{4}x + 6$; Given that $x = -4$, solve for y.

R.10. $y = \frac{3}{2}x - 5$; Given that $x = 6$, solve for y.

For Exercises R.11–R.16,

a. Identify the equation as a conditional equation, an identity, or a contradiction.
b. Solve the equation.

R.11. $2(x - 5) + 1 = 2x - 6$

R.12. $3y + 7 - 9y = 6(4 - y)$

R.13. $4w - (2 - w) + 8 = 4w + 6 + w$

R.14. $4(1 - n) + 3n = 4n + 4 - 5n$

R.15. $-11x + 4(3x - 1) = 7$

R.16. $2(y + 3) - 5y = -6$

Vocabulary and Key Concepts

1. Fill in the parentheses to show the first step of the substitution method to solve the given system.

$$\begin{aligned} 2x - 4y &= 2 \\ x &= 5 - 2y \end{aligned} \longrightarrow 2(\quad\quad) - 4y = 2$$

2. Fill in the parentheses to show the first step of the substitution method to solve the given system.

$$\begin{aligned} y &= -2x + 7 \\ 3x + 2y &= 10 \end{aligned} \longrightarrow 3x + 2(\quad\quad) = 10$$

3. For the given system, if the value of y in the solution is 1, what is the value of x? Write the solution set.

$$\begin{aligned} 2x - 4y &= 2 \\ x &= 5 - 2y \end{aligned}$$

4. For the given system, if the value of x in the solution is 4, what is the value of y? Write the solution set.

$$\begin{aligned} y &= -2x + 7 \\ 3x + 2y &= 10 \end{aligned}$$

5. When using the substitution method to solve a system of equations and you encounter the statement $0 = 3$, how many solutions does the system have?

6. When using the substitution method to solve a system of equations and you encounter the statement $0 = 0$, how many solutions does the system have?

7. Is the ordered pair $(-2, -5)$ a solution to the system? Why or why not?

$$\begin{aligned} 3x - 4y &= 14 \\ x + 5y &= 6 \end{aligned}$$

Concept 1: The Substitution Method

For Exercises 8–22, solve by using the substitution method. **(See Examples 1–2.)**

8. $4x + 12y = 4$
$y = 5x + 11$

9. $y = -3x - 1$
$2x - 3y = -8$

10. $10y + 34 = x$
$-7x + y = -31$

11. $-3x + 8y = -1$
$4x - 11 = y$

12. $12x - 2y = 0$
$-7x + y = -1$

13. $3x + 12y = 36$
$x - 5y = 12$

14. $x - 3y = -3$
$2x + 3y = -6$

15. $x - y = 8$
$3x + 2y = 9$

16. $5x - 2y = 10$
$y = x - 1$

17. $2x - y = -1$
$y = -2x$

18. $1 + 3y = 10$
$5x + 2y = 6$

19. $2x + 3 = 7$
$3x - 4y = 6$

20. $2x + 3y = 7$
$-5x = 2y - 12$

21. $4x - 5y = 14$
$3y = x - 7$

22. $x + 2y = 0$
$2x - 6y = -15$

Concept 2: Solving Inconsistent Systems and Systems of Dependent Equations

For Exercises 23–30, solve the system using the substitution method. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent. **(See Examples 3–4.)**

23. $2x - 6y = -2$
$x = 3y - 1$

24. $-2x + 4y = 22$
$x = 2y - 11$

25. $y = \frac{1}{7}x + 3$
$x - 7y = -4$

26. $x = -\frac{3}{2}y + \frac{1}{2}$
$4x + 6y = 7$

27. $5x - y = 10$
$2y = 10x - 5$

28. $x + 4y = 8$
$3x = 3 - 12y$

29. $3x - y = 7$
$-14 + 6x = 2y$

30. $x = 4y + 1$
$-12y = -3x + 3$

31. When using the substitution method, explain how to determine whether two linear equations in a system are dependent.

32. When using the substitution method, explain how to determine whether a system of linear equations is inconsistent.

Mixed Exercises

For Exercises 33–56, solve the system using the substitution method. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent.

33. $x = 1.3y + 1.5$
$y = 1.2x - 4.6$

34. $y = 0.8x - 1.8$
$1.1x = -y + 9.6$

35. $y = \frac{2}{3}x - \frac{1}{3}$
$x = \frac{1}{4}y + \frac{17}{4}$

36. $x = \frac{1}{6}y - \frac{5}{3}$
$y = \frac{1}{5}x + \frac{21}{5}$

37. $-2x + y = 4$
$-\frac{1}{4}x + \frac{1}{8}y = \frac{1}{4}$

38. $8x - y = 8$
$\frac{1}{3}x - \frac{1}{24}y = \frac{1}{2}$

39. $3x + 2y = 6$
$y = x + 3$

40. $-x + 4y = -4$
$y = x - 1$

41. $-300x - 125y = 1350$
$y + 2 = 8$

42. $200y = 150x$
$y - 4 = 1$

43. $2x - y = 6$
$\frac{1}{6}x - \frac{1}{12}y = \frac{1}{2}$

44. $x - 4y = 8$
$\frac{1}{16}x - \frac{1}{4}y = \frac{1}{2}$

45. $y = -2.7x - 5.1$
$y = 3.1x - 63.1$

46. $y = 6.8x + 2.3$
$y = -4.1x + 56.8$

47. $4x + 4y = 5$
$x - 4y = -\frac{5}{2}$

48. $-2x + y = -6$
$6x - 13y = -12$

49. $2(x + 2y) = 12$
$-6x = 5y - 8$

50. $5x - 2y = -25$
$10x = 3(y - 10)$

51. $5(3y - 2) = x + 4$
$4y = 7x - 3$

52. $2x = -3(y + 3)$
$3x - 4y = -22$

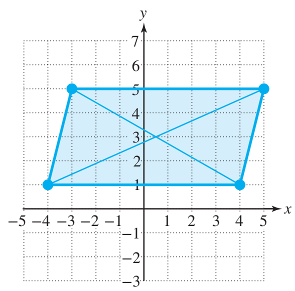
53. $2x - 5 = 7$
$4 = 3y + 1$

54. $-2 = 4 - 2y$
$7x - 5 = -5$

55. $0.01y = 0.02x - 0.11$
$0.3x - 0.5y = 2$

56. $0.3x - 0.4y = 1.3$
$0.01x = 0.03y + 0.01$

57. The **centroid** of a region is the geometric center. For the region shown, the centroid is the point of intersection of the diagonals of the parallelogram.

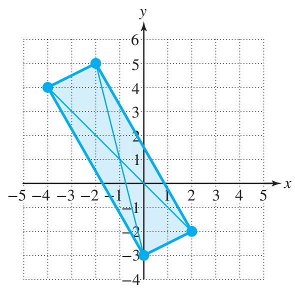


a. Find the slope-intercept form of the line through the points $(-4, 1)$ and $(5, 5)$.

b. Find the slope-intercept form of the line through the points $(-3, 5)$ and $(4, 1)$.

c. Find the centroid of the region.

58. **a.** Find the slope-intercept form of the line through the points $(0, -3)$ and $(-2, 5)$.



b. Find the slope-intercept form of the line through the points $(-4, 4)$ and $(2, -2)$.

c. Find the centroid of the region.

59. The cost to rent an apartment at Glendale Lakes is \$800 per month plus a one-time fee of \$250. An apartment at the Breakers requires a \$500 pet fee plus \$750 per month.

a. Write the cost y to rent for x months for each apartment.

b. Find the number of months for which the total amount spent for each apartment would be the same for renters with pets.

60. The Surfside Motel charges \$159.50 per night plus a one-time parking fee of \$24 for one or more nights. The Tropical Winds Motel charges \$165.50 per night with no parking fee.

a. Write the cost y to stay for x nights for each motel. Assume that the patron needs parking.

b. Find the number of nights for which the cost to stay at each motel would be the same.

Solving Systems of Linear Equations by the Addition Method

Section 3.3

Concepts

1. **The Addition Method**
2. **Solving Inconsistent Systems and Systems of Dependent Equations**

1. The Addition Method

The next method we present to solve systems of linear equations is the *addition method* (sometimes called the elimination method). With the addition method, begin by writing both equations in standard form $Ax + By = C$. Then multiply one or both equations by appropriate constants to create opposite coefficients on either the x or the y variable. Next, add the equations to eliminate the variable having opposite coefficients. This process is demonstrated in Example 1.

Example 1 Solving a System by the Addition Method

Solve the system by using the addition method.

$$3x - 4y = 2$$
$$4x + y = 9$$

Solution:

$$\begin{aligned} 3x - 4y &= 2 \\ 4x + y &= 9 \end{aligned} \xrightarrow{\text{Multiply by 4.}} \begin{aligned} 3x - 4y &= 2 \\ 16x + 4y &= 36 \end{aligned}$$

Multiply the second equation by 4. This makes the coefficients of the y variables *opposites*.

$$\begin{aligned} 3x - 4y &= 2 \\ 16x + 4y &= 36 \\ \hline 19x \quad\;\; &= 38 \end{aligned}$$

Now if the equations are added, the y variable will be eliminated.

$$x = 2$$

Solve for x.

$$\begin{aligned} 3x - 4y &= 2 \\ 3(2) - 4y &= 2 \\ 6 - 4y &= 2 \\ -4y &= -4 \\ y &= 1 \end{aligned}$$

Substitute $x = 2$ back into one of the original equations and solve for y.

Check the ordered pair (2, 1) in each original equation:

$$3x - 4y = 2 \qquad\qquad 4x + y = 9$$
$$3(2) - 4(1) \stackrel{?}{=} 2 \checkmark \text{ True} \qquad 4(2) + (1) \stackrel{?}{=} 9 \checkmark \text{ True}$$

The solution set is $\{(2, 1)\}$.

Skill Practice Solve by using the addition method.

1. $2x - 3y = 13$
$x + 2y = 3$

Avoiding Mistakes

Be sure to multiply *both* sides of the equation by 4.

$$4(4x + y) = 4(9)$$

FOR REVIEW

Recall that the sum of an expression and its opposite is zero. For example:

$$4y + (-4y) = 0$$

TIP: Substituting $x = 2$ into the other equation, $4x + y = 9$, produces the same value for y.

$$\begin{aligned} 4x + y &= 9 \\ 4(2) + y &= 9 \\ 8 + y &= 9 \\ y &= 1 \end{aligned}$$

Answer

1. $\{(5, -1)\}$

The steps to solve a system of linear equations in two variables by the addition method is outlined as follows:

Solving a System of Linear Equations by the Addition Method

Step 1 Write both equations in standard form: $Ax + By = C$.

Step 2 Clear fractions or decimals (optional).

Step 3 Multiply one or both equations by nonzero constants to create opposite coefficients for one of the variables.

Step 4 Add the equations from step 3 to eliminate one variable.

Step 5 Solve for the remaining variable.

Step 6 Substitute the known value found in step 5 into one of the original equations to solve for the other variable.

Step 7 Check the ordered pair in *both* equations and write the solution set.

Example 2 Solving a System by the Addition Method

Solve the system by using the addition method.

$$4x + 5y = 2$$

$$3x = 1 - 4y$$

Solution:

$4x + 5y = 2 \longrightarrow 4x + 5y = 2$ **Step 1:** Write both equations in standard form.

$3x = 1 - 4y \longrightarrow 3x + 4y = 1$ **Step 2:** There are no fractions or decimals.

TIP: The addition method works on the principle that adding the same quantity to both sides of an equation produces an equivalent equation. In step 4 of Example 2, the expressions $12x + 15y$ and 6 are equal. These expressions are added vertically with the equation below to produce an equivalent equation.

We may choose to eliminate either variable. To eliminate x, change the coefficients to 12 and -12.

$4x + 5y = 2 \xrightarrow{\text{Multiply by 3.}} 12x + 15y = 6$ **Step 3:** Multiply the first equation by 3.

$3x + 4y = 1 \xrightarrow{\text{Multiply by } -4.} -12x - 16y = -4$ Multiply the second equation by -4.

$$\begin{array}{r} 12x + 15y = 6 \\ -12x - 16y = -4 \\ \hline -y = 2 \\ y = -2 \end{array}$$

Step 4: Add the equations.

Step 5: Solve for y.

$$\begin{aligned} 4x + 5y &= 2 \\ 4x + 5(-2) &= 2 \\ 4x - 10 &= 2 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

Step 6: Substitute $y = -2$ back into one of the original equations and solve for x.

The solution set is $\{(3, -2)\}$. **Step 7:** Check the ordered pair $(3, -2)$ in both original equations.

TIP: To eliminate the x variable in Example 2, both equations were multiplied by appropriate constants to create $12x$ and $-12x$. We chose 12 because it is the *least common multiple* of 4 and 3.

We could have solved the system by eliminating the y variable. To eliminate y, we would multiply the top equation by 4 and the bottom equation by -5. This would make the coefficients of the y variable 20 and -20, respectively.

$$\begin{array}{rcl} 4x + 5y = 2 & \xrightarrow{\text{Multiply by 4.}} & 16x + 20y = 8 \\ 3x + 4y = 1 & \xrightarrow{\text{Multiply by } -5.} & -15x - 20y = -5 \end{array}$$

Skill Practice Solve by using the addition method.

2. $2y = 5x - 4$
$3x - 4y = 1$

Example 3 Solving a System by the Addition Method

Solve the system by using the addition method.

$$\begin{aligned} x - 2y &= 6 + y \\ 0.05y &= 0.02x - 0.10 \end{aligned}$$

Solution:

$$\begin{array}{rcl} x - 2y = 6 + y & \longrightarrow & x - 3y = 6 \\ 0.05y = 0.02x - 0.10 & \longrightarrow & -0.02x + 0.05y = -0.10 \end{array}$$

Step 1: Write both equations in standard form.

$$\begin{array}{rcl} x \quad -3y = 6 & & \\ -0.02x + 0.05y = -0.10 & \xrightarrow{\text{Multiply by 100.}} & -2x + 5y = -10 \end{array}$$

Step 2: Clear decimals.

$$\begin{array}{rcl} x - 3y = 6 & \xrightarrow{\text{Multiply by 2.}} & 2x - 6y = 12 \\ -2x + 5y = -10 & \longrightarrow & \underline{-2x + 5y = -10} \\ & & -y = 2 \\ & & y = -2 \end{array}$$

Step 3: Create opposite coefficients.

Step 4: Add the equations.

Step 5: Solve for y.

$$\begin{aligned} x - 2y &= 6 + y \\ x - 2(-2) &= 6 + (-2) \\ x + 4 &= 4 \\ x &= 0 \end{aligned}$$

Step 6: To solve for x, substitute $y = -2$ into one of the original equations.

Step 7: The ordered pair $(0, -2)$ checks in each original equation.

The solution set is $\{(0, -2)\}$.

Skill Practice Solve by using the addition method.

3. $0.2x + 0.3y = 1.5$
$5x + 3y = 20 - y$

Answers

2. $\left\{\left(1, \frac{1}{2}\right)\right\}$ **3.** $\{(0, 5)\}$

2. Solving Inconsistent Systems and Systems of Dependent Equations

Example 4 Solving a System of Dependent Equations

Solve the system by using the addition method.

$$\frac{1}{5}x - \frac{1}{2}y = 1$$
$$-4x + 10y = -20$$

Solution:

$$\frac{1}{5}x - \frac{1}{2}y = 1 \qquad -4x + 10y = -20$$

Step 1: Equations are in standard form.

$$10\left(\frac{1}{5}x - \frac{1}{2}y\right) = 10 \cdot 1 \xrightarrow{\text{Multiply by 10.}} 2x - 5y = 10$$
$$-4x + 10y = -20$$

Step 2: Clear fractions.

$$\begin{aligned} 2x - 5y &= 10 \xrightarrow{\text{Multiply by 2.}} & 4x - 10y &= 20 \\ -4x + 10y &= -20 \longrightarrow & -4x + 10y &= -20 \\ & & 0 &= 0 \end{aligned}$$

Step 3: Multiply the first equation by 2.

Step 4: Add the equations.

Notice that both variables were eliminated. The system of equations is reduced to the identity $0 = 0$. Therefore, the two original equations are dependent. The solution set consists of an infinite number of ordered pairs (x, y) that fall on the common line of intersection $-4x + 10y = -20$, or equivalently $\frac{1}{5}x - \frac{1}{2}y = 1$. The solution set is

$$\{(x, y) \mid -4x + 10y = -20\} \quad \text{or} \quad \left\{(x, y) \,\middle|\, \frac{1}{5}x - \frac{1}{2}y = 1\right\}$$

Skill Practice Solve by the addition method.

4. $3x + y = 4$

$x + \frac{1}{3}y = \frac{4}{3}$

Example 5 Solving an Inconsistent System

Solve the system by using the addition method.

$$2y = -3x + 4$$
$$20(6x + 5y) = 40 + 20y$$

Solution:

Step 1: Write the equations in standard form.

$$2y = -3x + 4 \longrightarrow 3x + 2y = 4$$
$$20(6x + 5y) = 40 + 20y \longrightarrow 120x + 100y = 40 + 20y \longrightarrow 120x + 80y = 40$$

With both equations now in standard form, we can proceed with the addition method.

Answer

4. Infinitely many solutions; $\{(x, y) \mid 3x + y = 4\}$; dependent equations

Step 2: The goal is to create opposite coefficients on either the x or y terms. The second equation can be divided by -40.

$$\begin{aligned} 3x + 2y &= 4 \\ 120x + 80y &= 40 \end{aligned} \xrightarrow[\text{Divide by } -40.]{} \begin{aligned} 3x + 2y &= 4 \\ \underline{-3x - 2y} &\underline{= -1} \\ 0 &= 3 \end{aligned}$$

Step 3: Divide the second equation by -40.

Step 4: Add the equations.

The equations reduce to a contradiction, indicating that the system has no solution. The system is inconsistent. The two equations represent parallel lines, as shown in Figure 3-6.

There is no solution, { }.

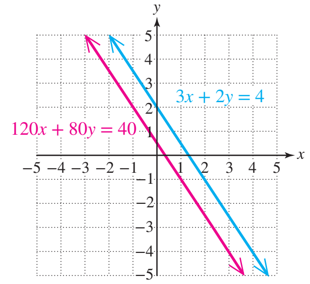


Figure 3-6

Skill Practice Solve by using the addition method.

5. $18 + 10x = 6y$
$5x - 3y = 9$

Answer

5. No solution; { }; inconsistent system

Section 3.3 Activity

A.1. **a.** Add the equations.

$$\begin{aligned} 2x - 3y &= -2 \\ 5x + 3y &= -26 \end{aligned}$$

b. Add the equations.

$$\begin{aligned} 3x + 7y &= 1 \\ -6x + 2y &= 14 \end{aligned}$$

c. In which system was a variable eliminated? Why?

A.2. **a.** Solve the system by using the addition method.

$$\begin{aligned} 2x - 3y &= -2 \\ 5x + 3y &= -26 \end{aligned}$$

b. Check the ordered pair in both equations.

A.3. **a.** To solve the system by using the addition method, which variable is easier to eliminate?

$$\begin{aligned} 3x + 7y &= 1 \\ -6x + 2y &= 14 \end{aligned}$$

b. Solve the system by using the addition method.
c. Check the ordered pair in both equations.

A.4. Consider the system.

$$\begin{aligned} 0.5x - 0.4y &= -0.7 \\ 0.02x + 0.03y &= 0.11 \end{aligned}$$

a. What is the smallest power of 10 (such as 10, 100, 1000, etc.) that could be used to clear the decimals in the first equation?
b. What is the smallest power of 10 that could be used to clear the decimals in the second equation?
c. Clear the decimals in both equations and rewrite the system of equations.
d. Solve the system of equations from part (c) using the addition method.
e. Check the ordered pair in both original equations.

A.5. Consider the system.

$$\begin{aligned} \frac{1}{5}x - 2y &= \frac{1}{3} \\ 6(x - 10y) &= 10 \end{aligned}$$

a. What is the smallest positive integer that can be used to clear the fractions in the first equation?
b. Clear the fractions in the first equation and write the second equation in standard form.

c. Solve the system from part (b) by using the addition method.
d. How many solutions are there to the system?

A.6. Consider the system.

$$5x = 2(3y + 1)$$
$$\frac{1}{2}x - \frac{3}{5}y = 0$$

a. Solve the system using the addition method.
b. How many solutions are there to the system?

Section 3.3 Practice Exercises

Study Skills Exercise

Often in mathematics, more than one method can be used to solve a problem. Solving a system of linear equations with two variables is one such example. Solve the given system using the graphing method, the substitution method, and the addition method. There are two advantages to this exercise. One advantage is to check your answer. You should get the same answer for each of the three methods. The second advantage is to show which method is the easiest for you to use.

- Solve the system by using the graphing method, the substitution method, and the addition method.

$$2x + y = -7$$
$$x - 10 = 4y$$

Prerequisite Review

For Exercises R.1–R.2, determine whether the given ordered pair is a solution to the system.

R.1. $2x = 4y + 2$
$5x - 8y = 1$
a. $(1, 0)$ **b.** $(-3, -2)$

R.2. $3x - y = 6$
$-2x + 3y = 17$
a. $(5, 9)$ **b.** $(1, -3)$

For Exercises R.3–R.8, solve the equation.

R.3. $2(y + 3) + y = 3y + 8$

R.4. $4 - 3(1 - t) - 2t = t - 3$

R.5. $\frac{2}{7}m - \frac{3}{14} = m - \frac{1}{2}$

R.6. $\frac{1}{3} + \frac{5}{6}n = \frac{2}{9} - \frac{1}{2}n$

R.7. $3x - 5 = 4(x - 1) - x - 1$

R.8. $w + 5(w + 3) = 17 + 6w - 2$

Vocabulary and Key Concepts

1. **a.** By what nonzero constant would the second equation be multiplied to eliminate the x variable from the system of equations?

$$3x - 4y = 8$$
$$x + 5y = -2$$

b. By what nonzero constant would the first equation be multiplied to eliminate the y variable from the system of equations?

$$3x - y = 8$$
$$-8x + 5y = 12$$

Concept 1: The Addition Method

For Exercises 2–4, answers will vary.

2. How would you eliminate x from the following system of equations?

$$\begin{aligned} x + 2y &= 2 \\ 3x - 4y &= 16 \end{aligned}$$

3. How would you eliminate y from the following system of equations?

$$\begin{aligned} 5x + 4y &= 14 \\ 3x + 5y &= 24 \end{aligned}$$

4. How would you eliminate x from the following system of equations?

$$\begin{aligned} 2x - 3y &= 8 \\ -7x + 2y &= -11 \end{aligned}$$

For Exercises 5–16, solve the system by using the addition method. **(See Examples 1–3.)**

5. $\begin{aligned} 3x - y &= -1 \\ -3x + 4y &= -14 \end{aligned}$

6. $\begin{aligned} 5x - 2y &= 15 \\ 3x + 2y &= -7 \end{aligned}$

7. $\begin{aligned} 2x + 3y &= 3 \\ -10x + 2y &= -32 \end{aligned}$

8. $\begin{aligned} 2x - 5y &= 7 \\ 3x - 10y &= 13 \end{aligned}$

9. $\begin{aligned} 3x + 7y &= -20 \\ -5x + 3y &= -84 \end{aligned}$

10. $\begin{aligned} 6x - 9y &= -15 \\ 5x - 2y &= -40 \end{aligned}$

11. $\begin{aligned} 3x &= 10y + 13 \\ 7y &= 4x - 11 \end{aligned}$

12. $\begin{aligned} -5x &= 6y - 4 \\ 5y &= 1 - 3x \end{aligned}$

13. $\begin{aligned} 1.2x - 0.6y &= 3 \\ 0.8x - 1.4y &= 3 \end{aligned}$

14. $\begin{aligned} 1.8x + 0.8y &= 1.4 \\ 1.2x + 0.6y &= 1.2 \end{aligned}$

15. $\begin{aligned} 3x + 2 &= 4y + 2 \\ 7x &= 3y \end{aligned}$

16. $\begin{aligned} -4y - 3 &= 2x - 3 \\ 5y &= 3x \end{aligned}$

Concept 2: Solving Inconsistent Systems and Systems of Dependent Equations

For Exercises 17–24, solve the system. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent. **(See Examples 4–5.)**

17. $\begin{aligned} 3x - 2y &= 1 \\ -6x + 4y &= -2 \end{aligned}$

18. $\begin{aligned} 3x - y &= 4 \\ 6x - 2y &= 8 \end{aligned}$

19. $\begin{aligned} 6y &= 14 - 4x \\ 2x &= -3y - 7 \end{aligned}$

20. $\begin{aligned} 2x &= 4 - y \\ -y &= 2x - 2 \end{aligned}$

21. $\begin{aligned} 12x - 4y &= 2 \\ 6x &= 1 + 2y \end{aligned}$

22. $\begin{aligned} 10x - 15y &= 5 \\ 3y &= 2x - 1 \end{aligned}$

23. $\begin{aligned} \frac{1}{2}x + y &= \frac{7}{6} \\ x + 2y &= 4.5 \end{aligned}$

24. $\begin{aligned} 0.2x - 0.1y &= -1.2 \\ x - \frac{1}{2}y &= 3 \end{aligned}$

Mixed Exercises

25. Describe a situation in which you would prefer to use the substitution method over the addition method.

26. If you used the addition method to solve the given system, would it be easier to eliminate the x or y variable? Explain.

$$\begin{aligned} 3x - 5y &= 4 \\ 7x + 10y &= 31 \end{aligned}$$

For Exercises 27–32, answer True or False.

27. A system of dependent equations has exactly one solution.

28. An inconsistent system has infinitely many solutions.

29. If solving a system of equations results in the contradiction $0 = 4$, then the system is inconsistent.

30. If solving a system of equations results in the identity $-3 = -3$, then the equations are dependent.

31. If a system of linear equations represents two intersecting lines, then the system has exactly one solution.

32. If a system of linear equations represents two parallel lines, then the system has no solution.

For Exercises 33–56, solve the system. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent.

33. $2x - 4y = 8$
$y = 2x + 1$

34. $8x + 6y = -8$
$x = 6y - 10$

35. $2x + 5y = 9$
$4x - 7y = -16$

36. $0.1x + 0.5y = 0.7$
$0.2x + 0.7y = 0.8$

37. $0.2x - 0.1y = 0.8$
$0.1x - 0.1y = 0.4$

38. $y = \frac{1}{2}x - 3$
$4x + y = -3$

39. $4x - 6y = 5$
$2x - 3y = 7$

40. $3x + 6y = 7$
$2x + 4y = 5$

41. $\frac{1}{4}x - \frac{1}{6}y = -2$
$-\frac{1}{6}x + \frac{1}{5}y = 4$

42. $\frac{1}{3}x + \frac{1}{5}y = 7$
$\frac{1}{6}x - \frac{2}{5}y = -4$

43. $\frac{1}{3}x - \frac{1}{2}y = 0$
$x = \frac{3}{2}y$

44. $\frac{2}{5}x - \frac{2}{3}y = 0$
$y = \frac{3}{5}x$

45. $2(x + 2y) = 20 - y$
$-7(x - y) = 16 + 3y$

46. $-3(x + y) = 10 - 4y$
$4(x + 2y) = 50 + 3y$

47. $-4y = 10$
$4x + 3 = 1$

48. $-9x = 15$
$3y + 2 = 1$

49. $0.04x = -0.05y + 1.7$
$-0.03y = -2.4 + 0.07x$

50. $-0.01x = -0.06y + 3.2$
$0.08y = 0.03x + 4.6$

51. $3x - 2 = \frac{1}{3}(11 + 5y)$
$x + \frac{2}{3}(2y - 3) = -2$

52. $2(2y + 3) - 2x = 1 - x$
$x + y = \frac{1}{5}(7 + y)$

53. $\frac{1}{4}x + \frac{1}{2}y = \frac{11}{4}$
$\frac{2}{3}x + \frac{1}{3}y = \frac{7}{3}$

54. $\frac{1}{10}x - \frac{1}{2}y = -\frac{8}{5}$
$x + \frac{1}{4}y = -\frac{11}{2}$

55. $4x = 3y$
$y = \frac{4}{3}x + 2$

56. $4x - 2y = 6$
$x = \frac{1}{2}y + \frac{3}{2}$

57. A luxury car gets 16 mpg in the city and 24 mpg on the highway. A sport-utility vehicle (SUV) gets 14 mpg in the city and 20 mpg on the highway. Suppose that each vehicle travels c miles in the city and h miles on the highway. The luxury car uses 12 gal of gasoline and the SUV uses 14 gal. Solve the system of equations to determine the number of city miles driven and the number of highway miles driven.

Luxury car: $\frac{1}{16}c + \frac{1}{24}h = 12$

SUV: $\frac{1}{14}c + \frac{1}{20}h = 14$

58. A sedan gets 21 mpg in the city and 27 mpg on the highway. A truck gets 15 mpg in the city and 21 mpg on the highway. Suppose that each vehicle travels c miles in the city and h miles on the highway. The sedan uses 12 gal of gasoline and the truck uses 16 gal. Solve the system of equations to determine the number of city miles driven and the number of highway miles driven.

$$\text{Sedan:}\quad \frac{1}{21}c + \frac{1}{27}h = 12$$

$$\text{Truck:}\quad \frac{1}{15}c + \frac{1}{21}h = 16$$

Expanding Your Skills

Sometimes the solution to a system of equations is an ordered pair containing fractions. In such a case, it is often easier to solve for each coordinate separately using the addition method. That is, use the addition method to solve for x. Then, rather than substituting x back into one of the original equations, repeat the addition method and solve for y. For Exercises 59–61, solve each system.

59. $9x + 11y = 47$
$-5x + 3y = 23$

60. $-6x + 7y = -4$
$4x - 9y = 31$

61. $4x - 10y = 19$
$5x + 12y = -41$

Problem Recognition Exercises

Solving Systems of Linear Equations

For Exercises 1–4, solve each system by using three different methods.

a. Use the graphing method. **b.** Use the substitution method. **c.** Use the addition method.

1. $-3x + y = -2$
$4x - y = 4$

2. $3x - 2y = 4$
$x = \frac{2}{3}y + \frac{4}{3}$

3. $5x = 2y$
$y = \frac{5}{2}x + 1$

4. $2y = 3x + 1$
$4x = 4$

For Exercises 5–8, solve the system of equations by using the most efficient method.

5. $y = -4x - 9$
$8x + 3y = -29$

6. $5x - 2y = -17$
$x + 5y = 2$

7. $5x - 3y = 2$
$7x + 4y = -30$

8. $\frac{1}{10}x - \frac{2}{5}y = -\frac{3}{5}$
$\frac{3}{4}x + \frac{1}{3}y = \frac{13}{6}$

Applications of Systems of Linear Equations in Two Variables

Section 3.4

Concepts

1. Applications Involving Cost
2. Applications Involving Mixtures
3. Applications Involving Principal and Interest
4. Applications Involving Uniform Motion
5. Applications Involving Geometry

1. Applications Involving Cost

We have already solved numerous application problems using equations that contain one variable. However, when an application has more than one unknown, sometimes it is more convenient to use multiple variables. In this section, we will solve applications containing two unknowns. When two variables are present, the goal is to set up a system of two independent equations.

Example 1 Solving a Cost Application

At an amusement park, five hot dogs and one drink cost \$22. Two hot dogs and three drinks cost \$14. Find the cost per hot dog and the cost per drink.

Solution:

Let h represent the cost per hot dog. — Label the variables.

Let d represent the cost per drink.

$$\begin{pmatrix}\text{Cost of 5}\\ \text{hot dogs}\end{pmatrix} + \begin{pmatrix}\text{cost of 1}\\ \text{drink}\end{pmatrix} = \$22 \longrightarrow 5h + d = 22$$

$$\begin{pmatrix}\text{Cost of 2}\\ \text{hot dogs}\end{pmatrix} + \begin{pmatrix}\text{cost of 3}\\ \text{drinks}\end{pmatrix} = \$14 \longrightarrow 2h + 3d = 14$$

Write two equations.

This system can be solved by either the substitution method or the addition method. We will solve by using the substitution method. The d variable in the first equation is the easiest variable to isolate.

$5h + d = 22 \longrightarrow d = -5h + 22$ — Solve for d in the first equation.

$2h + 3d = 14$

$2h + 3(-5h + 22) = 14$ — Substitute the quantity $-5h + 22$ for d in the *second* equation.

$2h - 15h + 66 = 14$ — Clear parentheses.

$-13h + 66 = 14$ — Solve for h.

$-13h = -52$

$h = 4$

$d = -5(4) + 22 \longrightarrow d = 2$ — Substitute $h = 4$ in the equation $d = -5h + 22$.

Because $h = 4$, the cost per hot dog is \$4.00.

Because $d = 2$, the cost per drink is \$2.00.

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Skill Practice

1. At the movie theater, Tom spent \$15.50 on 3 soft drinks and 2 boxes of popcorn. Carly bought 5 soft drinks and 1 box of popcorn for a total of \$16.50. Use a system of equations to find the cost of a soft drink and the cost of a box of popcorn.

TIP: A word problem can be checked by verifying that the solution meets the conditions specified in the problem.

5 hot dogs + 1 drink = 5(\$4.00) + 1(\$2.00) = \$22.00 ✓

2 hot dogs + 3 drinks = 2(\$4.00) + 3(\$2.00) = \$14.00 ✓

Answer

1. Soft drinks cost \$2.50 and popcorn costs \$4.00.

2. Applications Involving Mixtures

Example 2 **Solving an Application Involving Chemistry**

One brand of cleaner used to etch concrete is 25% acid. A stronger industrial-strength cleaner is 50% acid. How many gallons of each cleaner should be mixed to produce 20 gal of a 40% acid solution?

Solution:

Let x represent the amount of 25% acid cleaner.

Let y represent the amount of 50% acid cleaner.

	25% Acid	50% Acid	40% Acid
Number of gallons of solution	x	y	20
Number of gallons of pure acid	$0.25x$	$0.50y$	0.40(20), or 8

Avoiding Mistakes

Do not forget to write the percent as a decimal.

From the first row of the table, we have

$$\begin{pmatrix}\text{Amount of}\\ \text{25\% solution}\end{pmatrix} + \begin{pmatrix}\text{amount of}\\ \text{50\% solution}\end{pmatrix} = \begin{pmatrix}\text{total amount}\\ \text{of solution}\end{pmatrix} \rightarrow x + y = 20$$

From the second row of the table we have

$$\begin{pmatrix}\text{Amount of}\\ \text{pure acid in}\\ \text{25\% solution}\end{pmatrix} + \begin{pmatrix}\text{amount of}\\ \text{pure acid in}\\ \text{50\% solution}\end{pmatrix} = \begin{pmatrix}\text{amount of}\\ \text{pure acid in}\\ \text{resulting solution}\end{pmatrix} \rightarrow 0.25x + 0.50y = 8$$

FOR REVIEW

Recall that the amount of a pure substance in a mixture is equal to the amount of mixture times the concentration rate of the substance. For example, in 20 gal of a 40% acid mixture, the amount of pure acid is (20 gal)(0.40) = 8 gal.

$$\begin{aligned} x + y &= 20 &\longrightarrow\quad x + y &= 20 \\ 0.25x + 0.50y &= 8 &\longrightarrow\quad 25x + 50y &= 800 \end{aligned}$$

Multiply by 100 to clear decimals.

$$\begin{aligned} x + y &= 20 &\xrightarrow{\text{Multiply by } -25.}\quad -25x - 25y &= -500 \\ 25x + 50y &= 800 &\longrightarrow\quad \underline{25x + 50y} &\underline{= 800} \\ & & 25y &= 300 \\ & & y &= 12 \end{aligned}$$

Create opposite coefficients of x.

Add the equations to eliminate x.

$$\begin{aligned} x + y &= 20 \\ x + (12) &= 20 \\ x &= 8 \end{aligned}$$

Substitute $y = 12$ back into one of the original equations.

Therefore, 8 gal of 25% acid solution must be added to 12 gal of 50% acid solution to create 20 gal of a 40% acid solution.

Skill Practice

2. A pharmacist needs 8 ounces (oz) of a solution that is 50% saline. How many ounces of 60% saline solution and 20% saline solution must be mixed to obtain the mixture needed?

To check your answer to an application problem, be sure that your solution satisfies both criteria in the problem. From Example 2, we have

<u>Amount of solution</u>:
8 gal + 12 gal = 20 gal ✓

<u>Amount of acid</u>:
(8 gal)(0.25) + (12 gal)(0.50) = (20 gal)(0.40)
2 gal + 6 gal = 8 gal ✓

Answer

2. The pharmacist should mix 6 oz of 60% solution and 2 oz of 20% solution.

3. Applications Involving Principal and Interest

Example 3 Solving an Application Involving Finance

Serena invested money in two mutual funds. One had a return of 4.5% and the other had a return of 7%. Twice as much was invested at 7% as at 4.5%. If the amount earned on the original principal at the end of 1 yr was \$1017.50, determine the amount invested in each fund.

Solution:

Let x represent the amount invested at 4.5%.

Let y represent the amount invested at 7%.

	4.5% Account	7% Account	Total
Principal	x	y	
Amount earned	$0.045x$	$0.07y$	1017.50

Because the amount invested at 7% was twice the amount invested at 4.5%, we have

$$\begin{pmatrix}\text{Amount}\\\text{invested}\\\text{at }7\%\end{pmatrix} = 2\begin{pmatrix}\text{amount}\\\text{invested}\\\text{at }4.5\%\end{pmatrix} \rightarrow y = 2x$$

From the second row of the table, we have

$$\begin{pmatrix}\text{Amount}\\\text{earned from}\\4.5\%\text{ account}\end{pmatrix} + \begin{pmatrix}\text{amount}\\\text{earned from}\\7\%\text{ account}\end{pmatrix} = \begin{pmatrix}\text{total}\\\text{amount}\\\text{earned}\end{pmatrix} \rightarrow 0.045x + 0.07y = 1017.50$$

$$y = 2x$$

$$45x + 70y = 1{,}017{,}500$$ Multiply by 1000 to clear decimals.

Because the y variable in the first equation is isolated, we will use the substitution method.

$$45x + 70(2x) = 1{,}017{,}500$$ Substitute the quantity $2x$ into the second equation.

$$45x + 140x = 1{,}017{,}500$$ Solve for x.

$$185x = 1{,}017{,}500$$

$$x = \frac{1{,}017{,}500}{185}$$

$$x = 5500$$

$$y = 2x$$

$$y = 2(5500)$$ Substitute $x = 5500$ into the equation $y = 2x$ to solve for y.

$$y = 11{,}000$$

Because $x = 5500$, the amount invested at 4.5% was \$5500.

Because $y = 11{,}000$, the amount invested at 7% was \$11,000.

Avoiding Mistakes

To check the result of Example 3, we have:

Amount of principal:

\$11,000 = 2(\$5500) ✓

Amount of interest:

0.045(\$5500) + 0.07(\$11,000)
= \$247.50 + \$770
= \$1017.50 ✓

Skill Practice

3. Seth invested money in two accounts, one paying 5% simple interest and the other paying 6% simple interest. The amount invested at 6% was \$1000 more than the amount invested at 5%. He earned a total of \$830 interest in 1 yr. Determine the amount invested in each account.

Answer

3. Seth invested \$7000 at 5% and \$8000 at 6%.

TIP: To check Example 3, note that \$11,000 is twice \$5500. Furthermore,

$$\begin{pmatrix}\text{Amount}\\ \text{earned from}\\ \text{4.5\% account}\end{pmatrix} + \begin{pmatrix}\text{amount}\\ \text{earned from}\\ \text{7\% account}\end{pmatrix} = \$5500(0.045) + \$11{,}000(0.07) = 1017.50\ \checkmark$$

4. Applications Involving Uniform Motion

Example 4 Solving a Distance, Rate, and Time Application

A plane flies 660 mi from Atlanta to Miami in 1.2 hr when traveling with the wind. The return flight against the same wind takes 1.5 hr. Find the speed of the plane in still air and the speed of the wind.

Solution:

Let p represent the speed of the plane in still air.

Let w represent the speed of the wind.

The speed of the plane *with* the wind:

$$(\text{Plane's still airspeed}) + (\text{wind speed}) \rightarrow p + w$$

The speed of the plane *against* the wind:

$$(\text{Plane's still airspeed}) - (\text{wind speed}) \rightarrow p - w$$

Set up a chart to organize the given information:

	Distance	Rate	Time
With the wind	660	$p + w$	1.2
Against the wind	660	$p - w$	1.5

Two equations can be found by using the relationship $d = rt$.

$$\begin{pmatrix}\text{Distance}\\ \text{with}\\ \text{wind}\end{pmatrix} = \begin{pmatrix}\text{speed}\\ \text{with}\\ \text{wind}\end{pmatrix}\begin{pmatrix}\text{time}\\ \text{with}\\ \text{wind}\end{pmatrix} \longrightarrow 660 = (p + w)(1.2)$$

$$\begin{pmatrix}\text{Distance}\\ \text{against}\\ \text{wind}\end{pmatrix} = \begin{pmatrix}\text{speed}\\ \text{against}\\ \text{wind}\end{pmatrix}\begin{pmatrix}\text{time}\\ \text{against}\\ \text{wind}\end{pmatrix} \longrightarrow 660 = (p - w)(1.5)$$

$660 = (p + w)(1.2)$

$660 = (p - w)(1.5)$

Notice that the first equation may be *divided* by 1.2 and still leave integer coefficients. Similarly, the second equation may be simplified by dividing by 1.5.

$$660 = (p + w)(1.2) \xrightarrow{\text{Divide by 1.2.}} \frac{660}{1.2} = \frac{(p + w)\cancel{1.2}}{\cancel{1.2}} \longrightarrow 550 = p + w$$

$$660 = (p - w)(1.5) \xrightarrow{\text{Divide by 1.5.}} \frac{660}{1.5} = \frac{(p - w)\cancel{1.5}}{\cancel{1.5}} \longrightarrow 440 = p - w$$

TIP: When using the addition method, we use the multiplication property of equality to create opposite coefficients. Example 4 demonstrates that we can also use the division property of equality to create opposite coefficients.

$$550 = p + w$$
$$\underline{440 = p - w}$$
$$990 = 2p \qquad \text{Add the equations.}$$
$$p = 495 \qquad \text{Divide by 2 to isolate } p.$$

$$550 = (495) + w \qquad \text{Substitute } p = 495 \text{ into the equation } 550 = p + w.$$
$$55 = w \qquad \text{Solve for } w.$$

The speed of the plane in still air is 495 mph, and the speed of the wind is 55 mph.

Skill Practice

4. A plane flies 1200 mi from Orlando to New York in 2 hr with the wind. The return flight against the same wind takes 2.5 hr. Find the speed of the plane in still air and the speed of the wind.

5. Applications Involving Geometry

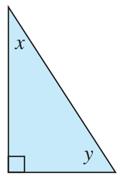
Example 5 Solving a Geometry Application

The sum of the two acute angles in a right triangle is 90°. The measure of one angle is 6° less than 2 times the measure of the other angle. Find the measure of each angle.

Solution:

Let x represent one acute angle.

Let y represent the other acute angle.



The sum of the two acute angles is 90° $\longrightarrow$ $x + y = 90$

One angle is 6° less than 2 times the other angle $\longrightarrow$ $x = 2y - 6$

$$x + y = 90$$
$$x = \underbrace{2y - 6}$$
Because one variable is already isolated, we will use the substitution method.

$$(2y - 6) + y = 90 \qquad \text{Substitute } x = 2y - 6 \text{ into the first equation.}$$
$$3y - 6 = 90$$
$$3y = 96$$
$$y = 32$$

$$x = 2y - 6 \qquad \text{Substitute } y = 32 \text{ into the equation } x = 2y - 6.$$
$$x = 2(32) - 6$$
$$x = 64 - 6$$
$$x = 58$$

The two acute angles in the triangle measure 32° and 58°.

Skill Practice

5. Two angles are supplementary. The measure of one angle is 16° less than 3 times the measure of the other. Find the measure of each angle.

FOR REVIEW

Recall these important facts from geometry.

- The sum of the measures of the angles in a triangle is 180°.
- Two angles are **supplementary** if the sum of their measures is 180°.
- Two angles are **complementary** if the sum of their measures is 90°.

Answers

4. The speed of the plane is 540 mph, and the speed of the wind is 60 mph.
5. The angles are 49° and 131°.

Section 3.4 Activity

To solve an application with two unknowns, it is often helpful to use two variables such as x and y. When two variables are used to solve an application, we need to have two independent equations that relate the variables. That is, for two unknowns, set up a system of two equations.

A.1. A food cart in an artisan mall serves burritos and tacos. The burritos cost \$4 each and the tacos cost \$1.50 each. At the end of one business day, the food cart sold 790 items for a total of \$1785 in revenue. How many burritos and how many tacos were sold? To solve this application, follow these steps.

a. Set up an equation that represents the fact that the number of burritos sold, x, plus the number of tacos sold, y, equals 790.

Set up an equation that represents the fact that the revenue from burritos plus the revenue from tacos equals the total revenue of \$1785.

b. Solve the system of equations from part (a).

c. Interpret the meaning of the solution from part (b).

A.2. An alloy is a metal made by combining two or more metallic elements.

a. If a chemist has 50 g of a 24% silver alloy, how much is pure silver?

b. If a chemist has x grams of a 24% silver alloy, how much is pure silver?

c. If a chemist has y grams of a 54% silver alloy, how much is pure silver?

A.3. Suppose that a 24% silver alloy is to be mixed with a 54% silver alloy to get 40 g of a 30% silver alloy. How much of the 24% alloy and how much of the 54% alloy should be used? Follow these steps.

a. Let x represent the amount of 24% alloy. What should y represent?

b. Refer to Exercises A.2(b) and A.2(c) and fill in the table.

	24% Alloy	54% Alloy	30% Alloy
Amount of alloy (g)			
Amount of pure silver (g)			

c. Set up an equation that indicates that the amount of 24% alloy plus the amount of 54% alloy equals 40 g of a 30% alloy.

Set up an equation that indicates that the amount of pure silver in the 24% alloy plus the amount of pure silver in the 54% alloy equals the amount of pure silver in 40 g of the 30% alloy.

d. Solve the system of equations from part (c).

e. Interpret the meaning of the solution from part (d).

A.4. **a.** If \$5000 is invested in an account earning 2.5% annual interest, how much interest is earned in one year?

b. Write an expression for the amount of interest earned in 1 year on x dollars at 2.5% annual interest.

c. Write an expression for the amount of interest earned in 1 year on y dollars at 4% annual interest.

A.5. Feliciano invests money in two different accounts. One account yields a 2.5% annual return and the other yields a 4% annual return. He invests three times as much in the 4% account as he does in the 2.5% account, and his total return for one year is \$652.50. How much does he invest in each account?

a. Let x represent the amount invested in the account paying 2.5% interest. What should y represent?

b. In this scenario, there are two "types" of money: principal (amount invested) and interest. Refer to Exercises A.4(b) and A.4(c) and fill in the table with expressions for the principal and interest in each account.

	2.5% Account	4% Account	Total
Amount invested (\$)			
Interest earned (\$)			

c. Set up an equation that indicates that he invests three times as much in the 4% account as the 2.5% account.

Set up an equation that indicates that the total annual interest generated by the 2.5% account plus the interest generated by the 4% account is \$652.50.

d. Solve the system of equations from part (c).
e. Interpret the meaning of the solution from part (d).

A.6. **a.** Suppose that a boat normally travels 13 mph in still water (without a current). If the boat travels *with* a current of 3 mph, what is the net speed? In such a case, how far would the boat travel in 2 hr?
b. Suppose that a boat normally travels 13 mph in still water. If the boat travels *against* a current of 3 mph, what is the net speed? In such a case, how far would the boat travel in 2 hr?
c. Suppose that a boat normally travels b mph in still water. If the boat travels *with* a current of c mph, write an expression for the net speed. Write an expression representing the distance the boat would travel in 1.5 hr.
d. Suppose that a boat normally travels b mph in still water. If the boat travels *against* a current of c mph, write an expression for the net speed. Write an expression representing the distance the boat would travel in 2.4 hr.

A.7. A boat travels 24 mi in 1.5 hr with the current. Against the current, the same trip takes 2.4 hr. What is the speed of the current and the speed of the boat in still water?
a. Let b represent the speed of the boat in still water. What should c represent?
b. Refer to Exercises A.6(c) and A.6(d) and complete the table.

	Distance (mi)	Rate (mph)	Time (hr)
With the current			
Against the current			

c. Set up an equation that indicates that the distance traveled with the current equals the rate of speed with the current times the time of travel with the current.

Set up an equation that indicates that the distance traveled against the current equals the rate of speed against the current times the time of travel against the current.
d. Solve the system of equations from part (c).
e. Interpret the meaning of the solution from part (d).

Section 3.4 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, solve the system using two different methods:
a. The substitution method.
b. The addition method.

R.1. $0.36x + 0.2y = 0.3(40)$
$x + y = 40$

R.2. $0.3x + 0.1y = 0.12(200)$
$x + y = 200$

R.3. The cost for a personal trainer is \$55 per session.
a. What is the cost for 5 sessions?
b. What is the cost for x sessions?

R.4. The cost to rent office space is \$840 per month.
a. What is the cost for 12 months?
b. What is the cost for y months?

R.5. Mark borrows money at a rate of 3.5% simple interest.
a. How much interest will he owe on \$8000 for 3 years?
b. How much interest will he owe if he borrows y dollars for 1 year?

R.6. Louise invests in an account that earns simple interest at a rate of 2.4%.
a. How much interest will she earn if she invests \$5000 for 4 years?
b. How much interest will she earn if she invests x dollars for 1 year?

R.7. A solution is made up of 25% bleach and 75% water.
a. How much bleach is in a 2-L container of solution? How much is water?
b. How much bleach is in x liters of solution?

R.8. A gardener makes up a nutrient solution that is 5% liquid fertilizer and 95% water.

a. How much fertilizer is present in 50 gal of the nutrient solution? How much is water?

b. How much fertilizer is in y gallons of solution?

R.9. Captain Coronel drives her boat at 4 mph. The current is 1 mph.

a. What is the net speed of the boat traveling against the current?

b. What is the net speed of the boat traveling with the current?

c. How far will Captain Coronel travel with the current in 1.5 hr?

d. How far will Captain Coronel travel against the current in 1.5 hr?

e. If the speed of the boat in still water is b miles per hour and the speed of the current is c miles per hour, write an expression for the net speed of the boat traveling against the current.

R.10. Jorge rides his bicycle at an average of 15 mph when there is no wind present. If the speed of the wind is 3 mph,

a. What is the net speed that Jorge travels riding against the wind?

b. What is the net speed that Jorge travels riding with the wind?

c. How far will Jorge travel with the wind in 1 hr and 20 min $\left(\frac{4}{3}\text{ hr}\right)$?

d. How far will Jorge travel against the wind in 2 hr?

e. If Jorge rides x miles per hour in still air and if the speed of the wind is y miles per hour, write an expression for the net speed when Jorge rides with the wind.

Vocabulary and Key Concepts

1. The sum of the measures of the angles within a triangle is ____________.
2. If the measure of an angle is x and the measure of its supplement is y, then $x + y =$ ____________.
3. If the measure of an angle is x and the measure of its complement is y, then $x + y =$ ____________.
4. If the measure of one acute angle in a right triangle is a and the other acute angle has measure b, then $a + b =$ ____________.

Concept 1: Applications Involving Cost

5. A 1200-seat theater sells two types of tickets for a concert. Premium seats sell for $30 each and regular seats sell for $20 each. At one event $30,180 was collected in ticket sales with 10 seats left unsold. How many of each type of ticket was sold? **(See Example 1.)**

6. John and Ariana bought school supplies. John spent $10.65 on 4 notebooks and 5 pens. Ariana spent $7.50 on 3 notebooks and 3 pens. What is the cost of 1 notebook and what is the cost of 1 pen?

7. Mickey bought lunch for his fellow office workers on Monday. He spent $24.20 on 3 hamburgers and 2 fish sandwiches. Chloe bought lunch on Tuesday and spent $23.60 for 4 hamburgers and 1 fish sandwich. What is the price of 1 hamburger, and what is the price of 1 fish sandwich?

John Flournoy/McGraw Hill

8. A group of four golfers paid $300 to play a round of golf. Of the golfers, one is a member of the club and three are nonmembers. Another group of golfers consists of two members and one nonmember. They paid a total of $150. What is the cost for a member to play a round of golf, and what is the cost for a nonmember?

Corbis

9. Jen has 2 scoops of vanilla ice cream and 1 scoop of mud pie ice cream for a total of 40 g of fat. Jim has 1 scoop of vanilla and 2 scoops of mud pie for a total of 44 g of fat. How much fat is in 1 scoop of each ice cream?

10. Roselle has 2 cups of popcorn and 8 oz of soda for a total of 216 calories. Carmel has 1 cup of popcorn and 12 oz of soda for a total of 204 calories. Determine the number of calories per cup of popcorn and per ounce of soda.

Concept 2: Applications Involving Mixtures

11. A jar of one face cream contains 18% moisturizer, and another type contains 24% moisturizer. How many ounces of each should be combined to get 12 oz of a cream that is 22% moisturizer? **(See Example 2.)**

12. Logan wants to mix an 18% acid solution with a 45% acid solution to get 16 L of a 36% acid solution. How many liters of the 18% solution and how many liters of the 45% solution should be mixed?

13. How much fertilizer containing 8% nitrogen should be mixed with a fertilizer containing 12% nitrogen to get 8 L of a fertilizer containing 11% nitrogen?

14. How much 30% acid solution should be added to 10% acid solution to make 100 mL of a 12% acid solution?

15. How much pure bleach should Tim combine with a solution that is 4% bleach to make 12 oz of a 12% bleach solution? (*Hint:* Pure bleach is 100% bleach.)

16. A fruit punch that contains 25% fruit juice is combined with 100% fruit juice. How many ounces of each should be used to make 48 oz of a mixture that is 75% fruit juice?

Concept 3: Applications Involving Principal and Interest

17. Mr. Coté invested 3 times as much money in a stock fund that returned 8% interest after 1 yr as he did in a bond fund that earned 5% interest. If his total earnings came to $435 after 1 yr, how much did he invest in each fund? **(See Example 3.)**

18. Aliya deposited half as much money in an account earning 2.5% simple interest as she invested in an account that earns 3.5% simple interest. If the total interest after 1 yr is $247, how much did she invest in each account?

19. Mr. Levy borrowed money from two lenders. One lender charged 5.5% simple interest and the other charged 3.5% simple interest. Mr. Levy had to borrow $200 more at 5.5% than he did at 3.5%. If the total interest after the first year was $245, how much did he borrow at each rate?

20. Jody invested $5000 less in an account paying 3% simple interest than she did in an account paying 4% simple interest. At the end of the first year, the total interest from both accounts was $725. Find the amount invested in each account.

21. Alina borrowed a total of $15,000 from two banks to buy a new boat. Because of her excellent credit, one bank charged only 6% simple interest and the other charged 7% simple interest. At the end of 5 yr, the total amount of money she paid in interest was $4750. How much did she borrow from each bank?

22. Didi plans to take a trip to the Galapagos Islands in 4 yr and knows that she needs approximately $3500 for the trip. She invests a total of $15,500 in two funds. One fund has a 6% return, and the other has a 5% return. How much should she invest in each fund so that she earns $3500 after 4 yr?

Concept 4: Applications Involving Uniform Motion

23. It takes a boat 2 hr to travel 16 mi downstream with the current and 4 hr to return against the current. Find the speed of the boat in still water and the speed of the current. **(See Example 4.)**

24. A plane flew 720 mi in 3 hr with the wind. It would take 4 hr to travel the same distance against the wind. What is the speed of the plane in still air and the speed of the wind?

25. A plane flies from Atlanta to Los Angeles against the wind in 5 hr. The return trip back to Atlanta with the wind takes only 4 hr. If the distance between Atlanta and Los Angeles is 3200 km, find the speed of the plane in still air and the speed of the wind.

26. The Gulf Stream is a warm ocean current that extends from the eastern side of the Gulf of Mexico up through the Florida Straits and along the southeastern coast of the United States to Cape Hatteras, North Carolina. A boat travels with the current 100 mi from Miami, Florida, to Freeport, Bahamas, in 2.5 hr. The return trip against the same current takes $3\frac{1}{3}$ hr. Find the speed of the boat in still water and the speed of the current.

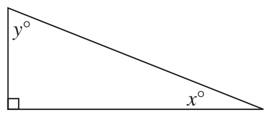
27. A moving sidewalk in the Atlanta airport moves people between gates. It takes Molly's 8-year-old son Stephen 20 sec to travel 100 ft walking with the sidewalk. It takes him 30 sec to travel 60 ft walking against the moving sidewalk (in the opposite direction). Find the speed of the moving sidewalk and Stephen's walking speed on nonmoving ground.

28. Kim rides a total of 48 km in the bicycle portion of a triathlon. The course is an "out and back" route. It takes her 3 hr on the way out against the wind. The ride back takes her 2 hr with the wind. Find the speed of the wind and Kim's speed riding her bike in still air.

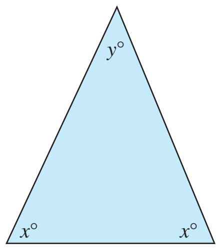
Concept 5: Applications Involving Geometry

For Exercises 29–34, solve the applications involving geometry.

29. In a right triangle, one acute angle measures 6° more than 3 times the other. If the sum of the measures of the two acute angles must equal 90°, find the measures of the acute angles. **(See Example 5.)**



30. An isosceles triangle has two angles of the same measure. If the angle represented by y measures 3° less than the angle x, find the measures of all angles of the triangle.



31. Two angles are supplementary. One angle measures 2° less than 3 times the other. What are the measures of the two angles?

32. The measure of one angle is 5 times the measure of another. If the two angles are supplementary, find the measures of the angles.

33. One angle measures 6° more than twice another. If the two angles are complementary, find the measures of the angles.

34. Two angles are complementary. One angle measures 15° more than 2 times the measure of the other. What are the measures of the two angles?

Mixed Exercises

35. How much pure gold (24K) must be mixed with 60% gold to get 20 grams (g) of 75% gold?

36. Connie is the head of maintenance at a large hospital. She received news of a new state mandate indicating that the minimum strength for disinfectant was to be 17%, up from the old requirement of 15%. Connie had plenty of barrels of 15% disinfectant left over, and also lots of the strong 55% disinfectant used in rooms for patients with highly contagious diseases. How many gallons of each disinfectant should be mixed to get 50 gal of 17% disinfectant?

37. A rowing team trains on the Halifax River. It can row upstream 10 mi against the current in 2.5 hr and 16 mi downstream with the current in the same amount of time. Find the speed of the boat in still water and the speed of the current.

Comstock Images

38. In her kayak, Taylor can travel 31.5 mi downstream with the current in 7 hr. The return trip against the current takes 9 hr. Find the speed of the kayak in still water and the speed of the current.

Karl Weatherly/Getty Images

39. There are two types of tickets sold at the Canadian Formula One Grand Prix race. The price of 6 grandstand tickets and 2 general admissions tickets is \$2330. The price of 4 grandstand tickets and 4 general admission tickets is \$2020. What is the price of each type of ticket?

40. A basketball player scored 19 points by shooting two-point and three-point baskets. If she made a total of eight baskets, how many of each type did she make?

41. A bank offers two accounts, a money market account at 2% simple interest and a regular savings account at 1.3% interest. If Svetlana deposits \$3000 between the two accounts and receives \$51.25 in total interest in the first year, how much did she invest in each account?

42. Angelo invested \$8000 in two accounts: one that pays 3% and one that pays 1.8%. At the end of the first year, his total interest earned was \$222. How much did he deposit in the account that pays 3%?

43. The perimeter of a rectangle is 42 m. The length is 1 m longer than the width. Find the dimensions of the rectangle.

44. In a right triangle, the measure of one acute angle is one-fourth the measure of the other. Find the measures of the acute angles.

45. A coin collection consists of 50¢ pieces and \$1 coins. If there are 21 coins worth \$15.50, how many 50¢ pieces and \$1 coins are there?

46. Jacob has a piggy bank consisting of nickels and dimes. If there are 30 coins worth \$1.90, how many nickels and dimes are in the bank?

47. One storage company charges \$60 per month to rent a small storage unit. Another charges \$50 per month but has an initial fee of \$100.

a. Write a linear function representing the cost $f(x)$ to rent a storage unit from the first company for x months.

b. Write a linear function representing the cost $g(x)$ to rent a storage unit from the second company for x months.

c. Find the number of months for which the cost to rent a storage unit would be the same for each company.

48. A rental car company rents a compact car for \$20 a day, plus \$0.25 per mile. A midsize car rents for \$30 a day, plus \$0.20 per mile.

a. Write a linear function representing the cost $c(x)$ to rent the compact car for x miles.

b. Write a linear function representing the cost $m(x)$ to rent a midsize car for x miles.

c. Find the number of miles at which the cost to rent either car would be the same.

Linear Inequalities and Systems of Linear Inequalities in Two Variables

Section 3.5

Concepts

1. Graphing Linear Inequalities in Two Variables
2. Systems of Linear Inequalities in Two Variables
3. Graphing a Feasible Region

1. Graphing Linear Inequalities in Two Variables

A **linear inequality in two variables** x and y is an inequality that can be written in one of the following forms: $Ax + By < C$, $Ax + By > C$, $Ax + By \leq C$, or $Ax + By \geq C$, provided A and B are not both zero.

A solution to a linear inequality in two variables is an ordered pair that makes the inequality true. For example, solutions to the inequality $x + y < 6$ are ordered pairs (x, y) such that the sum of the x- and y-coordinates is less than 6. This inequality has an infinite number of solutions, and therefore it is convenient to express the solution set as a graph.

To graph a linear inequality in two variables, we will follow these steps.

Graphing a Linear Inequality in Two Variables

Step 1 Write the inequality with the y variable isolated, if possible.

Step 2 Graph the related equation. Draw a dashed line if the inequality is strict, $<$ or $>$. Otherwise, draw a solid line.

Step 3 Shade above or below the line as follows:

- Shade *above* the line if the inequality is of the form $y > mx + b$ or $y \geq mx + b$.
- Shade *below* the line if the inequality is of the form $y < mx + b$ or $y \leq mx + b$.

Note: A dashed line indicates that the line is *not* included in the solution set. A solid line implies that the line *is* included in the solution set.

This process is demonstrated in Example 1.

Example 1 Graphing a Linear Inequality in Two Variables

Graph the solution set. $\quad -3x + y \leq 1$

Solution:

$-3x + y \leq 1$

$y \leq 3x + 1 \qquad$ Solve for y.

Next graph the line defined by the related equation $y = 3x + 1$.

Because the inequality is of the form $y \leq mx + b$, the solution to the inequality is the region *below* the line $y = 3x + 1$. So to indicate the solution set, shade the region on and below the line. See Figure 3-7.

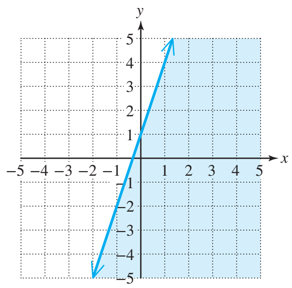


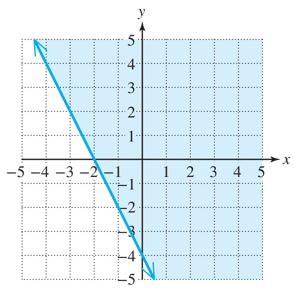
Figure 3-7

Skill Practice Graph the solution set.

1. $2x + y \geq -4$

Answer

1.



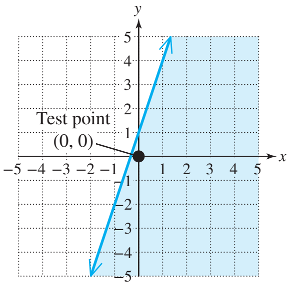


Figure 3-8

After graphing the solution to a linear inequality, we can verify that we have shaded the correct side of the line by using test points. In Example 1, we can pick an arbitrary ordered pair within the shaded region. Then substitute the x- and y-coordinates in the original inequality. If the result is a true statement, then that ordered pair is a solution to the inequality and suggests that other points from the same region are also solutions.

For example, the point (0, 0) lies within the shaded region (Figure 3-8).

$-3x + y \leq 1$ Substitute (0, 0) in the original inequality.

$-3(0) + (0) \stackrel{?}{\leq} 1$

$0 + 0 \stackrel{?}{\leq} 1$ ✓ True The point (0, 0) from the shaded region is a solution.

In Example 2, we will graph the solution set to a strict inequality. A strict inequality does not include equality and therefore, is expressed with the symbols < or >. In such a case, the boundary line will be drawn as a dashed line. This indicates that the boundary itself is *not* part of the solution set.

Example 2 Graphing a Linear Inequality in Two Variables

Graph the solution set. $-4y < 5x$

Solution:

$-4y < 5x$

$\dfrac{-4y}{-4} > \dfrac{5x}{-4}$ Solve for y. Recall that when we divide both sides by a negative number, reverse the inequality sign.

$y > -\dfrac{5}{4}x$

Graph the line defined by the related equation, $y = -\frac{5}{4}x$. The boundary line is drawn as a dashed line because the inequality is strict. Also note that the line passes through the origin.

Because the inequality is of the form $y > mx + b$, the solution to the inequality is the region *above* the line. See Figure 3-9.

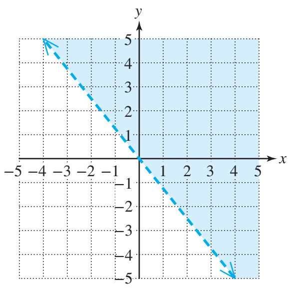


Figure 3-9

FOR REVIEW

Recall that to graph a linear equation in two variables, we can make a table of points and graph the line through the points. Alternatively, we can write the equation in slope-intercept form. Then graph the line using the y-intercept and slope.

Skill Practice Graph the solution set.

2. $-3y < x$

In Example 2, we cannot use the origin as a test point, because the point (0, 0) is on the boundary line. Be sure to select a test point strictly within the shaded region. In this case, we choose (2, 1). See Figure 3-10.

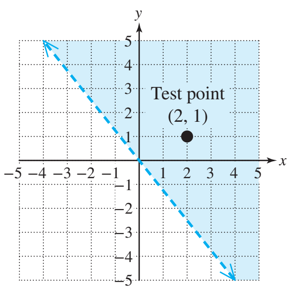


Figure 3-10

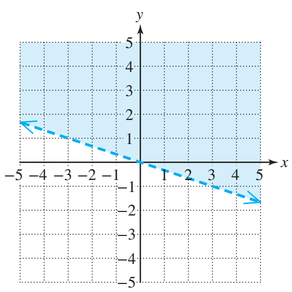
$-4y < 5x$ The test point (2, 1) is indeed a solution to the original inequality.

$-4(1) \stackrel{?}{<} 5(2)$

$-4 \stackrel{?}{<} 10$ ✓ True

Answer

2.



In Example 3, we encounter a situation in which we cannot solve for the y variable.

Example 3 Graphing a Linear Inequality in Two Variables

Graph the solution set. $4x \geq -12$

Solution:

$4x \geq -12$

$x \geq -3$

In this inequality, there is no y variable. However, we can simplify the inequality by solving for x.

Graph the related equation $x = -3$. This is a vertical line. The boundary is drawn as a solid line because the inequality is not strict, $\geq$.

To shade the appropriate region, refer to the inequality $x \geq -3$. The points for which x is greater than -3 are to the right of $x = -3$. Therefore, shade the region to the *right* of the line (Figure 3-11).

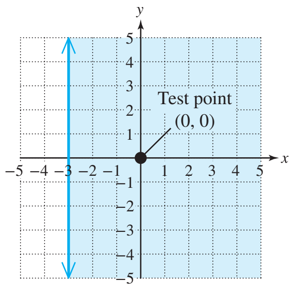


Figure 3-11

Selecting a test point such as (0, 0) from the shaded region indicates that we have shaded the correct side of the line.

$4x \geq -12$ Substitute $x = 0$.

$4(0) \stackrel{?}{\geq} -12$ ✓ True

FOR REVIEW

Recall that a vertical line is represented by an equation of the form $x = k$, such as $x = -3$. A horizontal line is represented by an equation of the form $y = k$, such as $y = 5$.

Skill Practice Graph the solution set.

3. $-2x \geq 2$

2. Systems of Linear Inequalities in Two Variables

Some applications require us to find the solutions to a system of linear inequalities.

Example 4 Graphing a System of Linear Inequalities

Graph the solution set.
$$y > \tfrac{1}{2}x + 1$$
$$x + y < 1$$

Solution:

Solve each inequality for y.

First inequality

$y > \frac{1}{2}x + 1$

The inequality is of the form $y > mx + b$. Shade *above* the boundary line. See Figure 3-12.

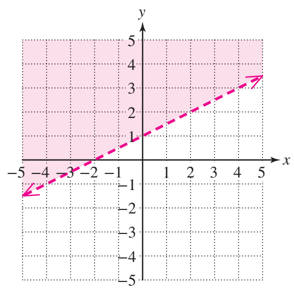
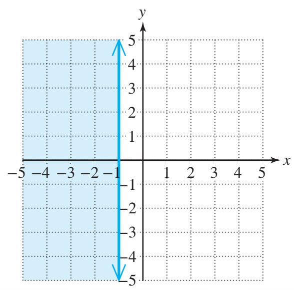


Figure 3-12

Answer

3.



Second inequality

$x + y < 1$

$y < -x + 1$

The inequality is of the form $y < mx + b$. Shade *below* the boundary line. See Figure 3-13.

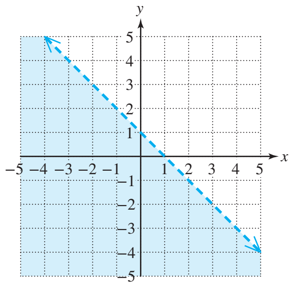


Figure 3-13

The region bounded by the inequalities is the region above the line $y = \frac{1}{2}x + 1$ and below the line $y = -x + 1$. This is the intersection or "overlap" of the two shaded regions (shown in purple in Figure 3-14).

The intersection is the solution set to the system of inequalities. See Figure 3-15.

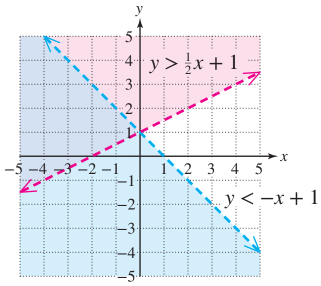


Figure 3-14

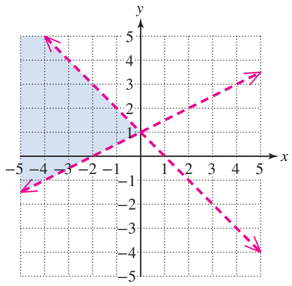


Figure 3-15

Skill Practice Graph the solution set.

4. $x - 3y > 3$
$y < -2x + 4$

Example 5 Graphing a System of Linear Inequalities

Graph the solution set. $3y \le 6$
$y - x \le 0$

Solution:

First inequality

$3y \le 6$

$y \le 2$

The graph of $y \le 2$ is the region on and below the horizontal line $y = 2$. See Figure 3-16.

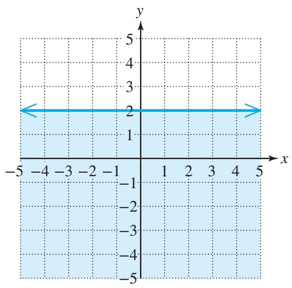
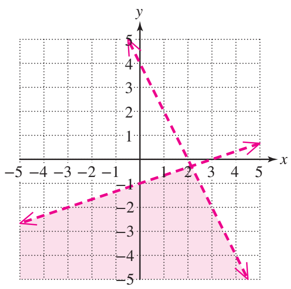


Figure 3-16

Answer

4.



<u>Second inequality</u>

$y - x \leq 0$

$y \leq x$

The inequality $y \leq x$ is of the form $y \leq mx + b$. Graph a solid line and shade the region below the line. See Figure 3-17.

The solution to the system of inequalities is the intersection of the shaded regions. Notice that the portions of the lines not bounding the solution are dashed. See Figure 3-18.

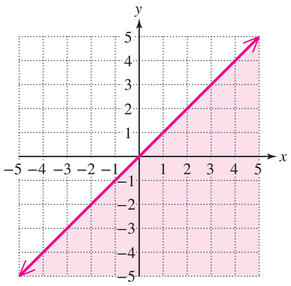


Figure 3-17

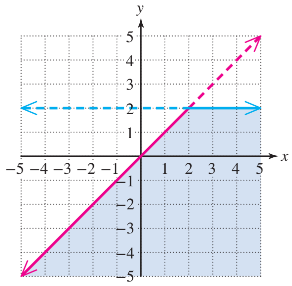


Figure 3-18

Skill Practice Graph the solution set.

5. $2y \leq 4$

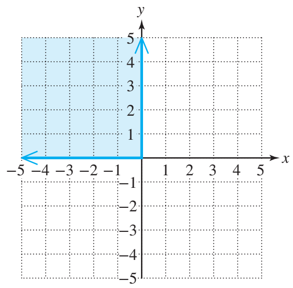
$y \leq x + 1$

Example 6 Graphing a System of Linear Inequalities

Describe the region of the plane defined by the system of inequalities.

$$x \leq 0$$
$$y \geq 0$$

Solution:



$x \leq 0$ — $x \leq 0$ for points on the y-axis and in the second and third quadrants.

$y \geq 0$ — $y \geq 0$ for points on the x-axis and in the first and second quadrants.

The intersection of these regions is the set of points in the second quadrant (with the boundaries included).

Skill Practice Graph the region defined by the system of inequalities.

6. $x \leq 0$

$y \leq 0$

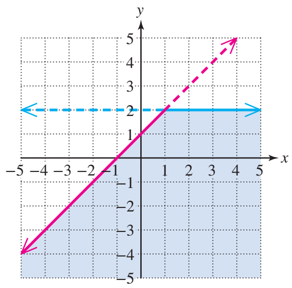
3. Graphing a Feasible Region

When two variables are related under certain constraints, a system of linear inequalities can be used to show a region of feasible values for the variables.

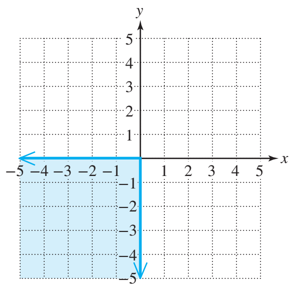
The feasible region represents the ordered pairs that are true for each inequality in the system.

Answers

5.



6.



Example 7 Graphing a Feasible Region

Susan has two tests on Friday: one in chemistry and one in psychology. Because the two classes meet in consecutive hours, she has no study time between tests. Susan estimates that she has a maximum of 12 hr of study time before the tests, and she must divide her time between chemistry and psychology.

Let x represent the number of hours Susan spends studying chemistry.

Let y represent the number of hours Susan spends studying psychology.

a. Find a set of inequalities to describe the constraints on Susan's study time.

b. Graph the constraints to find the feasible region defining Susan's study time.

Solution:

a. Because Susan cannot study chemistry or psychology for a negative period of time, we have $x \geq 0$ and $y \geq 0$. Furthermore, her total time studying cannot exceed 12 hr: $x + y \leq 12$.

A system of inequalities that defines the constraints on Susan's study time is:

$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 12$$

b. The first two conditions $x \geq 0$ and $y \geq 0$ represent the set of points in the first quadrant. The third condition $x + y \leq 12$ represents the set of points below and including the line $x + y = 12$ (Figure 3-19).

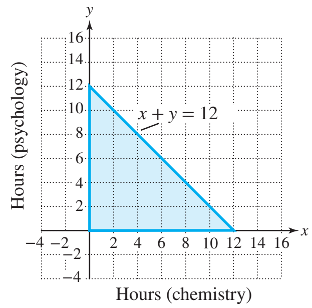


Figure 3-19

Discussion:

1. Refer to the feasible region drawn in Example 7(b). Is the ordered pair (8, 5) part of the feasible region?

No. The ordered pair (8, 5) indicates that Susan spent 8 hr studying chemistry and 5 hr studying psychology. This is a total of 13 hr, which exceeds the constraint that Susan only had 12 hr to study. The point (8, 5) lies outside the feasible region, above the line $x + y = 12$ (Figure 3-20).

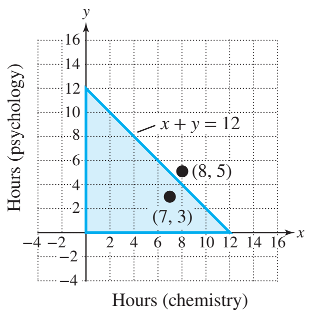


Figure 3-20

2. Is the ordered pair (7, 3) part of the feasible region?

Yes. The ordered pair (7, 3) indicates that Susan spent 7 hr studying chemistry and 3 hr studying psychology.

This point lies within the feasible region and satisfies all three constraints.

$$x \geq 0 \longrightarrow 7 \geq 0 \quad \text{True}$$
$$y \geq 0 \longrightarrow 3 \geq 0 \quad \text{True}$$
$$x + y \leq 12 \longrightarrow (7) + (3) \leq 12 \quad \text{True}$$

Notice that the ordered pair (7, 3) corresponds to a point where Susan is not making full use of the 12 hr of study time.

3. Suppose there was one additional constraint imposed on Susan's study time. She knows she needs to spend at least twice as much time studying chemistry as she does studying psychology. Graph the feasible region with this additional constraint.

Because the time studying chemistry must be at least twice the time studying psychology, we have $x \geq 2y$.

This inequality may also be written as $y \leq \frac{1}{2}x$.

Figure 3-21 shows the first quadrant with the constraint $y \leq \frac{1}{2}x$.

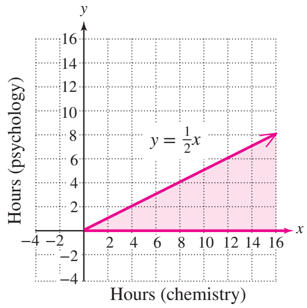


Figure 3-21

4. At what point in the feasible region is Susan making the most efficient use of her time for both classes?

First and foremost, Susan must make use of *all* 12 hr. This occurs for points along the line $x + y = 12$. Susan will also want to study for both classes with approximately twice as much time devoted to chemistry. Therefore, Susan will be deriving the maximum benefit at the point of intersection of the line

$x + y = 12$ and the line $y = \frac{1}{2}x$.

Using the substitution method, replace $y = \frac{1}{2}x$ into the equation $x + y = 12$.

$$x + \frac{1}{2}x = 12$$

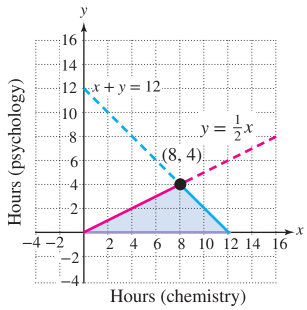
$$2x + x = 24 \qquad \text{Clear fractions.}$$

$$3x = 24$$

$$x = 8 \qquad \text{Solve for } x.$$

$$y = \frac{(8)}{2} \qquad \text{To solve for } y \text{, substitute } x = 8 \text{ into the equation } y = \tfrac{1}{2}x.$$

$$y = 4$$



Therefore, Susan should spend 8 hr studying chemistry and 4 hr studying psychology.

Skill Practice

7. A local pet rescue group has a total of 30 cages that can be used to hold cats and dogs. Let x represent the number of cages used for cats, and let y represent the number used for dogs.

a. Write a set of inequalities to express the fact that the number of cat and dog cages cannot be negative.

b. Write an inequality to describe the constraint on the total number of cages for cats and dogs.

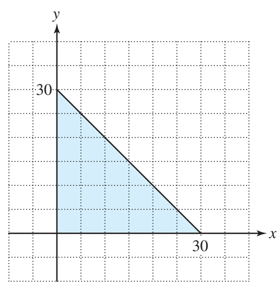
c. Graph the system of inequalities to find the feasible region describing the available cages.

Answer

7. a. $x \geq 0$ and $y \geq 0$

b. $x + y \leq 30$

c.

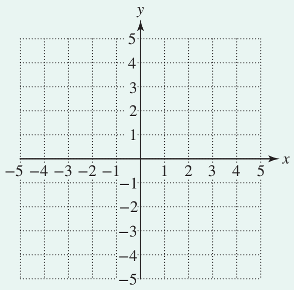


Section 3.5 Activity

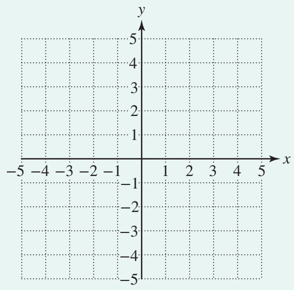
A.1. Determine if the ordered pair is a solution to the inequality $-3x + 5y < 15$.

a. $(2, 4)$ **b.** $(-4, 2)$ **c.** $(0, 3)$

A.2. **a.** Graph the line $-4x + 2y = 8$.



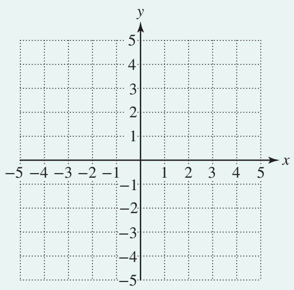
b. Graph the inequality $-4x + 2y \leq 8$.



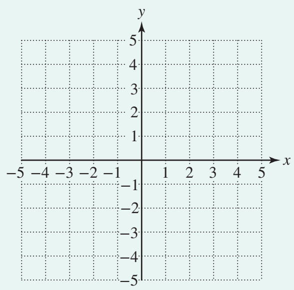
c. Explain how the graph of part (b) would change for the inequality $-4x + 2y \geq 8$.

d. Explain how the graph of part (b) would change for the inequality $-4x + 2y < 8$.

A.3. **a.** Graph the inequality $x < 0$.



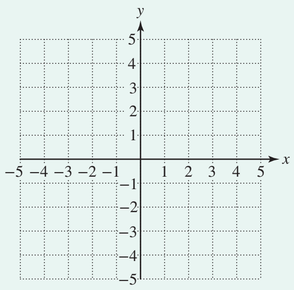
b. On the same coordinate system, graph the inequality $y > 0$.



c. Describe the area of overlap between the two inequalities.

d. Now graph the inequality $x - y > -3$ on the same coordinate system. Note that the region of overlap of the three inequalities is the solution set to the system:

$$\begin{aligned} &x < 0 \\ &y > 0 \\ &x - y > -3 \end{aligned}$$



Practice Exercises

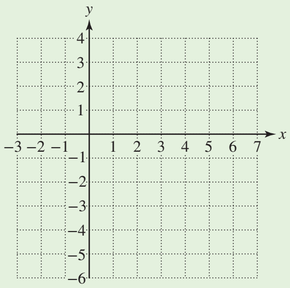
Section 3.5

Prerequisite Review

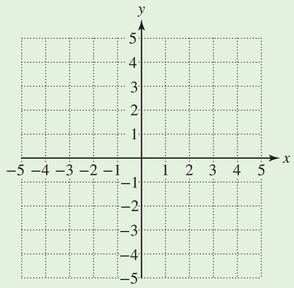
For Exercises R.1–R.4,

a. Find the x- and y-intercepts.
b. Graph the equation.

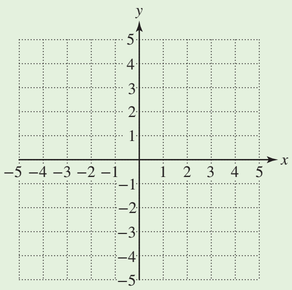
R.1. $4x - 3y = 12$



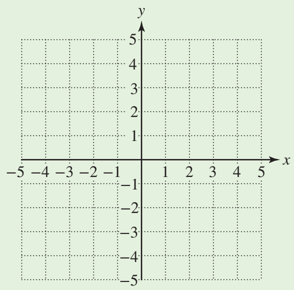
R.2. $3x = 4y$



R.3. $4 - y = 2$



R.4. $-3x = 9$



For Exercises R.5–R.6, determine whether the given value is a solution to the inequality.

R.5. $5x - 3 < 7$
a. $x = 2$ **b.** $x = -2$ **c.** $x = 6$

R.6. $-4x - 8 \geq 8$
a. $x = 4$ **b.** $x = -4$ **c.** $x = 0$

Vocabulary and Key Concepts

1. **a.** An inequality that can be written in the form $Ax + By > C$ is called a ____________ inequality in two variables.
   b. Given the inequality $3x - 4y < 6$, the boundary line $3x - 4y = 6$ (is/is not) included in the solution set. However, given $3x - 4y \leq 6$, then the boundary line (is/is not) included in the solution set.
2. **a.** Given the inequality $5x + y > 5$, the boundary line $5x + y = 5$ should be drawn as a (solid/dashed) line to indicate that the line (is/is not) part of the solution set.
   b. Given the inequality $y \geq -x + 4$, the boundary line $y = -x + 4$ should be drawn as a (solid/dashed) line to indicate that the line (is/is not) part of the solution set.
3. The graph of $y < x + 2$ is the set of points below the line $y = x + 2$. How would the graph of $y > x + 2$ be different?
4. The graph of $y \geq 3x + 1$ is the set of points on and above the line $y = 3x + 1$. How would the graph of $y > 3x + 1$ be different?
5. The line $y = -2x$ passes through the origin. Does the graph of $y < -2x$ include the origin?

Concept 1: Graphing Linear Inequalities in Two Variables

For Exercises 6–9, determine if the given point is a solution to the inequality.

6. $2x - y > 8$

a. $(3, -5)$ **b.** $(-1, -10)$ **c.** $(4, -2)$ **d.** $(0, 0)$

7. $3y + x < 5$

a. $(-1, 7)$ **b.** $(5, 0)$ **c.** $(0, 0)$ **d.** $(2, -3)$

8. $y \leq -2$

a. $(5, -3)$ **b.** $(-4, -2)$ **c.** $(0, 0)$ **d.** $(3, 2)$

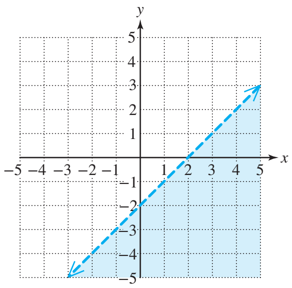
9. $x \geq 5$

a. $(4, 5)$ **b.** $(5, -1)$ **c.** $(8, 8)$ **d.** $(0, 0)$

10. When should you use a dashed line to graph the solution to a linear inequality?

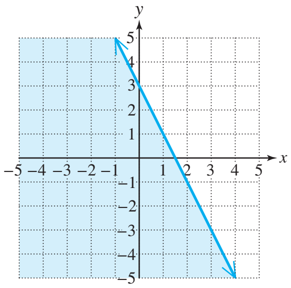
For Exercises 11–16, decide which inequality symbol should be used ($<, >, \geq, \leq$) by looking at the graph.

11.



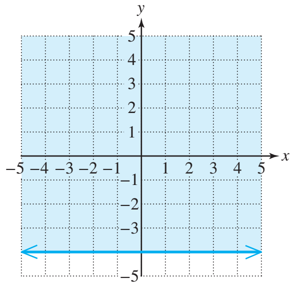
$x - y$ ____ 2

12.



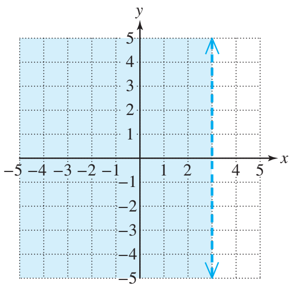
y ____ $-2x + 3$

13.



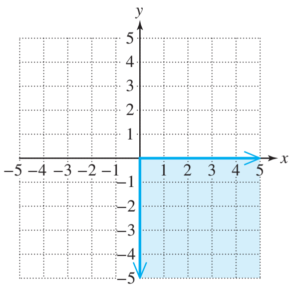
y ____ -4

14.



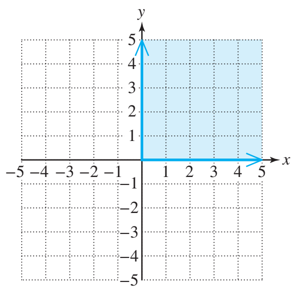
x ____ 3

15.



x ____ 0 and y ____ 0

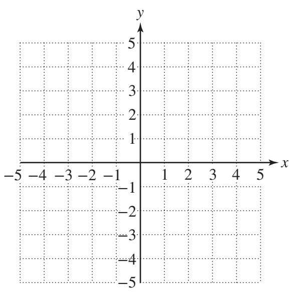
16.



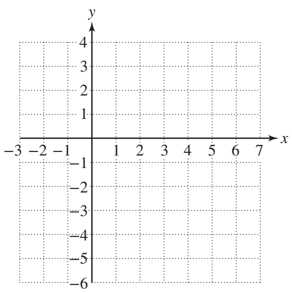
x ____ 0 and y ____ 0

For Exercises 17–40, graph the solution set. **(See Examples 1–3.)**

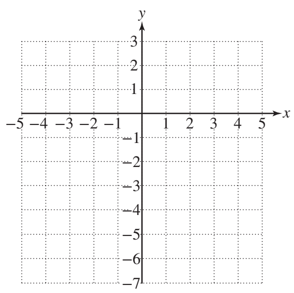
17. $x - 2y > 4$



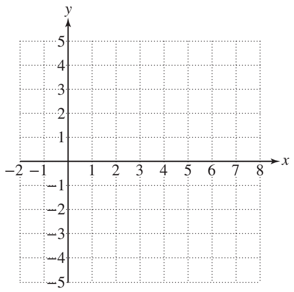
18. $x - 3y \geq 6$



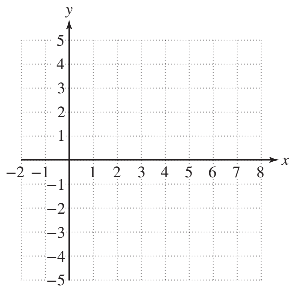
19. $5x - 2y < 10$



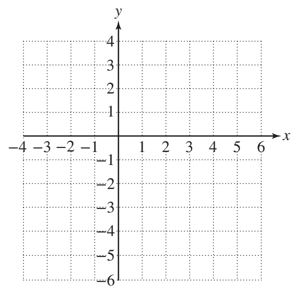
20. $x - 3y < 8$



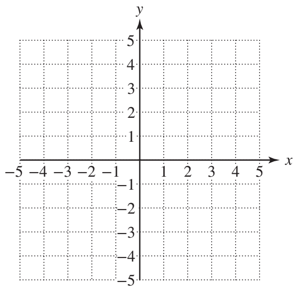
21. $2x \leq -6y + 12$



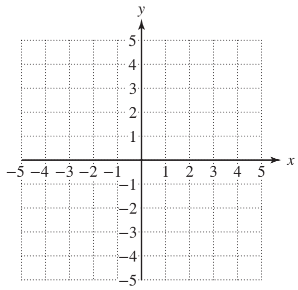
22. $4x < 3y + 12$



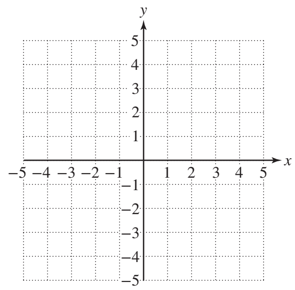
23. $2y \leq 4x$



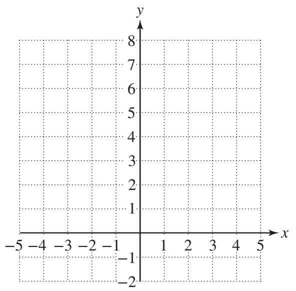
24. $-6x < 2y$



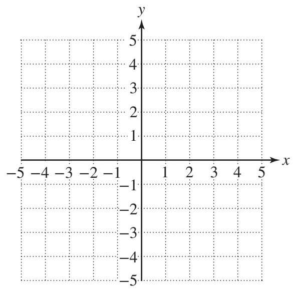
25. $y \geq -2$



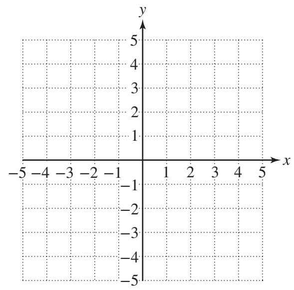
26. $y \geq 5$



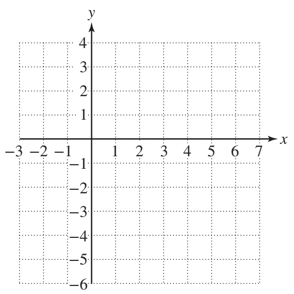
27. $4x < 5$



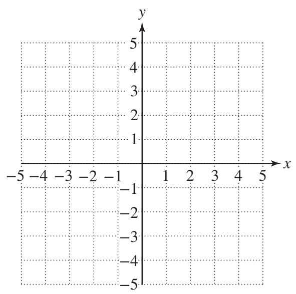
28. $x + 6 < 7$



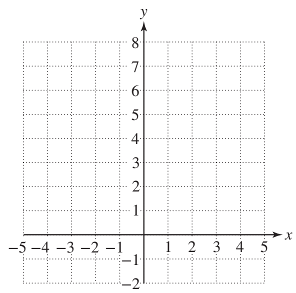
29. $y \geq \frac{2}{5}x - 4$



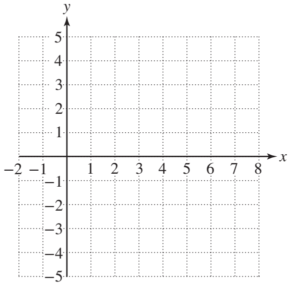
30. $y \geq -\frac{5}{2}x - 4$



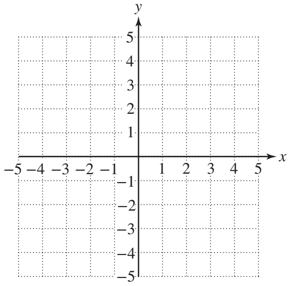
31. $y \leq \frac{1}{3}x + 6$



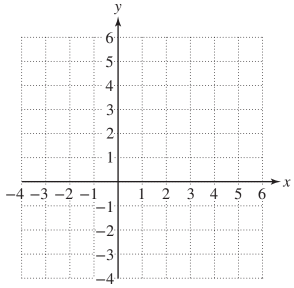
32. $y \leq -\frac{1}{4}x + 2$



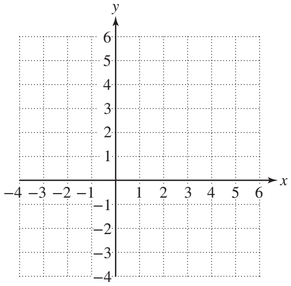
33. $y - 5x > 0$



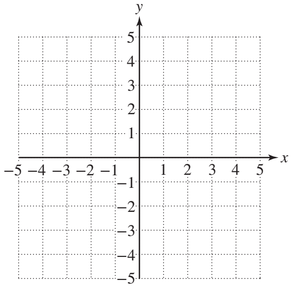
34. $y - \frac{1}{2}x > 0$



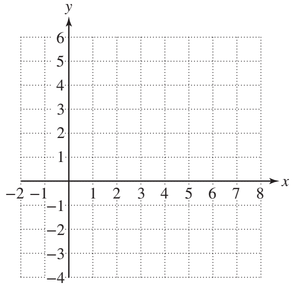
35. $\frac{x}{5} + \frac{y}{4} < 1$



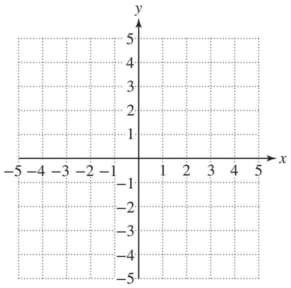
36. $x + \frac{y}{2} \geq 2$



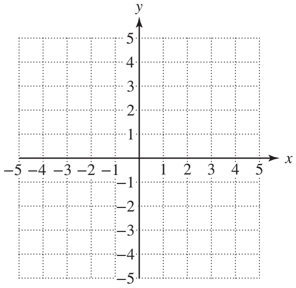
37. $0.1x + 0.2y \leq 0.6$



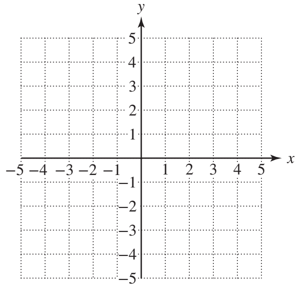
38. $0.3x - 0.2y < 0.6$



39. $x \leq -\frac{2}{3}y$



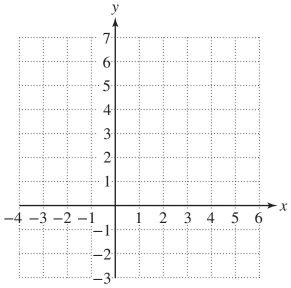
40. $x \geq -\frac{5}{4}y$



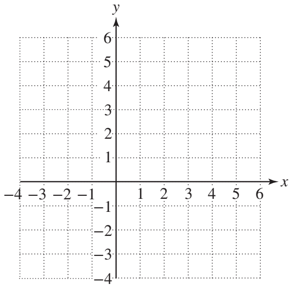
Concept 2: Systems of Linear Inequalities in Two Variables

For Exercises 41–55, graph the solution set. **(See Examples 4–6.)**

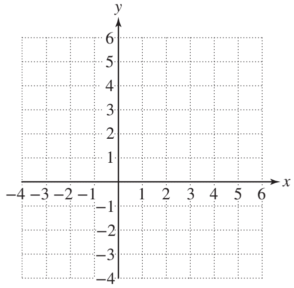
41. $y < 4$
$y > -x + 2$



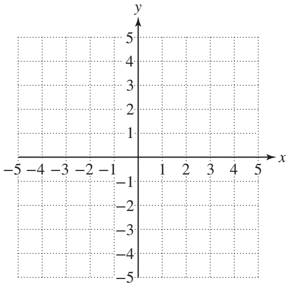
42. $y < 3$
$x + 2y < 6$



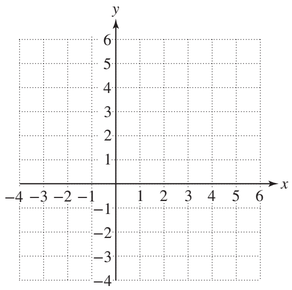
43. $2x + y \leq 5$
$x \geq 3$



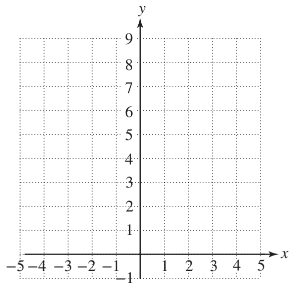
44. $x + 3y \geq 3$
$x \leq -2$



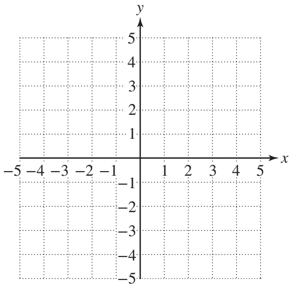
45. $x + y < 3$
$4x + y < 6$



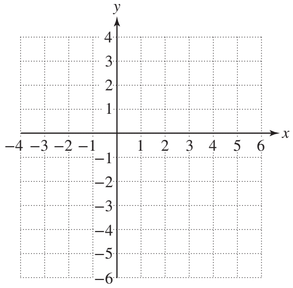
46. $x + y < 4$
$3x + y < 9$



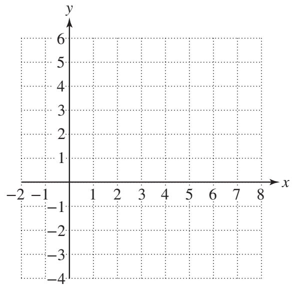
47. $2x - y \leq 2$
$2x + 3y \geq 6$



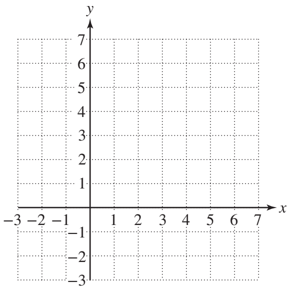
48. $3x + 2y \geq 4$
$x - y \leq 3$



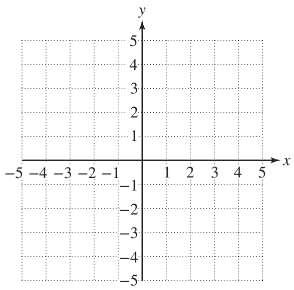
49. $x > 4$
$y < 2$



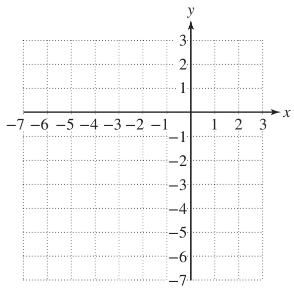
50. $x < 3$
$y > 4$



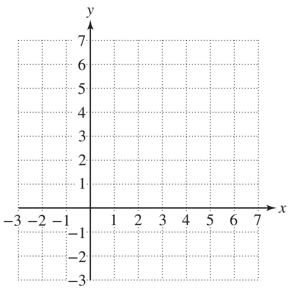
51. $x \leq -2$
$y \leq 0$



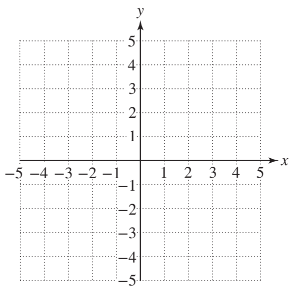
52. $x \geq 0$
$y \geq -3$



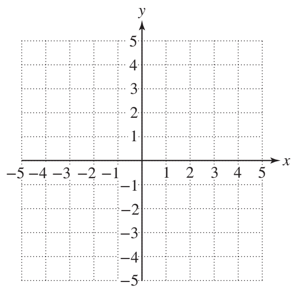
53. $x > 0$
$x + y < 6$



54. $x < 0$
$x + y < 2$



55. $y \geq 0$
$-2x + y \leq -4$



Concept 3: Graphing a Feasible Region

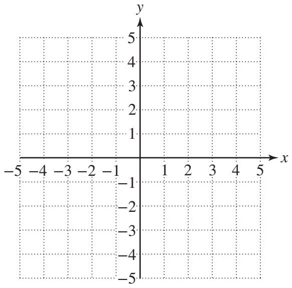
For Exercises 56–59, graph the feasible regions.

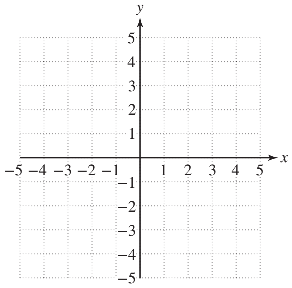
56. $x + y \leq 3$ and $x \geq 0, y \geq 0$

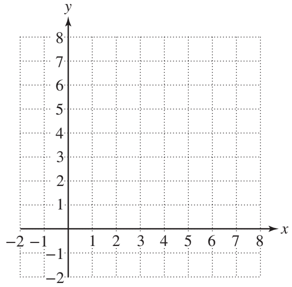
57. $x - y \leq 2$ and $x \geq 0, y \geq 0$

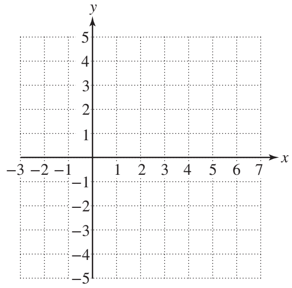
58. $x \geq 0, y \geq 0$, $x + y \leq 8$ and $3x + 5y \leq 30$

59. $x \geq 0, y \geq 0$, $x + y \leq 5$ and $x + 2y \leq 6$





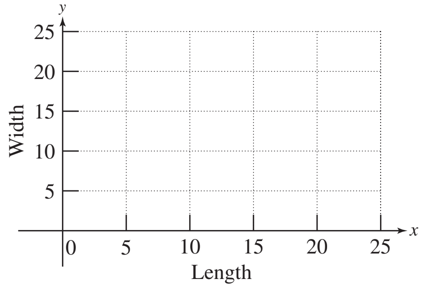




60. Suppose Sue has 50 ft of fencing with which she can build a rectangular dog run. Let x represent the length of the dog run and let y represent the width.

a. Write an inequality representing the fact that the total perimeter of the dog run is at most 50 ft.

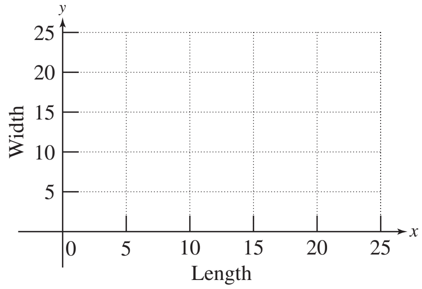
b. Sketch part of the solution set for this inequality that represents all possible values for the length and width of the dog run. (*Hint:* Note that both the length and the width must be positive.)



61. Suppose Rick has 40 ft of fencing with which he can build a rectangular garden. Let x represent the length of the garden and let y represent the width.

a. Write an inequality representing the fact that the total perimeter of the garden is at most 40 ft.

b. Sketch part of the solution set for this inequality that represents all possible values for the length and width of the garden. (*Hint:* Note that both the length and the width must be positive.)



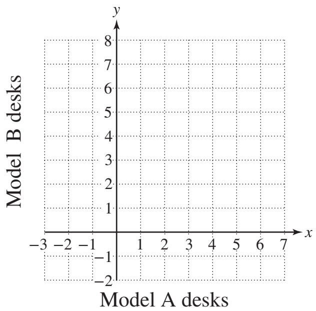
62. A manufacturer produces two models of desks. Model A requires 4 hr to stain and finish and 3 hr to assemble. Model B requires 3 hr to stain and finish and 1 hr to assemble. The total amount of time available for staining and finishing is 24 hr and for assembling is 12 hr. Let x represent the number of Model A desks produced, and let y represent the number of Model B desks produced.

a. Write two inequalities that express the fact that the number of desks to be produced cannot be negative.

b. Write an inequality in terms of the number of Model A and Model B desks that can be produced if the total time for staining and finishing is at most 24 hr.

c. Write an inequality in terms of the number of Model A and Model B desks that can be produced if the total time for assembly is no more than 12 hr.

d. Graph the feasible region formed by graphing the preceding inequalities.



e. Is the point (3, 1) in the feasible region? What does the point (3, 1) represent in the context of this problem?

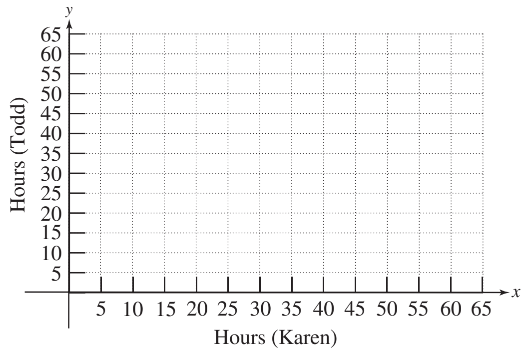
f. Is the point (5, 4) in the feasible region? What does the point (5, 4) represent in the context of this problem?

63. In scheduling two drivers for delivering pizza, James needs to have at least 65 hr scheduled this week. His two drivers, Karen and Todd, are not allowed to get overtime, so each one can work at most 40 hr. Let x represent the number of hours that Karen can be scheduled, and let y represent the number of hours Todd can be scheduled. **(See Example 7.)**

a. Write two inequalities that express the fact that Karen and Todd cannot work a negative number of hours.

b. Write two inequalities that express the fact that neither Karen nor Todd is allowed overtime (i.e., each driver can have at most 40 hr).

c. Write an inequality that expresses the fact that the total number of hours from both Karen and Todd needs to be at least 65 hr.



d. Graph the feasible region formed by graphing the inequalities.

e. Is the point (35, 40) in the feasible region? What does the point (35, 40) represent in the context of this problem?

f. Is the point (20, 40) in the feasible region? What does the point (20, 40) represent in the context of this problem?

Systems of Linear Equations in Three Variables and Applications

Section 3.6

Concepts

1. Solutions to Systems of Linear Equations in Three Variables
2. Solving Systems of Linear Equations in Three Variables
3. Applications of Linear Equations in Three Variables
4. Solving Inconsistent Systems and Systems of Dependent Equations

1. Solutions to Systems of Linear Equations in Three Variables

Thus far we have solved systems of linear equations in two variables. In this section, we will expand the discussion to solving systems involving three variables.

A **linear equation in three variables** can be written in the form $Ax + By + Cz = D$, where A, B, and C are not all zero. For example, the equation $2x + 3y + z = 6$ is a linear equation in three variables. Solutions to this equation are **ordered triples** of the form (x, y, z) that satisfy the equation. Some solutions to the equation $2x + 3y + z = 6$ are

Solution:	**Check:**
$(1, 1, 1) \longrightarrow$	$2(1) + 3(1) + (1) \stackrel{?}{=} 6$ ✓ True
$(2, 0, 2) \longrightarrow$	$2(2) + 3(0) + (2) \stackrel{?}{=} 6$ ✓ True
$(0, 1, 3) \longrightarrow$	$2(0) + 3(1) + (3) \stackrel{?}{=} 6$ ✓ True

Infinitely many ordered triples serve as solutions to the equation $2x + 3y + z = 6$.

The set of all ordered triples that are solutions to a linear equation in three variables may be represented graphically by a plane in space. Figure 3-22 shows a portion of the plane $2x + 3y + z = 6$ in a three-dimensional coordinate system.

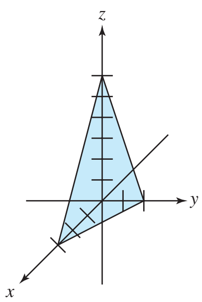


Figure 3-22

An example of a system of three linear equations in three variables is shown here.

$$\begin{aligned} 2x + y - 3z &= -7 \\ 3x - 2y + z &= 11 \\ -2x - 3y - 2z &= 3 \end{aligned}$$

A solution to a system of linear equations in three variables is an ordered triple that satisfies *each* equation. Geometrically, a solution is a point of intersection of the planes represented by the equations in the system.

A system of linear equations in three variables may have *one unique solution*, *infinitely many solutions*, or *no solution* (Table 3-2, Table 3-3, and Table 3-4).

Table 3-2

One unique solution (planes intersect at one point)

- The system is consistent.
- The equations are independent.

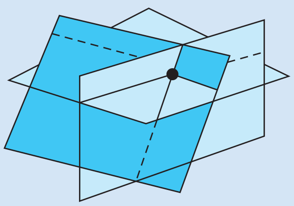


Table 3-3

No solution (the three planes do not all intersect)

- The system is inconsistent.
- The equations are independent.

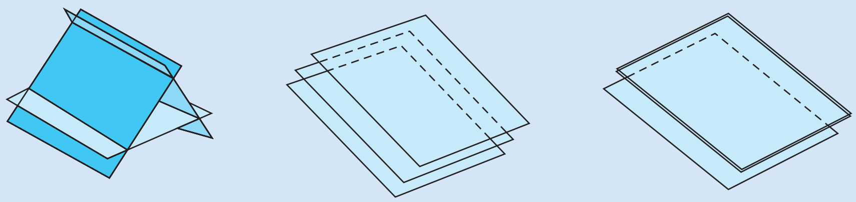
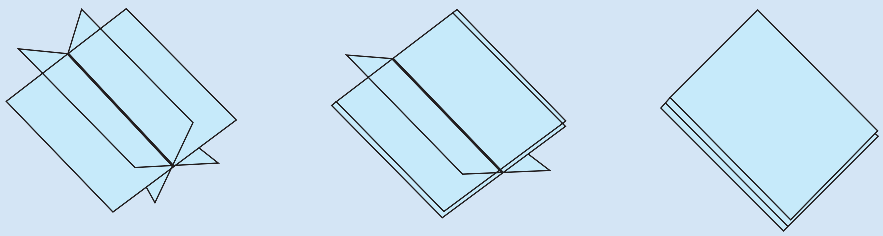


Table 3-4

Infinitely many solutions (planes intersect at infinitely many points)

- The system is consistent.
- The equations are dependent.



2. Solving Systems of Linear Equations in Three Variables

To solve a system involving three variables, the goal is to eliminate one variable. This reduces the system to two equations in two variables. One strategy for eliminating a variable is to pair up the original equations two at a time.

Solving a System of Three Linear Equations in Three Variables

Step 1 Write each equation in standard form $Ax + By + Cz = D$.

Step 2 Choose a pair of equations, and eliminate one of the variables by using the addition method.

Step 3 Choose a different pair of equations and eliminate the *same* variable.

Step 4 Once steps 2 and 3 are complete, you should have two equations in two variables. Solve this system by using the substitution method or the addition method.

Step 5 Substitute the values of the variables found in step 4 into any of the three original equations that contain the third variable. Solve for the third variable.

Step 6 Check the ordered triple in each of the original equations. Then write the solution as an ordered triple within set notation.

Example 1 Solving a System of Linear Equations in Three Variables

Solve the system.

$$\begin{aligned} 2x + y - 3z &= -7 \\ 3x - 2y + z &= 11 \\ -2x - 3y - 2z &= 3 \end{aligned}$$

Solution:

$\boxed{A}$ $2x + y - 3z = -7$

$\boxed{B}$ $3x - 2y + z = 11$

$\boxed{C}$ $-2x - 3y - 2z = 3$

Step 1: The equations are already in standard form.

- It is often helpful to label the equations.
- The y variable can be easily eliminated from equations $\boxed{A}$ and $\boxed{B}$ and from equations $\boxed{A}$ and $\boxed{C}$. This is accomplished by creating opposite coefficients for the y terms and then adding the equations.

Step 2: Eliminate the y variable from equations $\boxed{A}$ and $\boxed{B}$.

$$\begin{aligned} &\boxed{A}\ 2x + y - 3z = -7 \xrightarrow{\text{Multiply by 2.}} & 4x + 2y - 6z &= -14 \\ &\boxed{B}\ 3x - 2y + z = 11 \longrightarrow & 3x - 2y + z &= 11 \\ & & 7x \qquad - 5z &= -3 \ \boxed{D} \end{aligned}$$

Step 3: Eliminate the y variable again, this time from equations $\boxed{A}$ and $\boxed{C}$.

$$\begin{aligned} &\boxed{A}\ 2x + y - 3z = -7 \xrightarrow{\text{Multiply by 3.}} & 6x + 3y - 9z &= -21 \\ &\boxed{C}\ -2x - 3y - 2z = 3 \longrightarrow & -2x - 3y - 2z &= 3 \\ & & 4x \qquad - 11z &= -18 \ \boxed{E} \end{aligned}$$

TIP: It is important to note that in steps 2 and 3, the *same* variable is eliminated.

Step 4: Now equations $\boxed{\text{D}}$ and $\boxed{\text{E}}$ can be paired up to form a linear system in two variables. Solve this system.

$$\begin{aligned}
&\boxed{\text{D}}\; 7x - 5z = -3 \xrightarrow{\text{Multiply by } -4.} & -28x + 20z &= 12\\
&\boxed{\text{E}}\; 4x - 11z = -18 \xrightarrow[\text{Multiply by 7.}]{} & 28x - 77z &= -126\\
&& -57z &= -114\\
&& z &= 2
\end{aligned}$$

Once one variable has been found, substitute this value into either equation in the two-variable system, that is, either equation $\boxed{\text{D}}$ or $\boxed{\text{E}}$.

$$\begin{aligned}
\boxed{\text{D}}\; 7x - 5z &= -3\\
7x - 5(2) &= -3 \qquad \text{Substitute } z = 2 \text{ into equation } \boxed{\text{D}}.\\
7x - 10 &= -3\\
7x &= 7\\
x &= 1
\end{aligned}$$

$$\begin{aligned}
\boxed{\text{A}}\; 2x + y - 3z &= -7\\
2(1) + y - 3(2) &= -7\\
2 + y - 6 &= -7\\
y - 4 &= -7\\
y &= -3
\end{aligned}$$

Step 5: Now that two variables are known, substitute these values (x and z) into any of the original three equations to find the remaining variable y. Substitute $x = 1$ and $z = 2$ into equation $\boxed{\text{A}}$.

The solution set is $\{(1, -3, 2)\}$.

Step 6: Check the ordered triple in the three original equations.

Check:

$$\begin{aligned}
2x + y - 3z &= -7 \rightarrow 2(1) + (-3) - 3(2) \stackrel{?}{=} -7 \checkmark \text{ True}\\
3x - 2y + z &= 11 \rightarrow 3(1) - 2(-3) + (2) \stackrel{?}{=} 11 \checkmark \text{ True}\\
-2x - 3y - 2z &= 3 \rightarrow -2(1) - 3(-3) - 2(2) \stackrel{?}{=} 3 \checkmark \text{ True}
\end{aligned}$$

Skill Practice Solve the system.

1.
$$\begin{aligned}
x + 2y + z &= 1\\
3x - y + 2z &= 13\\
2x + 3y - z &= -8
\end{aligned}$$

3. Applications of Linear Equations in Three Variables

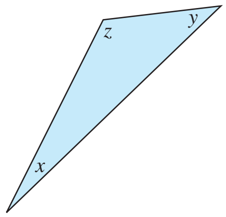
Example 2 Applying Systems of Linear Equations to Geometry

In a triangle, the smallest angle measures 10° more than one-half the measure of the largest angle. The middle angle measures 12° more than the measure of the smallest angle. Find the measure of each angle.

Answer

1. $\{(1, -2, 4)\}$

Solution:



Let x represent the measure of the smallest angle.
Let y represent the measure of the middle angle.
Let z represent the measure of the largest angle.

To solve for three variables, we need to establish three independent relationships among x, y, and z.

$\boxed{\text{A}}$ $x = \frac{1}{2}z + 10$	The smallest angle measures 10° more than one-half the measure of the largest angle.
$\boxed{\text{B}}$ $y = x + 12$	The middle angle measures 12° more than the measure of the smallest angle.
$\boxed{\text{C}}$ $x + y + z = 180$	The sum of the angles inscribed in a triangle is 180°.

Clear fractions and write each equation in standard form.

$$
\begin{array}{llll}
 & & & \textbf{Standard Form} \\
\boxed{\text{A}} & x = \frac{1}{2}z + 10 & \xrightarrow{\text{Multiply by 2.}} 2x = z + 20 \longrightarrow & 2x \quad - z = 20 \\
\boxed{\text{B}} & y = x + 12 & \longrightarrow & -x + y \quad = 12 \\
\boxed{\text{C}} & x + y + z = 180 & \longrightarrow & x + y + z = 180
\end{array}
$$

Notice equation $\boxed{\text{B}}$ is missing the z variable. Therefore, we can eliminate z again by pairing up equations $\boxed{\text{A}}$ and $\boxed{\text{C}}$.

$$
\begin{array}{lrl}
\boxed{\text{A}} & 2x \qquad - z = & 20 \\
\boxed{\text{C}} & x + y + z = & 180 \\
\hline
 & 3x + y \qquad = & 200 \quad \boxed{\text{D}}
\end{array}
$$

$$
\begin{array}{llrl}
\boxed{\text{B}} & -x + y = 12 & \xrightarrow{\text{Multiply by } -1.} & x - y = -12 \\
\boxed{\text{D}} & 3x + y = 200 & \longrightarrow & 3x + y = 200 \\
\hline
 & & & 4x \qquad = 188 \\
 & & & x = 47
\end{array}
$$

Pair up equations $\boxed{\text{B}}$ and $\boxed{\text{D}}$ to form a system of two variables.

Solve for x.

From equation $\boxed{\text{B}}$ we have $-x + y = 12 \longrightarrow -47 + y = 12 \rightarrow y = 59$

From equation $\boxed{\text{C}}$ we have $x + y + z = 180 \rightarrow 47 + 59 + z = 180 \rightarrow z = 74$

The smallest angle measures 47°, the middle angle measures 59°, and the largest angle measures 74°.

Skill Practice

2. The perimeter of a triangle is 30 in. The shortest side is 4 in. shorter than the longest side. The longest side is 6 in. less than the sum of the other two sides. Find the length of each side.

Answer

2. The sides are 8 in., 10 in., and 12 in.

Example 3 Applying Systems of Linear Equations to Nutrition

Doctors have become increasingly concerned about the sodium intake in the U.S. diet. Recommendations by the American Medical Association indicate that most individuals should not exceed 2400 mg of sodium per day.

Torie ate 1 slice of pizza, 1 serving of ice cream, and 1 glass of soda for a total of 1030 mg of sodium. David ate 3 slices of pizza, no ice cream, and 2 glasses of soda for a total of 2420 mg of sodium. Emilie ate 2 slices of pizza, 1 serving of ice cream, and 2 glasses of soda for a total of 1910 mg of sodium. How much sodium is in one serving of each item?

Solution:

Let x represent the sodium content of 1 slice of pizza.
Let y represent the sodium content of 1 serving of ice cream.
Let z represent the sodium content of 1 glass of soda.

From Torie's meal we have: $\boxed{\text{A}}\quad x + y + z = 1030$

From David's meal we have: $\boxed{\text{B}}\quad 3x + 2z = 2420$

From Emilie's meal we have: $\boxed{\text{C}}\quad 2x + y + 2z = 1910$

Equation $\boxed{\text{B}}$ is missing the y variable. Eliminating y from equations $\boxed{\text{A}}$ and $\boxed{\text{C}}$, we have

$$\begin{array}{ll}
\boxed{\text{A}}\quad x + y + z = 1030 & \xrightarrow{\text{Multiply by } -1.} \quad -x - y - z = -1030 \\
\boxed{\text{C}}\quad 2x + y + 2z = 1910 & \longrightarrow \quad \underline{2x + y + 2z = 1910} \\
 & \boxed{\text{D}}\quad x + z = 880
\end{array}$$

Solve the system formed by equations $\boxed{\text{B}}$ and $\boxed{\text{D}}$.

$$\begin{array}{ll}
\boxed{\text{B}}\quad 3x + 2z = 2420 & \longrightarrow \quad 3x + 2z = 2420 \\
\boxed{\text{D}}\quad x + z = 880 & \xrightarrow[\text{Multiply by } -2.]{} \quad \underline{-2x - 2z = -1760} \\
 & \quad x = 660
\end{array}$$

From equation $\boxed{\text{D}}$ we have $x + z = 800 \longrightarrow 660 + z = 880 \longrightarrow z = 220$

From equation $\boxed{\text{A}}$ we have $x + y + z = 1030 \rightarrow 660 + y + 220 = 1030 \rightarrow y = 150$

Therefore, 1 slice of pizza has 660 mg of sodium, 1 serving of ice cream has 150 mg of sodium, and 1 glass of soda has 220 mg of sodium.

Skill Practice

3. Annette, Barb, and Carlita work in a clothing shop. One day the three had combined sales of \$1480. Annette sold \$120 more than Barb. Barb and Carlita combined sold \$280 more than Annette. How much did each person sell?

Answer

3. Annette sold \$600, Barb sold \$480, and Carlita sold \$400.

4. Solving Inconsistent Systems and Systems of Dependent Equations

Example 4 **Solving a System of Dependent Equations**

Solve the system. If the system does not have one unique solution, state the number of solutions and whether the system is inconsistent or the equations are dependent.

$$\begin{aligned} \boxed{A}\quad 3x + y - z &= 8 \\ \boxed{B}\quad 2x - y + 2z &= 3 \\ \boxed{C}\quad x + 2y - 3z &= 5 \end{aligned}$$

Solution:

The first step is to make a decision regarding the variable to eliminate. The y variable is particularly easy to eliminate because the coefficients of y in equations $\boxed{A}$ and $\boxed{B}$ are already opposites. The y variable can be eliminated from equations $\boxed{B}$ and $\boxed{C}$ by multiplying equation $\boxed{B}$ by 2.

$$\begin{array}{lrcl} \boxed{A} & 3x + y - z &=& 8 \\ \boxed{B} & 2x - y + 2z &=& 3 \\ \hline & 5x \quad + z &=& 11 \quad \boxed{D} \end{array}$$

Pair up equations $\boxed{A}$ and $\boxed{B}$ to eliminate y.

$$\begin{array}{lrcl} \boxed{B} & 2x - y + 2z = 3 & \xrightarrow{\text{Multiply by 2.}} & 4x - 2y + 4z = 6 \\ \boxed{C} & x + 2y - 3z = 5 & \longrightarrow & x + 2y - 3z = 5 \\ \hline & & & 5x \quad + z = 11 \quad \boxed{E} \end{array}$$

Pair up equations $\boxed{B}$ and $\boxed{C}$ to eliminate y.

Because equations $\boxed{D}$ and $\boxed{E}$ are equivalent equations, it appears that this is a dependent system. By eliminating variables we obtain the identity $0 = 0$.

$$\begin{array}{lrcl} \boxed{D} & 5x + z = 11 & \xrightarrow{\text{Multiply by } -1.} & -5x - z = -11 \\ \boxed{E} & 5x + z = 11 & \longrightarrow & 5x + z = 11 \\ \hline & & & 0 = 0 \end{array}$$

The result $0 = 0$ indicates that there are infinitely many solutions and that the equations are dependent.

Skill Practice Solve the system. If the system does not have one unique solution, state the number of solutions and whether the system is inconsistent or the equations are dependent.

4.
$$\begin{aligned} x + y + z &= 8 \\ 2x - y + z &= 6 \\ -5x - 2y - 4z &= -30 \end{aligned}$$

Answer

4. Infinitely many solutions; dependent equations

Example 5 Solving an Inconsistent System of Linear Equations

Solve the system. If the system does not have one unique solution, state the number of solutions and whether the system is inconsistent or the equations are dependent.

$$\begin{aligned} 2x + 3y - 7z &= 4 \\ -4x - 6y + 14z &= 1 \\ 5x + y - 3z &= 6 \end{aligned}$$

Solution:

We will eliminate the x variable from equations $\boxed{A}$ and $\boxed{B}$.

$$\begin{array}{lrcr} \boxed{A} & 2x + 3y - 7z = 4 & \xrightarrow{\text{Multiply by 2.}} & 4x + 6y - 14z = 8 \\ \boxed{B} & -4x - 6y + 14z = 1 & \longrightarrow & \underline{-4x - 6y + 14z = 1} \\ & & & 0 = 9 \quad \text{(contradiction)} \end{array}$$

The result $0 = 9$ is a contradiction, indicating that the system has no solution. The system is inconsistent.

Skill Practice

5. Solve the system. If the system does not have one unique solution, state the number of solutions and whether the system is inconsistent or the equations are dependent.

$$\begin{aligned} x - 2y + z &= 5 \\ x - 3y + 2z &= -7 \\ -2x + 4y - 2z &= 6 \end{aligned}$$

Answer

5. No solution; inconsistent system

Section 3.6 Activity

A.1. **a.** The graph of a linear equation in two variables $Ax + By = C$ is a line, whereas the graph of a linear equation in three variables $Ax + By + Cz = D$ is a ________ in space.

b. A solution to a linear equation in two variables $Ax + By = C$ is an ordered ______ of the form (x, y). A solution to a linear equation in three variables $Ax + By + Cz = D$ is an ordered ______ of the form (x, y, z).

A.2. Determine if the ordered triple is a solution to the system.

$$\begin{aligned} x + 2y + 3z &= 1 \\ 3x - y + z &= -1 \\ 2x - 3y - 4z &= 4 \end{aligned}$$

a. $(1, -3, 2)$ **b.** $(2, 4, -3)$

The basic premise to solve a system of linear equations in three variables is to eliminate one of the variables to create a system of two-variable equations. In Exercise A.3, we walk through this process.

A.3. For the given system, we must first choose a variable to eliminate. We will arbitrarily choose variable z to eliminate.

$$\begin{aligned} x + 2y + 3z &= 1 \quad \boxed{A} \\ 3x - y + z &= -1 \quad \boxed{B} \\ 2x - 3y - 4z &= 4 \quad \boxed{C} \end{aligned}$$

a. Choose two equations from the original system and eliminate the z variable. For example, pair up equations $\boxed{A}$ and $\boxed{B}$. Multiply equation $\boxed{B}$ by –3, then add the equations. Label the resulting equation as $\boxed{D}$.

$$\begin{aligned} x + 2y + 3z &= 1 \quad \boxed{A} \\ 3x - y + z &= -1 \quad \boxed{B} \end{aligned}$$

b. Repeat part (a) with a different pair of equations from the original system. For example, pair up equations $\boxed{\text{B}}$ and $\boxed{\text{C}}$. Then eliminate the z variable. Label the resulting equation, $\boxed{\text{E}}$.

$$\begin{aligned} 3x - y + z &= -1 \quad \boxed{\text{B}} \\ 2x - 3y - 4z &= 4 \quad \boxed{\text{C}} \end{aligned}$$

c. Solve for the remaining variables x and y by solving the system given by equations $\boxed{\text{D}}$ and $\boxed{\text{E}}$.

d. Substitute the values of x and y found in part (c) into one of the original equations in the system. Then solve the equation for z.

e. Write the solution set as an ordered triple. Be sure to check the ordered triple in all three original equations.

A.4. The following system has missing variables within the individual equations. This makes the system slightly easier to solve.

$$\begin{aligned} 3x + 4y + z &= -3 \quad \boxed{\text{A}} \\ 5x \qquad + 4z &= -10 \quad \boxed{\text{B}} \\ 2y - 3z &= 13 \quad \boxed{\text{C}} \end{aligned}$$

a. Equations $\boxed{\text{B}}$ and $\boxed{\text{C}}$ each have a missing variable. For example, y is missing from equation $\boxed{\text{B}}$. Therefore, eliminate y from equations $\boxed{\text{A}}$ and $\boxed{\text{C}}$. Label the resulting equation as $\boxed{\text{D}}$.

b. Pair up equation $\boxed{\text{B}}$ from the original system and equation $\boxed{\text{D}}$ to form a system of equations involving the variables x and z. Solve this system for x and z.

c. Substitute the values of x and z found in part (b) into one of the original equations in the system. Then solve the equation for y.

d. Write the solution set as an ordered triple. Be sure to check the ordered triple in all three original equations.

A system of linear equations in three variables may have no solution. In such a case, you will encounter a contradiction such as $0 = 4$ and will identify the system as inconsistent. Likewise, a system of three variables may have dependent equations. In such a case, you will encounter an identity such as $0 = 0$ and will note that there are infinitely many solutions to the system. In Exercise A.5 and A.6, we investigate these two scenarios.

A.5. Consider the system.

$$\begin{aligned} 4x \qquad + 6z &= 6 \quad \boxed{\text{A}} \\ 8x - 9y + 15z &= 9 \quad \boxed{\text{B}} \\ 4x - 3y + 7z &= 5 \quad \boxed{\text{C}} \end{aligned}$$

a. Eliminate y from equations $\boxed{\text{B}}$ and $\boxed{\text{C}}$. Label the resulting equation as $\boxed{\text{D}}$.

b. As you solve the system given by equations $\boxed{\text{A}}$ and $\boxed{\text{D}}$, do you encounter a contradiction or an identity? From this finding, does the original system have no solution or infinitely many solutions?

A.6. Solve the system.

$$\begin{aligned} x + 2y - 3z &= -1 \quad \boxed{\text{A}} \\ 3x - 9y + 3z &= 3 \quad \boxed{\text{B}} \\ 2x - y - 2z &= 2 \quad \boxed{\text{C}} \end{aligned}$$

To solve an application with three unknowns, it is often helpful to use three variables such as x, y, and z. When three variables are used to solve an application, we need three independent equations that relate the variables. That is, for three unknowns, set up a system of three equations.

A.7. The average of Randall's three test grades in economics is 92. His first test score is 6 points higher than his second test score. His third test score is 3 points lower than the first test score. Find his test scores by following these steps.

a. Let x represent Randall's first test score.

Let y represent ______________________________.

Let z represent ______________________________.

b. Write an equation in terms of x, y, and z that represents the fact that the average of Randall's three test scores is 92.

c. Write an equation that represents the fact that Randall's first test score is 6 points higher than his second test score.

d. Write an equation that represents the fact that Randall's third test score is 3 points lower than the first test score.

e. Solve the system made up of the equations from parts (b) – (d).

f. Interpret the answer from part (e).

Section 3.6 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, determine if the ordered pair is a solution to the system.

R.1. $$\begin{aligned} -5x + 3y &= -1 \\ 4x - 2y &= -2 \end{aligned}$$

a. $(2, 3)$ **b.** $(-4, -7)$

R.2. $$\begin{aligned} 2x &= 4y - 2 \\ y &= 4x + 4 \end{aligned}$$

a. $(-1, 0)$ **b.** $(3, 2)$

For Exercises R.3–R.8, solve the system. If the system does not have one unique solution, also state whether the system is inconsistent or whether the equations are dependent.

R.3. $$\begin{aligned} y &= 2x - 11 \\ 3x - 5y &= 20 \end{aligned}$$

R.4. $$\begin{aligned} 4x + 3y &= -7 \\ x &= 2y + 12 \end{aligned}$$

R.5. $$\begin{aligned} 4x + y &= 6 \\ 2y &= 12 - 8x \end{aligned}$$

R.6. $$\begin{aligned} x - 3y &= -12 \\ y &= \frac{1}{3}x + 4 \end{aligned}$$

R.7. $$\begin{aligned} 2x + 5y &= 4 \\ -6x - 15y &= 0 \end{aligned}$$

R.8. $$\begin{aligned} -4x + y &= 1 \\ 8x - 2y &= 6 \end{aligned}$$

Vocabulary and Key Concepts

1. An equation written in the form $Ax + By + Cz = D$, where A, B, and C are not all zero, is called a ______________ equation in three variables.

2. Solutions to a linear equation in three variables are of the form (x, y, z) and are called ______________.

3. When solving a system of linear equations in three variables and you encounter the statement $0 = 0$, how many solutions does the system have?

4. When solving a system of linear equations in three variables and you encounter the statement $0 = 3$, how many solutions does the system have?

5. For the given system, the value of z is 2. Find the values of x and y and write the solution set.

$$\begin{aligned} x + 3y - 2z &= -12 \\ y + z &= -1 \\ 2x - 3z &= -4 \end{aligned}$$

Concept 1: Solutions to Systems of Linear Equations in Three Variables

6. How many solutions are possible when solving a system of three equations with three variables?

7. Which of the following points are solutions to the system? (2, 1, 7), (3, −10, −6), (4, 0, 2)

$$\begin{aligned} 2x - y + z &= 10 \\ 4x + 2y - 3z &= 10 \\ x - 3y + 2z &= 8 \end{aligned}$$

8. Which of the following points are solutions to the system? (1, 1, 3), (0, 0, 4), (4, 2, 1)

$$\begin{aligned} -3x - 3y - 6z &= -24 \\ -9x - 6y + 3z &= -45 \\ 9x + 3y - 9z &= 33 \end{aligned}$$

9. Which of the following points are solutions to the system? (12, 2, −2), (4, 2, 1), (1, 1, 1)

$$\begin{aligned} -x - y - 4z &= -6 \\ x - 3y + z &= -1 \\ 4x + y - z &= 4 \end{aligned}$$

10. Which of the following points are solutions to the system? (0, 4, 3), (3, 6, 10), (3, 3, 1)

$$\begin{aligned} x + 2y - z &= 5 \\ x - 3y + z &= -5 \\ -2x + y - z &= -4 \end{aligned}$$

Concept 2: Solving Systems of Linear Equations in Three Variables

For Exercises 11–22, solve the system of equations. **(See Example 1.)**

11. $$\begin{aligned} 2x + y - 3z &= -12 \\ 3x - 2y - z &= 3 \\ -x + 5y + 2z &= -3 \end{aligned}$$

12. $$\begin{aligned} -3x - 2y + 4z &= -15 \\ 2x + 5y - 3z &= 3 \\ 4x - y + 7z &= 15 \end{aligned}$$

13. $$\begin{aligned} x - 3y - 4z &= -7 \\ 5x + 2y + 2z &= -1 \\ 4x - y - 5z &= -6 \end{aligned}$$

14. $$\begin{aligned} 6x - 5y + z &= 7 \\ 5x + 3y + 2z &= 0 \\ -2x + y - 3z &= 11 \end{aligned}$$

15. $$\begin{aligned} 4x + 2z &= 12 + 3y \\ 2y &= 3x + 3z - 5 \\ y &= 2x + 7z + 8 \end{aligned}$$

16. $$\begin{aligned} y &= 2x + z + 1 \\ -3x - 1 &= -2y + 2z \\ 5x + 3z &= 16 - 3y \end{aligned}$$

17. $$\begin{aligned} x + y + z &= 6 \\ -x + y - z &= -2 \\ 2x + 3y + z &= 11 \end{aligned}$$

18. $$\begin{aligned} x - y - z &= -11 \\ x + y - z &= 15 \\ 2x - y + z &= -9 \end{aligned}$$

19. $$\begin{aligned} 2x - 3y + 2z &= -1 \\ x + 2y &= -4 \\ x + z &= 1 \end{aligned}$$

20. $$\begin{aligned} x + y + z &= 2 \\ 2x - z &= 5 \\ 3y + z &= 2 \end{aligned}$$

21. $$\begin{aligned} 4x + 9y &= 8 \\ 8x + 6z &= -1 \\ 6y + 6z &= -1 \end{aligned}$$

22. $$\begin{aligned} 3x + 2z &= 11 \\ y - 7z &= 4 \\ x - 6y &= 1 \end{aligned}$$

Concept 3: Applications of Linear Equations in Three Variables

23. A triangle has one angle that measures 5° more than twice the smallest angle, and the third angle measures 11° less than 3 times the measure of the smallest angle. Find the measures of the three angles. **(See Example 2.)**

24. The largest angle of a triangle measures 4° less than 5 times the measure of the smallest angle. The middle angle measures twice that of the smallest angle. Find the measures of the three angles.

25. The perimeter of a triangle is 55 cm. The measure of the shortest side is 8 cm less than the middle side. The measure of the longest side is 1 cm less than the sum of the other two sides. Find the lengths of the sides.

26. The perimeter of a triangle is 5 ft. The longest side of the triangle measures 20 in. more than the shortest side. The middle side is 3 times the measure of the shortest side. Find the lengths of the three sides in *inches*.

27. Sean kept track of his fiber intake from three sources for 3 weeks. The first week he had 3 servings of a fiber supplement, 1 serving of oatmeal, and 4 servings of cereal, which totaled 19 g of fiber. The second week he had 2 servings of the fiber supplement, 4 servings of oatmeal, and 2 servings of cereal totaling 25 g. The third week he had 5 servings of the fiber supplement, 3 servings of oatmeal, and 2 servings of cereal for a total of 30 g. Find the amount of fiber in one serving of each of the following: the fiber supplement, the oatmeal, and the cereal. **(See Example 3.)**

28. Natalie kept track of her calcium intake from three sources for 3 days. The first day she had 1 glass of milk, 1 serving of ice cream, and 1 calcium supplement in pill form which totaled 1180 mg of calcium. The second day she had 2 glasses of milk, 1 serving of ice cream, and 1 calcium supplement totaling 1680 mg. The third day she had 1 glass of milk, 2 servings of ice cream, and 1 calcium supplement for a total of 1260 mg. Find the amount of calcium in one glass of milk, in one serving of ice cream, and in one calcium supplement.

29. Goofie Golf has 18 holes that are par 3, par 4, or par 5. Most of the holes are par 4. In fact, there are 3 times as many par 4s as par 3s. There are 3 more par 5s than par 3s. How many of each type are there?

SW Productions/Getty Images

30. Combining peanuts, pecans, and cashews makes a party mixture of nuts. If the amount of peanuts equals the amount of pecans and cashews combined, and if there are twice as many cashews as pecans, how many ounces of each nut is used to make 48 oz of party mixture?

31. Souvenir hats, T-shirts, and jackets are sold at a rock concert. Three hats, two T-shirts, and one jacket cost \$140. Two hats, two T-shirts, and two jackets cost \$170. One hat, three T-shirts, and two jackets cost \$180. Find the prices of the individual items.

32. Annie and Maria traveled overseas for 7 days and stayed in three different hotels in three different cities: Stockholm, Sweden; Oslo, Norway; and Paris, France.

The total bill for all seven nights (not including tax) was \$1040. The total tax was \$106. The nightly cost (excluding tax) to stay at the hotel in Paris was \$80 more than the nightly cost (excluding tax) to stay in Oslo. Find the cost per night for each hotel excluding tax.

City	Number of Nights	Cost/Night (\$)	Tax Rate
Paris, France	1	x	8%
Stockholm, Sweden	4	y	11%
Oslo, Norway	2	z	10%

33. Walter had \$25,000 to invest. He split the money into three types of investment: small caps earning 6%, global market investments earning 10%, and a balanced fund earning 9%. He put twice as much money in the global account as he did in the balanced fund. If his earnings for the first year totaled \$2160, how much did he invest in each account?

34. Raeann deposited \$8000 into three accounts at her credit union: a checking account that pays 1.2% interest, a savings account that pays 2.5% interest, and a money market account that pays 3% interest. If she put 3 times more money in the 3% account than she did in the 1.2% account, and her total interest for 1 yr was \$202, how much did she deposit into each account?

Concept 4: Solving Inconsistent Systems and Systems of Dependent Equations

For Exercises 35–46, solve the system. If the system does not have a unique solution, state whether the system is inconsistent or the equations are dependent. **(See Examples 1, 4, and 5.)**

35. $$\begin{aligned} 2x + y + 3z &= 2 \\ x - y + 2z &= -4 \\ -2x + 2y - 4z &= 8 \end{aligned}$$

36. $$\begin{aligned} x + y &= z \\ 2x + 4y - 2z &= 6 \\ 3x + 6y - 3z &= 9 \end{aligned}$$

37. $$\begin{aligned} 6x - 2y + 2z &= 2 \\ 4x + 8y - 2z &= 5 \\ -2x - 4y + z &= -2 \end{aligned}$$

38. $$\begin{aligned} 3x + 2y + z &= 3 \\ x - 3y + z &= 4 \\ -6x - 4y - 2z &= 1 \end{aligned}$$

39. $$\begin{aligned} \tfrac{1}{2}x + \tfrac{2}{3}y &= \tfrac{5}{2} \\ \tfrac{1}{5}x - \tfrac{1}{2}z &= -\tfrac{3}{10} \\ \tfrac{1}{3}y - \tfrac{1}{4}z &= \tfrac{3}{4} \end{aligned}$$

40. $$\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{4}y + z &= 3 \\ \tfrac{1}{8}x + \tfrac{1}{4}y + \tfrac{1}{4}z &= \tfrac{9}{8} \\ x - y - \tfrac{2}{3}z &= \tfrac{1}{3} \end{aligned}$$

41. $$\begin{aligned} -3x + y - z &= 8 \\ -4x + 2y + 3z &= -3 \\ 2x + 3y - 2z &= -1 \end{aligned}$$

42. $$\begin{aligned} 2x + 3y + 3z &= 15 \\ 3x - 6y - 6z &= -23 \\ -9x - 3y + 6z &= 8 \end{aligned}$$

43. $$\begin{aligned} 2x + y &= 3(z - 1) \\ 3x - 2(y - 2z) &= 1 \\ 2(2x - 3z) &= -6 - 2y \end{aligned}$$

44. $$\begin{aligned} 2x + y &= -3 \\ 2y + 16z &= -10 \\ -7x - 3y + 4z &= 8 \end{aligned}$$

45. $$\begin{aligned} -0.1y + 0.2z &= 0.2 \\ 0.1x + 0.1y + 0.1z &= 0.2 \\ -0.1x - 0.3z &= 0.2 \end{aligned}$$

46. $$\begin{aligned} -0.4x - 0.3y &= 0 \\ 0.3y + 0.1z &= -0.1 \\ 0.4x - 0.1z &= 1.2 \end{aligned}$$

Expanding Your Skills

The systems in Exercises 47–50 are called homogeneous systems because each system has (0, 0, 0) as a solution. However, if the equations are dependent, the system will have infinitely many more solutions. For each system, determine whether (0, 0, 0) is the only solution or if the equations are dependent.

47. $$\begin{aligned} 2x - 4y + 8z &= 0 \\ -x - 3y + z &= 0 \\ x - 2y + 5z &= 0 \end{aligned}$$

48. $$\begin{aligned} 2x - 4y + z &= 0 \\ x - 3y - z &= 0 \\ 3x - y + 2z &= 0 \end{aligned}$$

49. $$\begin{aligned} 4x - 2y - 3z &= 0 \\ -8x - y + z &= 0 \\ 2x - y - \frac{3}{2}z &= 0 \end{aligned}$$

50. $$\begin{aligned} 5x + y &= 0 \\ 4y - z &= 0 \\ 5x + 5y - z &= 0 \end{aligned}$$

Solving Systems of Linear Equations by Using Matrices

Section 3.7

Concepts

1. Introduction to Matrices
2. Solving Systems of Linear Equations by Using the Gauss-Jordan Method

1. Introduction to Matrices

We have already learned how to solve systems of linear equations by using the substitution method and the addition method. We now present a third method called the Gauss-Jordan method that uses matrices to solve a linear system.

A **matrix** is a rectangular array of numbers (the plural of *matrix* is *matrices*). The rows of a matrix are read horizontally, and the columns of a matrix are read vertically. Every number or entry within a matrix is called an element of the matrix.

The **order of a matrix** is determined by the number of rows and number of columns. A matrix with m rows and n columns is an $m \times n$ (read as "m by n") matrix. Notice that with the order of a matrix, the number of rows is given first, followed by the number of columns.

Example 1 Determining the Order of a Matrix

Determine the order of each matrix.

a. $\begin{bmatrix} 2 & -4 & 1 \\ 5 & \pi & \sqrt{7} \end{bmatrix}$ **b.** $\begin{bmatrix} 1.9 \\ 0 \\ 7.2 \\ -6.1 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **d.** $[a \quad b \quad c]$

Solution:

a. This matrix has two rows and three columns. Therefore, it is a 2×3 matrix.

b. This matrix has four rows and one column. Therefore, it is a 4×1 matrix. A matrix with one column is called a **column matrix**.

c. This matrix has three rows and three columns. Therefore, it is a 3×3 matrix. A matrix with the same number of rows and columns is called a **square matrix**.

d. This matrix has one row and three columns. Therefore, it is a 1×3 matrix. A matrix with one row is called a **row matrix**.

Skill Practice Determine the order of the matrix.

1. $\begin{bmatrix} -5 & 2 \\ 1 & 3 \\ 8 & 9 \end{bmatrix}$ **2.** $[4 \quad -8]$ **3.** $\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$ **4.** $\begin{bmatrix} 2 & -0.5 \\ -1 & 6 \end{bmatrix}$

A matrix can be used to represent a system of linear equations written in standard form. To do so, we extract the coefficients of the variable terms and the constants within the equation. For example, consider the system

$$\begin{aligned} 2x - y &= 5 \\ x + 2y &= -5 \end{aligned}$$

The matrix **A** is called the **coefficient matrix**.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

If we extract both the coefficients and the constants from the equations, we can construct the **augmented matrix** of the system:

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 1 & 2 & -5 \end{array}\right]$$

A vertical bar is inserted into an augmented matrix to designate the position of the equal signs.

Example 2 Writing an Augmented Matrix for a System of Linear Equations

Write the augmented matrix for each linear system.

a. $\begin{aligned} -3x - 4y &= 3 \\ 2x + 4y &= 2 \end{aligned}$ **b.** $\begin{aligned} 2x - 3z &= 14 \\ 2y + z &= 2 \\ x + y &= 4 \end{aligned}$

Answers

1. 3×2 **2.** 1×2
3. 3×1 **4.** 2×2

Solution:

a. $\left[\begin{array}{rr|r} -3 & -4 & 3 \\ 2 & 4 & 2 \end{array}\right]$ **b.** $\left[\begin{array}{rrr|r} 2 & 0 & -3 & 14 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 0 & 4 \end{array}\right]$

> **TIP:** Notice that zeros are inserted to denote the coefficient of each missing term.

Skill Practice Write the augmented matrix for each system.

5. $\begin{aligned} -x + y &= 4 \\ 2x - y &= 1 \end{aligned}$

6. $\begin{aligned} 2x - y + z &= 14 \\ -3x + 4y \phantom{{}+z} &= 8 \\ x - y + 5z &= 0 \end{aligned}$

Example 3 **Writing a Linear System From an Augmented Matrix**

Write a system of linear equations represented by each augmented matrix. Use x, y, and z as the variables.

a. $\left[\begin{array}{rr|r} 2 & -5 & -8 \\ 4 & 1 & 6 \end{array}\right]$ **b.** $\left[\begin{array}{rrr|r} 2 & -1 & 3 & 14 \\ 1 & 1 & -2 & -5 \\ 3 & 1 & -1 & 2 \end{array}\right]$ **c.** $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Solution:

a. $\begin{aligned} 2x - 5y &= -8 \\ 4x + y &= 6 \end{aligned}$

b. $\begin{aligned} 2x - y + 3z &= 14 \\ x + y - 2z &= -5 \\ 3x + y - z &= 2 \end{aligned}$

c. $\begin{aligned} x + 0y + 0z &= 4 \\ 0x + y + 0z &= -1 \\ 0x + 0y + z &= 0 \end{aligned}$ or $\begin{aligned} x &= 4 \\ y &= -1 \\ z &= 0 \end{aligned}$

Skill Practice Write a system of linear equations represented by each augmented matrix. Use x, y, and z as the variables.

7. $\left[\begin{array}{rr|r} 2 & 3 & 5 \\ -1 & 8 & 1 \end{array}\right]$ **8.** $\left[\begin{array}{rrr|r} -3 & 2 & 1 & 4 \\ 14 & 1 & 0 & 20 \\ -8 & 3 & 5 & 6 \end{array}\right]$ **9.** $\left[\begin{array}{rrr|r} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 15 \end{array}\right]$

2. Solving Systems of Linear Equations by Using the Gauss-Jordan Method

We know that interchanging two equations results in an equivalent system of linear equations. Interchanging two rows in an augmented matrix results in an equivalent augmented matrix. Similarly, because each row in an augmented matrix represents a linear equation, we can perform the following elementary row operations that result in an equivalent augmented matrix.

Elementary Row Operations

The following *elementary row operations* performed on an augmented matrix produce an equivalent augmented matrix:

- Interchange two rows.
- Multiply every element in a row by a nonzero real number.
- Add a multiple of one row to another row.

Answers

5. $\left[\begin{array}{rr|r} -1 & 1 & 4 \\ 2 & -1 & 1 \end{array}\right]$

6. $\left[\begin{array}{rrr|r} 2 & -1 & 1 & 14 \\ -3 & 4 & 0 & 8 \\ 1 & -1 & 5 & 0 \end{array}\right]$

7. $\begin{aligned} 2x + 3y &= 5 \\ -x + 8y &= 1 \end{aligned}$

8. $\begin{aligned} -3x + 2y + z &= 4 \\ 14x + y \phantom{{}+z} &= 20 \\ -8x + 3y + 5z &= 6 \end{aligned}$

9. $x = -8, y = 2, z = 15$

When we are solving a system of linear equations by any method, the goal is to write a series of simpler but equivalent systems of equations until the solution is obvious. The *Gauss-Jordan method* uses a series of elementary row operations performed on the augmented matrix to produce a simpler augmented matrix. In particular, we want to produce an augmented matrix that has 1's along the diagonal of the matrix of coefficients and 0's for the remaining entries in the matrix of coefficients. A matrix written in this way is said to be written in **reduced row echelon form**. For example, the augmented matrix from Example 3(c) is written in reduced row echelon form.

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution to the corresponding system of equations is easily recognized as $x = 4$, $y = -1$, and $z = 0$.

Similarly, matrix **B** represents a solution of $x = a$ and $y = b$ to a system of two linear equations.

$$\mathbf{B} = \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$$

Example 4 Solving a System by Using the Gauss-Jordan Method

Solve by using the Gauss-Jordan method.

$$\begin{aligned} 2x - y &= 5 \\ x + 2y &= -5 \end{aligned}$$

Solution:

$$\left[\begin{array}{rr|r} 2 & -1 & 5 \\ 1 & 2 & -5 \end{array}\right]$$ Set up the augmented matrix.

$$\xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{rr|r} 1 & 2 & -5 \\ 2 & -1 & 5 \end{array}\right]$$ Switch row 1 and row 2 to get a 1 in the upper left position.

$$\xrightarrow{-2R_1 + R_2 \Rightarrow R_2} \left[\begin{array}{rr|r} 1 & 2 & -5 \\ 0 & -5 & 15 \end{array}\right]$$ Multiply row 1 by -2 and add the result to row 2. This produces an entry of 0 below the upper left position.

$$\xrightarrow{-\frac{1}{5}R_2 \Rightarrow R_2} \left[\begin{array}{rr|r} 1 & 2 & -5 \\ 0 & 1 & -3 \end{array}\right]$$ Multiply row 2 by $-\frac{1}{5}$ to produce a 1 along the diagonal in the second row.

$$\xrightarrow{-2R_2 + R_1 \Rightarrow R_1} \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -3 \end{array}\right]$$ Multiply row 2 by -2 and add the result to row 1. This produces a 0 in the first row, second column.

$$\mathbf{C} = \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -3 \end{array}\right]$$

The matrix **C** is in reduced row echelon form. From the augmented matrix, we have $x = 1$ and $y = -3$. The solution set is $\{(1, -3)\}$.

FOR REVIEW

Recall that to check a solution to a system of equations in two variables, test the ordered pair in both equations.

Skill Practice

10. Solve by using the Gauss-Jordan method.

$$\begin{aligned} x - 2y &= -21 \\ 2x + y &= -2 \end{aligned}$$

The order in which we manipulate the elements of an augmented matrix to produce reduced row echelon form was demonstrated in Example 4. In general, the order is as follows:

- First produce a 1 in the first row, first column. Then use the first row to obtain 0's in the first column below this element.
- Next, if possible, produce a 1 in the second row, second column. Use the second row to obtain 0's above and below this element.
- Next, if possible, produce a 1 in the third row, third column. Use the third row to obtain 0's above and below this element.
- The process continues until reduced row echelon form is obtained.

Example 5 Solving a System by Using the Gauss-Jordan Method

Solve by using the Gauss-Jordan method.

$$\begin{aligned} x \qquad\qquad &= -y + 5 \\ -2x \qquad + 2z &= y - 10 \\ 3x + 6y + 7z &= 14 \end{aligned}$$

Solution:

First write each equation in the system in standard form.

$$\begin{aligned} x \qquad\qquad &= -y + 5 &\longrightarrow\quad x + y \qquad &= 5 \\ -2x + \qquad 2z &= y - 10 &\longrightarrow\quad -2x - y + 2z &= -10 \\ 3x + 6y + 7z &= 14 &\longrightarrow\quad 3x + 6y + 7z &= 14 \end{aligned}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 5 \\ -2 & -1 & 2 & -10 \\ 3 & 6 & 7 & 14 \end{array}\right]$$ Set up the augmented matrix.

$$\begin{array}{l} \\ 2R_1 + R_2 \Rightarrow R_2 \longrightarrow \\ -3R_1 + R_3 \Rightarrow R_3 \longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 5 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 7 & -1 \end{array}\right]$$

Multiply row 1 by 2 and add the result to row 2.
Multiply row 1 by -3 and add the result to row 3.

$$\begin{array}{l} -1R_2 + R_1 \Rightarrow R_1 \longrightarrow \\ \\ -3R_2 + R_3 \Rightarrow R_3 \longrightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -2 & 5 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Multiply row 2 by -1 and add the result to row 1.
Multiply row 2 by -3 and add the result to row 3.

Answer

10. $\{(-5, 8)\}$

$$\begin{array}{l} 2R_3 + R_1 \Rightarrow R_1 \longrightarrow \\ -2R_3 + R_2 \Rightarrow R_2 \longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Multiply row 3 by 2 and add the result to row 1.
Multiply row 3 by -2 and add the result to row 2.

From the reduced row echelon form of the matrix, we have $x = 3$, $y = 2$, and $z = -1$. The solution set is $\{(3, 2, -1)\}$.

Skill Practice Solve by using the Gauss-Jordan method.

11. $\begin{aligned} x + y + z &= 2 \\ x - y + z &= 4 \\ x + 4y + 2z &= 1 \end{aligned}$

It is particularly easy to recognize a system of dependent equations or an inconsistent system of equations from the reduced row echelon form of an augmented matrix. This is demonstrated in Examples 6 and 7.

Example 6 Solving a System of Dependent Equations by Using the Gauss-Jordan Method

Solve by using the Gauss-Jordan method.

$$x - 3y = 4$$
$$\frac{1}{2}x - \frac{3}{2}y = 2$$

FOR REVIEW

Recall that if a system of equations reduces to an identity such as $0 = 0$, the system has infinitely many solutions. If the system reduces to a contradiction such as $0 = 7$, the system has no solutions.

Solution:

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ \frac{1}{2} & -\frac{3}{2} & 2 \end{array}\right]$$

Set up the augmented matrix.

$$-\tfrac{1}{2}R_1 + R_2 \Rightarrow R_2 \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

Multiply row 1 by $-\frac{1}{2}$ and add the result to row 2.

The second row of the augmented matrix represents the equation $0 = 0$. The equations are dependent. The solution set is $\{(x, y) \mid x - 3y = 4\}$.

Skill Practice Solve by using the Gauss-Jordan method.

12. $\begin{aligned} 4x - 6y &= 16 \\ 6x - 9y &= 24 \end{aligned}$

Answers

11. $\{(1, -1, 2)\}$

12. Infinitely many solutions; $\{(x, y) \mid 4x - 6y = 16\}$; dependent equations

Example 7 Solving an Inconsistent System by Using the Gauss-Jordan Method

Solve by using the Gauss-Jordan method.

$$x + 3y = 2$$
$$-3x - 9y = 1$$

Solution:

$$\left[\begin{array}{rr|r} 1 & 3 & 2 \\ -3 & -9 & 1 \end{array}\right]$$ Set up the augmented matrix.

$$\xrightarrow{3R_1 + R_2 \Rightarrow R_2} \left[\begin{array}{rr|r} 1 & 3 & 2 \\ 0 & 0 & 7 \end{array}\right]$$ Multiply row 1 by 3 and add the result to row 2.

The second row of the augmented matrix represents the contradiction $0 = 7$. The system is inconsistent. There is no solution, { }.

Skill Practice Solve by using the Gauss-Jordan method.

13. $6x + 10y = 1$
$15x + 25y = 3$

Answer

13. No solution; { }; inconsistent system

Section 3.7 Activity

A.1. Match the description of the matrix with matrices **A**, **B**, **C**, **D**, or **E**.

$$\mathbf{A} = \left[\begin{array}{rr|r} 1 & 0 & 4 \\ 0 & 1 & -5 \end{array}\right] \quad \mathbf{B} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix} \quad \mathbf{C} = \left[\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \mathbf{D} = [4 \quad 5 \quad -8] \quad \mathbf{E} = \begin{bmatrix} 6 \\ -7 \\ 1 \end{bmatrix}$$

a. This matrix has 2 rows and 3 columns.
b. This matrix is a square matrix.
c. This matrix has 1 row and 3 columns.
d. This matrix has 3 rows and 1 column.
e. This matrix is a column matrix.
f. This matrix is a row matrix.
g. These matrices are augmented matrices.
h. This matrix is in reduced row echelon form.

A.2. Consider the augmented matrix $\mathbf{C} = \left[\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 4 \end{array}\right]$.

a. Write the corresponding system of equations using x, y, and z as the variables.
b. Write the solution set to the system.

A.3. Consider the augmented matrix $\mathbf{A} = \left[\begin{array}{rr|r} 1 & 0 & 4 \\ 0 & 1 & -5 \end{array}\right]$.

a. Write the corresponding system of equations using x and y as the variables.
b. Write the solution set to the system.

In Exercises A.4–A.5, we will step through the process to solve a system by using the Gauss-Jordan method.

A.4. Consider the system. $2x - 2y = -4$
$4x + 2y = 10$

a. Write the augmented matrix for this system.
b. To make the leading element in the first row equal to 1, multiply row 1 by $\frac{1}{2}$. Rewrite the matrix (row 2 is unchanged).

c. To make the element in the second row, first column equal to 0, multiply row 1 by –4 and add the result to row 2. Rewrite the matrix (row 1 is unchanged).

d. Next, to make the leading nonzero element in the second row equal to 1, multiply row 2 by $\frac{1}{6}$. Rewrite the matrix (row 1 is unchanged).

e. To make the element in the first row, second column equal to 0, add row 2 to row 1. Rewrite the matrix (row 2 is unchanged).

f. The matrix from part (e) should now be in reduced row echelon form. Write the solution set to the original system.

A.5. Consider the system.

$$2x - 4y + 2z = 10$$
$$3x - y - 2z = 5$$
$$2x + 6y + 3z = -10$$

a. Write the augmented matrix for this system.

b. Give a row operation that can be used to make the leading element in row 1 equal to 1. Then perform that operation and rewrite the matrix.

c. Perform suitable row operations to make the first elements in rows 2 and 3 equal to 0.

d. The matrix from part (c) should now be $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 5 & -5 & -10 \\ 0 & 10 & 1 & -20 \end{array}\right]$.

Explain how you can make the first nonzero element in row 2 equal to 1. Then perform that operation and rewrite the matrix.

e. Perform suitable row operations to make the elements in column 2 of rows 1 and 3 equal to 0. Then perform those operations and rewrite the matrix.

f. The matrix from part (e) should now be $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 11 & 0 \end{array}\right]$.

Perform a row operation to make the leading nonzero term in row 3 equal to 1.

g. Complete the process to write the matrix in reduced row echelon form by making the elements in column 3 of rows 1 and 2 equal to 0.

h. From the reduced row echelon form in part (g), write the solution set to the system of equations.

For Exercises A.6–A.7, a matrix in reduced row echelon form is given. What can you conclude about the solution to the corresponding system of equations? Explain your answer.

A.6. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 10 \end{array}\right]$

A.7. $\left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right]$

Section 3.7 Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, solve the system by using any method.

R.1. $\frac{1}{14}x - \frac{2}{7}y = 1$

$\frac{1}{3}x + \frac{1}{2}y = 1$

R.2. $x + \frac{5}{2}y = -9$

$\frac{3}{4}x - y = -1$

R.3. $x + y - z = 8$
$x - 2y + z = 3$
$x + 3y + 2z = 7$

R.4. $2x - y + z = -4$
$-x + y + 3z = -7$
$x + 3y - 4z = 22$

R.5. When solving a system of linear equations, if the system results in a statement such as $0 = 7$, what do you conclude about the solution set? Would you characterize the system as consistent or inconsistent?

R.6. When solving a system of linear equations, if the system results in the statement $0 = 0$, how many solutions does the system have? Would you characterize the equations as independent or dependent?

Vocabulary and Key Concepts

1. a. A __________ is a rectangular array of numbers. The order of a matrix is $m \times n$ where m is the number of __________ and n is the number of __________.

b. A matrix that has exactly one column is called a __________ matrix. A matrix that has exactly __________ row is called a row matrix, and a matrix with the same number of rows and columns is called a __________ matrix.

c. Given the system of equations shown, matrix **A** is called the __________ matrix. The matrix **B** is called the __________ matrix.

$$\begin{aligned} 3x - 2y &= 6 \\ 4x + 5y &= 9 \end{aligned} \qquad \mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \qquad \mathbf{B} = \left[\begin{array}{cc|c} 3 & -2 & 6 \\ 4 & 5 & 9 \end{array}\right]$$

2. The matrix **A** is said to be written in reduced __________ form.

$$\mathbf{A} = \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 2 \end{array}\right]$$

Concept 1: Introduction to Matrices

For Exercises 3–5, consider the matrix $\mathbf{A} = \begin{bmatrix} -3 & 4 & 1 & 9 \\ 11 & -5 & 6 & -8 \end{bmatrix}$. Identify the row and column of the given element.

3. 9 **4.** 11 **5.** 6

For Exercises 6–14, (**a**) determine the order of each matrix and (**b**) determine if the matrix is a row matrix, a column matrix, a square matrix, or none of these. **(See Example 1.)**

6. $\begin{bmatrix} 4 \\ 5 \\ -3 \\ 0 \end{bmatrix}$

7. $\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

8. $\begin{bmatrix} -9 & 4 & 3 \\ -1 & -8 & 4 \\ 5 & 8 & 7 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -9 \\ -1 & -3 \end{bmatrix}$

10. $\begin{bmatrix} 4 & -7 \end{bmatrix}$

11. $\begin{bmatrix} 0 & -8 & 11 & 5 \end{bmatrix}$

12. $\begin{bmatrix} 5 & -8.1 & 4.2 & 0 \\ 4.3 & -9 & 18 & 3 \end{bmatrix}$

13. $\begin{bmatrix} \frac{1}{3} & \frac{3}{4} & 6 \\ -2 & 1 & -\frac{7}{8} \end{bmatrix}$

14. $\begin{bmatrix} 5 & 1 \\ -1 & 2 \\ 0 & 7 \end{bmatrix}$

For Exercises 15–18, set up the augmented matrix. **(See Example 2.)**

15. $x - 2y = -1$
$2x + y = -7$

16. $x - 3y = 3$
$2x - 5y = 4$

17. $$\begin{aligned} x - 2y &= 5 - z \\ 2x + 6y + 3z &= -2 \\ 3x - y - 2z &= 1 \end{aligned}$$

18. $$\begin{aligned} 5x - 17 &= -2z \\ 8x + 6z &= 26 + y \\ 8x + 3y - 12z &= 24 \end{aligned}$$

For Exercises 19–22, write a system of linear equations represented by the augmented matrix. Use x, y, and z as the variables. **(See Example 3.)**

19. $\left[\begin{array}{cc|c} 4 & 3 & 6 \\ 12 & 5 & -6 \end{array}\right]$

20. $\left[\begin{array}{cc|c} -2 & 5 & -15 \\ -7 & 15 & -45 \end{array}\right]$

21. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 7 \end{array}\right]$

22. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 6.1 \\ 0 & 0 & 1 & 3.9 \end{array}\right]$

Concept 2: Solving Systems of Linear Equations by Using the Gauss-Jordan Method

23. Given the matrix **E**

$$\mathbf{E} = \left[\begin{array}{cc|c} 3 & -2 & 8 \\ 9 & -1 & 7 \end{array}\right]$$

a. What is the element in the second row and third column?

b. What is the element in the first row and second column?

24. Given the matrix **F**

$$\mathbf{F} = \left[\begin{array}{cc|c} 1 & 8 & 0 \\ 12 & -13 & -2 \end{array}\right]$$

a. What is the element in the second row and second column?

b. What is the element in the first row and third column?

25. Given the matrix **Z**

$$\mathbf{Z} = \left[\begin{array}{cc|c} 2 & 1 & 11 \\ 2 & -1 & 1 \end{array}\right]$$

write the matrix obtained by multiplying the elements in the first row by $\frac{1}{2}$.

26. Given the matrix **J**

$$\mathbf{J} = \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 3 & -6 \end{array}\right]$$

write the matrix obtained by multiplying the elements in the second row by $\frac{1}{3}$.

27. Given the matrix **K**

$$\mathbf{K} = \left[\begin{array}{cc|c} 5 & 2 & 1 \\ 1 & -4 & 3 \end{array}\right]$$

write the matrix obtained by interchanging rows 1 and 2.

28. Given the matrix **L**

$$\mathbf{L} = \left[\begin{array}{cc|c} 9 & 6 & 13 \\ -7 & 2 & 19 \end{array}\right]$$

write the matrix obtained by interchanging rows 1 and 2.

29. Given the matrix **M**

$$\mathbf{M} = \left[\begin{array}{cc|c} 1 & 5 & 2 \\ -3 & -4 & -1 \end{array}\right]$$

write the matrix obtained by multiplying the first row by 3 and adding the result to row 2.

30. Given the matrix **N**

$$\mathbf{N} = \left[\begin{array}{cc|c} 1 & 3 & -5 \\ -2 & 2 & 12 \end{array}\right]$$

write the matrix obtained by multiplying the first row by 2 and adding the result to row 2.

31. Given the matrix **R**

$$\mathbf{R} = \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 4 & 1 & -5 & 6 \\ -2 & 0 & -3 & 10 \end{array}\right]$$

a. Write the matrix obtained by multiplying the first row by -4 and adding the result to row 2.

b. Using the matrix obtained from part (a), write the matrix obtained by multiplying the first row by 2 and adding the result to row 3.

32. Given the matrix **S**

$$\mathbf{S} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 10 \\ 5 & 1 & -4 & 3 \\ -3 & 4 & 5 & 2 \end{array}\right]$$

a. Write the matrix obtained by multiplying the first row by -5 and adding the result to row 2.

b. Using the matrix obtained from part (a), write the matrix obtained by multiplying the first row by 3 and adding the result to row 3.

For Exercises 33–36, use the augmented matrices **A**, **B**, **C**, and **D** to answer true or false.

$$\mathbf{A} = \left[\begin{array}{cc|c} 6 & -4 & 2 \\ 5 & -2 & 7 \end{array}\right] \qquad \mathbf{B} = \left[\begin{array}{cc|c} 5 & -2 & 7 \\ 6 & -4 & 2 \end{array}\right] \qquad \mathbf{C} = \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{1}{3} \\ 5 & -2 & 7 \end{array}\right] \qquad \mathbf{D} = \left[\begin{array}{cc|c} 5 & -2 & 7 \\ -12 & 8 & -4 \end{array}\right]$$

33. The matrix **A** is a 2×3 matrix.

34. Matrix **B** is equivalent to matrix **A**.

35. Matrix **A** is equivalent to matrix **C**.

36. Matrix **B** is equivalent to matrix **D**.

37. What does the notation $R_2 \Leftrightarrow R_1$ mean when one is performing the Gauss-Jordan method?

38. What does the notation $2R_3 \Rightarrow R_3$ mean when one is performing the Gauss-Jordan method?

39. What does the notation $-3R_1 + R_2 \Rightarrow R_2$ mean when one is performing the Gauss-Jordan method?

40. What does the notation $4R_2 + R_3 \Rightarrow R_3$ mean when one is performing the Gauss-Jordan method?

For Exercises 41–56, solve the system by using the Gauss-Jordan method. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent. **(See Example 4–7.)**

41. $\begin{aligned} x - 2y &= -1 \\ 2x + y &= -7 \end{aligned}$

42. $\begin{aligned} x - 3y &= 3 \\ 2x - 5y &= 4 \end{aligned}$

43. $\begin{aligned} x + 3y &= 6 \\ -4x - 9y &= 3 \end{aligned}$

44. $\begin{aligned} 2x - 3y &= -2 \\ x + 2y &= 13 \end{aligned}$

45. $\begin{aligned} x + 3y &= 3 \\ 4x + 12y &= 12 \end{aligned}$

46. $\begin{aligned} 2x + 5y &= 1 \\ -4x - 10y &= -2 \end{aligned}$

47. $\begin{aligned} x - y &= 4 \\ 2x + y &= 5 \end{aligned}$

48. $\begin{aligned} 2x - y &= 0 \\ x + y &= 3 \end{aligned}$

49. $\begin{aligned} x + 3y &= -1 \\ -3x - 6y &= 12 \end{aligned}$

50. $\begin{aligned} x + y &= 4 \\ 2x - 4y &= -4 \end{aligned}$

51. $\begin{aligned} 3x + y &= -4 \\ -6x - 2y &= 3 \end{aligned}$

52. $\begin{aligned} 2x + y &= 4 \\ 6x + 3y &= -1 \end{aligned}$

53. $\begin{aligned} x + y + z &= 6 \\ x - y + z &= 2 \\ x + y - z &= 0 \end{aligned}$

54. $\begin{aligned} 2x - 3y - 2z &= 11 \\ x + 3y + 8z &= 1 \\ 3x - y + 14z &= -2 \end{aligned}$

55. $\begin{aligned} x - 2y &= 5 - z \\ 2x + 6y + 3z &= -10 \\ 3x - y - 2z &= 5 \end{aligned}$

56. $\begin{aligned} 5x &= 10z + 15 \\ x - y + 6z &= 23 \\ x + 3y - 12z &= 13 \end{aligned}$

Technology Connections

For Exercises 57–62, use the matrix features on a graphing calculator to express each augmented matrix in reduced row echelon form. Compare your results to the solution you obtained in the indicated exercise.

57. $\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & 1 & -7 \end{array}\right]$

Compare with Exercise 41.

58. $\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 2 & -5 & 4 \end{array}\right]$

Compare with Exercise 42.

59. $\left[\begin{array}{cc|c} 1 & 3 & 6 \\ -4 & -9 & 3 \end{array}\right]$

Compare with Exercise 43.

60. $\left[\begin{array}{cc|c} 2 & -3 & -2 \\ 1 & 2 & 13 \end{array}\right]$

Compare with Exercise 44.

61. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 0 \end{array}\right]$

Compare with Exercise 53.

62. $\left[\begin{array}{ccc|c} 2 & -3 & -2 & 11 \\ 1 & 3 & 8 & 1 \\ 3 & -1 & 14 & -2 \end{array}\right]$

Compare with Exercise 54.

Chapter 3 Summary

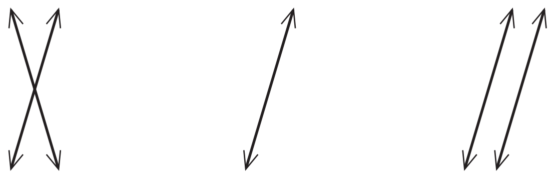
Section 3.1 Solving Systems of Linear Equations by the Graphing Method

Key Concepts

A **system of linear equations** in two variables can be solved by graphing.

A **solution to a system of linear equations** is an ordered pair that satisfies each equation in the system. Graphically, this represents a point of intersection of the lines.

There may be one solution, infinitely many solutions, or no solution.



One solution
Consistent
Independent

Infinitely many solutions
Consistent
Dependent

No solution
Inconsistent
Independent

A system of equations is **consistent** if there is at least one solution. A system is **inconsistent** if there is no solution.

Two linear equations in x and y are **dependent** if the equations represent the same line. The solution set is the set of all points on the line.

If two linear equations represent different lines, then the equations are **independent**.

Examples

Example 1

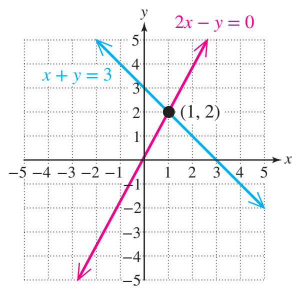
Solve by graphing.

$$x + y = 3$$
$$2x - y = 0$$

Write each equation in slope-intercept form ($y = mx + b$) to graph the lines.

$y = -x + 3$

$y = 2x$



The solution is the point of intersection, (1, 2).

The solution set is $\{(1, 2)\}$.

Section 3.2 Solving Systems of Linear Equations by the Substitution Method

Key Concepts

Substitution Method

1. Isolate one of the variables.
2. Substitute the quantity found in step 1 into the *other* equation.
3. Solve the resulting equation.
4. Substitute the value from step 3 back into the equation from step 1 to solve for the remaining variable.
5. Check the ordered pair in both equations, and write the answer as an ordered pair within set notation.

A system is consistent if there is at least one solution. A system is inconsistent if there is no solution. An inconsistent system is detected by a contradiction (such as $0 = 5$).

Two linear equations are independent if the equations represent different lines. The equations are dependent if they represent the same line. This produces infinitely many solutions. Two equations in a system of equations are dependent if the system reduces to an identity (such as $0 = 0$).

Examples

Example 1

$2y = -6x + 14$

$2x + y = 5 \xrightarrow{\text{Isolate a variable.}} y = \underbrace{-2x + 5}$

Substitute

$$\begin{aligned} 2(-2x + 5) &= -6x + 14 \\ -4x + 10 &= -6x + 14 \\ 2x + 10 &= 14 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

$y = -2x + 5$ Now solve for y.

$y = -2(2) + 5$

$y = 1$

The ordered pair (2, 1) checks in both equations.

The solution set is $\{(2, 1)\}$.

Example 2

$y = -2x + 3$

$$-4x - 2y = 1 \longrightarrow \begin{aligned} -4x - 2(-2x + 3) &= 1 \\ -4x + 4x - 6 &= 1 \\ -6 &= 1 \end{aligned}$$

Contradiction. The system is inconsistent.
There is no solution, { }.

Example 3

$x = -3y + 1$

$$2x + 6y = 2 \longrightarrow \begin{aligned} 2(-3y + 1) + 6y &= 2 \\ -6y + 2 + 6y &= 2 \\ 2 &= 2 \end{aligned}$$

Identity. The equations are dependent. There are infinitely many solutions.

$$\{(x, y) \mid x = -3y + 1\}$$

Section 3.3 Solving Systems of Linear Equations by the Addition Method

Key Concepts

Addition Method

1. Write both equations in standard form $Ax + By = C$.
2. Clear fractions or decimals (optional).
3. Multiply one or both equations by nonzero constants to create opposite coefficients for one of the variables.
4. Add the equations from step 3 to eliminate one variable.
5. Solve for the remaining variable.
6. Substitute the known value from step 5 back into one of the original equations to solve for the other variable.
7. Check the ordered pair in both equations and write the solution set.

Examples

Example 1

$$\begin{array}{rcl} 3x - 4y = 18 & \xrightarrow{\text{Mult. by 3.}} & 9x - 12y = 54 \\ -5x - 3y = -1 & \xrightarrow{\text{Mult. by } -4.} & \underline{20x + 12y = 4} \\ & & 29x = 58 \\ & & x = 2 \end{array}$$

$$\begin{aligned} 3(2) - 4y &= 18 \\ 6 - 4y &= 18 \\ -4y &= 12 \\ y &= -3 \end{aligned}$$

The ordered pair $(2, -3)$ checks in both equations.

The solution set is $\{(2, -3)\}$.

Section 3.4 Applications of Systems of Linear Equations in Two Variables

Key Concepts

Solve application problems by using systems of linear equations in two variables.

- Cost applications
- Mixture applications
- Principal and interest applications
- Uniform motion applications
- Geometry applications

Steps to Solve Applications:

1. Label two variables.
2. Construct two equations in words.
3. Write two equations.
4. Solve the system.
5. Write the answer in words.

Examples

Example 1

Mercedes borrowed \$1500 more from a lender that charges 6.5% interest than she did from a lender that charges 4% interest. If the total interest owed at the end of 1 yr is \$622.50, find the amount she borrowed at 6.5%.

Let x represent the amount borrowed at 6.5%.
Let y represent the amount borrowed at 4%.

$$\begin{pmatrix}\text{Amount borrowed}\\ \text{at } 6.5\%\end{pmatrix} = \begin{pmatrix}\text{amount borrowed}\\ \text{at } 4\%\end{pmatrix} + \$1500$$

$$\begin{pmatrix}\text{Interest owed}\\ \text{from } 6.5\%\\ \text{account}\end{pmatrix} + \begin{pmatrix}\text{interest owed}\\ \text{from } 4\%\\ \text{account}\end{pmatrix} = \$622.50$$

$$x = y + 1500$$
$$0.065x + 0.04y = 622.50$$

Using substitution gives

$$\begin{aligned} 0.065(y + 1500) + 0.04y &= 622.50 \\ 0.065y + 97.5 + 0.04y &= 622.50 \\ 0.105y &= 525 \\ y &= 5000 \end{aligned}$$

$$x = (5000) + 1500 = 6500$$

Mercedes borrowed \$6500 at 6.5%.

Section 3.5 Linear Inequalities and Systems of Linear Inequalities in Two Variables

Key Concepts

A **linear inequality in two variables** is an inequality of the form $Ax + By < C$, $Ax + By > C$, $Ax + By \leq C$, or $Ax + By \geq C$.

Graphing a Linear Inequality in Two Variables

1. Write the inequality with the y variable isolated, if possible.
2. Graph the related equation. Draw a dashed line if the inequality is strict, $<$ or $>$. Otherwise, draw a solid line.
3. Shade above or below the line according to the following convention.

   - Shade *above* the line if the inequality is of the form $y > mx + b$ or $y \geq mx + b$.
   - Shade *below* the line if the inequality is of the form $y < mx + b$ or $y \leq mx + b$.

You can use test points to check that you have shaded the correct region. Select an ordered pair from the proposed solution set and substitute the values of x and y in the original inequality. If the test point produces a true statement, then you have shaded the correct region.

Graphing a System of Linear Inequalities in Two Variables

Graph each linear inequality in the system. The solution to the system is the intersection of the shaded regions.

Examples

Example 1

Graph the solution to the inequality $2x - y < 4$.

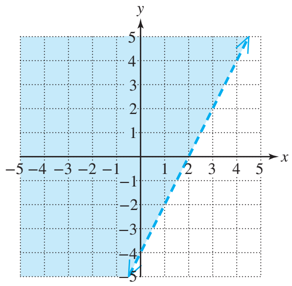
Graph the related equation, $y = 2x - 4$, with a dashed line.

Solve for y: $2x - y < 4$

$$-y < -2x + 4$$

$$y > 2x - 4$$

Shade above the line.

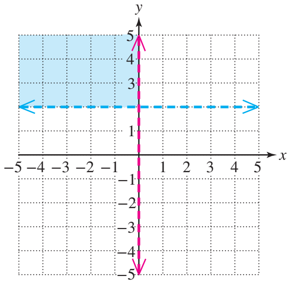


Example 2

Graph.

$x < 0$

$y > 2$



Section 3.6 Systems of Linear Equations in Three Variables and Applications

Key Concepts

A **linear equation in three variables** can be written in the form $Ax + By + Cz = D$, where A, B, and C are not all zero. The graph of a linear equation in three variables is a plane in space.

A solution to a system of linear equations in three variables is an **ordered triple** that satisfies each equation. Graphically, a solution is a point of intersection among three planes.

A system of linear equations in three variables may have one unique solution, infinitely many solutions (dependent equations), or no solution (inconsistent system).

Examples

Example 1

$$\begin{array}{ll} \boxed{A} & x + 2y - z = 4 \\ \boxed{B} & 3x - y + z = 5 \\ \boxed{C} & 2x + 3y + 2z = 7 \end{array}$$

$$\begin{array}{lrl} \boxed{A} & x + 2y - z = 4 & \\ \boxed{B} & \underline{3x - y + z = 5} & \\ & 4x + y \quad = 9 & \boxed{D} \end{array}$$

$$\begin{array}{lrl} 2 \cdot \boxed{A} & 2x + 4y - 2z = 8 & \\ \boxed{C} & \underline{2x + 3y + 2z = 7} & \\ & 4x + 7y \quad = 15 & \boxed{E} \end{array}$$

$$\begin{array}{llcr} \boxed{D} & 4x + y = 9 & \xrightarrow{\text{Multiply by } -1.} & -4x - y = -9 \\ \boxed{E} & 4x + 7y = 15 & & \underline{4x + 7y = 15} \\ & & & 6y = 6 \\ & & & y = 1 \end{array}$$

Substitute $y = 1$ into either equation $\boxed{D}$ or $\boxed{E}$.

$$\begin{array}{lr} \boxed{D} & 4x + (1) = 9 \\ & 4x = 8 \\ & x = 2 \end{array}$$

Substitute $x = 2$ and $y = 1$ into equation $\boxed{A}$, $\boxed{B}$, or $\boxed{C}$.

$$\begin{array}{lr} \boxed{A} & (2) + 2(1) - z = 4 \\ & -z = 0 \\ & z = 0 \end{array}$$

The solution set is $\{(2, 1, 0)\}$.

Section 3.7 Solving Systems of Linear Equations by Using Matrices

Key Concepts

A **matrix** is a rectangular array of numbers displayed in rows and columns. Every number or entry within a matrix is called an element of the matrix.

The **order of a matrix** is determined by the number of rows and number of columns. A matrix with m rows and n columns is an $m \times n$ matrix.

A system of equations written in standard form can be represented by an **augmented matrix** consisting of the coefficients of the terms of each equation in the system.

The Gauss-Jordan method can be used to solve a system of equations by using the following elementary row operations on an augmented matrix.

1. Interchange two rows.
2. Multiply every element in a row by a nonzero real number.
3. Add a multiple of one row to another row.

These operations are used to write the matrix in **reduced row echelon form**.

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$$

This represents the solution, $x = a$ and $y = b$.

Examples

Example 1

$[1 \quad 2 \quad 5]$ is a 1×3 matrix (a row matrix).

$\begin{bmatrix} -1 & 8 \\ 1 & 5 \end{bmatrix}$ is a 2×2 matrix (a square matrix).

$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is a 2×1 matrix (a column matrix).

Example 2

The augmented matrix for

$$\begin{aligned} 4x + y &= -12 \\ x - 2y &= 6 \end{aligned} \quad \text{is} \quad \left[\begin{array}{cc|c} 4 & 1 & -12 \\ 1 & -2 & 6 \end{array}\right]$$

Example 3

Solve the system from Example 2 by using the Gauss-Jordan method.

$$\left[\begin{array}{cc|c} 4 & 1 & -12 \\ 1 & -2 & 6 \end{array}\right]$$

$$R_1 \Leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & -2 & 6 \\ 4 & 1 & -12 \end{array}\right]$$

$$\xrightarrow{-4R_1 + R_2 \Rightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 6 \\ 0 & 9 & -36 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}R_2 \Rightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 6 \\ 0 & 1 & -4 \end{array}\right]$$

$$\xrightarrow{2R_2 + R_1 \Rightarrow R_1} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & -4 \end{array}\right]$$

From the reduced row echelon form of the matrix we have $x = -2$ and $y = -4$. The solution set is $\{(-2, -4)\}$.

Chapter 3 Review Exercises

Section 3.1

1. Determine if the ordered pair is a solution to the system.

$$-5x - 7y = 4$$
$$y = -\frac{1}{2}x - 1$$

a. $(2, 2)$ **b.** $(2, -2)$

For Exercises 2–4, answer true or false.

2. An inconsistent system has one solution.

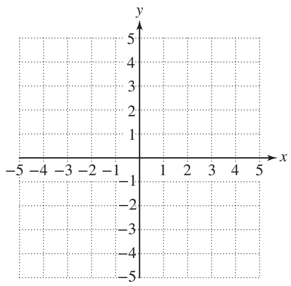
3. Parallel lines form an inconsistent system.

4. Lines with different slopes intersect in one point.

For Exercises 5–8, solve the system by graphing. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent.

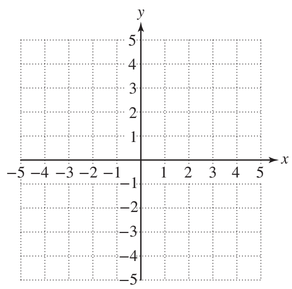
5. $f(x) = x - 1$

$g(x) = 2x - 4$



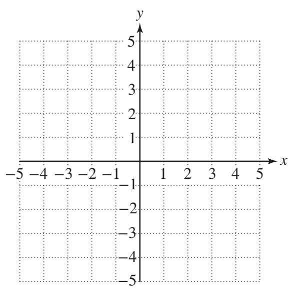
6. $y = 2x + 7$

$y = -x - 5$



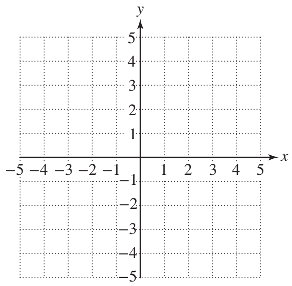
7. $6x + 2y = 4$

$3x = -y + 2$



8. $y = \frac{1}{2}x - 2$

$-4x + 8y = -8$



Section 3.2

For Exercises 9–14, solve the system by using the substitution method. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent.

9. $y = \frac{3}{4}x - 4$

$-x + 2y = -6$

10. $6x + y = 5$

$5x + y = 3$

11. $2(x + y) = 10 - 3y$

$0.4x + y = 1.2$

12. $3x = 11y - 9$

$y = \frac{3}{11}x + \frac{6}{11}$

13. $60(5x - y) = 90$

$10x = 2y + 3$

14. $4x + \phantom{\frac{1}{4}}y = 7$

$x + \frac{1}{4}y = \frac{7}{4}$

15. The cost y (in \$) to rent a 5-ft by 5-ft storage space for x months is given for two companies.

Company A: $y = 105 + 45x$
Company B: $y = 48.50x$

Determine the number of months for which the cost of renting a storage space would be the same for either company.

16. The cost y (in \$) to rent a car for x days is given for two rental car companies.

Company A: $y = 44 + 81.50x$
Company B: $y = 87x$

Determine the number of days for which the cost of renting a car would be the same for each company.

Section 3.3

For Exercises 17–26, solve the system by using the addition method. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent.

17. $\frac{2}{5}x + \frac{3}{5}y = 1$
$x - \frac{2}{3}y = \frac{1}{3}$

18. $4x + 3y = 5$
$3x - 4y = 10$

19. $3x + 4y = 2$
$2x + 5y = -1$

20. $3x + y = 1$
$-x - \frac{1}{3}y = -\frac{1}{3}$

21. $2y = 3x - 8$
$-6x = -4y + 4$

22. $3x + y = 16$
$3(x + y) = y + 2x + 2$

23. $-(y + 4x) = 2x - 9$
$-2x + 2y = -10$

24. $-(4x - 35) = 3y$
$-(x - 15) = y$

25. $-0.4x + 0.3y = 1.8$
$0.6x - 0.2y = -1.2$

26. $0.02x - 0.01y = -0.11$
$0.01x + 0.04y = 0.26$

Section 3.4

27. Melinda invested twice as much money in an account paying 5% simple interest as she did in an account paying 3.5% simple interest. If her total interest at the end of 1 yr is $303.75, find the amount she invested in the 5% account.

28. A school carnival sold tickets to ride on a Ferris wheel. The charge was $1.50 for adults and $1.00 for students. If 54 tickets were sold for a total of $70.50, how many of each type of ticket were sold?

Vadim Petrakov/Rawpixel.com/Shutterstock

29. How many liters of 20% saline solution must be mixed with 50% saline solution to produce 16 L of a 31.25% saline solution?

30. It takes a pilot $1\frac{3}{4}$ hr to travel with the wind from Jacksonville, Florida, to Myrtle Beach, South Carolina. Her return trip takes 2 hr flying against the wind. What is the speed of the wind and the speed of the plane in still air if the distance between Jacksonville and Myrtle Beach is 280 mi?

31. Josh wants to rent an apartment and he has a dog. The first apartment he found charges $915 per month with a one-time pet fee of $275. The second charges $940 per month for rent and an additional $25 per month for the dog.

a. Write a linear function describing the total cost $f(x)$ to rent the first apartment for x months.

b. Write a linear function describing the total cost $g(x)$ to rent the second apartment for x months.

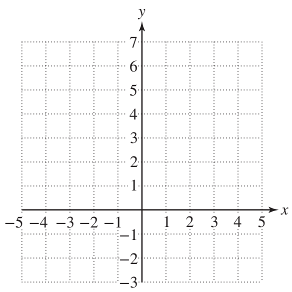
c. Find the number of months for which the cost to rent each apartment would be the same.

32. Two angles are complementary. One angle measures 6° more than 5 times the measure of the other. What are the measures of the two angles?

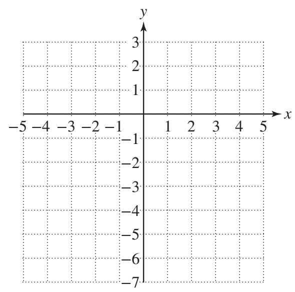
Section 3.5

For Exercises 33–40, solve the inequalities by graphing.

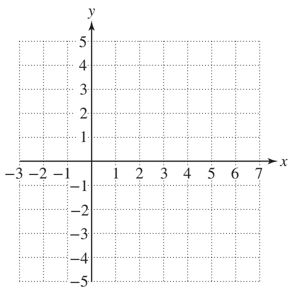
33. $2x > -y + 5$



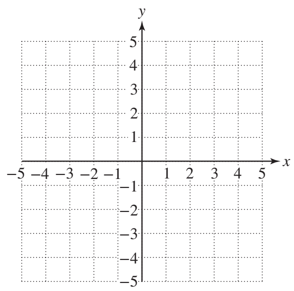
34. $2x \le -8 - 3y$



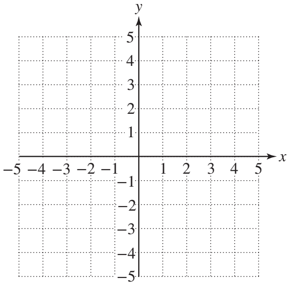
35. $y \ge -\frac{2}{3}x + 3$



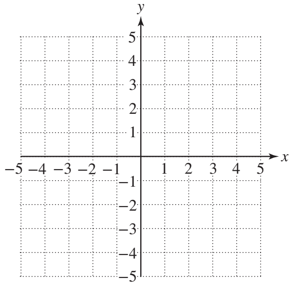
36. $y > \frac{3}{4}x - 2$



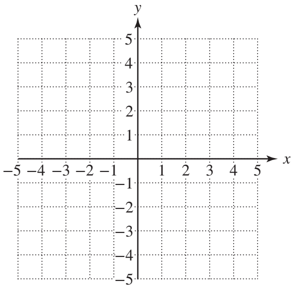
37. $x > -3$



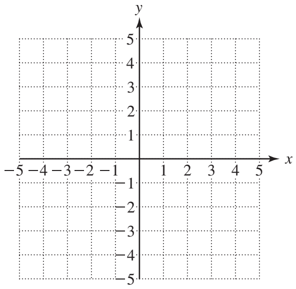
38. $x \le 2$



39. $x \ge \frac{1}{2}y$



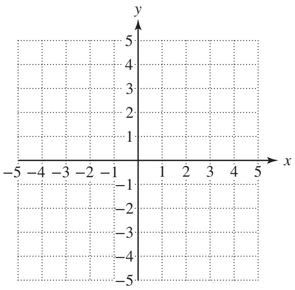
40. $x < \frac{2}{5}y$



For Exercises 41–44 graph the system of inequalities.

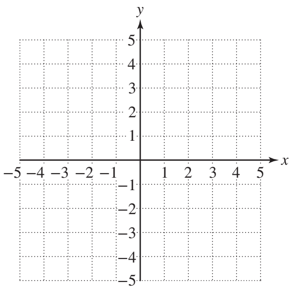
41. $2x - y > -2$

$2x - y \le 2$



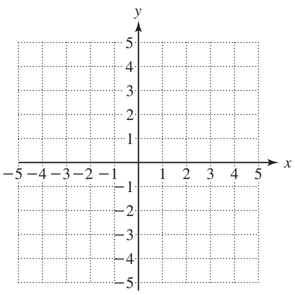
42. $3x + y < 6$

$-3x + y < -2$



43. $y \ge x$

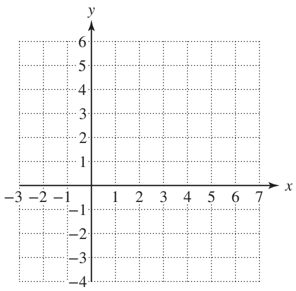
$y \le -x$



44. $x \ge 0$

$y \ge 0$

$y \le -\frac{2}{3}x + 4$



45. Suppose a farmer has 100 acres of land on which to grow oranges and grapefruit. Furthermore, because of demand from his customers, he wants to plant at least 4 times as many acres of orange trees as grapefruit trees.

Adalberto Rios Lanz/Sexto Sol/Photodisc/Getty Images

Let x represent the number of acres of orange trees.

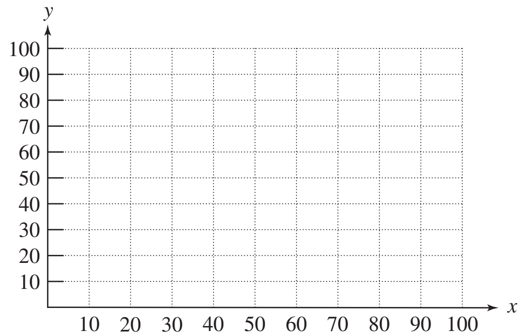
Let y represent the number of acres of grapefruit trees.

a. Write two inequalities that express the fact that the farmer cannot use a negative number of acres to plant orange and grapefruit trees.

b. Write an inequality that expresses the fact that the total number of acres used for growing orange and grapefruit trees is at most 100.

c. Write an inequality that expresses the fact that the farmer wants to plant at least 4 times as many orange trees as grapefruit trees.

d. Sketch the inequalities in parts (a)–(c) to find the feasible region for the farmer's distribution of orange and grapefruit trees.



Section 3.6

For Exercises 46–49, solve the system. If a system does not have a unique solution, state whether the system is inconsistent or the equations are dependent.

46. $5x + 5y + 5z = 30$

$-x + y + z = 2$

$10x + 6y - 2z = 4$

47. $5x + 3y - z = 5$

$x + 2y + z = 6$

$-x - 2y - z = 8$

48. $\begin{aligned} x + y + z &= 4 \\ -x - 2y - 3z &= -6 \\ 2x + 4y + 6z &= 12 \end{aligned}$

49. $\begin{aligned} 3x \quad + 4z &= 5 \\ 2y + 3z &= 2 \\ 2x - 5y \quad &= 8 \end{aligned}$

50. The perimeter of a right triangle is 30 ft. One leg is 2 ft longer than twice the shortest leg. The hypotenuse is 2 ft less than 3 times the shortest leg. Find the lengths of the sides of this triangle.

51. Three pumps are working to drain a construction site. Working together, the pumps can drain 950 gal/hr of water. The slowest pump drains 150 gal/hr less than the fastest pump. The fastest pump drains 150 gal/hr less than the sum of the other two pumps. How many gallons can each pump drain per hour?

52. The smallest angle in a triangle measures 9° less than the middle angle. The largest angle is 26° more than 3 times the measure of the smallest angle. Find the measure of each angle.

Section 3.7

For Exercises 53–56, determine the order of each matrix.

53. $\begin{bmatrix} 2 & 4 & -1 \\ 5 & 0 & -3 \\ -1 & 6 & 10 \end{bmatrix}$

54. $\begin{bmatrix} -5 & 6 \\ 9 & 2 \\ 0 & -3 \end{bmatrix}$

55. $\begin{bmatrix} 0 & 13 & -4 & 16 \end{bmatrix}$

56. $\begin{bmatrix} 7 \\ 12 \\ -4 \end{bmatrix}$

For Exercises 57–58, set up the augmented matrix.

57. $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$

58. $\begin{aligned} x - y + z &= 4 \\ 2x - y + 3z &= 8 \\ -2x + 2y - z &= -9 \end{aligned}$

For Exercises 59–60, write a corresponding system of equations from the augmented matrix. Use x, y, and z as the variables.

59. $\left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & -3 \end{array}\right]$

60. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -8 \end{array}\right]$

61. Given the matrix **C**

$$\mathbf{C} = \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 4 & -1 & 6 \end{array}\right]$$

a. What is the element in the second row and first column?

b. Write the matrix obtained by multiplying the first row by -4 and adding the result to row 2.

62. Given the matrix **D**

$$\mathbf{D} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 4 & -1 & 1 & 0 \\ -3 & 2 & 2 & 5 \end{array}\right]$$

a. Write the matrix obtained by multiplying the first row by -4 and adding the result to row 2.

b. Using the matrix obtained in part (a), write the matrix obtained by multiplying the first row by 3 and adding the result to row 3.

For Exercises 63–66, solve the system by using the Gauss-Jordan method.

63. $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$

64. $\begin{aligned} 4x + 3y &= 6 \\ 12x + 5y &= -6 \end{aligned}$

65. $\begin{aligned} x - y + z &= -4 \\ 2x + y - 2z &= 9 \\ x + 2y + z &= 5 \end{aligned}$

66. $\begin{aligned} x - y + z &= 4 \\ 2x - y + 3z &= 8 \\ -2x + 2y - z &= -9 \end{aligned}$

Chapter 3 Test

1. Determine if the ordered pair $(\frac{1}{4}, 2)$ is a solution to the system.

$$\begin{aligned} 4x - 3y &= -5 \\ 12x + 2y &= 7 \end{aligned}$$

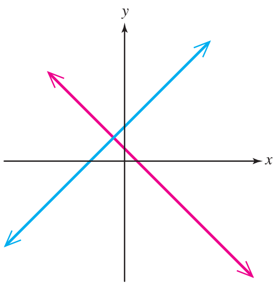
For Exercises 2–4, match each figure with the appropriate description, a, b, or c.

a. The system is consistent.
The equations are dependent.
There are infinitely many solutions.

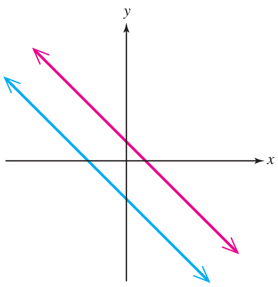
b. The system is consistent.
The equations are independent.
There is one solution.

c. The system is inconsistent.
The equations are independent.
There is no solution.

2.



3.



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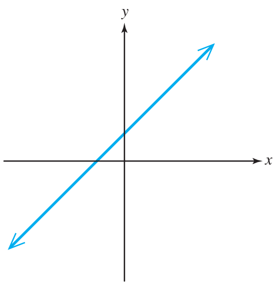
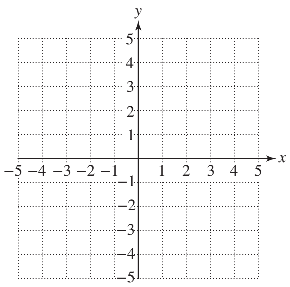


5. Solve the system by graphing.

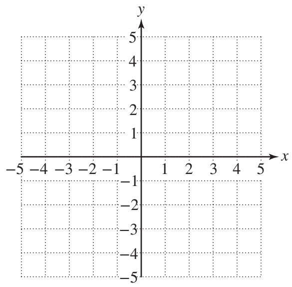
$$\begin{aligned} 4x - 2y &= -4 \\ 3x + y &= 7 \end{aligned}$$



6. Solve the system by graphing.

$$f(x) = x + 3$$

$$g(x) = -\frac{3}{2}x - 2$$



7. Solve the system by using the substitution method.

$$\begin{aligned} 3x + 5y &= 13 \\ y &= x + 9 \end{aligned}$$

8. Solve the system by using the addition method.

$$\begin{aligned} 6x + 8y &= 5 \\ 3x - 2y &= 1 \end{aligned}$$

For Exercises 9–13, solve the system. For systems that do not have one unique solution, also state the number of solutions and whether the system is inconsistent or the equations are dependent.

9. $7y = 5x - 21$
$9y + 2x = -27$

10. $3x - 5y = -7$
$-18x + 30y = 42$

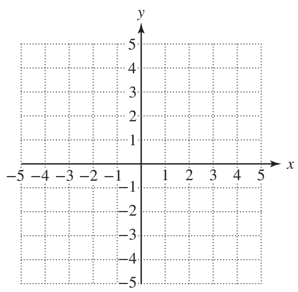
11. $\frac{1}{5}x = \frac{1}{2}y + \frac{17}{5}$
$\frac{1}{4}(x + 2) = -\frac{1}{6}y$

12. $4x = 5 - 2y$
$y = -2x + 4$

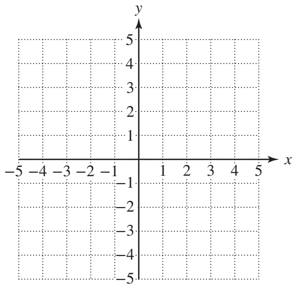
13. $-0.03y + 0.06x = 0.3$
$0.4x - 2 = -0.5y$

For Exercises 14–16, graph the solution set.

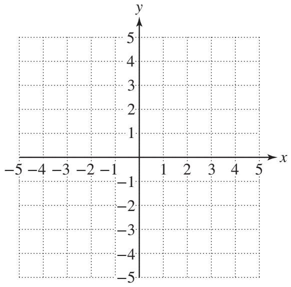
14. $2x - 5y \geq 10$



15. $x + y < 3$
$3x - 2y > -6$



16. $5x \le 5$
$x + y \le 0$

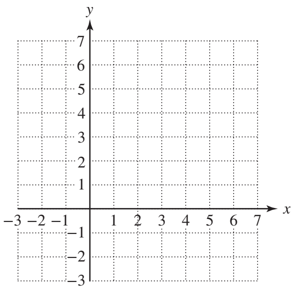


17. After menopause, women are at higher risk for hip fractures as a result of low calcium. As early as their teen years, women need at least 1200 mg of calcium per day (the USDA recommended daily allowance). One 8-oz glass of skim milk contains 300 mg of calcium, and one antacid tablet (regular strength) contains 400 mg of calcium. Let x represent the number of 8-oz glasses of milk that a woman drinks per day. Let y represent the number of antacid tablets (regular strength) that a woman takes per day.

a. Write two inequalities that express the fact that the number of glasses of milk and the number of antacid tablets taken each day cannot be negative.

b. Write a linear inequality in terms of x and y for which the daily calcium intake is at least 1200 mg.

c. Graph the inequalities.



For Exercises 18–19, solve the system.

18.
$$\begin{aligned} 2x + 2y + 4z &= -6 \\ 3x + y + 2z &= 29 \\ x - y - z &= 44 \end{aligned}$$

19.
$$\begin{aligned} 2(x + z) &= 6 + x - 3y \\ 2x &= 11 + y - z \\ x + 2(y + z) &= 8 \end{aligned}$$

20. Aiko borrows a total of \$5000 from two lenders. One charges 6.5% simple interest and the other charges 5% simple interest. If the total interest paid at the end of 1 yr is \$268, how much did she borrow from each lender?

21. How many liters of a 20% acid solution should be mixed with a 60% acid solution to produce 200 L of a 44% acid solution?

22. Two angles are complementary. Two times the measure of one angle is 60° less than the measure of the other. Find the measure of each angle.

23. Working together, Joanne, Kent, and Geoff can process 504 orders per day for their business. Kent can process 20 more orders per day than Joanne can process. Geoff can process 104 fewer orders per day than Kent and Joanne combined. Find the number of orders that each person can process per day.

24. Write an example of a 3×2 matrix.

25. Given the matrix **A**

$$\mathbf{A} = \left[\begin{array}{rrr|r} 1 & 2 & 1 & -3 \\ 4 & 0 & 1 & -2 \\ -5 & -6 & 3 & 0 \end{array}\right]$$

a. Write the matrix obtained by multiplying the first row by -4 and adding the result to row 2.

b. Using the matrix obtained in part (a), write the matrix obtained by multiplying the first row by 5 and adding the result to row 3.

For Exercises 26–27, solve by using the Gauss-Jordan method.

26.
$$\begin{aligned} 5x - 4y &= 34 \\ x - 2y &= 8 \end{aligned}$$

27.
$$\begin{aligned} x + y + z &= 1 \\ 2x + y &= 0 \\ -2y - z &= 5 \end{aligned}$$

Polynomials

4

CHAPTER OUTLINE

Mathematics and Modeling

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In this chapter we will define a mathematical structure called a **polynomial**. Polynomials and polynomial functions are important because they enable mathematicians to model data. They can also be used to approximate other, more complicated functions.

Some polynomials may be used to describe curves of various types such as the curves on a rollercoaster. Combinations of polynomials are also used in economics to perform cost and revenue analyses.

In physics, polynomials can be used to describe the trajectories of projectiles. For example, in a classic *Seinfeld* episode, Jerry tosses a loaf of bread (a marble rye) straight upward to his friend George who is leaning out of a third-story window. The bread leaves Jerry's hand at a height of 1 m with an initial velocity of 18 m/sec. The polynomial function defined by $s(t) = -4.9t^2 + 18t + 1$ approximates the height of the bread, $s(t)$, at a time t seconds after leaving Jerry's hand.

Section 4.1 Properties of Integer Exponents and Scientific Notation

Concepts

1. Simplifying Expressions with Exponents
2. Scientific Notation

1. Simplifying Expressions with Exponents

We have already learned that exponents are used to represent repeated multiplication. The following properties of exponents (Table 4-1) are often used to simplify algebraic expressions.

Table 4-1 Properties of Exponents*

Description	Property	Example	Expanded Form
Multiplication of like bases	$b^m \cdot b^n = b^{m+n}$	$b^2 \cdot b^4 = b^{2+4}$ $= b^6$	$b^2 \cdot b^4 = (b \cdot b)(b \cdot b \cdot b \cdot b)$ $= b^6$
Division of like bases	$\dfrac{b^m}{b^n} = b^{m-n}$	$\dfrac{b^5}{b^2} = b^{5-2}$ $= b^3$	$\dfrac{b^5}{b^2} = \dfrac{\cancel{b} \cdot \cancel{b} \cdot b \cdot b \cdot b}{\cancel{b} \cdot \cancel{b}}$ $= b^3$
Power rule	$(b^m)^n = b^{m \cdot n}$	$(b^4)^2 = b^{4 \cdot 2}$ $= b^8$	$(b^4)^2 = (b \cdot b \cdot b \cdot b)(b \cdot b \cdot b \cdot b)$ $= b^8$
Power of a product	$(ab)^m = a^m b^m$	$(ab)^3 = a^3b^3$	$(ab)^3 = (ab)(ab)(ab)$ $= (a \cdot a \cdot a)(b \cdot b \cdot b) = a^3b^3$
Power of a quotient	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$	$\left(\dfrac{a}{b}\right)^3 = \left(\dfrac{a}{b}\right)\left(\dfrac{a}{b}\right)\left(\dfrac{a}{b}\right)$ $= \dfrac{a \cdot a \cdot a}{b \cdot b \cdot b} = \dfrac{a^3}{b^3}$

*Assume that a and b are real numbers ($b \neq 0$) and that m and n represent integers.

In addition to the properties of exponents, two definitions are used to simplify algebraic expressions.

Definition of b^0 and b^{-n}

Let n be an integer, and b be a real number such that $b \neq 0$.

***1.** $b^0 = 1$ **Example:** $5^0 = 1$

2. $b^{-n} = \left(\dfrac{1}{b}\right)^n = \dfrac{1}{b^n}$ **Example:** $4^{-3} = \left(\dfrac{1}{4}\right)^3 = \dfrac{1}{4^3}$ or $\dfrac{1}{64}$

3. From the definition of b^{-n} we also have:

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n} \quad \text{for } a \neq 0, b \neq 0.$$

Example: $\left(\dfrac{3}{7}\right)^{-2} = \left(\dfrac{7}{3}\right)^2 = \dfrac{7^2}{3^2}$ or $\dfrac{49}{9}$

**Note:* The value of 0^0 is not defined by definition 1 because the base, b, must not equal 0.

The definition of b^0 is consistent with the properties of exponents. For example, if b is a nonzero real number and n is an integer, then

$$\frac{b^n}{b^n} = 1$$

The expression $b^0 = 1$

$$\frac{b^n}{b^n} = b^{n-n} = b^0$$

The definition of b^{-n} is also consistent with the properties of exponents. If b is a nonzero real number, then

$$\frac{b^3}{b^5} = \frac{\cancel{b} \cdot \cancel{b} \cdot \cancel{b}}{\cancel{b} \cdot \cancel{b} \cdot \cancel{b} \cdot b \cdot b} = \frac{1}{b^2}$$

$$\frac{b^3}{b^5} = b^{3-5} = b^{-2}$$

The expression $b^{-2} = \frac{1}{b^2}$

Example 1 Simplifying Expressions with Exponents

Simplify the expressions.

a. $(-2)^4$ **b.** -2^4 **c.** -2^{-4} **d.** $(-7x)^0$ **e.** $-7x^0$

Solution:

a. $(-2)^4 = (-2)(-2)(-2)(-2)$

$= 16$

b. $-2^4 = -1 \cdot 2^4$

$= -1 \cdot (2 \cdot 2 \cdot 2 \cdot 2)$

$= -16$

c. $-2^{-4} = -1 \cdot (2^{-4})$

$= -1 \cdot \left(\frac{1}{2^4}\right)$

$= -1 \cdot \frac{1}{2 \cdot 2 \cdot 2 \cdot 2}$

$= -\frac{1}{16}$

d. $(-7x)^0 = 1$ because $b^0 = 1$

e. $-7x^0 = -7 \cdot x^0$

$= -7 \cdot 1$

$= -7$

Skill Practice Simplify the expressions.

1. $(-3)^2$ **2.** -3^2 **3.** -3^{-2} **4.** $(-8y)^0$ **5.** $-8y^0$

FOR REVIEW

Recall that an odd number of negative factors produces a negative product. An even number of negative factors produces a positive product.

Example 2 Using Properties of Exponents

Simplify the expressions. Write the final answer with positive exponents only.

a. $x^3x^5x^{-2}$ **b.** $\frac{y^7}{y^4}$ **c.** $(b^2)^{-5}$

Answers

1. 9 **2.** -9 **3.** $-\frac{1}{9}$

4. 1 **5.** -8

Solution:

a. $x^3x^5x^{-2}$

$= x^{3+5+(-2)}$ Multiply like bases by adding exponents.

$= x^6$ Simplify.

b. $\dfrac{y^7}{y^4}$

$= y^{7-4}$ Divide like bases by subtracting exponents.

$= y^3$ Simplify.

c. $(b^2)^{-5}$

$= b^{2(-5)}$ Apply the power rule.

$= b^{-10}$ Multiply the exponents.

$= \dfrac{1}{b^{10}}$ Write the answer with positive exponents.

Skill Practice Simplify the expressions. Write the final answer with positive exponents only.

6. $w^7w^{-3}w$ **7.** $\dfrac{t^{11}}{t^6}$ **8.** $(p^{-4})^2$

Example 3 Simplifying an Expression with Exponents

Simplify the expression. $\left(\dfrac{1}{5}\right)^{-3} - (2)^{-2} + 3^0$

Solution:

$\left(\dfrac{1}{5}\right)^{-3} - (2)^{-2} + 3^0$

$= 5^3 - \left(\dfrac{1}{2}\right)^2 + 3^0$ Simplify negative exponents.

$= 125 - \dfrac{1}{4} + 1$ Evaluate expressions with exponents.

$= \dfrac{500}{4} - \dfrac{1}{4} + \dfrac{4}{4}$ Write the expressions with a common denominator.

$= \dfrac{503}{4}$ Simplify.

Skill Practice Simplify the expression.

9. $\left(\dfrac{2}{3}\right)^{-1} + 4^{-1} - \left(\dfrac{1}{4}\right)^0$

Answers

6. w^5 **7.** t^5 **8.** $\dfrac{1}{p^8}$ **9.** $\dfrac{3}{4}$

Example 4 Simplifying an Expression with Exponents

Simplify the expression. Write the answer with positive exponents only.

$$\frac{(2a^7b^{-4})^3}{(4a^3b^{-2})^2}$$

Solution:

$$\frac{(2a^7b^{-4})^3}{(4a^3b^{-2})^2}$$

$= \dfrac{2^3a^{21}b^{-12}}{4^2a^6b^{-4}}$ Apply the power rule.

$= \dfrac{8a^{21}b^{-12}}{16a^6b^{-4}}$ Simplify the coefficients.

$= \dfrac{8a^{21-6}b^{-12-(-4)}}{16}$ Divide like bases by subtracting exponents.

$= \dfrac{\overset{1}{\cancel{8}}a^{15}b^{-8}}{\underset{2}{\cancel{16}}}$ Simplify.

$= \dfrac{a^{15}}{2b^8}$ Simplify negative exponents.

Skill Practice Simplify the expression. Write the final answer with positive exponents only.

10. $\dfrac{(3x^3y^{-4})^2}{(x^{-2}y)^{-4}}$

Example 5 Simplifying an Expression with Exponents

Simplify the expression. Write the answer with positive exponents only.

$$4xy^{-3}\left(\frac{8x^2}{3x^5y^2}\right)^{-2}$$

Solution:

$4xy^{-3}\left(\dfrac{8x^2}{3x^5y^2}\right)^{-2} = 4xy^{-3} \cdot \left(\dfrac{8}{3x^3y^2}\right)^{-2}$ Simplify within parentheses.

$= \dfrac{4xy^{-3}}{1} \cdot \dfrac{8^{-2}}{3^{-2}x^{-6}y^{-4}}$ Raise the expression in parentheses to the -2 power.

$= \dfrac{4x}{y^3} \cdot \dfrac{3^2x^6y^4}{8^2}$ Simplify negative exponents.

$= \dfrac{4 \cdot 9 \cdot x \cdot x^6 \cdot y^4}{64y^3}$ Multiply the fractions and simplify the expressions 3^2 and 8^2.

$= \dfrac{\overset{1}{\cancel{4}} \cdot 9 \cdot x^{1+6}y^{4-3}}{\underset{16}{\cancel{64}}}$ Add the exponents on x. Subtract the exponents on y.

$= \dfrac{9x^7y}{16}$ Simplify.

Answer

10. $\dfrac{9}{x^2y^4}$

Skill Practice Simplify the expression. Write the answer with positive exponents only.

11. $-9m^5t\left(\dfrac{3m^{-2}t^3}{4t^6}\right)^{-3}$

2. Scientific Notation

Scientists in a variety of fields often work with very large or very small numbers. For instance, the distance between the Earth and the Sun is approximately 93,000,000 mi. The mass of an electron is 0.000 000 000 000 000 000 000 000 000 000 911 kg.

Scientific notation was devised as a shortcut method of expressing very large and very small numbers. The principle behind scientific notation is to use a power of 10 to express the magnitude of the number. For example, a number such as 50,000 can be written as $5 \times 10{,}000$ or equivalently as 5×10^4. Similarly, the number 0.0035 is equal to $3.5 \times \frac{1}{1000}$ or, equivalently, 3.5×10^{-3}.

Definition of Scientific Notation

A number expressed in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer, is said to be written in **scientific notation**.

Consider the following numbers in scientific notation:

The distance between the Sun and the Earth: $\underbrace{93{,}000{,}000}_{7\text{ places}}\text{ mi} = 9.3 \times 10^7\text{ mi}$

The mass of an electron: $0.\underbrace{000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 9}_{31\text{ places}}11\text{ kg}$

$= 9.11 \times 10^{-31}\text{ kg}$

In each case, the power of 10 corresponds to the number of place positions that the decimal point is moved. The power of 10 is sometimes called the order of magnitude (or simply the magnitude) of the number. The order of magnitude of the distance between the Earth and Sun is 10^7 mi (tens of millions). The mass of an electron has an order of magnitude of 10^{-31} kg.

Example 6 Writing Numbers in Scientific Notation

Fill in the table by writing the numbers in scientific notation or standard notation as indicated.

Quantity	Standard Notation	Scientific Notation
Number of NASCAR fans	75,000,000 people	
Width of an influenza virus	0.000000001 m	
Cost of Hurricane Katrina		$\$1.25 \times 10^{11}$
Probability of winning the Florida state lottery		$4.35587878 \times 10^{-8}$

Answer

11. $-\dfrac{64m^{11}t^{10}}{3}$

Solution:

Quantity	Standard Notation	Scientific Notation
Number of NASCAR fans	75,000,000 people	7.5×10^7 people
Width of an influenza virus	0.000000001 m	1×10^{-9} m
Cost of Hurricane Katrina	\$125,000,000,000	$\$1.25 \times 10^{11}$
Probability of winning the Florida state lottery	0.0000000435587878	$4.35587878 \times 10^{-8}$

Skill Practice Rewrite each number in either scientific notation or standard notation.

12. 2,600,000 **13.** 0.00088 **14.** -5.7×10^{-8} **15.** 1.9×10^5

Example 7 Applying Scientific Notation

a. During a recent economic crisis, the U.S. government lent money to troubled financial institutions that had a large number of mortgage-related assets. The government committed an estimated \$750,000,000,000. How much money does this represent per person if the U.S. population was 300,000,000 at that time?

b. The mean distance between the Earth and the Andromeda Galaxy is approximately 1.8×10^6 light-years. Assuming that 1 light-year is 6×10^{12} mi, what is the distance in miles to the Andromeda Galaxy?

Buras/S.E. Schneider/Shutterstock

Solution:

a. Express each value in scientific notation. Then divide the total amount to be paid off by the number of people.

$$\frac{7.5 \times 10^{11}}{3 \times 10^8}$$

$$= \left(\frac{7.5}{3}\right) \times \left(\frac{10^{11}}{10^8}\right) \qquad \text{Divide 7.5 by 3 and subtract the powers of 10.}$$

$$= 2.5 \times 10^3$$

In standard notation, this amounts to \$2500 per person.

b. Multiply the number of light-years by the number of miles per light-year.

$$(1.8 \times 10^6)(6 \times 10^{12})$$

$$= (1.8)(6) \times (10^6)(10^{12})$$

$$= 10.8 \times 10^{18}$$

Multiply 1.8 and 6 and add the powers of 10.

The number 10.8×10^{18} is not in "proper" scientific notation because 10.8 is not between 1 and 10.

$$= (1.08 \times 10^1) \times 10^{18} \qquad \text{Rewrite 10.8 as } 1.08 \times 10^1.$$

$$= 1.08 \times (10^1 \times 10^{18}) \qquad \text{Apply the associative property of multiplication.}$$

$$= 1.08 \times 10^{19}$$

The distance between the Earth and the Andromeda Galaxy is 1.08×10^{19} mi.

Answers

12. 2.6×10^6 **13.** 8.8×10^{-4}
14. -0.000000057 **15.** 190,000

Skill Practice

16. The thickness of a penny is 6×10^{-2} in. How many pennies would have to be stacked to equal the height of the Empire State Building (1.5×10^4 in.)?

17. The distance from the Earth to the "nearby" star, Barnard's Star, is 7.6 light-years (where 1 light-year $= 6 \times 10^{12}$ mi). How many miles away is Barnard's Star?

Answers

16. Approximately 2.5×10^5 pennies

17. 4.56×10^{13} mi

Section 4.1 Activity

A.1. Consider the expression $x^5 \cdot x^2$.

a. Expand the factors in the expression and simplify.

b. Write a rule to simplify $x^m \cdot x^n$.

For Exercises A.2–A.8, expand the factors for the given expression. Then simplify the expression, write an observation, and complete the general rule.

Example	Expanded Form	Simplified Form	Observation	General Rule
A.2. $x^4 \cdot x^5$	$(x \cdot x \cdot x \cdot x)(x \cdot x \cdot x \cdot x \cdot x)$	x^9	$x^4 \cdot x^5 = x^{4+5} = x^9$	$b^m \cdot b^n = b^{m+n}$
A.3. $\dfrac{w^6}{w^3}$				$\dfrac{b^m}{b^n} =$
A.4. $(t^4)^3$				$(b^m)^n =$
A.5. $(2c)^3$				$(ab)^m =$
A.6. $\left(\dfrac{x^4}{3}\right)^2$				$\left(\dfrac{a}{b}\right)^m =$
A.7. $\dfrac{b^4}{b^4}$				$b^0 =$
A.8. $\dfrac{b^2}{b^5}$				$\dfrac{b^m}{b^n} =$

For Exercises A.9–A.11, simplify the expression.

A.9. $\left(\dfrac{1}{2}\right)^{-3} + \left(\dfrac{1}{5}\right)^{-2} + \left(\dfrac{2}{3}\right)^0$ **A.10.** $(4a^3b^{-4})^{-2}$ **A.11.** $\left(\dfrac{x^5y^2}{3x^{-5}y^{-3}}\right)^3 \cdot \left(\dfrac{6x^{-1}y^0}{x^6y^8}\right)$

For Exercises A.12–A.13, simplify the expressions.

A.12. **a.** $(-4)^2$ **b.** -4^2 **c.** $(-4)^3$ **d.** -4^3

e. $(-4)^{-2}$ **f.** -4^{-2} **g.** $(-4)^{-3}$ **h.** -4^{-3}

A.13. **a.** $(-6)^0$ **b.** -6^0 **c.** $(-6x)^0$ **d.** $-6x^0$

A.14. Complete the table. The first two rows are done for you.

Standard Form	Expanded Form	Scientific Notation
63,000	$6.3 \times 10{,}000$	6.3×10^4
0.008	$8 \times \dfrac{1}{1000}$ or $8 \times \dfrac{1}{10^3}$	8×10^{-3}
8,120,000		
		2×10^3
0.034		
		5.6×10^{-1}

A.15. An adult female has approximately 5 million red blood cells per 1 μL(1 microliter) of blood.

a. Write 5 million in scientific notation.

b. If 1 L = 1,000,000 μL, determine the number of red blood cells present in 1 L. Write the answer in scientific notation.

c. Suppose a woman donates 1 pint of blood to a blood bank. How many red blood cells are present in 1 pint? Use the fact that 1 pint $\approx$ 0.47 L.

A.16. The mass of Earth is 5.98×10^{24} kg and the mass of the Moon is 7.36×10^{22} kg. Find the ratio of the mass of Earth to the mass of the Moon. This tells us how much more massive Earth is than the Moon.

Practice Exercises — Section 4.1

Study Skills Exercise

As you study properties of exponents, you will find that there are many rules for simplifying expressions. Identifying when and how to apply each rule is crucial. Good note-taking will help you master these skills.

Use your notes and simplify each expression. Identify the rule that you applied to simplify the expression. Choose from:

- Multiplication of like bases
- Division of like bases
- Power Rule
- Power of a Product
- Power of a Quotient

Also give another example of a similar expression that would be simplified using the same rule.

Expression to Simplify	Rule	New Example
$\dfrac{y^8}{y^4}$		
$\left(\dfrac{r}{s}\right)^2$		
a^3a^5		
$(3xy)^2$		
$(b^6)^2$		

Include this exercise in your class notes for future reference. Detailed notes will help you as you progress through the course.

Prerequisite Review

For Exercises R.1–R.8, simplify the exponential expression.

R.1. 8^2 **R.2.** 10^2 **R.3.** $(-8)^2$ **R.4.** $(-10)^2$

R.5. 5^3 **R.6.** 2^5 **R.7.** $(-5)^3$ **R.8.** $(-2)^5$

R.9. **a.** Simplify 10^2.

b. Write 10,000 as a power of 10.

R.10. **a.** Simplify 10^6.

b. Write 10,000,000 as a power of 10.

R.11. **a.** Simplify 10^{-2}.

b. Write $\dfrac{1}{10{,}000}$ as a power of 10.

c. Write 0.01 as a power of 10.

R.12. **a.** Simplify 10^{-6}.

b. Write $\dfrac{1}{10{,}000{,}000}$ as a power of 10.

c. Write 0.1 as a power of 10.

Vocabulary and Key Concepts

1. **a.** A(n) ____________ is used to show repeated multiplication of the base.
   b. For $b \neq 0$, the expression b^0 is defined to be ____________.
   c. For $b \neq 0$, the expression b^{-n} is defined as ____________.
   d. A number expressed in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer is said to be written in ____________ ____________.

Concept 1: Simplifying Expressions with Exponents

2. Write the expressions in expanded form and simplify.
   $$b^4 \cdot b^3 \text{ and } (b^4)^3$$

3. Write the expressions in expanded form and simplify.
   $$ab^3 \text{ and } (ab)^3$$

For Exercises 4–9, write an example of each property. (Answers may vary.)

4. $b^n \cdot b^m = b^{n+m}$
5. $(ab)^n = a^n b^n$
6. $(b^n)^m = b^{nm}$
7. $\dfrac{b^n}{b^m} = b^{n-m} \quad (b \neq 0)$
8. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} \quad (b \neq 0)$
9. $b^0 = 1 \quad (b \neq 0)$

For Exercises 10–28, simplify. **(See Example 1.)**

10. $\left(\dfrac{2}{3}\right)^{-1}$
11. $\left(\dfrac{1}{3}\right)^{-1}$
12. 3^{-1}
13. 5^{-2}
14. 8^{-2}
15. -5^{-2}
16. -8^{-2}
17. $(-5)^{-2}$
18. $(-8)^{-2}$
19. $\left(-\dfrac{1}{4}\right)^{-3}$
20. $\left(-\dfrac{3}{8}\right)^{-1}$
21. $\left(-\dfrac{3}{2}\right)^{-4}$
22. $\left(-\dfrac{1}{9}\right)^{-2}$
23. $-\left(\dfrac{2}{5}\right)^{-3}$
24. $-\left(\dfrac{1}{2}\right)^{-5}$
25. $(10ab)^0$
26. $(13x)^0$
27. $10ab^0$
28. $13x^0$

For Exercises 29–80, simplify and write the answer with positive exponents only. **(See Examples 2–5.)**

29. $y^3 \cdot y^5$
30. $x^4 \cdot x^8$
31. $\dfrac{13^8}{13^6}$
32. $\dfrac{5^7}{5^3}$
33. $(y^2)^4$
34. $(z^3)^4$
35. $(3x^2)^4$
36. $(2y^5)^3$
37. p^{-3}
38. q^{-5}
39. $7^{10} \cdot 7^{-13}$
40. $11^{-9} \cdot 11^7$
41. $\dfrac{w^3}{w^5}$
42. $\dfrac{t^4}{t^8}$
43. $a^{-2}a^{-5}$
44. $b^{-1}b^{-8}$
45. $\dfrac{r}{r^{-1}}$
46. $\dfrac{s^{-1}}{s}$
47. $\dfrac{z^{-6}}{z^{-2}}$
48. $\dfrac{w^{-8}}{w^{-3}}$
49. $\dfrac{a^3}{b^{-2}}$
50. $\dfrac{c^4}{d^{-1}}$
51. $(6xyz^2)^0$
52. $(-7ab^3)^0$

53. $2^4 + 2^{-2}$

54. $3^2 + 3^{-1}$

55. $1^{-2} + 5^{-2}$

56. $4^{-2} + 2^{-2}$

57. $\left(\frac{2}{3}\right)^{-2} - \left(\frac{1}{2}\right)^{2} + \left(\frac{1}{3}\right)^{0}$

58. $\left(\frac{1}{6}\right)^{-1} + \left(\frac{2}{3}\right)^{0} - \left(\frac{1}{4}\right)^{-2}$

59. $\left(\frac{4}{5}\right)^{-1} + \left(\frac{3}{2}\right)^{2} - \left(\frac{2}{7}\right)^{0}$

60. $\left(\frac{4}{5}\right)^{0} - \left(\frac{2}{3}\right)^{2} + \left(\frac{9}{5}\right)^{-1}$

61. $\dfrac{p^2q}{p^5q^{-1}}$

62. $\dfrac{m^{-1}n^3}{m^4n^{-2}}$

63. $\dfrac{-48ab^{10}}{32a^4b^3}$

64. $\dfrac{25x^2y^{12}}{10x^5y^7}$

65. $(-3x^{-4}y^5z^2)^{-4}$

66. $(-6a^{-2}b^3c)^{-2}$

67. $(4m^{-2}n)(-m^6n^{-3})$

68. $(-6pq^{-3})(2p^4q)$

69. $(p^{-2}q)^3(2pq^4)^2$

70. $(mn^3)^2(5m^{-2}n^2)$

71. $\left(\dfrac{x^2}{y}\right)^3(5x^2y)$

72. $\left(\dfrac{a}{b^2}\right)^2(3a^2b^3)$

73. $\dfrac{(-8a^2b^2)^4}{(16a^3b^7)^2}$

74. $\dfrac{(-3x^2y^3)^2}{(-2xy^4)^3}$

75. $\left(\dfrac{-2x^6y^{-5}}{3x^{-2}y^4}\right)^{-3}$

76. $\left(\dfrac{-6a^2b^{-3}}{5a^{-1}b}\right)^{-2}$

77. $\left(\dfrac{2x^{-3}y^0}{4x^6y^{-5}}\right)^{-2}$

78. $\left(\dfrac{a^3b^2c^0}{a^{-1}b^{-2}c^{-3}}\right)^{-2}$

79. $3xy^5\left(\dfrac{2x^4y}{6x^5y^3}\right)^{-2}$

80. $7x^{-3}y^{-4}\left(\dfrac{3x^{-1}y^5}{9x^3y^{-2}}\right)^{-3}$

Concept 2: Scientific Notation

81. The European Organization for Nuclear Research (known by the acronym CERN) has built the world's largest high-energy particle accelerator, called the Large Hadron Collider (LHC). Scientists hope the LHC will answer many open questions in physics. Write the following numbers in scientific notation. **(See Example 6.)**

a. The LHC cost \$8,000,000,000 to build.

b. 3,000,000 DVDs worth of data will be produced each year.

c. 14,000,000,000,000 electron volts (eV) of energy will be produced to smash the protons together.

d. 1 eV is equivalent to 0.000 000 000 000 000 000 1602 joules (J).

82. Write the numbers in scientific notation.

a. The estimated population of the United States for a recent year was 311,000,000.

b. The size of the smallest visible object in an optical microscope is 0.0000002 m.

c. A trillion is defined as 1,000,000,000,000.

83. Write the numbers in standard notation.

a. The Andromeda Galaxy contains at least 2×10^{11} stars.

b. The diameter of a capillary is 4×10^{-6} m.

c. The mean distance between Venus and the Sun is 1.082×10^{11} m.

84. Write the numbers in standard notation.

a. At the end of a recent year, the Department of Energy's inventory of high-level radioactive waste was approximately 3.784×10^5 m^3.

b. The diameter of a water molecule is 3×10^{-10} m.

c. The distance a bullet will travel in 1 sec when fired from a 0.22 caliber gun is 4.1×10^2 m.

For Exercises 85–90, determine which numbers are in “proper” scientific notation. If the number is not in “proper” scientific notation, correct it.

85. 35×10^4 **86.** 0.469×10^{-7} **87.** 7×10^0

88. 8.12×10^1 **89.** 9×10^1 **90.** 6.9×10^0

For Exercises 91–98, perform the indicated operations and write the answer in scientific notation.

91. $(6.5 \times 10^3)(5.2 \times 10^{-8})$ **92.** $(3.26 \times 10^{-6})(8.2 \times 10^9)$ **93.** $(0.0000024)(6{,}700{,}000{,}000)$

94. $(3{,}400{,}000{,}000)(70{,}000{,}000{,}000{,}000)$ **95.** $(8.5 \times 10^{-2}) \div (2.5 \times 10^{-15})$ **96.** $(3 \times 10^9) \div (1.5 \times 10^{13})$

97. $(900{,}000{,}000) \div (360{,}000)$ **98.** $(0.0000000002) \div (8{,}000{,}000)$

99. If one H_2O molecule contains 2 hydrogen atoms and 1 oxygen atom, and 10 H_2O molecules contain 20 hydrogen atoms and 10 oxygen atoms, how many hydrogen atoms and oxygen atoms are contained in 6.02×10^{23} H_2O molecules? **(See Example 7.)**

100. The star named Alpha Centauri is 4.3 light-years from the Earth. If 1 light-year is approximately 6×10^9 mi, how far (in miles) is Alpha Centauri?

101. If the county of Queens, New York, has a population of approximately 2,200,000 and the area is 110 mi^2, how many people are there per square mile? **(See Example 7.)**

102. If the county of Catawba, North Carolina, has a population of approximately 150,000 and the area is 400 mi^2, how many people are there per square mile?

103. According to the Federal Emergency Management Agency (FEMA), the annual loss due to earthquakes in California is approximately $\$3.5 \times 10^9$. If this is representative as a yearly average, find the loss over 15 yr.

104. Avogadro’s number $N_a = 6.02 \times 10^{23}$ is the number of atoms in 1 *mole* of an element.

a. How many atoms are in 5 moles of carbon-12?

b. If 75 g of carbon-12 has 4.515×10^{25} atoms, how many moles is this?

Expanding Your Skills

105. A 20-yr-old starts a savings plan for her retirement. She will put \$20 per month into a mutual fund that she hopes will average 6% growth annually.

a. If she plans to retire at age 65, for how many months will she be depositing money?

b. By age 65, how much money will she have deposited?

c. The value of an account built in this fashion is given by

$$A = P \cdot \left[\left(1 + \frac{r}{12}\right)^N - 1\right] \cdot \left(1 + \frac{12}{r}\right)$$

where A is the final amount of money in the account, P is the amount of the monthly deposit, and N is the number of months. Use a calculator to find the total amount in the woman’s retirement account at age 65.

For Exercises 106–111, simplify each expression. Assume that a and b represent positive integers and x and y are nonzero real numbers.

106. $x^{a+1}x^{a+5}$ **107.** $y^{a-5}y^{a+7}$ **108.** $\dfrac{y^{2a+1}}{y^{a-1}}$

109. $\dfrac{x^{3a-3}}{x^{a+1}}$ **110.** $\dfrac{x^{3b-2}y^{b+1}}{x^{2b+1}y^{2b+2}}$ **111.** $\dfrac{x^{2a-2}y^{a+3}}{x^{a+4}y^{a-3}}$

Addition and Subtraction of Polynomials and Polynomial Functions

Section 4.2

Concepts

1. Polynomials: Basic Definitions
2. Addition of Polynomials
3. Subtraction of Polynomials
4. Polynomial Functions

1. Polynomials: Basic Definitions

One commonly used algebraic expression is called a polynomial. A **polynomial** in x is defined as a finite sum of terms of the form ax^n, where a is a real number and the exponent n is a whole number. For each term, a is called the **coefficient**, and n is called the **degree of the term**. For example:

Term (Expressed in the Form ax^n)	Coefficient	Degree
$3x^5$	3	5
$x^{14} \rightarrow$ rewrite as $1x^{14}$	1	14
$7 \rightarrow$ rewrite as $7x^0$	7	0
$\frac{1}{2}p \rightarrow$ rewrite as $\frac{1}{2}p^1$	$\frac{1}{2}$	1

If a polynomial has exactly one term, it is called a **monomial**. A two-term polynomial is called a **binomial**, and a three-term polynomial is called a **trinomial**. Usually the terms of a polynomial are written in descending order according to degree. In descending order, the highest-degree term is written first and is called the **leading term**. Its coefficient is called the **leading coefficient**. The **degree of a polynomial** is the greatest degree of all its terms. Thus, the leading term determines the degree of the polynomial.

	Expression	Descending Order	Leading Coefficient	Degree of Polynomial
Monomials	$2x^9$	$2x^9$	2	9
	-49	-49	-49	0
Binomials	$10y - 7y^2$	$-7y^2 + 10y$	-7	2
	$6 - \frac{2}{3}b$	$-\frac{2}{3}b + 6$	$-\frac{2}{3}$	1
Trinomials	$w + 2w^3 + 9w^6$	$9w^6 + 2w^3 + w$	9	6
	$2.5a^4 - a^8 + 1.3a^3$	$-a^8 + 2.5a^4 + 1.3a^3$	-1	8

Polynomials may have more than one variable. In such a case, the degree of a term is the sum of the exponents of the variables contained in the term. For example, the term $2x^3y^4z$ has degree 8 because the exponents applied to x, y, and z are 3, 4, and 1, respectively.

The following polynomial has a degree of 12 because the highest degree of its terms is 12.

$$\underbrace{11x^4y^3z}_{\text{degree } 8} - \underbrace{5x^3y^2z^7}_{\text{degree } 12} + \underbrace{2x^2y}_{\text{degree } 3} + \underbrace{7}_{\text{degree } 0}$$

FOR REVIEW

The **degree** of a term in a polynomial is the sum of the exponents of the variable factors in the term. A polynomial is written in **descending order** if the terms are written in order by degree, with the term of highest degree written first.

$$\underbrace{-8p}_{\text{degree } 1} - \underbrace{48}_{\text{degree } 0} + \underbrace{p^2}_{\text{degree } 2}$$
$$= p^2 - 8p - 48$$

2. Addition of Polynomials

To add or subtract two polynomials, we combine *like* terms. Recall that two terms are ***like terms*** if they each have the same variables and the corresponding variables are raised to the same powers.

Example 1 Adding Polynomials

Add the polynomials.

a. $(3t^3 + 2t^2 - 5t) + (t^3 - 6t)$ **b.** $\left(\frac{2}{3}w^2 - w + \frac{1}{8}\right) + \left(\frac{4}{3}w^2 + 8w - \frac{1}{4}\right)$

Solution:

a. $(3t^3 + 2t^2 - 5t) + (t^3 - 6t)$

$= 3t^3 + t^3 + 2t^2 + (-5t) + (-6t)$ Group *like* terms.

$= 4t^3 + 2t^2 - 11t$ Add *like* terms.

b. $\left(\frac{2}{3}w^2 - w + \frac{1}{8}\right) + \left(\frac{4}{3}w^2 + 8w - \frac{1}{4}\right)$

$= \frac{2}{3}w^2 + \frac{4}{3}w^2 + (-w) + 8w + \frac{1}{8} + \left(-\frac{1}{4}\right)$ Group *like* terms.

$= \frac{6}{3}w^2 + 7w + \left(\frac{1}{8} - \frac{2}{8}\right)$ Add fractions with common denominators.

$= 2w^2 + 7w - \frac{1}{8}$ Simplify.

Skill Practice Add the polynomials.

1. $(2x^2 + 5x - 2) + (6x^2 - 8x - 8)$

2. $\left(-\frac{1}{4}m^2 - 2m + \frac{1}{3}\right) + \left(\frac{3}{4}m^2 + 7m - \frac{1}{12}\right)$

Example 2 Adding Polynomials

Add the polynomials. $(a^2b + 7ab + 6) + (5a^2b - 2ab - 7)$

Solution:

Polynomials can be added vertically. Be sure to line up the *like* terms.

$$\begin{array}{r} a^2b + 7ab + 6 \\ +\ 5a^2b - 2ab - 7 \\ \hline 6a^2b + 5ab - 1 \end{array}$$

Add *like* terms.

Skill Practice Add the polynomials.

3. $(-5a^2b - 6ab^2 + 2) + (2a^2b + ab^2 - 3)$

3. Subtraction of Polynomials

Subtraction of two polynomials is similar to subtracting real numbers. Add the opposite of the second polynomial to the first polynomial.

The opposite (or additive inverse) of a real number a is $-a$. Similarly, if A is a polynomial, then $-A$ is its opposite.

Answers

1. $8x^2 - 3x - 10$

2. $\frac{1}{2}m^2 + 5m + \frac{1}{4}$

3. $-3a^2b - 5ab^2 - 1$

Example 3 Finding the Opposite of a Polynomial

Find the opposite of the polynomials.

a. $5a - 2b - c$ **b.** $-5.5y^4 - 2.4y^3 + 1.1y$

Solution:

	Expression	Opposite	Simplified Form
a.	$5a - 2b - c$	$-(5a - 2b - c)$	$-5a + 2b + c$
b.	$-5.5y^4 - 2.4y^3 + 1.1y$	$-(-5.5y^4 - 2.4y^3 + 1.1y)$	$5.5y^4 + 2.4y^3 - 1.1y$

TIP: Notice that the sign of each term is changed when finding the opposite of a polynomial.

Skill Practice Find the opposite of the polynomials.

4. $-7z + 6w$ **5.** $2p - 3q + r + 1$

Subtraction of Polynomials

If A and B are polynomials, then $A - B = A + (-B)$.

Example 4 Subtracting Polynomials

Subtract the polynomials. $(3x^2 + 2x - 5) - (4x^2 - 7x + 2)$

Solution:

$(3x^2 + 2x - 5) - (4x^2 - 7x + 2)$

$= (3x^2 + 2x - 5) + (-4x^2 + 7x - 2)$ Add the opposite of the second polynomial.

$= 3x^2 + (-4x^2) + 2x + 7x + (-5) + (-2)$ Group *like* terms.

$= -x^2 + 9x - 7$ Combine *like* terms.

FOR REVIEW

Recall that to subtract two real numbers, we add the opposite. For example,

$$3 - 4 = 3 + (-4).$$

The same is true for subtracting polynomials.

Skill Practice Subtract the polynomials.

6. $(6a^2 - 2a) - (-3a^2 + 2a + 3)$

Example 5 Subtracting Polynomials

Subtract the polynomials. $(6x^2y - 2xy + 5) - (x^2y - 3)$

Solution:

$(6x^2y - 2xy + 5) - (x^2y - 3)$

Subtraction of polynomials can be performed vertically by vertically aligning *like* terms. Then add the opposite of the second polynomial. "Placeholders" (shown in red) may be used to help line up *like* terms.

$$\begin{array}{r} 6x^2y - 2xy + 5 \\ \underline{-(x^2y + 0xy - 3)} \end{array} \xrightarrow[\text{opposite.}]{\text{Add the}} \begin{array}{r} 6x^2y - 2xy + 5 \\ \underline{+ -x^2y - 0xy + 3} \\ 5x^2y - 2xy + 8 \end{array}$$

Skill Practice Subtract the polynomials.

7. $(7p^2q - 6) - (2p^2q + 4pq + 4)$

Answers

4. $7z - 6w$
5. $-2p + 3q - r - 1$
6. $9a^2 - 4a - 3$
7. $5p^2q - 4pq - 10$

Example 6 Subtracting Polynomials

Subtract $\frac{1}{2}x^4 - \frac{3}{4}x^2 + \frac{1}{5}$ from $\frac{3}{2}x^4 + \frac{1}{2}x^2 - 4x$.

Solution:

In general, to subtract a from b, we write $b - a$. Therefore, to subtract

$$\frac{1}{2}x^4 - \frac{3}{4}x^2 + \frac{1}{5} \quad \text{from} \quad \frac{3}{2}x^4 + \frac{1}{2}x^2 - 4x$$

we have

$$\left(\frac{3}{2}x^4 + \frac{1}{2}x^2 - 4x\right) - \left(\frac{1}{2}x^4 - \frac{3}{4}x^2 + \frac{1}{5}\right)$$

$$= \frac{3}{2}x^4 + \frac{1}{2}x^2 - 4x - \frac{1}{2}x^4 + \frac{3}{4}x^2 - \frac{1}{5}$$ Subtract the polynomials by adding the opposite of the second polynomial.

$$= \frac{3}{2}x^4 - \frac{1}{2}x^4 + \frac{1}{2}x^2 + \frac{3}{4}x^2 - 4x - \frac{1}{5}$$ Group *like* terms.

$$= \frac{3}{2}x^4 - \frac{1}{2}x^4 + \frac{2}{4}x^2 + \frac{3}{4}x^2 - 4x - \frac{1}{5}$$ Write *like* terms with a common denominator.

$$= \frac{2}{2}x^4 + \frac{5}{4}x^2 - 4x - \frac{1}{5}$$ Combine *like* terms.

$$= x^4 + \frac{5}{4}x^2 - 4x - \frac{1}{5}$$ Simplify.

Skill Practice

8. Subtract $\frac{1}{2}p^3 + \frac{1}{3}p^2 + \frac{1}{2}p$ from $\frac{1}{3}p^3 + \frac{3}{4}p^2 - p$

4. Polynomial Functions

A **polynomial function** is a function defined by a finite sum of terms of the form ax^n, where a is a real number and n is a whole number. For example, the functions defined here are polynomial functions:

$$f(x) = 3x - 8$$

$$g(x) = 4x^5 - 2x^3 + 5x - 3$$

$$h(x) = -\frac{1}{2}x^4 + \frac{3}{5}x^3 - 4x^2 + \frac{5}{9}x - 1$$

$k(x) = 7$ $\quad$ ($7 = 7x^0$, which is of the form ax^n, where $n = 0$ is a whole number)

The following functions are *not* polynomial functions:

$m(x) = \frac{1}{x} - 8$ $\quad$ $\left(\frac{1}{x} = x^{-1}\text{, the exponent } -1 \text{ is not a whole number}\right)$

$q(x) = |x|$ $\quad$ ($|x|$ is not of the form ax^n)

Answer

8. $-\frac{1}{6}p^3 + \frac{5}{12}p^2 - \frac{3}{2}p$

Example 7 Evaluating a Polynomial Function

Given $P(x) = x^3 + 2x^2 - x - 2$, find the function values.

a. $P(-3)$ **b.** $P(-1)$ **c.** $P(0)$ **d.** $P(2)$

Solution:

a. $P(x) = x^3 + 2x^2 - x - 2$

$$\begin{aligned} P(-3) &= (-3)^3 + 2(-3)^2 - (-3) - 2 \\ &= -27 + 2(9) + 3 - 2 \\ &= -27 + 18 + 3 - 2 \\ &= -8 \end{aligned}$$

b. $$\begin{aligned} P(-1) &= (-1)^3 + 2(-1)^2 - (-1) - 2 \\ &= -1 + 2(1) + 1 - 2 \\ &= -1 + 2 + 1 - 2 \\ &= 0 \end{aligned}$$

c. $$\begin{aligned} P(0) &= (0)^3 + 2(0)^2 - (0) - 2 \\ &= -2 \end{aligned}$$

d. $$\begin{aligned} P(2) &= (2)^3 + 2(2)^2 - (2) - 2 \\ &= 8 + 2(4) - 2 - 2 \\ &= 8 + 8 - 2 - 2 \\ &= 12 \end{aligned}$$

The function values can be confirmed from the graph of $y = P(x)$ (Figure 4-1).

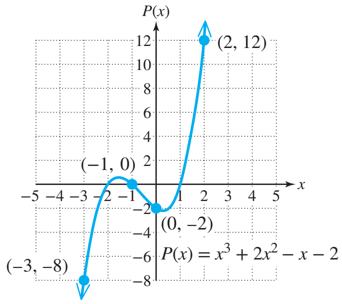


Figure 4-1

Skill Practice Given: $P(x) = -2x^3 - 4x + 6$

9. Find $P(0)$. **10.** Find $P(-2)$.
11. Find $P(-1)$. **12.** Find $P(2)$.

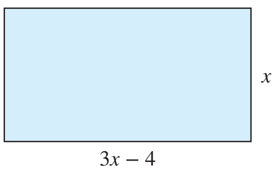
Example 8 Using Polynomial Functions in an Application

The length of a rectangle is 4 m less than 3 times the width. Let x represent the width. Write a polynomial function P that represents the perimeter of the rectangle and simplify the result.

Solution:

Let x represent the width. Then $3x - 4$ is the length.
The perimeter of a rectangle is given by $P = 2L + 2W$.
Thus,

$$\begin{aligned} P(x) &= 2(3x - 4) + 2(x) \\ &= 6x - 8 + 2x \\ &= 8x - 8 \end{aligned}$$



Answers

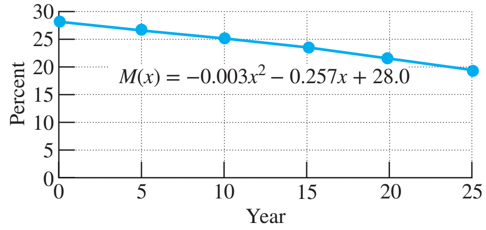
9. $P(0) = 6$ **10.** $P(-2) = 30$
11. $P(-1) = 12$ **12.** $P(2) = -18$

Skill Practice

13. The longest side of a triangle is 2 ft less than 4 times the shortest side. The middle side is 3 ft more than twice the shortest side. Let x represent the shortest side. Find a polynomial function P that represents the perimeter of the triangle, and simplify the result.

Example 9 Using a Polynomial Function in an Application

The percent of male smokers in the United States has decreased in recent years. The function defined by $M(x) = -0.003x^2 - 0.257x + 28.0$ approximates the percent of male smokers, $M(x)$, where x represents the number of years since the study began. See Figure 4-2.



Source: U.S. National Center for Health Statistics.

Figure 4-2

a. Evaluate $M(5)$ and interpret the meaning in the context of this problem.

b. Determine the percent of male smokers in year 18.

Solution:

a. $M(5) = -0.003(5)^2 - 0.257(5) + 28.0$ Substitute $x = 5$ into the function.

≈ 26.6

$M(5) \approx 26.6$ means that in year 5, 26.6% of males in the United States smoked.

b. Evaluate $M(18)$.

$M(18) = -0.003(18)^2 - 0.257(18) + 28.0$ Substitute $x = 18$ into the function.

≈ 22.4

$M(18) \approx 22.4$ means that in year 18, 22.4% of males in the United States smoked.

Skill Practice The population of Kenya can be modeled by $P(t) = 0.017t^2 + 1.04t + 38$. The value t is the number of years since the study began, and $P(t)$ is the population of Kenya (in millions). (*Source:* Central Intelligence Agency)

14. Evaluate $P(3)$ and interpret its meaning.

15. If this trend continues, predict the population of Kenya for year 8.

Answers

13. $P(x) = x + (4x - 2) + (2x + 3)$
$= 7x + 1$

14. $P(3) = 41.273$ means that in year 3 the population of Kenya was 41.273 million.

15. $P(8) = 47.408$; In year 8, the population of Kenya will be 47.408 million.

Section 4.2 Activity

A.1. **a.** Write a monomial with variable t, coefficient 6, and degree 3.

b. Write a trinomial of degree 2 in descending order. Use the variable x and coefficients –4, 1, and –5, respectively.

A.2. **a.** Write a monomial of degree zero. (Answers may vary.)

b. Write a binomial of degree 4 and leading coefficient $\frac{1}{2}$. (Answers may vary.)

A.3. **a.** Find the sum of $(4x^2 - x) + (2x^2 - 8x - 7)$.
b. Explain how to add two polynomials.

A.4. **a.** Simplify $-(-7y^2 + 3y - 4)$.
b. Explain how to find the opposite of a polynomial.

A.5. **a.** Find the difference of $\left(\frac{2}{5}w^2 - w + \frac{1}{3}\right)$ and $\left(-\frac{1}{10}w^2 - 2w + \frac{5}{6}\right)$.
b. Subtract $(3x^2y^2 - 5xy + 10)$ from $(8x^2y^2 - 2xy + 3)$.
c. Summarize how the difference in wording in parts (a) and (b) affects the order in which the polynomials are subtracted.

A.6. The length of a rectangle is 4 cm longer than the width.
a. Let x represent the width of the rectangle. Write an expression in terms of x that represents the length.
b. Write a function that represents the perimeter of the rectangle $P(x)$.
c. Write a function that represents the area of the rectangle $A(x)$.
d. Evaluate $P(12)$ and interpret its meaning.
e. Evaluate $A(7)$ and interpret its meaning.

Practice Exercises

Section 4.2

Study Skills Exercise

Reading comprehension is the skill of understanding mathematical operations and symbols in an expression. This skill is required to add and subtract polynomials.

- One technique that might help your understanding is to read each term of the polynomial as if it were a word in a sentence. For example, read the expression out loud: $(4x^2 + 5x) + (-3x^2 - 2x)$
- Recall that only *like* terms can be combined within a polynomial. As you read the expression $(4x^2 + 5x) + (-3x^2 - 2x)$, ask yourself what the parentheses imply and how you might regroup terms to add the polynomial.
- As you read the expression $(4x^2 + 5x) + (-3x^2 - 2x)$, you recognize that the terms $4x^2$ and $-3x^2$ can be combined. Ask yourself whether the exponents change when combining *like* terms. If necessary, expand each term to visualize the similarity between the terms.

Prerequisite Review

For Exercises R.1–R.4, combine *like* terms.

R.1. $2t + 5t - 10t$

R.2. $-3m + 4m + 6m$

R.3. $-\frac{3}{2}x + \frac{1}{3}x + \frac{5}{12}x$

R.4. $\frac{3}{2}y - \frac{4}{3}y + \frac{2}{5}y$

For Exercises R.5–R.6, apply the distributive property.

R.5. $-2(4y + 8z - 11)$

R.6. $-7(3t - 4s + 6)$

For Exercises R.7–R.8, clear parentheses and combine *like* terms.

R.7. $(5m - 6) - (-m + 7)$

R.8. $(-4n + 3) - (-2n - 8)$

R.9. **a.** Find the difference of 4 and -7.
b. Subtract 4 from -7.

R.10. **a.** Subtract -15 from -2.
b. Find the difference of -15 and -2.

For Exercises R.11–R.12, find the function values for $f(x) = -3x + 6$ and $g(x) = 4x - 10$.

R.11. **a.** $f(-4)$
b. $f(0)$
c. $f\left(\frac{1}{3}\right)$

R.12. **a.** $g(-6)$
b. $g(0)$
c. $g\left(\frac{3}{2}\right)$

Vocabulary and Key Concepts

1. **a.** A ________ in the variable, x, is a single term or a sum of terms of the form ax^n, where a is a real number and n is a nonnegative integer.

b. Given the term ax^n, a is called the ________, and ________ is called the degree of the term.

c. Given the term x, the coefficient of the term is ________ and the degree is ________.

d. The term in a polynomial with the highest degree is called the ________ term and its coefficient is called the ________ ________.

e. The degree of a polynomial is the ________ degree of all of its terms.

f. The degree of a nonzero constant such as 7 is ________.

g. If a term of a polynomial has more than one variable, then the degree of the term is the sum of the ________ of the variables contained in the term.

h. A ________ function is a function defined by a finite sum of terms of the form ax^n, where a is a real number and n is a whole number.

2. A monomial is a polynomial with exactly ________ term(s).

3. A ________ is a polynomial with exactly two terms.

4. A ________ is a polynomial with exactly three terms.

5. What is the degree of the term $3a^2bc^3$?

6. What is the degree of the term $-5xyz^8$?

Concept 1: Polynomials: Basic Definitions

For Exercises 7–12, write the polynomial in descending order. Then identify the leading coefficient and the degree.

7. $a^2 - 6a^3 - a$

8. $2b - b^4 + 5b^2$

9. $6x^2 - x + 3x^4 - 1$

10. $8 - 4y + y^5 - y^2$

11. $100 - t^2$

12. $-51 + s^2$

For Exercises 13–18, write a polynomial in one variable that is described by the following. (Answers may vary.)

13. A monomial of degree 5

14. A monomial of degree 4

15. A trinomial of degree 2

16. A trinomial of degree 3

17. A binomial of degree 4

18. A binomial of degree 2

Concept 2: Addition of Polynomials

For Exercises 19–30, add the polynomials and simplify. **(See Examples 1–2.)**

19. $(-4m^2 + 4m) + (5m^2 + 6m)$

20. $(3n^3 + 5n) + (2n^3 - 2n)$

21. $(3x^4 - x^3 - x^2) + (3x^3 - 7x^2 + 2x)$

22. $(6x^3 - 2x^2 - 12) + (x^2 + 3x + 9)$

23. $\left(\frac{1}{2}w^3 + \frac{2}{9}w^2 - 1.8w\right) + \left(\frac{3}{2}w^3 - \frac{1}{9}w^2 + 2.7w\right)$

24. $\left(2.9t^4 - \frac{7}{8}t + \frac{5}{3}\right) + \left(-8.1t^4 - \frac{1}{8}t - \frac{1}{3}\right)$

25. Add $(9x^2y - 5xy + 1)$ to $(8x^2y + xy - 15)$.

26. Add $(-x^3y^2 + 5xy)$ to $(10x^3y^2 + x^2y - 10)$.

27. Add $(-7a + 6a^2 + 1)$ to $(-8 - 4a - 2a^2)$.

28. Add $(1 - 12p + 8p^3)$ to $(6p^2 + p^3 - 14)$.

29. $$\begin{array}{r} 12x^3 \qquad\quad + 6x - 8 \\ \underline{+ (-3x^3 - 5x^2 - 4x \qquad)} \end{array}$$

30. $$\begin{array}{r} -8y^4 - 8y^3 - 6y^2 \qquad\quad - 9 \\ \underline{+ \ (4y^4 + 5y^3 \qquad\quad - 10y - 3)} \end{array}$$

Concept 3: Subtraction of Polynomials

For Exercises 31–36, write the opposite of the given polynomial. **(See Example 3.)**

31. $-30y^3$

32. $-2x^2$

33. $4p^3 + 2p - 12$

34. $8t^2 - 4t - 3$

35. $-11ab^2 + a^2b$

36. $-23rs - 4r + 9s$

For Exercises 37–46, subtract the polynomials and simplify. **(See Examples 4–5.)**

37. $(13z^5 - z^2) - (7z^5 + 5z^2)$

38. $(8w^4 + 3w^2) - (12w^4 - w^2)$

39. $(-3x^3 + 3x^2 - x + 6) - (1 - x - x^2 - x^3)$

40. $(-8x^3 + 6x + 7) - (-4 - 2x - 5x^3)$

41. $(-3xy^3 + 3x^2y - x + 6) - (-xy^3 - xy - x + 1)$

42. $(-8x^2y^2 + 6xy^2 + 7xy) - (5xy^2 - 2xy - 4)$

43. $$\begin{array}{r} 4t^3 - 6t^2 \qquad\quad - 18 \\ \underline{-\ (3t^3 + 7t^2 + 9t - \ \ 5)} \end{array}$$

44. $$\begin{array}{r} 5w^3 - 9w^2 + \ \ 6w + 13 \\ \underline{-\ (7w^3 \qquad\quad - 10w - \ \ 8)} \end{array}$$

45. $\left(\frac{1}{5}a^2 - \frac{1}{2}ab + \frac{1}{10}b^2 + 3\right) - \left(-\frac{3}{10}a^2 + \frac{2}{5}ab - \frac{1}{2}b^2 - 5\right)$

46. $\left(\frac{4}{7}a^2 - \frac{1}{7}ab + \frac{1}{14}b^2 - 7\right) - \left(\frac{1}{2}a^2 - \frac{2}{7}ab - \frac{9}{14}b^2 + 1\right)$

47. Subtract $(9x^2 - 5x + 1)$ from $(8x^2 + x - 15)$. **(See Example 6.)**

48. Subtract $(-x^3 + 5x)$ from $(10x^3 + x^2 - 10)$.

49. Find the difference of $(3x^5 - 2x^3 + 4)$ and $(x^4 + 2x^3 - 7)$.

50. Find the difference of $(7x^{10} - 2x^4 - 3x)$ and $(-4x^3 - 5x^4 + x + 5)$.

Mixed Exercises

For Exercises 51–74, add or subtract as indicated. Write the answers in descending order, if possible.

51. $(8y^2 - 4y^3) - (3y^2 - 8y^3)$

52. $(-9y^2 - 8) - (4y^2 + 3)$

53. $(-2r - 6r^4) + (-r^4 - 9r)$

54. $(-8s^9 + 7s^2) + (7s^9 - s^2)$

55. $(5xy + 13x^2 + 3y) - (4x^2 - 8y)$

56. $(6p^2q - 2q) - (-2p^2q + 13)$

57. $(11ab - 23b^2) + (7ab - 19b^2)$

58. $(-4x^2y + 9) + (8x^2y - 12)$

59. $[2p - (3p + 5)] + (4p - 6) + 2$

60. $-(q - 2) - [4 - (2q - 3) + 5]$

61. $5 - [2m^2 - (4m^2 + 1)]$

62. $[4n^3 - (n^3 + 4)] + 3n^3$

63. $(6x^3 - 5) - (-3x^3 + 2x) - (2x^3 - 6x)$

64. $(9p^4 - 2) + (7p^4 + 1) - (8p^4 - 10)$

65. $(-ab + 5a^2b) - [7ab^2 - 2ab - (7a^2b + 2ab^2)]$

66. $(m^3n^2 + 4m^2n) - [-5m^3n^2 - 4mn - (7m^2n - 6mn)]$

67. $(8x^3 - x^2 + 3) - [5x^2 + x - (4x^3 + x - 2)]$

68. $(y^2 + 6y - 6) - [(2y^3 - 4y) - (3y^2 + y + 1)]$

69.
$$\begin{array}{r} 12a^2b - 4ab^2 - \quad ab \\ \underline{-\ (4a^2b + \quad ab^2 - 5ab)} \end{array}$$

70.
$$\begin{array}{r} 2x^2 - \quad 7xy + 3y^2 \\ \underline{-\ (9x^2 - 10xy - \quad y^2)} \end{array}$$

71.
$$\begin{array}{r} -5x^4 \qquad\qquad - 11x^2 \qquad\qquad + 6 \\ \underline{-\ (-5x^4 + 3x^3 + \quad 5x^2 - 10x + 5)} \end{array}$$

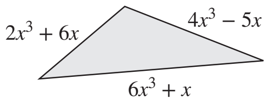
72.
$$\begin{array}{r} 9z^4 \qquad\qquad + 2z^2 \qquad\qquad + 11 \\ \underline{-\ (9z^4 - 4z^3 + 8z^2 - 9z - \quad 4)} \end{array}$$

73.
$$\begin{array}{l} \quad -2.2p^5 - 9.1p^4 \qquad\qquad + 5.3p^2 \quad - 7.9p \\ \underline{+ \qquad\qquad - 6.4p^4 - 8.5p^3 - 10.3p^2 \qquad\qquad} \end{array}$$

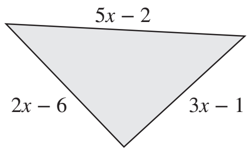
74.
$$\begin{array}{l} \quad 5.5w^4 \qquad\qquad\qquad + 4.6w^2 - 9.3w - 8.3 \\ \underline{+ 0.4w^4 - 7.3w^3 \qquad\qquad\qquad - 5.8w + 4.6} \end{array}$$

For Exercises 75–76, find the perimeter.

75.



76.



Concept 4: Polynomial Functions

For Exercises 77–84, determine whether the given function is a polynomial function. If it is a polynomial function, state the degree. If not, state the reason why.

77. $h(x) = \dfrac{2}{3}x^2 - 5$

78. $k(x) = -7x^4 - 0.3x + x^3$

79. $p(x) = 8x^3 + 2x^2 - \dfrac{3}{x}$

80. $q(x) = x^2 - 4x^{-3}$

81. $g(x) = -7$

82. $g(x) = 4x$

83. $M(x) = |x| + 5x$

84. $N(x) = x^2 + |x|$

85. Given $P(x) = -x^4 + 2x - 5$, find the function values. **(See Example 7.)**

a. $P(2)$ **b.** $P(-1)$ **c.** $P(0)$ **d.** $P(1)$

86. Given $N(x) = -x^2 + 5x$, find the function values.

a. $N(1)$ **b.** $N(-1)$ **c.** $N(2)$ **d.** $N(0)$

87. Given $H(x) = \frac{1}{2}x^3 - x + \frac{1}{4}$, find the function values.

a. $H(0)$ **b.** $H(2)$ **c.** $H(-2)$ **d.** $H(-1)$

88. Given $K(x) = \frac{2}{3}x^2 + \frac{1}{9}$, find the function values.

a. $K(0)$ **b.** $K(3)$ **c.** $K(-3)$ **d.** $K(-1)$

89. A rectangular garden is designed to be 3 ft longer than it is wide. Let x represent the width of the garden. Find a function P that represents the perimeter in terms of x. **(See Example 8.)**

Sandra Ivany/Brand X Pictures/Getty Images

90. Pauline measures a rectangular conference room and finds that the length is 4 yd greater than twice the width. Let x represent the width. Find a function P that represents the perimeter in terms of x.

91. The cost in dollars of producing x calendars is $C(x) = 5.40x + 99$. The revenue for selling x calendars is $R(x) = 12x$. To calculate profit, subtract the cost from the revenue.

a. Write and simplify a function P that represents profit in terms of x.

b. Find the profit of producing and selling 50 calendars.

92. The cost in dollars of producing x lawn chairs is $C(x) = 4.5x + 10.1$. The revenue for selling x chairs is $R(x) = 12.99x$. To calculate profit, subtract the cost from the revenue.

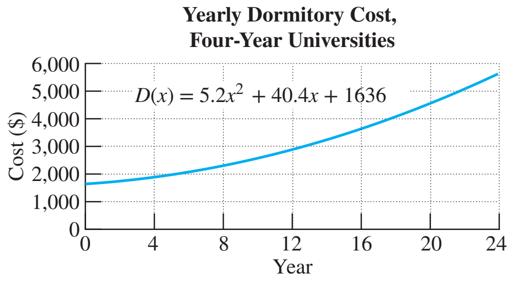
a. Write and simplify a function P that represents profit in terms of x.

b. Find the profit of producing and selling 100 lawn chairs.

93. The function defined by $D(x) = 5.2x^2 + 40.4x + 1636$ approximates the average yearly dormitory charge for 4-yr universities. $D(x)$ is the cost in dollars and x represents the number of years since the study began. **(See Example 9.)**

a. Evaluate $D(0)$ and $D(18)$, and interpret their meaning in the context of this problem.

b. If this trend continues, what will the annual dormitory charge be in year 25?

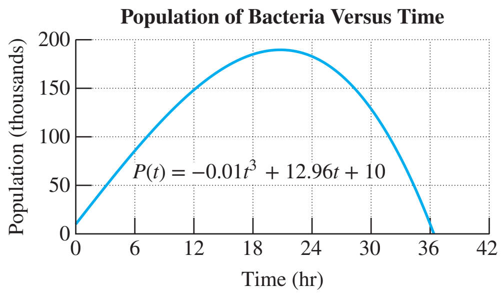


Source: U.S. National Center for Education Statistics

94. The population of bacteria in a culture can be modeled by $P(t) = -0.01t^3 + 12.96t + 10$, where t is the time in hours after the culture was started and $P(t)$ is the population in thousands.

a. Evaluate $P(0)$ and $P(14)$, and interpret their meaning in the context of this problem.

b. Predict the population of bacteria 24 hr after the culture was started.



95. The polynomial function defined by $G(x) = -0.03x^2 + 2.4x - 12$ for x between 20 and 60 mph, gives the gas mileage $G(x)$ in miles per gallon (mpg) for a compact car based on the speed of the car in miles per hour (mph). Evaluate $G(20)$, $G(40)$, and $G(50)$, and interpret their meanings in the context of this problem.

96. The total yearly amount of child support due (in billions of dollars) in the United States can be approximated by

$$D(t) = 0.925t + 4.625$$

where t is the number of years since the study began, and $D(t)$ is the amount due (in billions of dollars).

a. Evaluate $D(0)$, $D(4)$, and $D(8)$.

b. Interpret the meaning of the function value $D(8)$.

Expanding Your Skills

97. A toy rocket is shot from ground level at an angle of 60° from the horizontal. See the figure. The x- and y-positions of the rocket (measured in feet) vary with time t (in seconds) according to

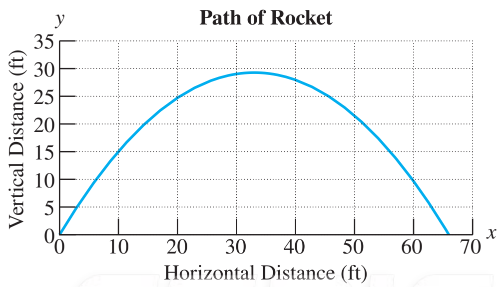
$$x(t) = 25t$$

$$y(t) = -16t^2 + 43.3t$$

a. Evaluate $x(0)$ and $y(0)$, and write the values as an ordered pair. Interpret the meaning of these function values in the context of this problem. Match the ordered pair with a point on the graph.

b. Evaluate $x(1)$ and $y(1)$, and write the values as an ordered pair. Interpret the meaning of these function values in the context of this problem. Match the ordered pair with a point on the graph.

c. Evaluate $x(2)$ and $y(2)$, and write the values as an ordered pair. Match the ordered pair with a point on the graph.



Section 4.3 Multiplication of Polynomials

Concepts

1. Multiplying Polynomials
2. Special Case Products: Difference of Squares and Perfect Square Trinomials
3. Translations Involving Polynomials
4. Applications Involving a Product of Polynomials

1. Multiplying Polynomials

The properties of exponents can be used to simplify many algebraic expressions, including the multiplication of monomials. To multiply monomials, first use the associative and commutative properties of multiplication to group coefficients and like bases. Then simplify the result by using the properties of exponents.

Example 1 Multiplying Monomials

Multiply the monomials. $(3x^2y^7)(5x^3y)$

Solution:

$(3x^2y^7)(5x^3y)$

$= (3 \cdot 5)(x^2 \cdot x^3)(y^7 \cdot y)$ Group coefficients and like bases.

$= 15x^5y^8$ Add exponents and simplify.

Skill Practice Multiply the monomials.

1. $(-8r^3s)(-4r^4s^4)$

The distributive property is used to multiply polynomials: $a(b + c) = ab + ac$.

Example 2 Multiplying a Polynomial by a Monomial

Multiply the polynomials.

a. $5y^3(2y^2 - 7y + 6)$ **b.** $-4a^3b^7c\left(2ab^2c^4 - \frac{1}{2}a^5b\right)$

Solution:

a. $5y^3(2y^2 - 7y + 6)$

$= (5y^3)(2y^2) + (5y^3)(-7y) + (5y^3)(6)$ Apply the distributive property.

$= 10y^5 - 35y^4 + 30y^3$ Simplify each term.

b. $-4a^3b^7c\left(2ab^2c^4 - \frac{1}{2}a^5b\right)$

$= (-4a^3b^7c)(2ab^2c^4) + (-4a^3b^7c)\left(-\frac{1}{2}a^5b\right)$ Apply the distributive property.

$= -8a^4b^9c^5 + 2a^8b^8c$ Simplify each term.

Skill Practice Multiply the polynomials.

2. $-6b^2(2b^2 + 3b - 8)$ **3.** $8st^3\left(\frac{1}{2}s^2 - \frac{1}{4}st\right)$

Answers

1. $32r^7s^5$
2. $-12b^4 - 18b^3 + 48b^2$
3. $4s^3t^3 - 2s^2t^4$

Thus far, we have illustrated polynomial multiplication involving monomials. Next, the distributive property will be used to multiply polynomials with more than one term. For example:

$$(x+3)(x+5) = (x+3)x + (x+3)5$$ Apply the distributive property.

$$= (x+3)x + (x+3)5$$ Apply the distributive property again.

$$= x^2 + 3x + 5x + 15$$

$$= x^2 + 8x + 15$$ Combine *like* terms.

Note: Using the distributive property results in multiplying each term of the first polynomial by each term of the second polynomial:

$$(x+3)(x+5) = x^2 + 5x + 3x + 15$$

$$= x^2 + 8x + 15$$

Example 3 **Multiplying Polynomials**

Multiply the polynomials. $(2x^2+4)(3x^2-x+5)$

Solution:

$$(2x^2+4)(3x^2-x+5)$$ Multiply each term in the first polynomial by each term in the second.

$$= (2x^2)(3x^2) + (2x^2)(-x) + (2x^2)(5) + (4)(3x^2) + (4)(-x) + (4)(5)$$ Apply the distributive property.

$$= 6x^4 - 2x^3 + 10x^2 + 12x^2 - 4x + 20$$ Simplify each term.

$$= 6x^4 - 2x^3 + 22x^2 - 4x + 20$$ Combine *like* terms.

TIP: Multiplication of polynomials can be performed vertically by a process similar to column multiplication of real numbers.

$$(2x^2+4)(3x^2-x+5) \longrightarrow \begin{array}{rrrrr} & & 3x^2 & -\;x & +\;5 \\ & \times & 2x^2 & & +\;4 \\ \hline & & 12x^2 & -\;4x & +\;20 \\ 6x^4 & -\;2x^3 & +\;10x^2 & & \\ \hline 6x^4 & -\;2x^3 & +\;22x^2 & -\;4x & +\;20 \end{array}$$

Note: When multiplying by the column method, it is important to align *like* terms vertically before adding terms.

Skill Practice Multiply the polynomials.

4. $(2y-1)(3y^2-2y-1)$

Answer

4. $6y^3 - 7y^2 + 1$

Example 4 **Multiplying Polynomials**

Multiply the polynomials. $(3y + 2)(7y - 6)$

Solution:

$(3y + 2)(7y - 6)$	Multiply each term in the first polynomial by each term in the second.
$= (3y)(7y) + (3y)(-6) + (2)(7y) + (2)(-6)$	Apply the distributive property.
$= 21y^2 - 18y + 14y - 12$	Simplify each term.
$= 21y^2 - 4y - 12$	Combine *like* terms.

TIP: The acronym FOIL (first outer inner last) can be used as a memory device to multiply the two binomials.

Outer terms First Outer Inner Last

First terms

$(3y + 2)(7y - 6) = (3y)(7y) + (3y)(-6) + (2)(7y) + (2)(-6)$

Inner terms $= 21y^2 - 18y + 14y - 12$

Last terms $= 21y^2 - 4y - 12$

Note: It is important to realize that the acronym FOIL may only be used when finding the product of two *binomials.*

Skill Practice Multiply the polynomials.

5. $(4t + 5)(2t + 3)$

2. Special Case Products: Difference of Squares and Perfect Square Trinomials

In some cases, the product of two binomials takes on a special pattern.

I. The first special case occurs when multiplying the sum and difference of the same two terms. For example:

$$\left.\begin{aligned}&(2x + 3)(2x - 3)\\ &\quad = 4x^2 - 6x + 6x - 9\\ &\quad = 4x^2 - 9\end{aligned}\right\}$$

Notice that the "middle terms" are opposites. This leaves only the difference between the square of the first term and the square of the second term. For this reason, the product is called a *difference of squares*.

Note: The binomials $2x + 3$ and $2x - 3$ are called **conjugates**. In one expression, $2x$ and 3 are added, and in the other, $2x$ and 3 are subtracted. In general, $a + b$ and $a - b$ are conjugates of each other.

Answer

5. $8t^2 + 22t + 15$

II. The second special case involves the square of a binomial. For example:

$$\left.\begin{aligned} &(3x+7)^2 \\ &\quad = (3x+7)(3x+7) \\ &\quad = 9x^2 + 21x + 21x + 49 \\ &\quad = 9x^2 \ + \ 42x \ + \ 49 \\ &\quad = (3x)^2 + 2(3x)(7) + (7)^2 \end{aligned}\right\}$$

When squaring a binomial, the product will be a trinomial called a *perfect square trinomial*. The first and third terms are formed by squaring the terms of the binomial. The middle term is twice the product of the terms in the binomial.

Note: The expression $(3x - 7)^2$ also results in a perfect square trinomial, but the middle term is negative.

$$(3x-7)(3x-7) = 9x^2 - 21x - 21x + 49 = 9x^2 - 42x + 49$$

The special case products are summarized as follows.

Special Case Product Formulas

1. $(a+b)(a-b) = a^2 - b^2$ The product is called a **difference of squares**.

2. $\left.\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ (a-b)^2 &= a^2 - 2ab + b^2\end{aligned}\right\}$ The product is called a **perfect square trinomial**.

It is advantageous for you to become familiar with these special case products because they will be presented again when we factor polynomials.

Example 5 Using Special Products

Use the special product formulas to multiply the polynomials.

a. $(6c - 7d)(6c + 7d)$ **b.** $(5x - 2)^2$ **c.** $(4x^3 + 3y^2)^2$

Solution:

a. $(6c - 7d)(6c + 7d)$ $a = 6c,\ b = 7d$

$= (6c)^2 - (7d)^2$ Apply the formula $a^2 - b^2$.

$= 36c^2 - 49d^2$ Simplify each term.

b. $(5x - 2)^2$ $a = 5x,\ b = 2$

$= (5x)^2 - 2(5x)(2) + (2)^2$ Apply the formula $a^2 - 2ab + b^2$.

$= 25x^2 - 20x + 4$ Simplify each term.

c. $(4x^3 + 3y^2)^2$ $a = 4x^3,\ b = 3y^2$

$= (4x^3)^2 + 2(4x^3)(3y^2) + (3y^2)^2$ Apply the formula $a^2 + 2ab + b^2$.

$= 16x^6 + 24x^3y^2 + 9y^4$ Simplify each term.

Skill Practice Multiply the polynomials.

6. $(5x - 4y)(5x + 4y)$ **7.** $(c - 3)^2$ **8.** $(7s^2 + 2t)^2$

Answers

6. $25x^2 - 16y^2$
7. $c^2 - 6c + 9$
8. $49s^4 + 28s^2t + 4t^2$

The special case products can be used to simplify more complicated algebraic expressions.

Example 6 Using Special Products

Multiply. $[x + (y + z)][x - (y + z)]$

Solution:

$[x + (y + z)][x - (y + z)]$	This product is in the form $(a + b)(a - b)$, where $a = x$ and $b = (y + z)$.
$= (x)^2 - (y + z)^2$	Apply the formula $a^2 - b^2$.
$= (x)^2 - (y^2 + 2yz + z^2)$	Expand $(y + z)^2$ by using the special case product formula.
$= x^2 - y^2 - 2yz - z^2$	Apply the distributive property.

Skill Practice Multiply.

9. $[a + (c + 5)][a - (c + 5)]$

Example 7 Using Special Products

Multiply. $(x + y)^3$

Solution:

$(x + y)^3$	
$= (x + y)^2(x + y)$	Rewrite as the square of a binomial and another factor.
$= (x^2 + 2xy + y^2)(x + y)$	Expand $(x + y)^2$ by using the special case product formula.
$= (x^2)(x) + (x^2)(y) + (2xy)(x) + (2xy)(y) + (y^2)(x) + (y^2)(y)$	Apply the distributive property.
$= x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3$	Simplify each term.
$= x^3 + 3x^2y + 3xy^2 + y^3$	Combine *like* terms.

Skill Practice Multiply.

10. $(t + 2)^3$

3. Translations Involving Polynomials

Example 8 Translating Between English Form and Algebraic Form

Complete the table.

English Form	Algebraic Form
The square of the sum of x and y	
	$x^2 + y^2$
The square of the product of 3 and x	

Answers

9. $a^2 - c^2 - 10c - 25$
10. $t^3 + 6t^2 + 12t + 8$

Solution:

English Form	Algebraic Form	Notes
The square of the sum of x and y	$(x + y)^2$	The *sum* is squared, not the individual terms.
The sum of the squares of x and y	$x^2 + y^2$	The individual terms x and y are squared first. Then the sum is taken.
The square of the product of 3 and x	$(3x)^2$	The product of 3 and x is taken. Then the result is squared.

Skill Practice Translate to algebraic form:

11. The square of the difference of a and b

12. The difference of the square of a and the square of b

13. Translate to English form: $a - b^2$

4. Applications Involving a Product of Polynomials

Example 9 Applying a Product of Polynomials

A box is created from a sheet of cardboard 20 in. on a side by cutting a square from each corner and folding up the sides (Figures 4-3 and 4-4). Let x represent the length of the sides of the squares removed from each corner.

a. Write a function V that represents the volume of the box in terms of x.

b. Find the volume if a 4-in. square is removed.

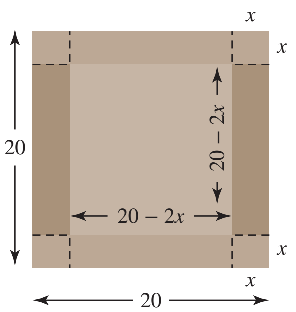


Figure 4-3

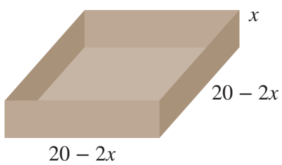


Figure 4-4

Solution:

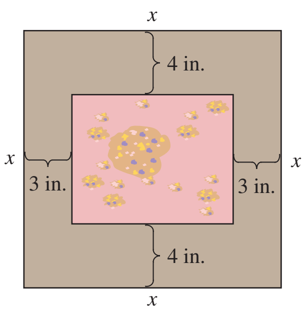
a. The volume of a rectangular box is given by the formula $V = lwh$. The length and width can both be expressed as $20 - 2x$. The height of the box is x. The volume is given by

$$\begin{aligned} V &= l \cdot w \cdot h \\ V(x) &= (20 - 2x)(20 - 2x)x \\ &= (20 - 2x)^2 x \\ &= (400 - 80x + 4x^2)x \\ &= 400x - 80x^2 + 4x^3 \\ &= 4x^3 - 80x^2 + 400x \end{aligned}$$

Answers

11. $(a - b)^2$ **12.** $a^2 - b^2$

13. The difference of a and the square of b



Answers

14. $A(x) = (x - 8)(x - 6)$;
$A(x) = x^2 - 14x + 48$

15. 24 in.^{2}

b. If a 4-in. square is removed from the corners of the box, we have $x = 4$. Thus, the volume is

$$\begin{aligned} V(x) &= 4x^3 - 80x^2 + 400x \\ V(4) &= 4(4)^3 - 80(4)^2 + 400(4) \\ &= 4(64) - 80(16) + 400(4) \\ &= 256 - 1280 + 1600 \\ &= 576 \end{aligned}$$

The volume is 576 in.^{3}

Skill Practice A rectangular photograph is mounted on a square piece of cardboard whose sides have length x. The border that surrounds the photo is 3 in. on each side and 4 in. on both top and bottom.

14. Write a function A for the area of the photograph in terms of x.

15. Determine the area of the photograph if x is 12 in.

Section 4.3 Activity

A.1. Multiply the monomials by applying the properties of exponents.

$$\left(\frac{2}{7}a^3b\right)(14ab^4)$$

A.2. To multiply polynomials, multiply each term in the first polynomial by each term in the second polynomial.

a. Given the product $(2w - 3)(w^2 - 5w + 8)$, how many terms are in the first polynomial? How many terms are in the second polynomial? How many terms should be in the product before combining *like* terms?

b. Multiply the polynomials. $(2w - 3)(w^2 - 5w + 8)$,

A.3. **a.** The polynomials $(5c - 2)$ and $(5c + 2)$ are called _______________ of each other.

b. Multiply the polynomials by multiplying each term in the first binomial by each term in the second binomial.

$$(5c - 2)(5c + 2)$$

c. Multiply the polynomials by applying the formula $(a - b)(a + b) = a^2 - b^2$.

$$(5c - 2)(5c + 2) = \underline{\hspace{4cm}}$$

A.4. **a.** Square the binomial by expanding the product of binomials and multiplying each term in the first binomial by each term in the second.

$$(3x - 7)^2 = (3x - 7)(3x - 7)$$

b. Square the binomial by applying the formula $(a - b)^2 = a^2 - 2ab + b^2$.

$$(3x - 7)^2 = \underline{\hspace{8cm}}$$

A.5. The expression $(x + 2y)^3$ can be written as $(x + 2y)^3 = (x + 2y)(x + 2y)(x + 2y)$. Use the order of operations to find the product. That is, multiply the polynomials from left to right.

Practice Exercises

Section 4.3

Prerequisite Review

For Exercises R.1–R.14, simplify the expression. Assume that all variables represent nonzero real numbers.

R.1. $3y + 6y$

R.2. $6x^2 - 2x^2$

R.3. $(3y)(6y)$

R.4. $(6x^2)(-2x^2)$

R.5. $-7a^4b + 3a^4b$

R.6. $6kn^5 - 11kn^5$

R.7. $(-7a^4b)(3a^4b)$

R.8. $(6kn^5)(-11kn^5)$

R.9. $5k^2 + 4k - 6k + 8$

R.10. $10m^2 - 5m + 8m - 11$

R.11. $-5(3a - 6b - 7c)$

R.12. $-3(-x + 3y - 5z)$

R.13. $\frac{3}{2}(6x + 8y)$

R.14. $\frac{4}{5}(10c - 25d)$

Vocabulary and Key Concepts

1. a. To multiply $2(4x - 5)$, apply the __________ property.

b. The conjugate of $4x + 7$ is __________.

2. a. When two conjugates are multiplied the resulting binomial is a difference of __________. This is given by the formula $(a + b)(a - b) =$ __________.

b. When a binomial is squared, the resulting trinomial is a __________ square trinomial. This is given by the formula $(a + b)^2 =$ __________.

Concept 1: Multiplying Polynomials

For Exercises 3–40, multiply the polynomials. **(See Examples 1–4.)**

3. $7(2x^2)$

4. $-5(3t^4)$

5. $\frac{2}{3}(-6xy^2)$

6. $-\frac{3}{5}(-10c^3d)$

7. $(7x^4y)(-6xy^5)$

8. $(-4a^3b^7)(-2ab^3)$

9. $(2.2a^6b^4c^7)(5ab^4c^3)$

10. $(8.5c^4d^5e)(6cd^2e)$

11. $\frac{1}{5}(2a - 3)$

12. $\frac{1}{3}(6b + 4)$

13. $2m^3n^2(m^2n^3 - 3mn^2 + 4n)$

14. $3p^2q(p^3q^3 - pq^2 - 4p)$

15. $6xy^2\left(\frac{1}{2}x - \frac{2}{3}xy\right)$

16. $12ab\left(\frac{5}{6}a + \frac{1}{4}ab^2\right)$

17. $(x + y)(x - 2y)$

18. $(3a + 5)(a - 2)$

19. $(6x - 1)(5 + 2x)$

20. $(7 + 3x)(x - 8)$

21. $(y^2 - 12)(2y^2 + 3)$

22. $(4p^2 - 1)(2p^2 + 5)$

23. $(5s + 3t)(5s - 2t)$

24. $(4a + 3b)(4a - b)$

25. $(n^2 + 10)(5n + 3)$

26. $(m^2 + 8)(3m + 7)$

27. $(1.3a - 4b)(2.5a + 7b)$

28. $(2.1x - 3.5y)(4.7x + 2y)$

29. $(2x + y)(3x^2 + 2xy + y^2)$

30. $(h - 5k)(h^2 - 2hk + 3k^2)$

31. $(x - 7)(x^2 + 7x + 49)$

32. $(x + 3)(x^2 - 3x + 9)$

33. $(4a - b)(a^3 - 4a^2b + ab^2 - b^3)$

34. $(3m + 2n)(m^3 + 2m^2n - mn^2 + 2n^3)$

35. $\left(\frac{1}{2}a - 2b + c\right)(a + 6b - c)$

36. $\left(\frac{1}{2}a^2 - 2ab + b^2\right)(2a + b)$

37. $(-x^2 + 2x + 1)(3x - 5)$

38. $(x + y - 2z)(5x - y + z)$

39. $\left(\frac{1}{5}y - 10\right)\left(\frac{1}{2}y - 15\right)$

40. $\left(\frac{2}{3}x + 6\right)\left(\frac{1}{2}x - 9\right)$

Concept 2: Special Case Products: Difference of Squares and Perfect Square Trinomials

For Exercises 41–60, multiply by using the special case products. **(See Example 5.)**

41. $(a - 8)(a + 8)$

42. $(b + 2)(b - 2)$

43. $(3p + 1)(3p - 1)$

44. $(5q - 3)(5q + 3)$

45. $\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$

46. $\left(\frac{1}{2}x + \frac{1}{3}\right)\left(\frac{1}{2}x - \frac{1}{3}\right)$

47. $(3h - k)(3h + k)$

48. $(x - 7y)(x + 7y)$

49. $(3h - k)^2$

50. $(x - 7y)^2$

51. $(t - 7)^2$

52. $(w + 9)^2$

53. $(u + 3v)^2$

54. $(a - 4b)^2$

55. $\left(h + \frac{1}{6}k\right)^2$

56. $\left(\frac{2}{5}x + 1\right)^2$

57. $(2z^2 - w^3)(2z^2 + w^3)$

58. $(a^4 - 2b^3)(a^4 + 2b^3)$

59. $(5x^2 - 3y)^2$

60. $(4p^3 - 2m)^2$

For Exercises 61–62, the product of two binomials is shown. Determine if the binomials are conjugates.

61. **a.** $(-5x + 4)(5x + 4)$

b. $(-5x + 4)(5x - 4)$

62. **a.** $(-3 - 7x)(3 + 7x)$

b. $(-3 + 7x)(3 + 7x)$

63. Multiply the expressions. Explain their similarities.

a. $(A - B)(A + B)$

b. $[(x + y) - B][(x + y) + B]$

64. Multiply the expressions. Explain their similarities.

a. $(A + B)(A - B)$

b. $[A + (3h + k)][A - (3h + k)]$

For Exercises 65–70, multiply the expressions. **(See Example 6.)**

65. $[(w + v) - 2][(w + v) + 2]$

66. $[(x + y) - 6][(x + y) + 6]$

67. $[2 - (x + y)][2 + (x + y)]$

68. $[a - (b + 1)][a + (b + 1)]$

69. $[(3a - 4) + b][(3a - 4) - b]$

70. $[(5p - 7) - q][(5p - 7) + q]$

71. Explain how to multiply $(x + y)^3$.

72. Explain how to multiply $(a - b)^3$.

For Exercises 73–76, multiply the expressions. **(See Example 7.)**

73. $(2x + y)^3$

74. $(x - 5y)^3$

75. $(4a - b)^3$

76. $(3a + 4b)^3$

77. Explain how you would multiply the binomials.

$$(x - 2)(x + 6)(2x + 1)$$

78. Explain how you would multiply the binomials.

$$(a + b)(a - b)(2a + b)(2a - b)$$

For Exercises 79–86, simplify the expressions.

79. $2a^2(a + 5)(3a + 1)$

80. $-5y(2y - 3)(y + 3)$

81. $(x + 3)(x - 3)(x + 5)$

82. $(t + 2)(t - 3)(t + 1)$

83. $-3(2x + 7) - (4x - 1)^2$

84. $(p + 10)^2 - 4(p + 6)^2$

85. $(y + 1)^2 - (2y + 3)^2$

86. $(b - 3)^2 - (3b - 1)^2$

Concept 3: Translations Involving Polynomials

For Exercises 87–90, translate from English form to algebraic form. **(See Example 8.)**

87. The square of the sum of r and t

88. The square of a plus the cube of b

89. The difference of x squared and y cubed

90. The square of the product of 3 and a

For Exercises 91–94, translate from algebraic form to English form. **(See Example 8.)**

91. $p^3 + q^2$

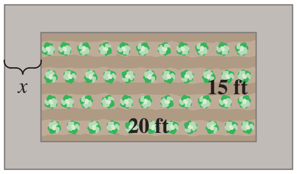
92. $a^3 - b^3$

93. xy^2

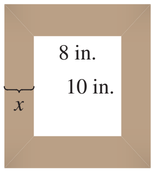
94. $(c + d)^3$

Concept 4: Applications Involving a Product of Polynomials

95. A rectangular garden has a sidewalk around it of width x. The garden is 20 ft by 15 ft. Write a function representing the combined area $A(x)$ of the garden and sidewalk. Simplify the result.



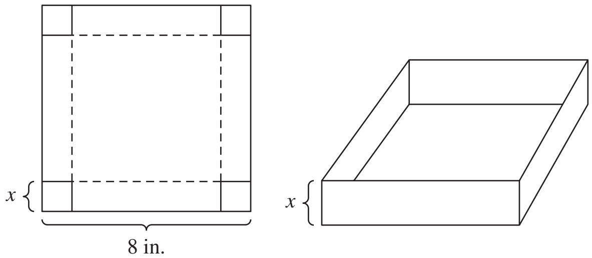
96. An 8-in. by 10-in. photograph is in a frame of width x. Write a function that represents the area $A(x)$ of the frame alone. Simplify the result.



97. A box is created from a square piece of cardboard 8 in. on a side by cutting a square from each corner and folding up the sides. Let x represent the length of the sides of the squares removed from each corner. **(See Example 9.)**

a. Write a function representing the volume of the box.

b. Find the volume if 1-in. squares are removed from the corners.



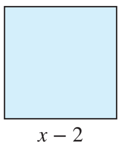
98. A box is created from a rectangular piece of metal with dimensions 12 in. by 9 in. by removing a square from each corner of the metal sheet and folding up the sides. Let x represent the length of the sides of the squares removed from each corner.

a. Write a function representing the volume of the box.

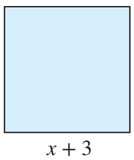
b. Find the volume if 2-in. squares are removed from the corners.

For Exercises 99–104, write an expression for the area and simplify your answer.

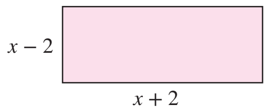
99. Square



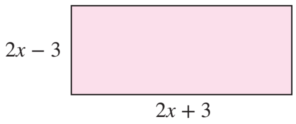
100. Square



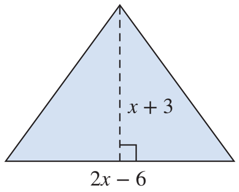
101. Rectangle



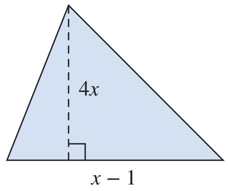
102. Rectangle



103. Triangle

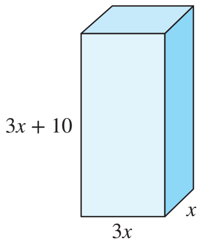


104. Triangle

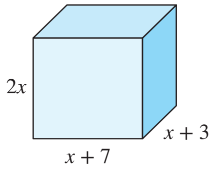


For Exercises 105–106, write an expression for the volume and simplify your answer.

105.



106.



Expanding Your Skills

For Exercises 107–108, simplify completely.

107. $\dfrac{[(x+h)^2 - 3(x+h) - 5] - (x^2 - 3x - 5)}{h}$

108. $\dfrac{[(x+h)^2 - 4(x+h) + 2] - (x^2 - 4x + 2)}{h}$

109. Explain how to multiply $(x + 2)^4$.

110. Explain how to multiply $(y - 3)^4$.

111. $(2x - 3)$ multiplied by what binomial will result in the trinomial $10x^2 - 27x + 18$? Check your answer by multiplying the binomials.

112. $(4x + 1)$ multiplied by what binomial will result in the trinomial $12x^2 - 5x - 2$? Check your answer by multiplying the binomials.

113. $(4y + 3)$ multiplied by what binomial will result in the trinomial $8y^2 + 2y - 3$? Check your answer by multiplying the binomials.

114. $(3y - 2)$ multiplied by what binomial will result in the trinomial $3y^2 - 17y + 10$? Check your answer by multiplying the binomials.

Section 4.4 Division of Polynomials

Concepts

1. Division by a Monomial
2. Long Division
3. Synthetic Division

Division of polynomials will be presented in this section as two separate cases: The first case illustrates division by a monomial divisor. The second case illustrates long division by a polynomial with two or more terms.

1. Division by a Monomial

To divide a polynomial by a monomial, divide each individual term in the polynomial by the divisor and simplify the result.

Dividing a Polynomial by a Monomial

If a, b, and c are polynomials such that $c \neq 0$, then

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \qquad \text{Similarly,} \qquad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

Example 1 Dividing a Polynomial by a Monomial

Divide the polynomials.

a. $\dfrac{5a^3 - 10a^2 + 20}{5a}$ **b.** $(12y^2z^3 - 15yz^2 + 6y^2z) \div (-6y^2z)$

Solution:

a. $\dfrac{5a^3 - 10a^2 + 20}{5a}$

$= \dfrac{5a^3}{5a} - \dfrac{10a^2}{5a} + \dfrac{20}{5a}$ Divide each term in the numerator by $5a$.

$= a^2 - 2a + \dfrac{4}{a}$ Simplify each term using the properties of exponents.

b. $(12y^2z^3 - 15yz^2 + 6y^2z) \div (-6y^2z)$

$= \dfrac{12y^2z^3 - 15yz^2 + 6y^2z}{-6y^2z}$

$= \dfrac{12y^2z^3}{-6y^2z} - \dfrac{15yz^2}{-6y^2z} + \dfrac{6y^2z}{-6y^2z}$ Divide each term by $-6y^2z$.

$= -2z^2 + \dfrac{5z}{2y} - 1$ Simplify each term.

Skill Practice Divide the polynomials.

1. $(36a^4 - 48a^3 + 12a^2) \div (6a^3)$ **2.** $\dfrac{-15x^3y^4 + 25x^2y^3 - 5xy^2}{-5xy^2}$

2. Long Division

If the divisor has two or more terms, a *long division* process similar to the division of real numbers is used. Take a minute to review the long division process for real numbers by dividing 2273 by 5.

$$\begin{array}{rl}
454 & \leftarrow \text{Quotient} \\
5\overline{)2273} & \\
\underline{-20}\downarrow & \\
27 & \\
\underline{-25}\downarrow & \\
23 & \\
\underline{-20} & \\
3 & \leftarrow \text{Remainder}
\end{array}$$

Therefore, $2273 \div 5 = 454\frac{3}{5}$.

A similar procedure is used for long division of polynomials as shown in Example 2.

FOR REVIEW

Recall the terminology associated with division. For the given division statement we have:

dividend quotient remainder

$$2273 \div 5 = 454\frac{3}{5}$$

divisor divisor

Answers

1. $6a - 8 + \dfrac{2}{a}$

2. $3x^2y^2 - 5xy + 1$

Example 2 Using Long Division to Divide Polynomials

Divide the polynomials using long division. $(2x^2 - x + 3) \div (x - 3)$

Solution:

$$x - 3\overline{)2x^2 - x + 3}$$

Divide the leading term in the dividend by the leading term in the divisor.

$$\frac{2x^2}{x} = 2x$$

This is the first term in the quotient.

$$\begin{array}{r} 2x \phantom{{}-x+3} \\ x - 3\overline{)2x^2 - x + 3} \\ -\underline{(2x^2 - 6x)} \phantom{{}+3} \end{array}$$

Multiply $2x$ by the divisor: $2x(x - 3) = 2x^2 - 6x$. Then subtract the result.

$$\begin{array}{r} 2x \phantom{{}-x+3} \\ x - 3\overline{)2x^2 - x + 3} \\ \underline{-2x^2 + 6x} \phantom{{}+3} \\ 5x \phantom{{}+3} \end{array}$$

Subtract the quantity $2x^2 - 6x$. To do this, add the opposite.

$$\begin{array}{r} 2x + 5 \\ x - 3\overline{)2x^2 - x + 3} \\ \underline{-2x^2 + 6x} \phantom{{}+3} \\ 5x + 3 \end{array}$$

Bring down the next column, and repeat the process.

Divide the leading term by x: $(5x)/x = 5$

Place 5 in the quotient.

$$\begin{array}{r} 2x + 5 \\ x - 3\overline{)2x^2 - x + 3} \\ \underline{-2x^2 + 6x} \phantom{{}+3)} \\ 5x + 3 \\ -\underline{(5x - 15)} \end{array}$$

Multiply the divisor by 5: $5(x - 3) = 5x - 15$

Subtract the result.

$$\begin{array}{r} 2x + 5 \\ x - 3\overline{)2x^2 - x + 3} \\ \underline{-2x^2 + 6x} \phantom{{}+3} \\ 5x + 3 \\ \underline{-5x + 15} \\ 18 \end{array}$$

Subtract the quantity $5x - 15$ by adding the opposite.

The remainder is 18.

Summary:

The quotient is	$2x + 5$
The remainder is	18
The divisor is	$x - 3$
The dividend is	$2x^2 - x + 3$

The solution to a long division problem is usually written in the form:

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Hence,

$$(2x^2 - x + 3) \div (x - 3) = 2x + 5 + \frac{18}{x - 3}$$

Skill Practice Divide the polynomials using long division.

3. $(3x^2 + 2x - 5) \div (x + 2)$

TIP: Recall that taking the opposite of a polynomial changes the sign of each term of the polynomial.

Answer

3. $3x - 4 + \dfrac{3}{x + 2}$

The division of polynomials can be checked in the same fashion as the division of real numbers. To check Example 2, we use the **division algorithm**:

$$\text{Dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

$$\begin{aligned} 2x^2 - x + 3 &\stackrel{?}{=} (x-3)(2x+5) + (18) \\ &\stackrel{?}{=} 2x^2 + 5x - 6x - 15 + (18) \\ &= 2x^2 - x + 3 \ ✓ \end{aligned}$$

Example 3 Using Long Division to Divide Polynomials

Divide the polynomials using long division: $(3w^3 + 26w^2 - 3) \div (3w - 1)$

Solution:

First note that the dividend has a missing power of w and can be written as $3w^3 + 26w^2 + 0w - 3$. The term $0w$ is a placeholder for the missing term. It is helpful to use the placeholder to keep the powers of w lined up.

$$\begin{array}{r} w^2\phantom{{}+0w-3} \\ 3w-1\overline{)3w^3+26w^2+0w-3} \\ -(3w^3-w^2)\phantom{{}+0w-3} \end{array}$$

Divide $3w^3 \div 3w = w^2$. This is the first term of the quotient.
Then multiply $w^2(3w-1) = 3w^3 - w^2$.

$$\begin{array}{r} w^2\phantom{{}+0w-3} \\ 3w-1\overline{)3w^3+26w^2+0w-3} \\ \underline{-3w^3+w^2}\phantom{{}+0w-3} \\ 27w^2+0w\phantom{{}-3} \end{array}$$

Subtract by adding the opposite.

Bring down the next column, and repeat the process.

$$\begin{array}{r} w^2+9w\phantom{{}-3} \\ 3w-1\overline{)3w^3+26w^2+0w-3} \\ \underline{-3w^3+w^2}\phantom{{}+0w-3} \\ 27w^2+0w\phantom{{}-3} \\ -(27w^2-9w)\phantom{{}-3} \end{array}$$

Divide $27w^2$ by the leading term in the divisor. $27w^2 \div 3w = 9w$.
Place $9w$ in the quotient.
Multiply $9w(3w-1) = 27w^2 - 9w$.

$$\begin{array}{r} w^2+9w\phantom{{}-3} \\ 3w-1\overline{)3w^3+26w^2+0w-3} \\ \underline{-3w^3+w^2}\phantom{{}+0w-3} \\ 27w^2+0w\phantom{{}-3} \\ \underline{-27w^2+9w}\phantom{{}-3} \\ 9w-3 \end{array}$$

Subtract by adding the opposite.

Bring down the next column, and repeat the process.

$$\begin{array}{r} w^2+9w+3 \\ 3w-1\overline{)3w^3+26w^2+0w-3} \\ \underline{-3w^3+w^2}\phantom{{}+0w-3} \\ 27w^2+0w\phantom{{}-3} \\ \underline{-27w^2+9w}\phantom{{}-3} \\ 9w-3 \\ -(9w-3) \end{array}$$

Divide $9w$ by the leading term in the divisor. $9w \div 3w = 3$.
Place 3 in the quotient.
Multiply $3(3w-1) = 9w - 3$.

$$
\begin{array}{r}
w^2 + 9w + 3 \\
3w - 1 \overline{\big)\, 3w^3 + 26w^2 + 0w - 3} \\
\underline{-3w^3 + \quad w^2} \\
27w^2 + 0w \\
\underline{-27w^2 + 9w} \\
9w - 3 \\
\underline{-9w + 3} \\
0
\end{array}
$$

Subtract by adding the opposite.

The remainder is 0.

The quotient is $w^2 + 9w + 3$, and the remainder is 0.

Skill Practice Divide the polynomials using long division.

4. $\dfrac{9x^3 + 11x + 10}{3x + 2}$

In Example 3, the remainder is zero. Therefore, we say that $3w - 1$ divides evenly into $3w^3 + 26w^2 - 3$. For this reason, the divisor and quotient are factors of $3w^3 + 26w^2 - 3$. To check, we have

$$
\begin{aligned}
\text{Dividend} &= (\text{divisor})(\text{quotient}) + \text{remainder} \\
3w^3 + 26w^2 - 3 &\stackrel{?}{=} (3w - 1)(w^2 + 9w + 3) + 0 \\
&\stackrel{?}{=} 3w^3 + 27w^2 + 9w - w^2 - 9w - 3 \\
&= 3w^3 + 26w^2 - 3 \checkmark
\end{aligned}
$$

Example 4 Using Long Division to Divide Polynomials

Divide the polynomials using long division.

$$\frac{2y + y^4 - 5}{1 + y^2}$$

Solution:

First note that both the dividend and divisor should be written in descending order:

$$\frac{y^4 + 2y - 5}{y^2 + 1}$$

Also note that the dividend and the divisor have missing powers of y. Leave placeholders.

$$y^2 + 0y + 1 \overline{\big)\, y^4 + 0y^3 + 0y^2 + 2y - 5}$$

$$
\begin{array}{r}
y^2 \\
y^2 + 0y + 1 \overline{\big)\, y^4 + 0y^3 + 0y^2 + 2y - 5} \\
\underline{-(y^4 + 0y^3 + y^2)}
\end{array}
$$

Divide $y^4 \div y^2 = y^2$. This is the first term of the quotient.

Multiply $y^2(y^2 + 0y + 1) = y^4 + 0y^3 + y^2$.

$$
\begin{array}{r}
y^2 \\
y^2 + 0y + 1 \overline{\big)\, y^4 + 0y^3 + 0y^2 + 2y - 5} \\
\underline{-y^4 - 0y^3 - y^2} \\
-y^2 + 2y - 5
\end{array}
$$

Subtract by adding the opposite.

Bring down the next columns.

Answer

4. $3x^2 - 2x + 5$

$$
\begin{array}{r}
y^2 \qquad -1 \\
y^2 + 0y + 1\overline{)y^4 + 0y^3 + 0y^2 + 2y - 5} \\
\underline{-y^4 - 0y^3 - y^2} \qquad\qquad \\
-y^2 + 2y - 5 \\
\underline{-(-y^2 - 0y - 1)}
\end{array}
$$

Divide $-y^2 \div y^2 = -1$.

Multiply $-1(y^2 + 0y + 1) =$
$-y^2 - 0y - 1$.

$$
\begin{array}{r}
y^2 \qquad -1 \\
y^2 + 0y + 1\overline{)y^4 + 0y^3 + 0y^2 + 2y - 5} \\
\underline{-y^4 - 0y^3 - \ y^2} \qquad\qquad \\
-y^2 + 2y - 5 \\
\underline{y^2 + 0y + 1} \\
2y - 4
\end{array}
$$

Subtract by adding the opposite.

Remainder

Therefore, $\dfrac{y^4 + 2y - 5}{y^2 + 1} = y^2 - 1 + \dfrac{2y - 4}{y^2 + 1}$.

Skill Practice Divide the polynomials using long division.

5. $(4 - x^2 + x^3) \div (2 + x^2)$

Example 5 Determining Whether Long Division Is Necessary

Determine whether long division is necessary for each division of polynomials.

a. $\dfrac{2p^5 - 8p^4 + 4p - 16}{p^2 - 2p + 1}$ **b.** $\dfrac{2p^5 - 8p^4 + 4p - 16}{2p^2}$

c. $(3z^3 - 5z^2 + 10) \div (15z^3)$ **d.** $(3z^3 - 5z^2 + 10) \div (3z + 1)$

Solution:

- Long division is used when the divisor has *two or more terms*.
- If the divisor has *one term*, then divide each term in the dividend by the monomial divisor.

a. $\dfrac{2p^5 - 8p^4 + 4p - 16}{p^2 - 2p + 1}$ The divisor has three terms. Use long division.

b. $\dfrac{2p^5 - 8p^4 + 4p - 16}{2p^2}$ The divisor has one term. Long division is not necessary.

c. $(3z^3 - 5z^2 + 10) \div (15z^3)$ The divisor has one term. Long division is not necessary.

d. $(3z^3 - 5z^2 + 10) \div (3z + 1)$ The divisor has two terms. Use long division.

Skill Practice Divide the polynomials using the appropriate method of division.

6. $\dfrac{6x^3 - x^2 + 3x - 5}{2x + 3}$ **7.** $\dfrac{9w^3 - 18w^2 + 6w + 12}{3w}$

Answers

5. $x - 1 + \dfrac{-2x + 6}{x^2 + 2}$

6. $3x^2 - 5x + 9 + \dfrac{-32}{2x + 3}$

7. $3w^2 - 6w + 2 + \dfrac{4}{w}$

3. Synthetic Division

In this section, we introduced the process of long division to divide two polynomials. Next, we will learn another technique called **synthetic division** to divide two polynomials. Synthetic division can be used when dividing a polynomial by a first-degree divisor of the form $x - r$, where r is a constant. Synthetic division is considered a "shortcut" because it uses the coefficients of the divisor and dividend without writing the variables.

Consider dividing the polynomials $(3x^2 - 14x - 10) \div (x - 2)$.

$$
\begin{array}{r}
3x - 8 \\
x - 2\overline{)3x^2 - 14x - 10} \\
-(\underline{3x^2 - 6x)} \\
-8x - 10 \\
-(\underline{-8x + 16)} \\
-26
\end{array}
$$

First note that the divisor $x - 2$ is in the form $x - r$, where $r = 2$. Therefore, synthetic division can be used to find the quotient and remainder.

Step 1: Write the value of r in a box.

Step 2: Write the coefficients of the dividend to the right of the box.

$$
\begin{array}{c|ccc}
2 & 3 & -14 & -10 \\
 & & & \\
\hline
 & 3 & &
\end{array}
$$

Step 3: Skip a line and draw a horizontal line below the list of coefficients.

Step 4: Bring down the leading coefficient from the dividend and write it below the line.

$$
\begin{array}{c|ccc}
2 & 3 & -14 & -10 \\
 & & 6 & \\
\hline
 & 3 & -8 &
\end{array}
$$

Step 5: Multiply the value of r by the number below the line $(2 \times 3 = 6)$. Write the result in the next column above the line.

Step 6: Add the numbers in the column above the line $(-14 + 6)$, and write the result below the line.

Repeat steps 5 and 6 until all columns have been completed.

Step 7: To get the final result, we use the numbers below the line. The number in the last column is the remainder. The other numbers are the coefficients of the quotient.

$$
\begin{array}{c|ccc}
2 & 3 & -14 & -10 \\
 & & 6 & -16 \\
\hline
 & 3 & -8 & \boxed{-26}
\end{array}
$$

A box is usually drawn around the remainder.

Quotient: $3x - 8$, remainder: -26

The degree of the quotient will always be 1 less than that of the dividend. Because the dividend is a second-degree polynomial, the quotient will be a first-degree polynomial. In this case, the quotient is $3x - 8$ and the remainder is -26.

Example 6 Using Synthetic Division to Divide Polynomials

Divide the polynomials $(5x + 4x^3 - 6 + x^4) \div (x + 3)$ by using synthetic division.

Solution:

As with long division, the terms of the dividend and divisor should be written in descending order. Furthermore, missing powers must be accounted for by using placeholders (shown here in red).

$$5x + 4x^3 - 6 + x^4$$
$$= x^4 + 4x^3 + 0x^2 + 5x - 6$$

To use synthetic division, the divisor must be in the form $(x - r)$. The divisor $x + 3$ can be written as $x - (-3)$. Hence, $r = -3$.

Step 1: Write the value of r in a box.

Step 2: Write the coefficients of the dividend to the right of the box.

Step 3: Skip a line and draw a horizontal line below the list of coefficients.

Step 4: Bring down the leading coefficient from the dividend and write it below the line.

$$\begin{array}{r|rrrrr} \underline{-3} & 1 & 4 & 0 & 5 & -6 \\ & & & & & \\ \hline & 1 & & & & \end{array}$$

Step 5: Multiply the value of r by the number below the line $(-3 \times 1 = -3)$. Write the result in the next column above the line.

Step 6: Add the numbers in the column above the line: $4 + (-3) = 1$.

$$\begin{array}{r|rrrrr} \underline{-3} & 1 & 4 & 0 & 5 & -6 \\ & & -3 & & & \\ \hline & 1 & 1 & & & \end{array}$$

Repeat steps 5 and 6:

$$\begin{array}{r|rrrrr} \underline{-3} & 1 & 4 & 0 & 5 & -6 \\ & & -3 & -3 & 9 & -42 \\ \hline & 1 & 1 & -3 & 14 & \boxed{-48} \end{array}$$

remainder

constant

x term coefficient

x^2 term coefficient

x^3 term coefficient

The quotient is $x^3 + x^2 - 3x + 14$.

The remainder is -48.

The answer is $x^3 + x^2 - 3x + 14 + \dfrac{-48}{x + 3}$.

Avoiding Mistakes

It is important to check that the divisor is in the form $(x - r)$ before applying synthetic division. The variable x in the divisor must be of first degree, and its coefficient must be 1.

Skill Practice Divide the polynomials by using synthetic division.

8. $(5y^2 + 2y^3 - 5) \div (y + 2)$

Answer

8. $2y^2 + y - 2 + \dfrac{-1}{y + 2}$

TIP: It is interesting to compare the long division process to the synthetic division process. For Example 5, long division is shown on the left, and synthetic division is shown on the right. Notice that the same pattern of coefficients used in long division appears in the synthetic division process.

$$\begin{array}{r}
x^3 + x^2 - 3x + 14 \\
x+3\overline{)x^4 + 4x^3 + 0x^2 + 5x - 6} \\
-(x^4 + 3x^3) \quad\quad\quad\quad\quad\quad \\
x^3 + 0x^2 \quad\quad\quad\quad \\
-(x^3 + 3x^2) \quad\quad\quad\quad \\
-3x^2 + 5x \quad\quad \\
-(-3x^2 - 9x) \quad\quad \\
14x - 6 \\
-(14x + 42) \\
-48
\end{array}$$

$$\begin{array}{r|rrrrr}
-3 & 1 & 4 & 0 & 5 & -6 \\
 & & -3 & -3 & 9 & -42 \\
\hline
 & 1 & 1 & -3 & 14 & \boxed{-48} \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & x^3 & x^2 & x & \text{constant} & \text{remainder}
\end{array}$$

Quotient: $x^3 + x^2 - 3x + 14$
Remainder: -48

Example 7 Using Synthetic Division to Divide Polynomials

Divide the polynomials by using synthetic division. $(p^4 - 81) \div (p - 3)$

Solution:

$(p^4 - 81) \div (p - 3)$

$(p^4 + 0p^3 + 0p^2 + 0p - 81) \div (p - 3)$ Insert placeholders (red) for missing powers of p.

$$\begin{array}{r|rrrrr}
3 & 1 & 0 & 0 & 0 & -81 \\
 & & 3 & 9 & 27 & 81 \\
\hline
 & 1 & 3 & 9 & 27 & \boxed{0}
\end{array}$$

Quotient: $p^3 + 3p^2 + 9p + 27$

Remainder: 0

The answer is $p^3 + 3p^2 + 9p + 27$.

Skill Practice Divide the polynomials by using synthetic division.

9. $(x^3 + 1) \div (x + 1)$

Answer

9. $x^2 - x + 1$

Section 4.4 Activity

A.1. **a.** Write the division expression as a sum.

$$\frac{12a^3 - 6a^2 + 18a}{-6a} = \frac{\quad}{-6a} + \frac{\quad}{-6a} + \frac{\quad}{-6a}$$

b. Simplify the expression from part (a).

A.2. Divide. $(12k^5 - 2k^3 + 8k^2 - 3) \div (2k^2)$

A.3. How do you determine when to use long division when dividing polynomials?

A.4. Divide by using long division. $31\overline{)50,417}$

Write the answer in the form quotient $+ \dfrac{\text{remainder}}{\text{divisor}}$.

A.5. **a.** Divide the polynomials. $\dfrac{2x^4 - 9x^3 + 10x^2 - 7x + 10}{x - 3}$

b. Identify the quotient.

c. Identify the divisor.

d. Identify the dividend.

e. Identify the remainder.

f. Check the result of the division by verifying that Dividend = (Divisor)(Quotient) + Remainder.

A.6. Consider the division expression. $(27x^3 + 8) \div (3x + 2)$

a. Rewrite the expression with the dividend written in descending order and with placeholders for missing powers of x.

b. Divide the polynomials.

c. Identify the quotient.

d. Identify the divisor.

e. Identify the dividend.

f. Identify the remainder.

g. Check the result of the division.

A.7. Consider the division expression. $\dfrac{2y^3 - 6y + 3y^4 + 1}{y^2 - 2}$

a. Rewrite the expression with the dividend written in descending order and with placeholders for missing powers of y in the dividend and divisor.

b. Divide the polynomials.

c. Check the result of the division.

A.8. Consider the division expression from Exercise A.5.

$$\frac{2x^4 - 9x^3 + 10x^2 - 7x + 10}{x - 3}$$

a. Perform synthetic division.

$$\begin{array}{r|rrrrr} & 2 & -9 & 10 & -7 & 10 \\ & & & & & \\ \hline & & & & & \end{array}$$

b. From the results of part (a), identify the quotient.

c. Identify the remainder.

d. Write the result of the division. Is this the same result as in Exercise A.5?

e. Was long division or synthetic division easier?

A.9. **a.** Divide using synthetic division. $(t^3 - 125) \div (t - 5)$
(*Hint:* Be sure to leave 0's for missing powers of t in the dividend.)

b. Check the result of the division.

A.10. Given two polynomials in the variable x, explain when synthetic division can be used to divide two polynomials.

For Exercises A.11–A.14, determine which technique of division could be used to perform the indicated operation. Choose from monomial division, long division, and synthetic division.

A.11. $(8y^3 + y^2 - 5y + 3) \div (2y + 1)$

A.12. $(4t^3 - 12t^2 - 6t + 3) \div (2t^2)$

A.13. $(5p^4 - 3p^2 + p - 6) \div (p + 4)$

A.14. $(2t^5 - 3t^4 + 5t + 8) \div (t^2 + 2t - 3)$

Section 4.4 Practice Exercises

Study Skills Exercise

Now is a good time to review your schedule to reflect on how you spend your time outside the classroom. Ask yourself the following questions:

- How many hours are being spent studying mathematics outside of class?
- Is my study time spread out over the course of a week, or do I cram for long periods of time in one day?
- Am I retaining material that I've practiced, or am I forgetting the majority of the material after a day or two?

If you have not done so already, now is the time to plan frequent, short intervals of time into your weekly schedule dedicated to practicing mathematics.

Prerequisite Review

R.1. Divide $8945 \div 9$ using long division. Identify the dividend, divisor, quotient, and remainder.

R.2. Divide $25{,}461 \div 24$ using long division. Identify the dividend, divisor, quotient, and remainder.

For Exercises R.3–R.10, perform the indicated operations.

R.3. $(4x^2 - 2x + 3)(2x - 5) + 7$

R.4. $(5x^2 - 3x - 2)(7x - 1) - 9$

R.5. $8b^2(2b^2 - 5b + 12)$

R.6. $-4x^2(3x^2 + 6x - 7)$

R.7. $\begin{array}{r} 5x^3 - 4x^2 \\ \underline{-(5x^3 - 15x^2)} \end{array}$

R.8. $\begin{array}{r} 2x^5 - 3x^4 \\ \underline{-(2x^5 + 4x^4)} \end{array}$

R.9. $\begin{array}{r} x^3 + 4x^2 - 5x \\ \underline{-(x^3 + 2x^2 - 3x)} \end{array}$

R.10. $\begin{array}{r} x^4 - 3x^3 + 9x^2 \\ \underline{-(x^4 - 7x^3 + 2x^2)} \end{array}$

Vocabulary and Key Concepts

1. a. The __________ algorithm states that: Dividend = (divisor)(__________) + (__________).

b. __________ division or long division can be used when dividing a polynomial by a divisor of the form $x - r$, where r is a constant.

2. Divide 313 by 6 and identify the dividend, divisor, quotient, and remainder.

Concept 1: Division by a Monomial

For Exercises 3–24, divide the polynomials. Check your answer by multiplication. **(See Example 1.)**

3. $\dfrac{p^{12}}{p^7}$

4. $\dfrac{c^{15}}{c^{12}}$

5. $\dfrac{14a^5}{2a}$

6. $\dfrac{12b^7}{3b}$

7. $\dfrac{24x^3y^4}{36xy^5}$

8. $\dfrac{15t^2v^3}{25t^8v^2}$

9. $\dfrac{16t^4 - 4t^2 + 20t}{-4t}$

10. $\dfrac{2x^3 + 8x^2 - 2x}{-2x}$

11. $(36y + 24y^2 + 6y^3) \div (3y)$

12. $(6p^2 - 18p^4 + 30p^5) \div (6p)$

13. $(4x^3y + 12x^2y^2 - 4xy^3) \div (4xy)$

14. $(25m^5n - 10m^4n + m^3n) \div (5m^3n)$

15. $(-8y^4 - 12y^3 + 32y^2) \div (-4y^2)$

16. $(12y^5 - 8y^6 + 16y^4 - 10y^3) \div (2y^3)$

17. $(3p^4 - 6p^3 + 2p^2 - p) \div (-6p)$

18. $(-4q^3 + 8q^2 - q) \div (-12q)$

19. $(a^3 + 5a^2 + a - 5) \div (a)$

20. $(2m^5 - 3m^4 + m^3 - m^2 + 9m) \div (m^2)$

21. $\dfrac{6s^3t^5 - 8s^2t^4 + 10st^2}{-2st^4}$

22. $\dfrac{-8r^4w^2 - 4r^3w + 2w^3}{-4r^3w}$

23. $(8p^4q^7 - 9p^5q^6 - 11p^3q - 4) \div (p^2q)$

24. $(20a^5b^5 - 20a^3b^2 + 5a^2b + 6) \div (a^2b)$

Concept 2: Long Division

25. **a.** Divide $(2x^3 - 7x^2 + 5x - 1) \div (x - 2)$ and identify the divisor, quotient, and remainder.

b. Explain how to check by using multiplication.

26. **a.** Divide $(x^3 + 4x^2 + 7x - 3) \div (x + 3)$ and identify the divisor, quotient, and remainder.

b. Explain how to check by using multiplication.

For Exercises 27–48, divide the polynomials by using long division. Check your answer by multiplication. **(See Examples 2–4.)**

27. $(x^2 + 11x + 19) \div (x + 4)$

28. $(x^3 - 7x^2 - 13x + 3) \div (x + 2)$

29. $(3y^3 - 7y^2 - 4y + 3) \div (y - 3)$

30. $(z^3 - 2z^2 + 2z - 5) \div (z - 4)$

31. $(-12a^2 + 77a - 121) \div (3a - 11)$

32. $(28x^2 - 29x + 6) \div (4x - 3)$

33. $(9y + 18y^2 - 20) \div (3y + 4)$

34. $(-2y + 3y^2 - 1) \div (y - 1)$

35. $(18x^3 + 7x + 12) \div (3x - 2)$

36. $(8x^3 - 6x + 22) \div (2x - 1)$

37. $(8a^3 + 1) \div (2a + 1)$

38. $(81x^4 - 1) \div (3x + 1)$

39. $(x^4 - x^3 - x^2 + 4x - 2) \div (x^2 + x - 1)$

40. $(2a^5 - 7a^4 + 11a^3 - 22a^2 + 29a - 10) \div (2a^2 - 5a + 2)$

41. $(2x^3 - 10x + x^4 - 25) \div (x^2 - 5)$

42. $(-5x^3 + x^4 - 4 - 10x) \div (x^2 + 2)$

43. $(x^4 - 3x^2 + 10) \div (x^2 - 2)$

44. $(3y^4 - 25y^2 - 18) \div (y^2 - 3)$

45. $(n^4 - 16) \div (n - 2)$

46. $(m^3 + 27) \div (m + 3)$

47. $\dfrac{3y^4 + 2y + 3}{1 + y^2}$

48. $\dfrac{2x^4 + 6x + 4}{2 + x^2}$

Concept 3: Synthetic Division

49. Explain the conditions under which you may use synthetic division to divide polynomials.

50. Can synthetic division be used directly to divide $(4x^4 + 3x^3 - 7x + 9)$ by $(2x + 5)$? Explain why or why not.

51. Can synthetic division be used to divide $(6x^5 - 3x^2 + 2x - 14)$ by $(x^2 - 3)$? Explain why or why not.

52. Can synthetic division be used to divide $(3x^4 - x + 1)$ by $(x - 5)$? Explain why or why not.

53. The following table represents the result of a synthetic division.

$$\begin{array}{r|rrrr} 5 & 1 & -2 & -4 & 3 \\ & & 5 & 15 & 55 \\ \hline & 1 & 3 & 11 & \underline{|58} \end{array}$$

Use x as the variable.

a. Identify the divisor.

b. Identify the quotient.

c. Identify the remainder.

54. The following table represents the result of a synthetic division.

$$\begin{array}{r|rrrrr} -2 & 2 & 3 & 0 & -1 & 6 \\ & & -4 & 2 & -4 & 10 \\ \hline & 2 & -1 & 2 & -5 & \underline{|16} \end{array}$$

Use x as the variable.

a. Identify the divisor.

b. Identify the quotient.

c. Identify the remainder.

For Exercises 55–70, divide by using synthetic division. Check your answer by multiplication. **(See Examples 6–7.)**

55. $(x^2 - 2x - 48) \div (x - 8)$

56. $(x^2 - 4x - 12) \div (x - 6)$

57. $(t^2 - 3t - 4) \div (t + 1)$

58. $(h^2 + 7h + 12) \div (h + 3)$

59. $(5y^2 + 5y + 1) \div (y - 1)$

60. $(3w^2 + w - 5) \div (w + 2)$

61. $(3 + 7y^2 - 4y + 3y^3) \div (y + 3)$

62. $(2z - 2z^2 + z^3 - 5) \div (z + 3)$

63. $(x^3 - 3x^2 + 4) \div (x - 2)$

64. $(3y^4 - 25y^2 - 18) \div (y - 3)$

65. $(a^5 - 32) \div (a - 2)$

66. $(b^3 + 27) \div (b + 3)$

67. $(x^3 - 216) \div (x - 6)$

68. $(y^4 - 16) \div (y + 2)$

69. $(4w^4 - w^2 + 6w - 3) \div \left(w - \frac{1}{2}\right)$

70. $(-12y^4 - 5y^3 - y^2 + y + 3) \div \left(y + \frac{3}{4}\right)$

Mixed Exercises

For Exercises 71–82, divide the polynomials by using an appropriate method. **(See Examples 5.)**

71. $(-x^3 - 8x^2 - 3x - 2) \div (x + 4)$

72. $(8xy^2 - 9x^2y + 6x^2y^2) \div (x^2y^2)$

73. $(22x^2 - 11x + 33) \div (11x)$

74. $(2m^3 - 4m^2 + 5m - 33) \div (m - 3)$

75. $(12y^3 - 17y^2 + 30y - 10) \div (3y^2 - 2y + 5)$

76. $(90h^{12} - 63h^9 + 45h^8 - 36h^7) \div (9h^9)$

77. $(4x^4 + 6x^3 + 3x - 1) \div (2x^2 + 1)$

78. $(y^4 - 3y^3 - 5y^2 - 2y + 5) \div (y + 2)$

79. $(16k^{11} - 32k^{10} + 8k^8 - 40k^4) \div (8k^8)$

80. $(4m^3 - 18m^2 + 22m - 10) \div (2m^2 - 4m + 3)$

81. $(5x^3 + 9x^2 + 10x) \div (5x^2)$

82. $(15k^4 + 3k^3 + 4k^2 + 4) \div (3k^2 - 1)$

Problem Recognition Exercises

Operations on Polynomials

Perform the indicated operations.

1. **a.** $(3x + 1)^2$
   b. $(3x + 1)(3x - 1)$
   c. $(3x + 1) - (3x - 1)$

2. **a.** $(9m - 5) - (9m + 5)$
   b. $(9m - 5)(9m + 5)$
   c. $(9m - 5)^2$

3. **a.** $\dfrac{4x^2 + 8x - 10}{2x}$
   b. $\dfrac{4x^2 + 8x - 10}{2x - 1}$
   c. $(4x^2 + 8x - 10) \div (x - 1)$

4. **a.** $\dfrac{3y^2 - 15y + 4}{3y}$
   b. $\dfrac{3y^2 - 15y + 4}{3y + 6}$
   c. $(3y^2 - 15y + 4) \div (y + 6)$

5. **a.** $(p - 5)(p + 5) - (p^2 + 5)$
   b. $(p - 5)(p + 5) - (p + 5)^2$
   c. $(p - 5)(p + 5) - (p^2 - 25)$

6. **a.** $(x + 4)(x - 4) - (x + 4)^2$
   b. $(x + 4)(x - 4) - (x^2 + 4)$
   c. $(x + 4)(x - 4) - (x^2 - 16)$

7. $(5t^2 - 6t + 2) - (3t^2 - 7t + 3)$

8. $-5x^2(3x^2 + x - 2)$

9. $(6z + 5)(6z - 5)$

10. $(6y^3 + 2y^2 + y - 2) + (3y^3 - 4y + 3)$

11. $(3b - 4)(2b - 1)$

12. $(5a + 2)(2a^2 + 3a + 1)$

13. $(t^3 - 4t^2 + t - 9) + (t + 12) - (2t^2 - 6t)$

14. $(2b^3 - 3b - 10) \div (b - 2)$

15. $(k + 4)^2 + (-4k + 9)$

16. $(3x^4 - 11x^3 - 4x^2 - 5x + 20) \div (x - 4)$

17. $-2t(t^2 + 6t - 3) + t(3t + 2)(3t - 2)$

18. $\dfrac{7x^2y^3 - 14xy^2 - x^2}{-7xy}$

19. $\left(\frac{1}{4}p^3 - \frac{1}{6}p^2 + 5\right) - \left(-\frac{2}{3}p^3 + \frac{1}{3}p^2 - \frac{1}{5}p\right)$

20. $-6w^3(1.2w - 2.6w^2 + 5.1w^3)$

21. $(6a^2 - 4b)^2$

22. $\left(\frac{1}{2}z^2 - \frac{1}{3}\right)\left(\frac{1}{2}z^2 + \frac{1}{3}\right)$

23. $(m - 3)^2 - 2(m + 8)$

24. $(2x - 5)(x + 1) - (x - 3)^2$

25. $(m^2 - 6m + 7)(2m^2 + 4m - 3)$

26. $(x^3 - 64) \div (x - 4)$

27. $[5 - (a + b)]^2$

28. $[a - (x - y)][a + (x - y)]$

29. $(x + y)^2 - (x - y)^2$

30. $(a - 4)^3$

31. $\left(-\frac{1}{2}x + \frac{1}{3}\right)\left(\frac{1}{4}x - \frac{1}{2}\right)$

32. $-3x^2y^3z^4\left(\frac{1}{6}x^4yzw^3\right)$

Section 4.5 Greatest Common Factor and Factoring by Grouping

Concepts

1. Factoring Out the Greatest Common Factor
2. Factoring Out a Negative Factor
3. Factoring Out a Binomial Factor
4. Factoring by Grouping

1. Factoring Out the Greatest Common Factor

In this section, we begin our study of a mathematical operation called factoring. To factor an integer means to write the integer as a product of two or more integers. To factor a polynomial means to express the polynomial as a product of two or more polynomials. For example, in the product $5 \cdot 7 = 35$, the numbers 5 and 7 are factors of 35. In the product $(2x + 1)(x - 6) = 2x^2 - 11x - 6$, the quantities $(2x + 1)$ and $(x - 6)$ are factors of $2x^2 - 11x - 6$.

The **greatest common factor (GCF)** of a polynomial is the greatest factor that divides each term of the polynomial evenly. For example, the greatest common factor of $9x^4 + 18x^3 - 6x^2$ is $3x^2$. To factor out the greatest common factor from a polynomial, follow these steps:

Factoring Out the Greatest Common Factor

Step 1 Identify the greatest common factor of all terms of the polynomial.

Step 2 Write each term as the product of the GCF and another factor.

Step 3 Use the distributive property to factor out the greatest common factor.

Note: To check the factorization, multiply the polynomials.

Example 1 Factoring Out the Greatest Common Factor

Factor out the greatest common factor.

a. $12x^3 + 30x^2$ **b.** $12c^2d^3 - 30c^3d^2 - 3cd$

Solution:

a. $12x^3 + 30x^2$ The GCF is $6x^2$.

$= 6x^2(2x) + 6x^2(5)$ Write each term as the product of the GCF and another factor.

$= 6x^2(2x + 5)$ Factor out $6x^2$ by using the distributive property.

TIP: Any factoring problem can be checked by multiplying the factors.

Check: $6x^2(2x + 5) = 12x^3 + 30x^2$ ✓

b. $12c^2d^3 - 30c^3d^2 - 3cd$ The GCF is $3cd$.

$= 3cd(4cd^2) - 3cd(10c^2d) - 3cd(1)$ Write each term as the product of the GCF and another factor.

$= 3cd(4cd^2 - 10c^2d - 1)$ Factor out $3cd$ by using the distributive property.

Check: $3cd(4cd^2 - 10c^2d - 1) = 12c^2d^3 - 30c^3d^2 - 3cd$ ✓

Avoiding Mistakes

In Example 1(b), the GCF of $3cd$ is equal to one of the terms of the polynomial. In such a case, you must leave a 1 in place of that term after the GCF is factored out.

$3cd(4cd^2 - 10c^2d - 1)$

Skill Practice Factor out the greatest common factor.

1. $45y^5 - 15y^2 + 30y$ **2.** $16a^2b^5 + 12a^3b^3 + 4a^3b^2$

Answers

1. $15y(3y^4 - y + 2)$
2. $4a^2b^2(4b^3 + 3ab + a)$

2. Factoring Out a Negative Factor

Sometimes it is advantageous to factor out the *opposite* of the GCF, particularly when the leading coefficient of the polynomial is negative. This is demonstrated in Example 2. Notice that this *changes the signs* of the remaining terms inside the parentheses.

Example 2 Factoring Out a Negative Factor

Factor out the quantity $-5a^2b$ from the polynomial $-5a^4b - 10a^3b^2 + 15a^2b^3$.

Solution:

$-5a^4b - 10a^3b^2 + 15a^2b^3$ — The GCF is $5a^2b$. However, in this case we will factor out the opposite of the GCF, $-5a^2b$.

$= -5a^2b(a^2) + -5a^2b(2ab) + -5a^2b(-3b^2)$ — Write each term as the product of $-5a^2b$ and another factor.

$= -5a^2b(a^2 + 2ab - 3b^2)$ — Factor out $-5a^2b$ by using the distributive property.

Skill Practice Factor out the quantity $-6xy$ from the polynomial.

3. $24x^4y^3 - 12x^2y + 18xy^2$

3. Factoring Out a Binomial Factor

The distributive property may also be used to factor out a common factor that consists of more than one term. This is shown in Example 3.

Example 3 Factoring Out a Binomial Factor

Factor out the greatest common factor.

$$x^3(x + 2) - x(x + 2) - 9(x + 2)$$

Solution:

$x^3(x + 2) - x(x + 2) - 9(x + 2)$ — The GCF is the quantity $(x + 2)$.

$= (x + 2)(x^3) - (x + 2)(x) - (x + 2)(9)$ — Write each term as the product of $(x + 2)$ and another factor.

$= (x + 2)(x^3 - x - 9)$ — Factor out $(x + 2)$ by using the distributive property.

Skill Practice Factor out the greatest common factor.

4. $a^2(b + 2) + 5(b + 2)$

Answers

3. $-6xy(-4x^3y^2 + 2x - 3y)$

4. $(b + 2)(a^2 + 5)$

4. Factoring by Grouping

When two binomials are multiplied, the product before simplifying contains four terms. For example:

$$\begin{aligned}(3a+2)(2b-7) &= (3a+2)(2b)+(3a+2)(-7)\\ &= (3a+2)(2b)+(3a+2)(-7)\\ &= 6ab+4b-21a-14\end{aligned}$$

In Example 4, we learn how to reverse this process. That is, given a four-term polynomial, we will factor it as a product of two binomials. The process is called **factoring by grouping**.

Factoring by Grouping

To factor a four-term polynomial by grouping:

Step 1 Identify and factor out the GCF from all four terms.

Step 2 Factor out the GCF from the first pair of terms. Factor out the GCF from the second pair of terms. (Sometimes it is necessary to factor out the *opposite* of the GCF.)

Step 3 If the two terms share a common binomial factor, factor out the binomial factor.

Example 4 Factoring by Grouping

Factor by grouping. $6ab - 21a + 4b - 14$

Solution:

$6ab - 21a + 4b - 14$	**Step 1:** Identify and factor out the GCF from all four terms. In this case the GCF is 1.
$= 6ab - 21a \;\vert\; + 4b - 14$	Group the first pair of terms and the second pair of terms.
$= 3a(2b-7) + 2(2b-7)$	**Step 2:** Factor out the GCF from each pair of terms.
	Note: The two terms now share a common binomial factor of $(2b - 7)$.
$= (2b-7)(3a+2)$	**Step 3:** Factor out the common binomial factor.

Check: $(2b-7)(3a+2) = 2b(3a) + 2b(2) - 7(3a) - 7(2)$

$= 6ab + 4b - 21a - 14$ ✓

Skill Practice Factor by grouping.

5. $7c^2 + cd + 14c + 2d$

Avoiding Mistakes

In step 2, the expression $3a(2b-7) + 2(2b-7)$ is not yet factored because it is a *sum*, not a product. To factor the expression, you must carry it one step further.

$3a(2b-7) + 2(2b-7)$

$= (2b-7)(3a+2)$

The factored form must be represented as a product.

Answer

5. $(7c + d)(c + 2)$

Example 5 Factoring by Grouping

Factor by grouping. $x^3 + 3x^2 - 3x - 9$

Solution:

$x^3 + 3x^2 - 3x - 9$ **Step 1:** Identify and factor out the GCF from all four terms. In this case the GCF is 1.

$= x^3 + 3x^2 \;\vdots\; - 3x - 9$ Group the first pair of terms and the second pair of terms.

$= x^2(x + 3) - 3(x + 3)$ **Step 2:** Factor out x^2 from the first pair of terms.

Factor out -3 from the second pair of terms (this causes the signs to change in the second parentheses). The terms now contain a common binomial factor.

$= (x + 3)(x^2 - 3)$ **Step 3:** Factor out the common binomial $(x + 3)$.

TIP: One frequent question is, can the order be switched between factors? The answer is yes. Because multiplication is commutative, the order in which two or more factors are written does not matter. Thus, the following factorizations are equivalent:

$$(x + 3)(x^2 - 3) = (x^2 - 3)(x + 3)$$

Skill Practice Factor by grouping.

6. $a^3 - 4a^2 - 3a + 12$

Example 6 Factoring by Grouping

Factor by grouping. $24p^2q^2 - 18p^2q + 60pq^2 - 45pq$

Solution:

$24p^2q^2 - 18p^2q + 60pq^2 - 45pq$

$= 3pq(8pq - 6p + 20q - 15)$ **Step 1:** Remove the GCF $3pq$ from all four terms.

$= 3pq(8pq - 6p \;\vdots\; + 20q - 15)$ Group the first pair of terms and the second pair of terms.

Answer

6. $(a^2 - 3)(a - 4)$

$= 3pq[2p(4q - 3) + 5(4q - 3)]$ **Step 2:** Factor out the GCF from each pair of terms. The terms share the binomial factor $(4q - 3)$.

$= 3pq(4q - 3)(2p + 5)$ **Step 3:** Factor out the common binomial $(4q - 3)$.

Skill Practice Factor the polynomial.

7. $24x^2y - 12x^2 + 20xy - 10x$

Notice that in step 3 of factoring by grouping, a common binomial is factored from the two terms. These binomials must be *exactly* the same in each term. If the two binomial factors differ, try rearranging the original four terms.

Example 7 Factoring by Grouping Where Rearranging Terms Is Necessary

Factor the polynomial. $4x + 6pa - 8a - 3px$

Solution:

$4x + 6pa - 8a - 3px$ **Step 1:** Identify and factor out the GCF from all four terms. In this case the GCF is 1.

$= 4x + 6pa \mid - 8a - 3px$

$= 2(2x + 3pa) - 1(8a + 3px)$ **Step 2:** The binomial factors in each term are different.

Avoiding Mistakes

Remember that when factoring by grouping, the binomial factors must be *exactly* the same.

$= 4x - 8a \mid - 3px + 6pa$ *Try rearranging the original four terms* in such a way that the first pair of coefficients is in the same ratio as the second pair of coefficients. Notice that the ratio 4 to −8 is the same as the ratio −3 to 6.

$= 4(x - 2a) - 3p(x - 2a)$ **Step 2:** Factor out 4 from the first pair of terms.

Factor out $-3p$ from the second pair of terms.

$= (x - 2a)(4 - 3p)$ **Step 3:** Factor out the common binomial factor.

Skill Practice Factor the polynomial.

8. $3ry + 2s + sy + 6r$

Answers

7. $2x(6x + 5)(2y - 1)$
8. $(3r + s)(2 + y)$

Section 4.5 Activity

A.1. Consider the terms $18x^3y^2$, $12x^2y^3$, and $9xy^4$. The terms can be written in expanded form as:

$$18x^3y^2 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot y \cdot y$$
$$12x^2y^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y \cdot y \cdot y$$
$$9xy^4 = 3 \cdot 3 \cdot x \cdot y \cdot y \cdot y \cdot y$$

a. Circle the common factors that appear in all three lists.
b. Identify the greatest common factor of the terms $18x^3y^2$, $12x^2y^3$, and $9xy^4$ by multiplying the common factors identified in part (a).
c. Factor out the greatest common factor. $18x^3y^2 + 12x^2y^3 + 9xy^4 =$ ____ (____ ____ ____)

A.2. **a.** Identify the greatest common factor of the term $4x(a + 2b)$ and $6(a + 2b)$.
b. Factor out the greatest common factor. $4x(a + 2b) + 6(a + 2b) =$ ____ (____ ____)

A.3. **a.** Factor out $8x$ from the polynomial. $-8x^2 + 24x$
b. Factor out $-8x$ from the polynomial. $-8x^2 + 24x$
c. When a negative factor is factored out of a polynomial, what is the effect on the signs of the terms?

A.4. Multiply. $(5x + 3y)(2x + 9)$

A.5. To factor $10x^2 + 45x + 6xy + 27y$, follow these steps.
a. Factor out the greatest common factor from the first two terms.
Factor out the greatest common factor from the last two terms.
b. Factor out the greatest common factor from the expression $5x(2x + 9) + 3y(2x + 9)$.
c. Compare your answer to the result from Exercise A.4.

A.6. Factor $c^2 + 5c - 2cd - 10d$ by grouping.

A.7. Given $8x^2 + 15y + 10x + 12xy$,
a. Factor out the GCF from the first two terms.
Factor out the GCF from the last two terms.
b. Why does the process of factoring by grouping fail at this stage?
c. Return to the original polynomial and factor the polynomial by first rearranging the terms and then factoring by grouping.
d. Check the answer by multiplying factors.

Practice Exercises — Section 4.5

Study Skills Exercise

As you study mathematics you must believe that you are capable of achieving success despite any past difficulties. This requires the right frame of mind. It requires a growth mindset. One strategy for developing a growth mindset is to make connections between existing knowledge and new concepts. For example, as a child you learned to count to ten and then applied that knowledge to add whole numbers. As you learn about factoring polynomials, recall what you already know about greatest common factors. Notice the similarities between the two examples.

- Find the greatest common factor between 16 and 24: 8

$$16 = 8 \cdot 2 \qquad 24 = 8 \cdot 3$$

- Find the greatest common factor between $16x^3$ and $24x^2$: $8x^2$

$$16x^3 = 8 \cdot 2 \cdot x \cdot x \cdot x \qquad 24x^2 = 8 \cdot 3 \cdot x \cdot x$$

Practice making connections between skills you already know and those you are learning. Your overall understanding will grow.

Prerequisite Review

For Exercises R.1–R.2, write the prime factorization for the given real number.

R.1. 240

R.2. 350

For Exercises R.3–R.10, perform the indicated operations.

R.3. $-(3x^2 - 7x + 8)$

R.4. $-(-10x + 13)$

R.5. $-5d^2(2d - 8)$

R.6. $-4n^3(3n + 6)$

R.7. $5x^2y(3x^2 - 2x + 1)$

R.8. $11a^2b^4(3a - 5ab + b^2)$

R.9. $(6x + 5)(3a - 11)$

R.10. $(y - 2)(7x - 5)$

Vocabulary and Key Concepts

1. **a.** Factoring a polynomial means to write it as a __________ of two or more polynomials.

b. The __________ __________ __________ (GCF) of a polynomial is the greatest factor that divides each term of the polynomial evenly.

2. **a.** The first step toward factoring a polynomial is to factor out the __________ __________ __________.

d. To factor a four-term polynomial, we try the process of factoring by __________.

Concept 1: Factoring Out the Greatest Common Factor

For Exercises 3–18, factor out the greatest common factor. **(See Example 1.)**

3. $3x + 12$

4. $15x - 10$

5. $6z^2 + 4z$

6. $49y^3 - 35y^2$

7. $4p^6 - 4p$

8. $5q^2 - 5q$

9. $12x^4 - 36x^2$

10. $51w^4 - 34w^3$

11. $9st^2 + 27t$

12. $8a^2b^3 + 12a^2b$

13. $9a^4b^3 + 27a^3b^4 - 18a^2b^5$

14. $3x^5y^4 - 15x^4y^5 + 9x^2y^7$

15. $10x^2y + 15xy^2 - 5xy$

16. $12c^3d - 15c^2d + 3cd$

17. $13b^2 - 11a^2b - 12ab$

18. $6a^3 - 2a^2b + 5a^2$

Concept 2: Factoring Out a Negative Factor

For Exercises 19–24, factor out the indicated quantity. **(See Example 2.)**

19. $-x^2 - 10x + 7$: Factor out -1.

20. $-5y^2 + 10y + 3$: Factor out -1.

21. $-12x^3y - 6x^2y - 3xy$: Factor out $-3xy$.

22. $-32a^4b^2 + 24a^3b + 16a^2b$: Factor out $-8a^2b$.

23. $-2t^3 + 11t^2 - 3t$: Factor out $-t$.

24. $-7y^2z - 5yz - z$: Factor out $-z$.

Concept 3: Factoring Out a Binomial Factor

For Exercises 25–32, factor out the GCF. **(See Example 3.)**

25. $2a(3z - 2b) - 5(3z - 2b)$

26. $5x(3x + 4) + 2(3x + 4)$

27. $2x^2(2x - 3) + (2x - 3)$

28. $z(w-9)+(w-9)$

29. $y(2x+1)^2-3(2x+1)^2$

30. $a(b-7)^2+5(b-7)^2$

31. $3y(x-2)^2+6(x-2)^2$

32. $10z(z+3)^2-2(z+3)^2$

33. Construct a polynomial that has a greatest common factor of $3x^2$. (Answers may vary.)

34. Construct two different trinomials that have a greatest common factor of $5x^2y^3$. (Answers may vary.)

35. Construct a binomial that has a greatest common factor of $(c+d)$. (Answers may vary.)

Concept 4: Factoring by Grouping

36. If a polynomial has four terms, what technique would you use to factor it?

37. Factor the polynomials by grouping.

a. $2ax-ay+6bx-3by$

b. $10w^2-5w-6bw+3b$

c. Explain why you factored out $3b$ from the second pair of terms in part (a) but factored out the quantity $-3b$ from the second pair of terms in part (b).

38. Factor the polynomials by grouping.

a. $3xy+2bx+6by+4b^2$

b. $15ac+10ab-6bc-4b^2$

c. Explain why you factored out $2b$ from the second pair of terms in part (a) but factored out the quantity $-2b$ from the second pair of terms in part (b).

For Exercises 39–58, factor each polynomial by grouping (if possible). **(See Examples 4–7.)**

39. $y^3+4y^2+3y+12$

40. $ab+b+2a+2$

41. $6p-42+pq-7q$

42. $2t-8+st-4s$

43. $2mx+2nx+3my+3ny$

44. $4x^2+6xy-2xy-3y^2$

45. $10ax-15ay-8bx+12by$

46. $35a^2-15a+14a-6$

47. x^3-x^2-3x+3

48. $2rs+4s-r-2$

49. $6p^2q+18pq-30p^2-90p$

50. $5s^2t+20st-15s^2-60s$

51. $100x^3-300x^2+200x-600$

52. $2x^5-10x^4+6x^3-30x^2$

53. $6ax-by+2bx-3ay$

54. $5pq-12-4q+15p$

55. $4a-3b-ab+12$

56. $x^2y+6x-3x^3-2y$

57. $7y^3-21y^2+5y-10$

58. $5ax+10bx-2ac+4bc$

59. Explain why the grouping method failed for Exercise 57.

60. Explain why the grouping method failed for Exercise 58.

Mixed Exercises

61. Solve the equation $U = Av + Acw$ for A by first factoring out A.

62. Solve the equation $S = rt + wt$ for t by first factoring out t.

63. Solve the equation $ay + bx = cy$ for y.

64. Solve the equation $cd + 2x = ac$ for c.

65. The area of a rectangle of width w is given by $A = 2w^2 + w$. Factor the right-hand side of the equation to find an expression for the length of the rectangle.

66. The amount in a savings account bearing simple interest at an annual interest rate r for t years is given by $A = P + Prt$ where P is the principal amount invested.

a. Solve the equation for P.

b. Compute the amount of principal originally invested if the account is worth \$12,705 after 3 yr at a 7% interest rate.

Expanding Your Skills

For Exercises 67–74, factor out the greatest common factor and simplify.

67. $(a + 3)^4 + 6(a + 3)^5$

68. $(4 - b)^4 - 2(4 - b)^3$

69. $24(3x + 5)^3 - 30(3x + 5)^2$

70. $10(2y + 3)^2 + 15(2y + 3)^3$

71. $(t + 4)^2 - (t + 4)$

72. $(p + 6)^2 - (p + 6)$

73. $15w^2(2w - 1)^3 + 5w^3(2w - 1)^2$

74. $8z^4(3z - 2)^2 + 12z^3(3z - 2)^3$

Section 4.6 Factoring Trinomials

Concepts

1. Factoring Trinomials: AC-Method
2. Factoring Trinomials: Trial-and-Error Method
3. Factoring Perfect Square Trinomials
4. Factoring by Using Substitution

1. Factoring Trinomials: AC-Method

In this section, we present two methods to factor trinomials. The first method is called the ac-method. The second method is called the trial-and-error method.

The product of two binomials results in a four-term expression that can sometimes be simplified to a trinomial. To factor the trinomial, we want to reverse the process.

$$\text{Multiply:} \quad (2x + 3)(x + 2) = \xrightarrow{\text{Multiply the binomials.}} 2x^2 + 4x + 3x + 6$$
$$= \xrightarrow{\text{Add the middle terms.}} 2x^2 + 7x + 6$$

$$\text{Factor:} \quad 2x^2 + 7x + 6 = \xrightarrow[\text{a sum or difference of terms.}]{\text{Rewrite the middle term as}} 2x^2 + 4x + 3x + 6$$
$$= \xrightarrow{\text{Factor by grouping.}} (2x + 3)(x + 2)$$

To factor a trinomial $ax^2 + bx + c$ by the ac-method, we rewrite the middle term bx as a sum or difference of terms. The goal is to produce a four-term polynomial that can be factored by grouping. The process is outlined as follows.

The AC-Method to Factor $ax^2 + bx + c$ $(a \neq 0)$

Step 1 After factoring out the GCF, multiply the coefficients of the first and last terms, ac.

Step 2 Find two integers whose product is ac and whose sum is b. (If no pair of integers can be found, then the trinomial cannot be factored further and is called a **prime polynomial**.)

Step 3 Rewrite the middle term bx as the sum of two terms whose coefficients are the integers found in step 2.

Step 4 Factor by grouping.

The ac-method for factoring trinomials is illustrated in Example 1. Before we begin, however, keep these two important guidelines in mind.

- For any factoring problem you encounter, always factor out the GCF from all terms first.
- To factor a trinomial, write the trinomial in the form $ax^2 + bx + c$.

Example 1 Factoring a Trinomial by the AC-Method

Factor by using the ac-method. $12x^2 - 5x - 2$

Solution:

$12x^2 - 5x - 2$ — The GCF is 1.

$a = 12 \quad b = -5 \quad c = -2$

Step 1: The expression is written in the form $ax^2 + bx + c$. Find the product $ac = 12(-2) = -24$.

Factors of −24	Factors of −24
$(1)(-24)$	$(-1)(24)$
$(2)(-12)$	$(-2)(12)$
$(3)(-8)$	$(-3)(8)$
$(4)(-6)$	$(-4)(6)$

Step 2: List all the factors of -24, and find the pair whose sum equals -5.

The numbers 3 and -8 produce a product of -24 and a sum of -5.

$12x^2 - 5x - 2$

$= 12x^2 + 3x - 8x - 2$

Step 3: Write the middle term of the trinomial as two terms whose coefficients are the selected numbers 3 and -8.

$= 12x^2 + 3x \quad - 8x - 2$

$= 3x(4x + 1) - 2(4x + 1)$

$= (4x + 1)(3x - 2)$

Step 4: Factor by grouping.

The check is left for the reader.

Skill Practice

1. Factor by using the ac-method. $10x^2 + x - 3$

FOR REVIEW

Recall that the factored form of a polynomial can be checked by multiplying factors.

$(4x + 1)(3x - 2)$
$= 12x^2 - 8x + 3x - 2$
$= 12x^2 - 5x - 2$ ✓

Answer

1. $(5x + 3)(2x - 1)$

TIP: One frequently asked question is whether the order matters when we rewrite the middle term of the trinomial as two terms (step 3). The answer is no. From Example 1, the two middle terms in step 3 could have been reversed.

$$\begin{aligned} 12x^2 - 5x - 2 &= 12x^2 - 8x + 3x - 2 \\ &= 4x(3x - 2) + 1(3x - 2) \\ &= (3x - 2)(4x + 1) \end{aligned}$$

This example also shows that the order in which two factors are written does not matter. The expression $(3x - 2)(4x + 1)$ is equivalent to $(4x + 1)(3x - 2)$ because multiplication is a commutative operation.

Example 2 Factoring a Trinomial by the AC-Method

Factor the trinomial by using the ac-method. $-20c^3 + 34c^2d - 6cd^2$

Solution:

$-20c^3 + 34c^2d - 6cd^2$

$= -2c(10c^2 - 17cd + 3d^2)$ Factor out $-2c$.

Step 1: Find the product $a \cdot c = (10)(3) = 30$

Factors of 30	**Factors of 30**
$1 \cdot 30$	$(-1)(-30)$
$2 \cdot 15$	$(-2)(-15)$
$3 \cdot 10$	$(-3)(-10)$
$5 \cdot 6$	$(-5)(-6)$

Step 2: The numbers -2 and -15 form a product of 30 and a sum of -17.

$= -2c(10c^2 - 17cd + 3d^2)$

$= -2c(10c^2 - 2cd - 15cd + 3d^2)$

Step 3: Write the middle term of the trinomial as two terms whose coefficients are -2 and -15.

$= -2c[2c(5c - d) - 3d(5c - d)]$ **Step 4:** Factor by grouping.

$= -2c(5c - d)(2c - 3d)$

Avoiding Mistakes

For factoring by grouping to work, the two binomial factors must be the same.

$$\begin{aligned} &10c^2 - 17cd + 3d^2 \\ &= 10c^2 - 2cd - 15cd + 3d^2 \\ &= 2c(5c - d) - 3d(5c - d) \\ &= (5c - d)(2c - 3d) \ ✓ \end{aligned}$$

Skill Practice

2. Factor by using the ac-method. $-4wz^3 - 2w^2z^2 + 20w^3z$

TIP: In Example 2, removing the GCF from the original trinomial produced a new trinomial with smaller coefficients. This makes the factoring process simpler because the product ac is smaller.

Original trinomial	**With the GCF factored out**
$-20c^3 + 34c^2d - 6cd^2$	$-2c(10c^2 - 17cd + 3d^2)$
$ac = (-20)(-6) = 120$	$ac = (10)(3) = 30$

Answer

2. $-2wz(2z + 5w)(z - 2w)$

2. Factoring Trinomials: Trial-and-Error Method

Another method that is widely used to factor trinomials of the form $ax^2 + bx + c$ is the trial-and-error method. To understand how the trial-and-error method works, first consider the multiplication of two binomials:

Product of $2 \cdot 1$ Product of $3 \cdot 2$

$$(2x + 3)(1x + 2) = 2x^2 + \underbrace{4x + 3x}_{\substack{\text{sum of products}\\ \text{of inner terms}\\ \text{and outer terms}}} + 6 = 2x^2 + 7x + 6$$

In Example 3, we will factor this trinomial by reversing this process.

Example 3 Factoring a Trinomial by the Trial-and-Error Method

Factor by the trial-and-error method. $2x^2 + 7x + 6$

Solution:

To factor by the trial-and-error method, we must fill in the blanks to create the correct product.

Factors of 2

$$2x^2 + 7x + 6 = (\square x \quad \square)(\square x \quad \square)$$

Factors of 6

- The first terms in the binomials must be $2x$ and x. This creates a product of $2x^2$, which is the first term in the trinomial.
- The second terms in the binomials must form a product of 6. This means that the factors must both be positive or both be negative. Because the middle term of the trinomial is positive, we will consider only *positive* factors of 6. The options are $1 \cdot 6$, $2 \cdot 3$, $6 \cdot 1$, and $3 \cdot 2$.
- Test each combination of factors until the correct product of binomials is found.

$(2x + 1)(x + 6) = 2x^2 + 12x + 1x + 6$
$= 2x^2 + 13x + 6$ *Incorrect.* Wrong middle term.

$(2x + 2)(x + 3) = 2x^2 + 6x + 2x + 6$
$= 2x^2 + 8x + 6$ *Incorrect.* Wrong middle term.

$(2x + 6)(x + 1) = 2x^2 + 2x + 6x + 6$
$= 2x^2 + 8x + 6$ *Incorrect.* Wrong middle term.

$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6$
$= 2x^2 + 7x + 6$ *Correct.*

The factored form of $2x^2 + 7x + 6$ is $(2x + 3)(x + 2)$.

Skill Practice Factor by the trial-and-error method.

3. $5y^2 - 9y + 4$

Answer

3. $(5y - 4)(y - 1)$

When applying the trial-and-error method, sometimes it is not necessary to test all possible combinations of factors. For the trinomial $2x^2 + 7x + 6$, the GCF is 1. Therefore, any binomial factor that shares a common factor greater than 1 will not work and does not need to be tested. For example, the following binomials cannot work:

$$(\underbrace{2x + 2}_{\text{Common factor of 2}})(x + 3) \qquad (\underbrace{2x + 6}_{\text{Common factor of 2}})(x + 1)$$

Although the trial-and-error method is tedious, its principle is generally easy to remember. We reverse the process of multiplying binomials.

The Trial-and-Error Method to Factor $ax^2 + bx + c$

Step 1 Factor out the greatest common factor.

Step 2 List all pairs of positive factors of a and pairs of positive factors of c. Consider the reverse order for either list of factors.

Step 3 Construct two binomials of the form

Factors of a

$$(\square x \quad \square)(\square x \quad \square)$$

Factors of c

Step 4 Test each combination of factors and signs until the correct product is found.

Step 5 If no combination of factors produces the correct product, the trinomial cannot be factored further and is a **prime polynomial**.

Example 4 Factoring a Trinomial by the Trial-and-Error Method

Factor by the trial-and-error method. $13y - 6 + 8y^2$

Solution:

$8y^2 + 13y - 6$ — Write in the form $ax^2 + bx + c$.

$(\square y \quad \square)(\square y \quad \square)$ — **Step 1:** The GCF is 1.

Factors of 8	**Factors of 6**
$1 \cdot 8$	$1 \cdot 6$
$2 \cdot 4$	$2 \cdot 3$
	$3 \cdot 2$
	$6 \cdot 1$

$\left.\begin{matrix} 3 \cdot 2 \\ 6 \cdot 1 \end{matrix}\right\}$ (reverse order)

Step 2: List the positive factors of 8 and positive factors of 6. Consider the reverse order in one list of factors.

Step 3: Construct all possible binomial factors by using different combinations of the factors of 8 and 6.

$$\left.\begin{array}{l} (2y \quad 1)(4y \quad 6) \\ (2y \quad 2)(4y \quad 3) \\ (2y \quad 3)(4y \quad 2) \\ (2y \quad 6)(4y \quad 1) \\ (1y \quad 1)(8y \quad 6) \\ (1y \quad 3)(8y \quad 2) \end{array}\right\}$$

Without regard to signs, these factorizations cannot work because the terms in the binomial share a common factor greater than 1.

Test the remaining factorizations. Keep in mind that to produce a product of −6, the signs within the parentheses must be opposite (one positive and one negative). Also, the sum of the products of the inner terms and outer terms must be combined to form 13*y*.

$(1y \quad 6)(8y \quad 1)$ *Incorrect.* Wrong middle term.

Regardless of signs, the product of inner terms 48*y* and the product of outer terms 1*y* cannot be combined to form the middle term 13*y*.

$(1y \quad 2)(8y \quad 3)$ *Correct.* The terms 16*y* and 3*y* can be combined to form the middle term 13*y*, provided the signs are applied correctly. We require $+16y$ and $-3y$.

The correct factorization of $8y^2 + 13y - 6$ is $(y + 2)(8y - 3)$.

Skill Practice Factor by the trial-and-error method.

4. $5t - 6 + 4t^2$

In Example 4, the factors of −6 must have opposite signs to produce a negative product. Therefore, one binomial factor is a sum and one is a difference. Determining the correct signs is an important aspect of factoring trinomials. We suggest the following guidelines:

TIP: Given the trinomial $ax^2 + bx + c$ $(a > 0)$, the signs can be determined as follows:

1. If *c* is *positive*, then the signs in the binomials must be the same (either both positive or both negative). The correct choice is determined by the middle term. If the middle term is positive, then both signs must be positive. If the middle term is negative, then both signs must be negative.

c is positive.

Example: $20x^2 + 43x + 21$

$(4x + 3)(5x + 7)$

same signs

c is positive.

Example: $20x^2 - 43x + 21$

$(4x - 3)(5x - 7)$

same signs

2. If *c* is *negative,* then the signs in the binomials must be different. The middle term in the trinomial determines which factor gets the positive sign and which factor gets the negative sign.

c is negative.

Example: $x^2 + 3x - 28$

$(x + 7)(x - 4)$

different signs

c is negative.

Example: $x^2 - 3x - 28$

$(x - 7)(x + 4)$

different signs

Answer

4. $(4t - 3)(t + 2)$

Example 5 Factoring a Trinomial by the Trial-and-Error Method

Factor by the trial-and-error method.

$$-80x^3y + 208x^2y^2 - 20xy^3$$

Solution:

$-80x^3y + 208x^2y^2 - 20xy^3$

$= -4xy(20x^2 - 52xy + 5y^2)$ **Step 1:** Factor out $-4xy$.

$= -4xy(\square x \ \square y)(\square x \ \square y)$

Factors of 20	**Factors of 5**
$1 \cdot 20$	$1 \cdot 5$
$2 \cdot 10$	$5 \cdot 1$
$4 \cdot 5$	

Step 2: List the positive factors of 20 and positive factors of 5. Consider the reverse order in one list of factors.

Step 3: Construct all possible binomial factors by using different combinations of the factors of 20 and factors of 5. The signs in the parentheses must both be negative.

$$\left.\begin{array}{l} -4xy(1x - 1y)(20x - 5y) \\ -4xy(2x - 1y)(10x - 5y) \\ -4xy(4x - 1y)(5x - 5y) \end{array}\right\}$$ *Incorrect.* These binomials contain a common factor.

$-4xy(1x - 5y)(20x - 1y)$ *Incorrect.* Wrong middle term.
$-4xy(x - 5y)(20x - 1y)$
$= -4xy(20x^2 - 101xy + 5y^2)$

$-4xy(4x - 5y)(5x - 1y)$ *Incorrect.* Wrong middle term.
$-4xy(4x - 5y)(5x - 1y)$
$= -4xy(20x^2 - 29x + 5y^2)$

$-4xy(2x - 5y)(10x - 1y)$ ***Correct.*** $-4xy(2x - 5y)(10x - 1y)$
$= \mathbf{-4xy(20x^2 - 52xy + 5y^2)}$
$= -80x^3y + 208x^2y^2 - 20xy^3$

The correct factorization of $-80x^3y + 208x^2y^2 - 20xy^3$ is $-4xy(2x - 5y)(10x - y)$.

Skill Practice Factor by the trial-and-error method.

5. $-4z^3 - 22z^2 - 30z$

Answer

5. $-2z(2z + 5)(z + 3)$

Example 6 **Factoring a Trinomial by the Trial-and-Error Method**

Factor completely. $2x^2 + 9x + 14$

Solution:

$2x^2 + 9x + 14$	The GCF is 1 and the trinomial is written in the form $ax^2 + bx + c$.
$(2x + 14)(x + 1)$	*Incorrect.* $(2x + 14)$ contains a common factor of 2.
$(2x + 2)(x + 7)$	*Incorrect.* $(2x + 2)$ contains a common factor of 2.
$(2x + 1)(x + 14) = 2x^2 + 15x + 14$	*Incorrect.* Wrong middle term.
$(2x + 7)(x + 2) = 2x^2 + 11x + 14$	*Incorrect.* Wrong middle term.

No combination of factors results in the correct product. Therefore, the trinomial is prime (cannot be factored).

Skill Practice Factor completely.

6. $6r^2 - 13r + 10$

If a trinomial has a leading coefficient of 1, the factoring process simplifies significantly. Consider the trinomial $x^2 + bx + c$. To produce a leading term of x^2, we can construct binomials of the form $(x + \square)(x + \square)$. The remaining terms may be satisfied by two numbers p and q whose product is c and whose sum is b:

$$(x + \overbrace{p)(x + q}^{\text{Factors of } c}) = x^2 + qx + px + pq = x^2 + \underbrace{(p + q)}_{\text{Sum} = b}x + \underbrace{pq}_{\text{Product} = c}$$

This process is demonstrated in Example 7.

Example 7 **Factoring a Trinomial with a Leading Coefficient of 1**

Factor completely. $x^2 - 10x + 16$

Solution:

$x^2 - 10x + 16$	Factor out the GCF from all terms. In this case, the GCF is 1.
$= (x \quad \square)(x \quad \square)$	The trinomial is written in the form $x^2 + bx + c$. To form the product x^2, use the factors x and x.

Answer

6. Prime

Next, look for two numbers whose product is 16 and whose sum is -10. Because the middle term is negative, we will consider only the negative factors of 16.

Factors of 16	Sum
$-1(-16)$	$-1 + (-16) = -17$
$-2(-8)$	$-2 + (-8) = -10$
$-4(-4)$	$-4 + (-4) = -8$

The numbers are -2 and -8.

Therefore, $x^2 - 10x + 16 = (x - 2)(x - 8)$.

Skill Practice Factor completely.

7. $c^2 + 6c - 27$

3. Factoring Perfect Square Trinomials

Recall that the square of a binomial always results in a **perfect square trinomial**.

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = (a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

For example, $(2x + 7)^2 = (2x)^2 + 2(2x)(7) + (7)^2 = 4x^2 + 28x + 49$

$a = 2x \quad b = 7 \qquad\qquad a^2 + 2ab + b^2$

To factor the trinomial $4x^2 + 28x + 49$, the ac-method or the trial-and-error method can be used. However, recognizing that the trinomial is a perfect square trinomial, we can use one of the following patterns to reach a quick solution.

Factored Form of a Perfect Square Trinomial

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

TIP: The following are perfect squares.

$1^2 = 1$	$(x^1)^2 = x^2$
$2^2 = 4$	$(x^2)^2 = x^4$
$3^2 = 9$	$(x^3)^2 = x^6$
$4^2 = 16$	$(x^4)^2 = x^8$
$\vdots$	$\vdots$

Any expression raised to an even power (multiple of 2) is a perfect square.

TIP: To determine if a trinomial is a perfect square trinomial, follow these steps:

1. Check if the first and third terms are both perfect squares with positive coefficients.
2. If this is the case, identify a and b, and determine if the middle term equals $2ab$ or $-2ab$.

Answer

7. $(c + 9)(c - 3)$

Example 8 Factoring a Perfect Square Trinomial

Factor completely. $x^2 + 12x + 36$

Solution:

$x^2 + 12x + 36$

The GCF is 1.

- The first and third terms are positive.
- The first term is a perfect square: $x^2 = (x)^2$
- The third term is a perfect square: $36 = (6)^2$
- The middle term is twice the product of x and 6:

$$12x = 2(x)(6)$$

Perfect squares

$= x^2 + 12x + 36$

$= (x)^2 + 2(x)(6) + (6)^2$

The trinomial is in the form $a^2 + 2ab + b^2$, where $a = x$ and $b = 6$.

$= (x + 6)^2$

Factor as $(a + b)^2$.

Skill Practice Factor completely.

8. $x^2 + 2x + 1$

Example 9 Factoring a Perfect Square Trinomial

Factor completely. $4x^2 - 36xy + 81y^2$

Solution:

$4x^2 - 36xy + 81y^2$

The GCF is 1.

- The first and third terms are positive.
- The first term is a perfect square: $4x^2 = (2x)^2$
- The third term is a perfect square: $81y^2 = (9y)^2$
- The middle term:

$$-36xy = -2(2x)(9y)$$

Perfect squares

$= 4x^2 - 36xy + 81y^2$

$= (2x)^2 - 2(2x)(9y) + (9y)^2$

The trinomial is in the form $a^2 - 2ab + b^2$, where $a = 2x$ and $b = 9y$.

$= (2x - 9y)^2$

Factor as $(a - b)^2$.

Skill Practice Factor completely.

9. $9y^2 - 12yz + 4z^2$

Answers

8. $(x + 1)^2$

9. $(3y - 2z)^2$

4. Factoring by Using Substitution

Sometimes it is convenient to use substitution to convert a polynomial into a simpler form before factoring.

Example 10 Using Substitution to Factor a Polynomial

Factor by using substitution. $(2x - 7)^2 - 3(2x - 7) - 40$

Solution:

$(2x - 7)^2 - 3(2x - 7) - 40$

$= u^2 - 3u - 40$	Substitute $u = 2x - 7$. The trinomial is simpler in form.
$= (u - 8)(u + 5)$	Factor the trinomial.
$= [(2x - 7) - 8][(2x - 7) + 5]$	Reverse substitute. Replace u by $2x - 7$.
$= (2x - 7 - 8)(2x - 7 + 5)$	Simplify.
$= (2x - 15)(2x - 2)$	The second binomial has a GCF of 2.
$= (2x - 15)(2)(x - 1)$	Factor out the GCF from the second binomial.
$= 2(2x - 15)(x - 1)$	

Skill Practice Factor by using substitution.

10. $(3x + 1)^2 + 2(3x + 1) - 15$

Example 11 Using Substitution to Factor a Polynomial

Factor by using substitution. $6y^6 - 5y^3 - 4$

TIP: The ac-method or trial-and-error method can also be used for Example 11 without using substitution.

Solution:

$6y^6 - 5y^3 - 4$

$= 6(y^3)^2 - 5(y^3) - 4$	Let $u = y^3$.
$= 6u^2 - 5u - 4$	Substitute u for y^3 in the trinomial.
$= (2u + 1)(3u - 4)$	Factor the trinomial.
$= (2y^3 + 1)(3y^3 - 4)$	Reverse substitute. Replace u with y^3.

Skill Practice Factor by using substitution.

11. $2x^4 + 7x^2 + 3$

Answers

10. $3(3x - 2)(x + 2)$
11. $(2x^2 + 1)(x^2 + 3)$

As you work through the exercises in this section, keep these guidelines in mind to factor trinomials.

Factoring Trinomials of the Form $ax^2 + bx + c$ $(a \neq 0)$

When factoring trinomials, the following guidelines should be considered:

Step 1 Factor out the greatest common factor.

Step 2 Check to see if the trinomial is a perfect square trinomial. If so, factor it as either $(a + b)^2$ or $(a - b)^2$. (With a perfect square trinomial, you do not need to use the ac-method or trial-and-error method.)

Step 3 If the trinomial is not a perfect square, use either the ac-method or the trial-and-error method to factor.

Step 4 Check the factorization by multiplication.

Note: Consider using substitution if a trinomial is in the form $au^2 + bu + c$, where u is an algebraic expression.

Section 4.6 Activity

A.1. To use the ac-method to factor the polynomial $6x^2 + x - 12$, follow these steps.

a. What is the greatest common factor?

b. Is the trinomial in descending order? If not, write it in descending order.

c. The trinomial is in the form $ax^2 + bx + c$. Identify a, b, and c.

d. What is the product of a and c? $ac = \_\_\_\_\_$

e. Find two numbers whose product is ac and whose sum is b. That is, the product is -72 and the sum is 1.

f. Write the trinomial as a four-term polynomial by splitting the middle term into the sum of two terms. The coefficients of the two terms should be the numbers you found in part (e).

$$6x^2 + x - 12$$

$$= 6x^2 + \square x + \square x - 12$$

g. Factor the expression from part (f) by grouping.

h. Switch the order of the middle terms found in part (f). Then factor by grouping. Compare your result to that found in part (g). Does the order of the middle terms matter?

A.2. Consider the polynomial $13x + 2x^2 + 6$.

a. What is the greatest common factor?

b. Is the trinomial in descending order? If not, write it in descending order.

c. Use the ac-method to factor the trinomial from part (b). Follow the steps given in Exercise A.1(c)–A.1(g).

d. Multiply the factors to check your answer.

A.3. Consider the trinomial $5x^2 + 34x + 24$.

a. What is the greatest common factor?

b. Is the polynomial written in descending order? If not, write it in descending order.

c. What is the leading coefficient?

d. To factor the trinomial by the trial-and-error method, we can set up the binomials as either $(5x \quad)(x \quad)$ or as $(x \quad)(5x \quad)$. In each case, what is the product of the leading terms in the binomials?

e. Using the binomials $(5x \quad)(x \quad)$, complete the factorization by filling in the second term of each binomial. This will require some trial-and-error. The two numbers you insert into the binomials must have a product of 24. But the overall product of the binomials should also produce a middle term of $34x$.

f. Check your answer by multiplying factors.

A.4. Consider the trinomial $8c^2 - 18cd + 9d^2$.

a. What is the greatest common factor?

b. Which product will produce a leading term of $8c^2$? There may be more than one answer.

$$8c^2 - 18cd + 9d^2 = (8c \quad)(c \quad)$$
$$8c^2 - 18cd + 9d^2 = (4c \quad)(2c \quad)$$
$$8c^2 - 18cd + 9d^2 = (c \quad)(8c \quad)$$

c. To factor the trinomial $8c^2 - 18cd + 9d^2$ by the trial-and-error method, would the signs in the binomials be both positive, both negative, or different?

d. Choose one of the products of binomials from part (b). Then try different pairs of factors of $9d^2$ until you find the correct factorization. If you do not find the correct factorization, try using a different combination from part (b).

e. Check your answer by multiplying factors.

A.5. Consider the trinomial $-3y^3 + 12y^2 + 96y$.

a. What is the greatest common factor?

b. Factor out $-3y$ so that the leading term of the trinomial within the parentheses is positive.

c. To factor the trinomial $y^2 - 4y - 32$, would the signs in the binomials be both positive, both negative, or different?

d. Complete the factorization by factoring the trinomial from part (b).

e. Check your answer by multiplying factors.

A.6. **a.** Identify the values that are perfect squares.

1, 2, 4, 6, 9, 15, 16, 18, 25, 30, 36, 40, 49, 64, 81, 100, 120, 121

b. Identify the expressions that are perfect squares.

$x, x^2, x^3, x^4, x^6, x^8, x^9, x^{10}, x^{12}$

c. For what positive values of n is the expression x^n a perfect square?

A.7. **a.** Multiply. $(a + b)^2$

b. Factor. $a^2 + 2ab + b^2$

c. Multiply. $(3p + 7)^2$

d. Factor. $9p^2 + 42p + 49$

e. How do you recognize a perfect square trinomial?

A.8. Factor completely. $8m^2 - 40mn + 50n^2$

Section 4.6 Practice Exercises

Prerequisite Review

R.1. Find two integers whose product is -40 and whose sum is 6.

R.2. Find two integers whose product is 48 and whose sum is -14.

For Exercises R.3–R.4, write the trinomial in descending order.

R.3. $14t + 5t^2 + 13$

R.4. $22 - 13b + 4b^2$

For Exercises R.5–R.6, factor out the greatest common factor.

R.5. $4x^3 + 20x^2 + 12x$

R.6. $6y^4 - 16y^3 + 8y^2$

For Exercises R.7–R.14, perform the indicated operations.

R.7. $(x - 3)(x - 11)$

R.8. $(t + 5)(t - 7)$

R.9. $(2p - 5)(8p + 3)$

R.10. $(4w + 3)(2w + 1)$

R.11. $2cd^2(c - 8)(3c + 2)$

R.12. $3xy(x - 4)(2x - 7)$

R.13. $(3c + 8)^2$

R.14. $(2d - 7)^2$

For Exercises R.15–R.18, factor completely.

R.15. $2x(3a - b) - (3a - b)$

R.16. $4y(2z + 5) - 3(2z + 5)$

R.17. $8x^2 + 6x - 20x - 15$

R.18. $12t^2 - 9t + 8t - 6$

Vocabulary and Key Concepts

1. **a.** Given a trinomial $x^2 + bx + c$, if c is positive, then the signs in the binomial factors are either both ____________ or both negative.

b. Given a trinomial $x^2 + bx + c$, if c is negative, then the signs in the binomial factors are (choose one: both positive, both negative, opposite).

c. Which is the correct factored form of $2x^2 - 5x - 12$, the product $(2x + 3)(x - 4)$ or $(x - 4)(2x + 3)$?

d. Which is the complete factorization of $6x^2 - 4x - 10$, the product $(3x - 5)(2x + 2)$ or $2(3x - 5)(x + 1)$?

e. A perfect square trinomial $a^2 + 2ab + b^2$ factors as ____________.
Likewise $a^2 - 2ab + b^2$ factors as ____________.

2. Explain how to check a factoring problem.

3. To factor $2x^2 - 11x + 12$ as a product of binomials, would the signs in the binomial factors be both positive, both negative, or different?

4. To factor $6x^2 - 11x - 35$ as a product of binomials, would the signs in the binomial factors be both positive, both negative, or different?

Concepts 1–2: Factoring Trinomials

For Exercises 5–8, fill in the blanks.

5. $2x^2 - 5x - 12 = (2x + \square)(x - \square)$

6. $3x^2 + 14x - 5 = (3x - \square)(x + \square)$

7. $10x^2 + 31x + 15 = (\square + 3)(\square + 5)$

8. $8x^2 - 38x + 35 = (\square - 5)(\square - 7)$

In Exercises 9–46, factor the trinomial completely by using any method. Remember to look for a common factor first. **(See Examples 1–7.)**

9. $b^2 - 12b + 32$

10. $a^2 - 12a + 27$

11. $y^2 + 10y - 24$

12. $w^2 + 3w - 54$

13. $x^2 + 13x + 30$

14. $t^2 + 9t + 8$

15. $c^2 - 6c - 16$

16. $z^2 - 3z - 28$

17. $2x^2 - 7x - 15$

18. $2y^2 - 13y + 15$

19. $a + 6a^2 - 5$

20. $10b^2 - 3 - 29b$

21. $s^2 + st - 6t^2$

22. $p^2 - pq - 20q^2$

23. $3x^2 - 60x + 108$

24. $4c^2 + 12c - 72$

25. $2c^2 - 2c - 24$

26. $3x^2 + 12x - 15$

27. $2x^2 + 8xy - 10y^2$

28. $20z^2 + 26zw - 28w^2$

29. $33t^2 - 18t + 2$

30. $5p^2 - 10p + 7$

31. $3x^2 + 14xy + 15y^2$

32. $2a^2 + 15ab - 27b^2$

33. $5u^3v - 30u^2v^2 + 45uv^3$

34. $3a^3 + 30a^2b + 75ab^2$

35. $x^3 - 5x^2 - 14x$

36. $p^3 + 2p^2 - 24p$

37. $-23z - 5 + 10z^2$

38. $3 + 16y^2 + 14y$

39. $b^2 + 2b + 15$

40. $x^2 - x - 1$

41. $-2t^2 + 12t + 80$

42. $-3c^2 + 33c - 72$

43. $14a^2 + 13a - 12$

44. $12x^2 - 16x + 5$

45. $6a^2b + 22ab + 12b$

46. $6cd^2 + 9cd - 42c$

Concept 3: Factoring Perfect Square Trinomials

47. **a.** Multiply the binomials $(x + 5)(x + 5)$.
b. Factor $x^2 + 10x + 25$.

48. **a.** Multiply the binomials $(2w - 5)(2w - 5)$.
b. Factor $4w^2 - 20w + 25$.

49. **a.** Multiply the binomials $(3x - 2y)^2$.
b. Factor $9x^2 - 12xy + 4y^2$.

50. **a.** Multiply the binomials $(x + 7y)^2$.
b. Factor $x^2 + 14xy + 49y^2$.

For Exercises 51–54, fill in the blank to make the trinomial a perfect square trinomial.

51. $9x^2 + (\_\_\_\_\_\_) + 25$

52. $16x^4 - (\_\_\_\_\_\_) + 1$

53. $64z^4 + (\_\_\_\_\_\_) + t^2$

54. $9m^4 - (\_\_\_\_\_\_) + 49n^2$

For Exercises 55–66, factor out the greatest common factor, if necessary. Then determine if the polynomial is a perfect square trinomial. If it is, factor it if possible. **(See Examples 8–9.)**

55. $y^2 - 8y + 16$

56. $x^2 + 10x + 25$

57. $64m^2 + 80m + 25$

58. $100c^2 - 140c + 49$

59. $w^2 - 5w + 9$

60. $2a^2 + 14a + 98$

61. $9a^2 - 30ab + 25b^2$

62. $16x^4 - 48x^2y + 9y^2$

63. $16t^2 - 80tv + 20v^2$

64. $12x^2 - 12xy + 3y^2$

65. $5b^4 - 20b^2 + 20$

66. $a^4 + 12a^2 + 36$

Concept 4: Factoring by Using Substitution

For Exercises 67–70, factor the polynomial in part (a). Then use substitution to help factor the polynomials in parts (b) and (c).

67. **a.** $u^2 - 10u + 25$
b. $x^4 - 10x^2 + 25$
c. $(a + 1)^2 - 10(a + 1) + 25$

68. **a.** $u^2 + 12u + 36$
b. $y^4 + 12y^2 + 36$
c. $(b - 2)^2 + 12(b - 2) + 36$

69. **a.** $u^2 + 11u - 26$
b. $w^6 + 11w^3 - 26$
c. $(y - 4)^2 + 11(y - 4) - 26$

70. **a.** $u^2 + 17u + 30$
b. $z^6 + 17z^3 + 30$
c. $(x + 3)^2 + 17(x + 3) + 30$

For Exercises 71–82, factor by using substitution. **(See Examples 10–11.)**

71. $(3x - 1)^2 - (3x - 1) - 6$

72. $(2x + 5)^2 - (2x + 5) - 12$

73. $2(x - 5)^2 + 9(x - 5) + 4$

74. $4(x - 3)^2 + 7(x - 3) + 3$

75. $3(y + 4)^2 + 5(y + 4) - 2$

76. $(3t - 2)^2 - (3t - 2) - 20$

77. $3y^6 + 11y^3 + 6$

78. $3x^4 - 5x^2 - 12$

79. $4p^4 + 5p^2 + 1$

80. $t^4 + 3t^2 + 2$

81. $x^4 + 15x^2 + 36$

82. $t^6 - 16t^3 + 63$

Mixed Exercises

83. A student factored $4y^2 - 10y + 4$ as $(2y - 1)(2y - 4)$ on her factoring test. Why did her professor deduct several points, even though $(2y - 1)(2y - 4)$ does multiply out to $4y^2 - 10y + 4$?

84. A student factored $9w^2 + 36w + 36$ as $(3w + 6)^2$ on his factoring test. Why did his instructor deduct several points, even though $(3w + 6)^2$ does multiply out to $9w^2 + 36w + 36$?

For Exercises 85–105, factor completely by using an appropriate method. (Be sure to note the number of terms in the polynomial.)

85. $w^4 + 12w^2 + 36$

86. $9 - 6t^2 + t^4$

87. $81w^2 + 90w + 25$

88. $49a^2 - 28ab + 4b^2$

89. $3x(a + b) - 6(a + b)$

90. $4p(t - 8) + 2(t - 8)$

91. $12a^2bc^2 + 4ab^2c^2 - 6abc^3$

92. $18x^2z - 6xyz + 30xz^2$

93. $-20x^3 + 74x^2 - 60x$

94. $-24y^3 + 90y^2 - 75y$

95. $2y^2 - 9y - 4$

96. $3w^2 - 12w + 4$

97. $2(w^2 - 5)^2 + (w^2 - 5) - 15$

98. $5(t^2 + 3)^2 + 21(t^2 + 3) + 4$

99. $1 - 4d + 3d^2$

100. $2 - 5a + 2a^2$

101. $ax - 5a^2 + 2bx - 10ab$

102. $my + y^2 - 3xm - 3xy$

103. $8z^2 + 24zw - 224w^2$

104. $9x^2 - 18xy - 135y^2$

105. $ay + ax - 5cy - 5cx$

For Exercises 106–114, factor the expression that defines each function.

106. $f(x) = 2x^2 + 13x - 7$

107. $g(x) = 3x^2 + 14x + 8$

108. $m(t) = t^2 - 22t + 121$

109. $n(t) = t^2 + 20t + 100$

110. $P(x) = x^3 + 4x^2 + 3x$

111. $Q(x) = x^4 + 6x^3 + 8x^2$

112. $h(a) = a^3 + 5a^2 - 6a - 30$

113. $k(a) = a^3 - 4a^2 + 2a - 8$

114. $f(x) = 3x^3 - 9x^2 + 5x - 15$

Factoring Binomials

Section 4.7

Concepts

1. Difference of Squares
2. Using a Difference of Squares in Grouping
3. Sum and Difference of Cubes
4. Summary of Factoring Binomials
5. Factoring Binomials of the Form $x^6 - y^6$

1. Difference of Squares

Up to this point we have learned how to

- Factor out the greatest common factor from a polynomial.
- Factor a four-term polynomial by grouping.
- Recognize and factor perfect square trinomials.
- Factor trinomials by the ac-method and by the trial-and-error method.

Next, we will learn how to factor binomials that fit the pattern of a difference of squares. Recall that the product of two conjugates results in a **difference of squares**.

$$(a + b)(a - b) = a^2 - b^2$$

Therefore, to factor a difference of squares, the process is reversed. Identify a and b and construct the conjugate factors.

Factored Form of a Difference of Squares

$$a^2 - b^2 = (a + b)(a - b)$$

FOR REVIEW

When multiplying the product of conjugates, the middle terms "cancel," resulting in a difference of squares.

$$(4x+3)(4x-3)$$
$$= 16x^2 - \cancel{12x} + \cancel{12x} - 9$$
$$= 16x^2 - 9$$

Example 1 Factoring a Difference of Squares

Factor the binomial completely. $16x^2 - 9$

Solution:

$16x^2 - 9$	The GCF is 1. The binomial is a difference of squares.
$= (4x)^2 - (3)^2$	Write in the form $a^2 - b^2$, where $a = 4x$ and $b = 3$.
$= (4x+3)(4x-3)$	Factor as $(a+b)(a-b)$.

Skill Practice Factor completely.

1. $4z^2 - 1$

Example 2 Factoring a Difference of Squares

Factor the binomial completely. $98c^2d - 50d^3$

Solution:

$98c^2d - 50d^3$	
$= 2d(49c^2 - 25d^2)$	The GCF is $2d$. The resulting binomial is a difference of squares.
$= 2d[(7c)^2 - (5d)^2]$	Write in the form $a^2 - b^2$, where $a = 7c$ and $b = 5d$.
$= 2d(7c+5d)(7c-5d)$	Factor as $(a+b)(a-b)$.

Skill Practice Factor completely.

2. $7y^3z - 63yz^3$

Example 3 Factoring a Difference of Squares

Factor the binomial completely. $z^4 - 81$

Solution:

$z^4 - 81$	The GCF is 1. The binomial is a difference of squares.
$= (z^2)^2 - (9)^2$	Write in the form $a^2 - b^2$, where $a = z^2$ and $b = 9$.
$= (z^2+9)(z^2-9)$	Factor as $(a+b)(a-b)$. $z^2 - 9$ is also a difference of squares.
$= (z^2+9)\overbrace{(z+3)(z-3)}$	

Skill Practice Factor completely.

3. $b^4 - 16$

Answers

1. $(2z-1)(2z+1)$
2. $7yz(y+3z)(y-3z)$
3. $(b^2+4)(b-2)(b+2)$

The difference of squares $a^2 - b^2$ factors as $(a - b)(a + b)$. However, the *sum* of squares is not factorable.

Sum of Squares

Suppose a and b have no common factors. Then the **sum of squares** $a^2 + b^2$ is *not* factorable over the real numbers.

That is, $a^2 + b^2$ is prime over the real numbers.

To see why $a^2 + b^2$ is not factorable, consider the product of binomials:

$$(a \quad b)(a \quad b) \stackrel{?}{=} a^2 + b^2$$

If all possible combinations of signs are considered, none produces the correct product.

$(a + b)(a - b) = a^2 - b^2$	Wrong sign
$(a + b)(a + b) = a^2 + 2ab + b^2$	Wrong middle term
$(a - b)(a - b) = a^2 - 2ab + b^2$	Wrong middle term

After exhausting all possibilities, we see that if a and b share no common factors, then the sum of squares $a^2 + b^2$ is a prime polynomial.

2. Using a Difference of Squares in Grouping

Sometimes a difference of squares can be used along with other factoring techniques.

Example 4 Using a Difference of Squares in Grouping

Factor completely. $\quad y^3 - 6y^2 - 4y + 24$

Solution:

$y^3 - 6y^2 - 4y + 24$	The GCF is 1.
$= y^3 - 6y^2 \;\vdots\; - 4y + 24$	The polynomial has four terms. Factor by grouping.
$= y^2(y - 6) - 4(y - 6)$	
$= (y - 6)(y^2 - 4)$	$y^2 - 4$ is a difference of squares.
$= (y - 6)\overbrace{(y + 2)(y - 2)}$	

Skill Practice Factor completely.

4. $a^3 + 5a^2 - 9a - 45$

Answer

4. $(a + 5)(a - 3)(a + 3)$

Example 5 Factoring a Four-Term Polynomial by Grouping Three Terms

Factor completely. $x^2 - y^2 - 6y - 9$

Solution:

Grouping "2 by 2" will not work to factor this polynomial. However, if we factor out -1 from the last three terms, the resulting trinomial will be a perfect square trinomial.

$x^2 \;\vdots\; - y^2 - 6y - 9$	Group the last three terms.
$= x^2 - 1(y^2 + 6y + 9)$	Factor out -1 from the last three terms.
$= x^2 - (y + 3)^2$	Factor the perfect square trinomial $y^2 + 6y + 9$ as $(y + 3)^2$.
	The quantity $x^2 - (y + 3)^2$ is a difference of squares, $a^2 - b^2$, where $a = x$ and $b = (y + 3)$.
$= [x - (y + 3)]\,[x + (y + 3)]$	Factor as $a^2 - b^2 = (a + b)(a - b)$.
$= (x - y - 3)(x + y + 3)$	Apply the distributive property to clear the inner parentheses.

Avoiding Mistakes

When factoring the expression $x^2 - (y + 3)^2$ as a difference of squares, be sure to use parentheses around the quantity $(y + 3)$. This will help you remember to "distribute the negative" in the expression $[x - (y + 3)]$.

$[x - (y + 3)] = (x - y - 3)$

Skill Practice Factor completely.

5. $x^2 + 10x + 25 - y^2$

TIP: From Example 5, the expression $x^2 - (y + 3)^2$ can also be factored by using substitution. Let $u = y + 3$.

$x^2 - (y + 3)^2$	
$= x^2 - u^2$	Substitution $u = y + 3$.
$= (x - u)(x + u)$	Factor as a difference of squares.
$= [x - (y + 3)][x + (y + 3)]$	Substitute back.
$= (x - y - 3)(x + y + 3)$	Apply the distributive property.

3. Sum and Difference of Cubes

For binomials that represent the sum or difference of cubes, factor by using the following formulas.

Factored Form of a Sum and Difference of Cubes

Sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Answer

5. $(x + 5 - y)(x + 5 + y)$

Multiplication can be used to confirm the formulas for factoring a sum or difference of cubes.

$$(a+b)(a^2-ab+b^2) = a^3 - \cancel{a^2b} + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} + b^3 = a^3 + b^3 \checkmark$$

$$(a-b)(a^2+ab+b^2) = a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{a^2b} - \cancel{ab^2} - b^3 = a^3 - b^3 \checkmark$$

To help you remember the formulas for factoring a sum or difference of cubes, keep the following guidelines in mind.

- The factored form is the product of a binomial and a trinomial.
- The first and third terms in the trinomial are the squares of the terms within the binomial factor. Therefore, these terms are always positive.
- Without regard to sign, the middle term in the trinomial is the product of terms in the binomial factor.

$$x^3 + 8 = (x)^3 + (2)^3 = (x+2)\Big[\overset{\text{Square the first term of the binomial.}}{(x)^2} - \overset{\text{Product of terms in the binomial}}{(x)(2)} + \underset{\text{Square the last term of the binomial.}}{(2)^2}\Big]$$

- The sign within the binomial factor is the same as the sign of the original binomial.
- The first and third terms in the trinomial are always positive.
- The sign of the middle term in the trinomial is opposite the sign within the binomial.

Same sign Positive

$$x^3 + 8 = (x)^3 + (2)^3 = (x+2)[(x)^2 - (x)(2) + (2)^2]$$

Opposite signs

TIP: The following are perfect cubes.

$1^3 = 1$	$(x^1)^3 = x^3$
$2^3 = 8$	$(x^2)^3 = x^6$
$3^3 = 27$	$(x^3)^3 = x^9$
$4^3 = 64$	$(x^4)^3 = x^{12}$
$\vdots$	$\vdots$

Any expression raised to a multiple of 3 is a perfect cube.

TIP: To help remember the placement of the signs in factoring the sum or difference of cubes, remember SOAP: Same sign, Opposite signs, Always Positive.

Example 6 Factoring a Difference of Cubes

Factor. $8x^3 - 27$

Solution:

$8x^3 - 27$	$8x^3$ and 27 are perfect cubes.
$= (2x)^3 - (3)^3$	Write as $a^3 - b^3$, where $a = 2x$ and $b = 3$.
$a^3 - b^3 = (a-b)(a^2+ab+b^2)$	
$(2x)^3 - (3)^3 = (2x-3)[(2x)^2 + (2x)(3) + (3)^2]$	Apply the difference of cubes formula.
$= (2x-3)(4x^2+6x+9)$	Simplify.

Skill Practice Factor completely.

6. $125p^3 - 8$

Answer

6. $(5p-2)(25p^2+10p+4)$

Example 7 Factoring a Sum of Cubes

Factor. $125t^3 + 64z^6$

Solution:

$125t^3 + 64z^6$ — $125t^3$ and $64z^6$ are perfect cubes.

$= (5t)^3 + (4z^2)^3$ — Write as $a^3 + b^3$, where $a = 5t$ and $b = 4z^2$.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ — Apply the sum of cubes formula.

$(5t)^3 + (4z^2)^3 = [(5t) + (4z^2)][(5t)^2 - (5t)(4z^2) + (4z^2)^2]$

$= (5t + 4z^2)(25t^2 - 20tz^2 + 16z^4)$ — Simplify.

Skill Practice Factor completely.

7. $x^3 + 1000y^6$

4. Summary of Factoring Binomials

After factoring out the greatest common factor, the next step in any factoring problem is to recognize what type of pattern it follows. Exponents that are divisible by 2 are perfect squares, and those divisible by 3 are perfect cubes. The formulas for factoring binomials are summarized here.

Summary of Factoring Binomials

- Difference of squares: $a^2 - b^2 = (a + b)(a - b)$
- Difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
- Sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Example 8 Review of Factoring Binomials

Factor the binomials.

a. $m^3 - \frac{1}{8}$ **b.** $9k^2 + 24m^2$ **c.** $128y^6 + 54x^3$ **d.** $50y^6 - 8x^2$

Solution:

a. $m^3 - \frac{1}{8}$ — m^3 is a perfect cube: $m^3 = (m)^3$. $\frac{1}{8}$ is a perfect cube: $\frac{1}{8} = (\frac{1}{2})^3$.

$= (m)^3 - \left(\frac{1}{2}\right)^3$ — This is a difference of cubes, where $a = m$ and $b = \frac{1}{2}$.

$= \left(m - \frac{1}{2}\right)\left(m^2 + \frac{1}{2}m + \frac{1}{4}\right)$ — $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Answer

7. $(x + 10y^2)(x^2 - 10xy^2 + 100y^4)$

b. $9k^2 + 24m^2$ — Factor out the GCF.

$= 3(3k^2 + 8m^2)$ — The resulting binomial is not a difference of squares or a sum or difference of cubes. It cannot be factored further over the real numbers.

c. $128y^6 + 54x^3$ — Factor out the GCF.

$= 2(64y^6 + 27x^3)$ — Both 64 and 27 are perfect cubes, and the exponents of both x and y are multiples of 3. This is a sum of cubes, where $a = 4y^2$ and $b = 3x$.

$= 2[(4y^2)^3 + (3x)^3]$

$= 2(4y^2 + 3x)(16y^4 - 12xy^2 + 9x^2)$ — $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

d. $50y^6 - 8x^2$ — Factor out the GCF.

$= 2(25y^6 - 4x^2)$ — Both 25 and 4 are perfect squares. The exponents of both x and y are multiples of 2. This is a difference of squares, where $a = 5y^3$ and $b = 2x$.

$= 2[(5y^3)^2 - (2x)^2]$

$= 2(5y^3 + 2x)(5y^3 - 2x)$ — $a^2 - b^2 = (a + b)(a - b)$

Skill Practice Factor the binomials.

8. $x^2 - \frac{1}{25}$ **9.** $16y^3 + 4y$ **10.** $24a^7 - 3a$ **11.** $18p^4 - 50t^2$

FOR REVIEW

Remember that the first step to factor any polynomial is to factor out the greatest common factor, GCF.

$128y^6 + 54x^3$
$= 2(64y^6 + 27x^3)$

5. Factoring Binomials of the Form $x^6 - y^6$

Example 9 Factoring Binomials

Factor the binomial $x^6 - y^6$ as

a. A difference of cubes **b.** A difference of squares

Solution:

Notice that the expressions x^6 and y^6 are both perfect squares and perfect cubes because the exponents are both multiples of 2 and of 3. Consequently, $x^6 - y^6$ can be interpreted initially as either a difference of cubes or a difference of squares.

a. $x^6 - y^6$

Difference of cubes

$= (x^2)^3 - (y^2)^3$ — Write as $a^3 - b^3$, where $a = x^2$ and $b = y^2$.

$= (x^2 - y^2)[(x^2)^2 + (x^2)(y^2) + (y^2)^2]$ — Apply the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$= (x^2 - y^2)(x^4 + x^2y^2 + y^4)$ — Factor $x^2 - y^2$ as a difference of squares.

$= \overbrace{(x + y)(x - y)}(x^4 + x^2y^2 + y^4)$ — The expression $x^4 + x^2y^2 + y^4$ cannot be factored by using the skills learned thus far.

Answers

8. $\left(x + \frac{1}{5}\right)\left(x - \frac{1}{5}\right)$
9. $4y(4y^2 + 1)$
10. $3a(2a^2 - 1)(4a^4 + 2a^2 + 1)$
11. $2(3p^2 + 5t)(3p^2 - 5t)$

b. $x^6 - y^6$

Difference of squares

$= (x^3)^2 - (y^3)^2$ — Write as $a^2 - b^2$, where $a = x^3$ and $b = y^3$.

$= (x^3 + y^3)(x^3 - y^3)$ — Apply the formula $a^2 - b^2 = (a + b)(a - b)$.

Factor $x^3 + y^3$ as a sum of cubes. Factor $x^3 - y^3$ as a difference of cubes.

$$= \overbrace{(x + y)(x^2 - xy + y^2)}^{\text{Sum of cubes}}\overbrace{(x - y)(x^2 + xy + y^2)}^{\text{Difference of cubes}}$$

TIP: Notice that the expressions x^6 and y^6 are both perfect squares and perfect cubes because both exponents are multiples of 2 and of 3. Consequently, $x^6 - y^6$ can be factored initially as either the difference of squares or as the difference of cubes. In such a case, it is recommended that you factor the expression as a difference of squares first because it factors more completely into polynomials of lower degree.

$$x^6 - y^6 = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$$

Answer

12. $(a - 2)(a + 2)(a^2 + 2a + 4)(a^2 - 2a + 4)$

Skill Practice Factor completely.

12. $a^6 - 64$

Section 4.7 Activity

A.1. **a.** Multiply. $(a - b)(a + b)$ **b.** Factor. $a^2 - b^2$

c. Multiply. $(5y - 4)(5y + 4)$ **d.** Factor. $25y^2 - 16$

e. How do you recognize a difference of squares?

For Exercises A.2–A.4, factor completely. Recall that you can check your answers by multiplying factors.

A.2. $64p^2 - 49$

A.3. $45m^4 - 20n^2$

A.4. $81t^4 - 16$

A.5. **a.** What technique of factoring is generally tried when factoring a four-term polynomial?

b. Use factoring by grouping to factor the polynomial $2x^3 + 3x^2 - 50x - 75$ into a product of two binomials.

c. Can either binomial from part (b) be factored further? Explain.

d. Factor $2x^3 + 3x^2 - 50x - 75$ completely.

A.6. Consider the polynomial $y^2 - z^2 - 8z - 16$.

a. Explain why factoring by grouping the first two terms with the second two terms will not work.

b. Instead, factor out –1 from the last three terms and rewrite the polynomial.

c. The purpose of factoring out –1 from the last three terms is that the resulting trinomial is a perfect square trinomial. Write the polynomial from part (b) with the trinomial in factored form.

d. Factor the expression from part (c) as a difference of squares.

A.7. **a.** Identify the values that are perfect cubes. 1, 4, 8, 9, 15, 16, 24, 27, 36, 48, 64, 100, 125
b. Identify the expressions that are perfect cubes. $x, x^2, x^3, x^4, x^6, x^8, x^9, x^{10}, x^{12}$
c. For what positive values of n is the expression x^n a perfect cube?

A.8. **a.** Multiply $(a - b)(a^2 + ab + b^2)$ **b.** Factor. $a^3 - b^3$

A.9. Factor $125x^3 + 64$ as a sum of cubes by using the following steps.
a. Write the formula to factor a sum of cubes. $a^3 + b^3 = (\quad)(\quad)$
b. Write $125x^3$ as a perfect cube. $125x^3 = (\quad)^3$
c. Write 64 as a perfect cube. $64 = (\quad)^3$
d. To factor the polynomial $125x^3 + 64$ as a sum of cubes, $a^3 + b^3$, identify the value of a and the value of b.
e. Factor $125x^3 + 64$ using the formula in part (a).
f. Check the factored form by multiplying the factors.

A.10. Factor $c^6 - 8d^3$ as a difference of cubes by using the following steps.
a. Write the formula to factor a difference of cubes. $a^3 - b^3 = (\quad)(\quad)$
b. Write c^6 as a perfect cube. $c^6 = (\quad)^3$
c. Write $8d^3$ as a perfect cube. $8d^3 = (\quad)^3$
d. To factor the polynomial $c^6 - 8d^3$ as a difference of cubes, $a^3 - b^3$, identify the value of a and the value of b.
e. Factor $c^6 - 8d^3$ using the formula in part (a).
f. Check the factored form by multiplying the factors.

Practice Exercises — Section 4.7

Study Skills Exercise

A valuable skill in mathematics is the ability to communicate your understanding of the steps required to solve a problem. This skill is especially important when factoring polynomials. In order to factor a polynomial successfully, you must recognize the strategy that is appropriate for the problem and how to properly apply that strategy. At this point in the text, we have covered several techniques to factor a polynomial. Suppose that you want to tutor a friend in how to factor a polynomial.

- Write a step-by-step approach that you would follow to assess and factor a polynomial. Consider all different scenarios, such as factoring two-term, three-term, and four-term polynomials. These guidelines will help your overall understanding (and your friend's) and will help you commit these strategies to long-term memory. Refer to this list when completing the Practice Exercises, and commit the strategies to memory.

Prerequisite Review

For Exercises R.1–R.6, fill in the blank.

R.1. $25y^4 = (\quad)^2$

R.2. $81b^6 = (\quad)^2$

R.3. $64p^3 = (\quad)^3$

R.4. $8q^3 = (\quad)^3$

R.5. $-27t^6 = (\quad)^3$

R.6. $-1000n^9 = (\quad)^3$

For Exercises R.7–R.14, perform the indicated operations.

R.7. $(3x - 5)(3x + 5)$

R.8. $(2y + 11)(2y - 11)$

R.9. $(t - 4)(t + 4)(t^2 + 16)$

R.10. $(m - 5)(m + 5)(m^2 + 25)$

R.11. $(x - 8)^2$

R.12. $(m + 4)^2$

R.13. $(y - 2)(y^2 + 2y + 4)$

R.14. $(p + 1)(p^2 - p + 1)$

Vocabulary and Key Concepts

1. **a.** The binomial $x^2 - 36$ is an example of a ___________ of squares. A difference of squares $a^2 - b^2$ factors as ___________.

   b. The binomial $y^2 + 9$ is an example of a ___________ of squares.

   c. A sum of squares with greatest common factor 1 (is/is not) factorable over the real numbers.

   d. The square of a binomial always results in a perfect ___________ trinomial.

2. The binomial $x^3 + 64$ is an example of a ___________ of ___________.
3. The binomial $c^3 - 27$ is an example of a ___________ of ___________.
4. A difference of cubes $a^3 - b^3$ factors as (   )(   ).
5. A sum of cubes $a^3 + b^3$ factors as (   )(   ).
6. Identify which expressions represent perfect squares. 2, 4, 8, 16, 25, 64, x^2, x^3, x^4, x^5, x^9
7. Identify which expressions represent perfect cubes. 2, 4, 8, 16, 25, 64, x^2, x^3, x^4, x^5, x^9

Concept 1: Difference of Squares

8. Explain how to identify and factor a difference of squares.
9. Can you factor $25x^2 + 4$?
10. What is the first step to factor $18x^2 - 50$?

For Exercises 11–22, factor the binomials. Identify the binomials that are prime. **(See Examples 1–3.)**

11. $x^2 - 9$
12. $y^2 - 25$
13. $16 - 49w^2$
14. $81 - 64b^2$
15. $8a^2 - 162b^2$
16. $50c^2 - 72d^2$
17. $25u^2 + 1$
18. $w^2 + 4$
19. $2a^4 - 32$
20. $5y^4 - 5$
21. $49 - k^6$
22. $4 - h^6$

Concept 2: Using a Difference of Squares in Grouping

For Exercises 23–36, use the difference of squares along with factoring by grouping. **(See Examples 4–5.)**

23. $x^3 - x^2 - 16x + 16$
24. $x^3 + 5x^2 - x - 5$
25. $4x^3 + 12x^2 - x - 3$
26. $5x^3 - x^2 - 45x + 9$
27. $9y^3 + 7y^2 - 36y - 28$
28. $9z^3 - 5z^2 - 36z + 20$
29. $49x^2 + 28x + 4 - y^2$
30. $100y^2 + 140y + 49 - z^2$
31. $w^2 - 9n^2 + 6n - 1$
32. $m^2 - 25c^2 + 20c - 4$
33. $p^4 - 10p^2 + 25 - t^4$
34. $m^4 - 14m^2 + 49 - z^4$
35. $9u^4 - 4v^4 + 20v^2 - 25$
36. $x^4 - 9y^4 - 42y^2 - 49$

Concept 3: Sum and Difference of Cubes

37. Explain how to identify and factor a sum of cubes.
38. Explain how to identify and factor a difference of cubes.

For Exercises 39–52, factor the sum or difference of cubes. **(See Examples 6–7.)**

39. $8x^3 - 1$ (Check by multiplying.)

40. $y^3 + 64$ (Check by multiplying.)

41. $125c^3 + 27$

42. $216u^3 - v^3$

43. $x^3 - 1000$

44. $y^3 - 27$

45. $64t^6 + 1$

46. $125r^6 + 1$

47. $2000y^6 + 2x^3$

48. $3a^6 + 24b^3$

49. $16z^4 - 54z$

50. $x^5 - 64x^2$

51. $p^{12} - 125$

52. $t^9 - 8$

Concept 4: Summary of Factoring Binomials

For Exercises 53–80, factor completely. **(See Example 8.)**

53. $36y^2 - \frac{1}{25}$

54. $16p^2 - \frac{1}{9}$

55. $18d^{12} - 32$

56. $3z^8 - 12$

57. $242v^2 + 32$

58. $8p^2 + 200$

59. $4x^2 - 16$

60. $9m^2 - 81n^2$

61. $25 - 49q^2$

62. $1 - 25p^2$

63. $(t + 2s)^2 - 36$

64. $(5x + 4)^2 - y^2$

65. $27 - t^3$

66. $8 + y^3$

67. $27a^3 + \frac{1}{8}$

68. $b^3 + \frac{27}{125}$

69. $2m^3 + 16$

70. $3x^3 - 375$

71. $x^4 - y^4$

72. $81u^4 - 16v^4$

73. $a^9 + b^9$

74. $27m^9 - 8n^9$

75. $\frac{1}{8}p^3 - \frac{1}{125}$

76. $1 - \frac{1}{27}d^3$

77. $4w^2 + 25$

78. $64 + a^2$

79. $\frac{1}{25}x^2 - \frac{1}{4}y^2$

80. $\frac{1}{100}a^2 - \frac{4}{49}b^2$

Concept 5: Factoring Binomials of the Form $x^6 - y^6$

For Exercises 81–88, factor completely. **(See Example 9.)**

81. $a^6 - b^6$ (*Hint:* First factor as a difference of squares.)

82. $64x^6 - y^6$

83. $64 - y^6$

84. $1 - p^6$

85. $h^6 + k^6$

86. $27q^6 + 125p^6$

87. $8x^6 + 125$

88. $t^6 + 1$

Mixed Exercises

89. Find a difference of squares that has $(2x + 3)$ as one of its factors.

90. Find a difference of squares that has $(4 - p)$ as one of its factors.

91. Find a difference of cubes that has $(4a^2 + 6a + 9)$ as its trinomial factor.

92. Find a sum of cubes that has $(25c^2 - 10cd + 4d^2)$ as its trinomial factor.

93. Find a sum of cubes that has $(4x^2 + y)$ as its binomial factor.

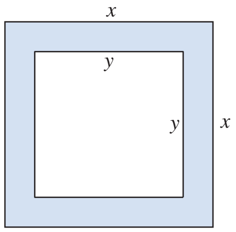
94. Find a difference of cubes that has $(3t - r^2)$ as its binomial factor.

95. Consider the shaded region.

a. Find an expression that represents the area of the shaded region.

b. Factor the expression found in part (a).

c. Find the area of the shaded region if $x = 6$ in. and $y = 4$ in.

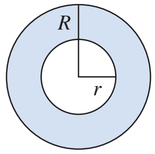


96. A manufacturer needs to know the area of a metal washer. The outer radius of the washer is R and the inner radius is r.

a. Find an expression that represents the area of the washer.

b. Factor the expression found in part (a).

c. Find the area of the washer if $R = \frac{1}{2}$ in. and $r = \frac{1}{4}$ in. (Round to the nearest 0.01 in.^{2})



Expanding Your Skills

For Exercises 97–102, factor the polynomials by using the difference of squares, sum of cubes, or difference of cubes with grouping.

97. $x^2 - y^2 + x + y$

98. $64m^2 - 25n^2 + 8m + 5n$

99. $x^3 + y^3 + x + y$

100. $4pu^3 - 4pv^3 - 7yu^3 + 7yv^3$

101. $576a^5 - 9a^2 - 64a^3c^2 + c^2$

102. $32t^5 - 108t^2 - 72t^3v^2 + 243v^2$

Problem Recognition Exercises

Factoring Summary

We now review the techniques of factoring presented thus far along with a general strategy for factoring polynomials.

Factoring Strategy

Step 1 Factor out the greatest common factor.

Step 2 Identify whether the polynomial has two terms, three terms, or more than three terms.

Step 3 If the polynomial has more than three terms, try factoring by grouping.

Step 4 If the polynomial has three terms, check first for a perfect square trinomial. Otherwise, factor the trinomial with the ac-method or the trial-and-error method.

Step 5 If the polynomial has two terms, determine if it fits the pattern for a difference of squares, difference of cubes, or sum of cubes. Remember, a sum of squares is not factorable over the real numbers.

Step 6 Be sure to factor the polynomial completely.

Step 7 Check by multiplying.

1. What is meant by a prime polynomial?

2. What is the first step in factoring any polynomial?

3. When factoring a binomial, what patterns do you look for?

4. When factoring a trinomial, what pattern do you look for first?

5. What do you look for when factoring a four-term polynomial?

6. How would you use substitution to factor $3(4x^2 + 1)^2 + 20(4x^2 + 1) + 12$.

For Exercises 7–66,

a. Factor out the GCF from each polynomial and identify the category in which the remaining polynomial best fits. Choose from

- difference of squares
- sum of squares
- difference of cubes
- sum of cubes
- perfect square trinomial
- trinomial (ac-method or trial-and-error)
- four terms—grouping
- none of these

b. Factor the polynomial completely.

7. $6x^2 - 21x - 45$

8. $8m^3 - 10m^2 - 3m$

9. $8a^2 - 50$

10. $ab + ay - b^2 - by$

11. $14u^2 - 11uv + 2v^2$

12. $9p^2 - 12pq + 4q^2$

13. $16x^3 - 2$

14. $9m^2 + 16n^2$

15. $27y^3 + 125$

16. $3x^2 - 16$

17. $128p^6 + 54q^3$

18. $5b^2 - 30b + 45$

19. $16a^4 - 1$

20. $81u^2 - 90uv + 25v^2$

21. $p^2 - 12p + 36 - c^2$

22. $4x^2 + 16$

23. $12ax - 6ay + 4bx - 2by$

24. $125y^3 - 8$

25. $5y^2 + 14y - 3$

26. $2m^4 - 128$

27. $t^2 - 100$

28. $4m^2 - 49n^2$

29. $y^3 + 27$

30. $x^3 + 1$

31. $d^2 + 3d - 28$

32. $c^2 + 5c - 24$

33. $x^2 - 12x + 36$

34. $p^2 + 16p + 64$

35. $2ax^2 - 5ax + 2bx - 5b$

36. $8x^2 - 4bx + 2ax - ab$

37. $10y^2 + 3y - 4$

38. $12z^2 + 11z + 2$

39. $10p^2 - 640$

40. $50a^2 - 72$

41. $z^4 - 64z$

42. $t^4 - 8t$

43. $b^3 - 4b^2 - 45b$

44. $y^3 - 14y^2 + 40y$

45. $9w^2 + 24wx + 16x^2$

46. $4k^2 - 20kp + 25p^2$ **47.** $60x^2 - 20x + 30ax - 10a$ **48.** $50x^2 - 200x + 10cx - 40c$

49. $w^4 - 16$ **50.** $k^4 - 81$ **51.** $t^6 - 8$

52. $p^6 + 27$ **53.** $8p^2 - 22p + 5$ **54.** $9m^2 - 3m - 20$

55. $36y^2 - 12y + 1$ **56.** $9a^2 + 42a + 49$ **57.** $2x^2 + 50$

58. $4y^2 + 64$ **59.** $12r^2s^2 + 7rs^2 - 10s^2$ **60.** $7z^2w^2 - 10zw^2 - 8w^2$

61. $x^2 + 8xy - 33y^2$ **62.** $s^2 - 9st - 36t^2$ **63.** $m^6 + n^3$

64. $a^3 - b^6$ **65.** $x^2 - 4x$ **66.** $y^2 - 9y$

For Exercises 67–101, factor completely.

67. $x^2(x + y) - y^2(x + y)$ **68.** $u^2(u - v) - v^2(u - v)$ **69.** $(a + 3)^4 + 6(a + 3)^5$

70. $(4 - b)^4 - 2(4 - b)^3$ **71.** $24(3x + 5)^3 - 30(3x + 5)^2$ **72.** $10(2y + 3)^2 + 15(2y + 3)^3$

73. $\frac{1}{100}x^2 + \frac{1}{35}x + \frac{1}{49}$ **74.** $\frac{1}{25}a^2 + \frac{1}{15}a + \frac{1}{36}$ **75.** $(5x^2 - 1)^2 - 4(5x^2 - 1) - 5$

76. $(x^3 + 4)^2 - 10(x^3 + 4) + 24$ **77.** $16p^4 - q^4$ **78.** $s^4t^4 - 81$

79. $y^3 + \frac{1}{64}$ **80.** $z^3 + \frac{1}{125}$ **81.** $6a^3 + a^2b - 6ab^2 - b^3$

82. $4p^3 + 12p^2q - pq^2 - 3q^3$ **83.** $\frac{1}{9}t^2 + \frac{1}{6}t + \frac{1}{16}$ **84.** $\frac{1}{25}y^2 + \frac{1}{5}y + \frac{1}{4}$

85. $x^2 + 12x + 36 - a^2$ **86.** $a^2 + 10a + 25 - b^2$ **87.** $p^2 + 2pq + q^2 - 81$

88. $m^2 - 2mn + n^2 - 9$ **89.** $b^2 - (x^2 + 4x + 4)$ **90.** $p^2 - (y^2 - 6y + 9)$

91. $4 - u^2 + 2uv - v^2$ **92.** $25 - a^2 - 2ab - b^2$ **93.** $6ax - by + 2bx - 3ay$

94. $5pq - 12 - 4q + 15p$ **95.** $u^6 - 64$ [*Hint:* Factor first as a difference of squares, $(u^3)^2 - (8)^2$.]

96. $1 - v^6$ **97.** $x^8 - 1$ **98.** $y^8 - 256$

99. $a^2 - b^2 + a + b$ **100.** $25c^2 - 9d^2 + 5c - 3d$ **101.** $5wx^3 + 5wy^3 - 2zx^3 - 2zy^3$

Solving Equations by Using the Zero Product Rule

Section 4.8

Concepts

1. **Solving Equations by Using the Zero Product Rule**
2. **Applications of Quadratic Equations**
3. **Definition of a Quadratic Function**
4. **Applications of Quadratic Functions**

1. Solving Equations by Using the Zero Product Rule

Previously we defined a linear equation in one variable as an equation of the form $ax + b = c$ $(a \neq 0)$. A linear equation in one variable is sometimes called a first-degree polynomial equation because the highest degree of all its terms is 1. A second-degree polynomial equation is called a quadratic equation.

Definition of a Quadratic Equation in One Variable

If a, b, and c are real numbers such that $a \neq 0$, then a **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

The following equations are quadratic because they can each be written in the form $ax^2 + bx + c = 0$ $(a \neq 0)$.

$$\begin{aligned} -4x^2 + 4x &= 1 \\ -4x^2 + 4x - 1 &= 0 \end{aligned} \qquad \begin{aligned} x(x-2) &= 3 \\ x^2 - 2x &= 3 \\ x^2 - 2x - 3 &= 0 \end{aligned} \qquad \begin{aligned} (x-4)(x+4) &= 9 \\ x^2 - 16 &= 9 \\ x^2 - 25 &= 0 \\ x^2 + 0x - 25 &= 0 \end{aligned}$$

One method to solve a quadratic equation is to factor and apply the zero product rule. The **zero product rule** states that if the product of two factors is zero, then one or both of its factors is equal to zero.

The Zero Product Rule

If $ab = 0$, then $a = 0$ or $b = 0$.

For example, the quadratic equation $x^2 - x - 12 = 0$ can be written in factored form as $(x - 4)(x + 3) = 0$. By the zero product rule, one or both factors must be zero: $x - 4 = 0$ or $x + 3 = 0$. Therefore, to solve the quadratic equation, set each factor to zero and solve for x.

$(x - 4)(x + 3) = 0$ Apply the zero product rule.

$x - 4 = 0$ or $x + 3 = 0$ Set each factor to zero.

$x = 4$ or $x = -3$ Solve each equation for x.

Quadratic equations, like linear equations, arise in many applications of mathematics, science, and business. The following steps summarize the factoring method to solve a quadratic equation.

Solving a Quadratic Equation by Factoring

Step 1 Write the equation in the form $ax^2 + bx + c = 0$.

Step 2 Factor completely.

Step 3 Apply the zero product rule. That is, set each factor equal to zero and solve the resulting equations.*

*The solution(s) found in step 3 may be checked by substitution in the original equation.

Avoiding Mistakes

The zero product rule tells us that if $ab = 0$, then $a = 0$ or $b = 0$. This property does not hold for other numbers. For example if $ab = 12$, then it is not necessary that a or b must equal 12.

FOR REVIEW

The purpose of factoring a quadratic equation is to rewrite the quadratic equation as two linear equations. Recall that linear equations are *first-degree* equations.

$$2x^2 - 5x - 12 = 0$$
$$(2x + 3)(x - 4) = 0$$
$$2x + 3 = 0 \qquad x - 4 = 0$$

Two linear equations

Example 1 Solving a Quadratic Equation

Solve. $\quad 2x^2 - 5x = 12$

Solution:

$$2x^2 - 5x = 12$$

$$2x^2 - 5x - 12 = 0 \qquad \text{Write the equation in the form } ax^2 + bx + c = 0.$$

$$(2x + 3)(x - 4) = 0 \qquad \text{Factor completely.}$$

$$2x + 3 = 0 \quad \text{or} \quad x - 4 = 0 \qquad \text{Set each factor equal to zero.}$$

$$2x = -3 \quad \text{or} \quad x = 4 \qquad \text{Solve each equation.}$$

$$x = -\frac{3}{2} \quad \text{or} \quad x = 4$$

Check: $x = -\frac{3}{2}$

$$2x^2 - 5x = 12$$
$$2\left(-\frac{3}{2}\right)^2 - 5\left(-\frac{3}{2}\right) \stackrel{?}{=} 12$$
$$2\left(\frac{9}{4}\right) + \frac{15}{2} \stackrel{?}{=} 12$$
$$\frac{9}{2} + \frac{15}{2} \stackrel{?}{=} 12$$
$$\frac{24}{2} \stackrel{?}{=} 12 \checkmark$$

Check: $x = 4$

$$2x^2 - 5x = 12$$
$$2(4)^2 - 5(4) \stackrel{?}{=} 12$$
$$2(16) - 20 \stackrel{?}{=} 12$$
$$32 - 20 \stackrel{?}{=} 12 \checkmark$$

The solution set is $\left\{-\frac{3}{2}, 4\right\}$.

Skill Practice Solve.

1. $y^2 - 2y = 35$

Answer

1. $\{7, -5\}$

Example 2 **Solving a Quadratic Equation**

Solve. $6x^2 + 8x = 0$

Solution:

$6x^2 + 8x = 0$	
$2x(3x + 4) = 0$	Factor completely.
$2x = 0$ or $3x + 4 = 0$	Set each factor equal to zero.
$x = 0$ $\quad$ $3x = -4$	Solve each equation for x.
$x = -\frac{4}{3}$	
The solution set is $\left\{0, -\frac{4}{3}\right\}$.	The solutions check.

Skill Practice Solve.

2. $9x^2 = 21x$

Example 3 **Solving a Quadratic Equation**

Solve. $9x(4x + 2) - 10x = 8x + 25$

Solution:

$9x(4x + 2) - 10x = 8x + 25$	
$36x^2 + 18x - 10x = 8x + 25$	Clear parentheses.
$36x^2 + 8x = 8x + 25$	Combine *like* terms.
$36x^2 - 25 = 0$	Make one side of the equation equal to zero. The equation is in the form $ax^2 + bx + c = 0$. (*Note:* $b = 0$.)
$(6x - 5)(6x + 5) = 0$	Factor completely.
$6x - 5 = 0$ or $6x + 5 = 0$	Set each factor equal to zero.
$6x = 5$ or $6x = -5$	Solve each equation.
$x = \frac{5}{6}$ or $x = -\frac{5}{6}$	The check is left to the reader.

The solution set is $\left\{\frac{5}{6}, -\frac{5}{6}\right\}$.

Skill Practice Solve.

3. $5a(2a - 3) + 4(a + 1) = 3a(3a - 2)$

Answers

2. $\left\{0, \frac{7}{3}\right\}$ **3.** $\{4, 1\}$

Example 4 Solving an Equation

Solve. $2x(x + 5) + 3 = 2x^2 - 5x + 1$

Solution:

$2x(x + 5) + 3 = 2x^2 - 5x + 1$

$2x^2 + 10x + 3 = 2x^2 - 5x + 1$ Clear parentheses.

$15x + 2 = 0$

$15x = -2$

$x = -\frac{2}{15}$

Make one side of the equation equal to zero. The equation is not quadratic. It is in the form $ax + b = 0$, which is linear. Solve by using the method for linear equations.

The solution set is $\left\{-\frac{2}{15}\right\}$. The check is left to the reader.

Skill Practice Solve.

4. $t^2 - 3t + 1 = t^2 + 2t + 11$

The zero product rule can be used to solve higher-degree polynomial equations provided one side of the equation is zero and the other is written in factored form.

Example 5 Solving a Higher-Degree Polynomial Equation

Solve. $-2(y + 7)(y - 1)(10y + 3) = 0$

Solution:

$-2(y + 7)(y - 1)(10y + 3) = 0$

One side of the equation is zero, and the other side is already factored.

$-2 = 0$ or $y + 7 = 0$ or $y - 1 = 0$ or $10y + 3 = 0$ Set each factor equal to zero.

No solution $y = -7$ or $y = 1$ or $y = -\frac{3}{10}$ Solve each equation for y.

Notice that when the constant factor is set to zero, the result is the contradiction $-2 = 0$. The constant factor does not produce a solution to the equation. Therefore, the only solutions are -7, 1, and $-\frac{3}{10}$. Each solution can be checked in the original equation.

The solution set is $\left\{-7, 1, -\frac{3}{10}\right\}$.

Skill Practice Solve.

5. $3(w + 2)(2w + 1)(w - 8) = 0$

Answers

4. $\{-2\}$

5. $\left\{-2, -\frac{1}{2}, 8\right\}$

Example 6 Solving a Higher-Degree Polynomial Equation

Solve. $z^3 + 3z^2 - 4z - 12 = 0$

Solution:

$z^3 + 3z^2 - 4z - 12 = 0$ This is a higher-degree polynomial equation.

$z^3 + 3z^2 \mid - 4z - 12 = 0$ One side of the equation is zero. Now factor. Because there are four terms, try factoring by grouping.

$z^2(z + 3) - 4(z + 3) = 0$

$(z + 3)(z^2 - 4) = 0$ $z^2 - 4$ can be factored further as a difference of squares.

$(z + 3)(z - 2)(z + 2) = 0$

$z + 3 = 0$ or $z - 2 = 0$ or $z + 2 = 0$ Set each factor equal to zero.

$z = -3$ or $z = 2$ or $z = -2$ Solve each equation.

The solution set is $\{-3, 2, -2\}$.

Skill Practice Solve.

6. $x^3 + x^2 - 9x - 9 = 0$

2. Applications of Quadratic Equations

Example 7 Solving an Application of a Quadratic Equation

The product of two consecutive odd integers is 35. Find the integers.

Solution:

Let x represent the smaller odd integer and $x + 2$ represent the next consecutive odd integer.

$\begin{pmatrix}\text{First odd}\\\text{integer}\end{pmatrix} \cdot \begin{pmatrix}\text{next odd}\\\text{integer}\end{pmatrix} = 35$ Verbal model

$x \cdot (x + 2) = 35$ Mathematical equation

$x^2 + 2x = 35$ Clear parentheses.

$x^2 + 2x - 35 = 0$ Set the equation equal to zero.

$(x + 7)(x - 5) = 0$ Factor.

$x + 7 = 0$ or $x - 5 = 0$ Set each factor equal to zero.

$x = -7$ or $x = 5$ Solve each equation.

If $x = -7$ then the next odd integer is $x + 2 = -5$.
If $x = 5$ then the next odd integer is $x + 2 = 7$.

There are two pairs of odd integers that are solutions, $-7, -5$ and $5, 7$.

Skill Practice

7. The product of two consecutive even integers is 48. Find the integers.

Answers

6. $\{-1, 3, -3\}$

7. The integers are -8 and -6 or 6 and 8.

Example 8 Solving an Application of a Quadratic Equation

The length of a basketball court is 6 ft less than 2 times the width. If the total area is 4700 ft^2, find the dimensions of the court.

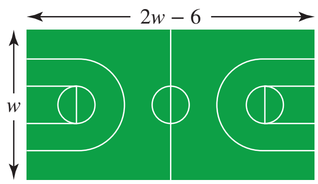


Figure 4-5

Solution:

If the width of the court is represented by w, then the length can be represented by $2w - 6$ (Figure 4-5).

FOR REVIEW

Recall that the area A of a rectangle of length l and width w is $A = lw$.

The area A of a triangle with base b and height h is $A = \frac{1}{2}bh$.

$A = (\text{length})(\text{width})$	Area of a rectangle
$4700 = (2w - 6)w$	Mathematical equation
$4700 = 2w^2 - 6w$	
$2w^2 - 6w - 4700 = 0$	Set the equation equal to zero and factor.
$2(w^2 - 3w - 2350) = 0$	Factor out the GCF.
$2(w - 50)(w + 47) = 0$	Factor the trinomial.
$2 \cancel{=} 0$ or $w - 50 = 0$ or $w + 47 = 0$	Set each factor equal to zero.
contradiction; $w = 50$ or $w \cancel{=} -47$	A negative width is not possible.

The width is 50 ft.

The length is $2w - 6 = 2(50) - 6 = 94$ ft.

Skill Practice

8. The width of a rectangle is 5 in. less than 3 times the length. The area is 2 $in.^2$ Find the length and width.

TIP: When applying the Pythagorean theorem, it does not matter which leg you label a and which you label b. Since the lengths of the legs are interchangeable you can also write the Pythagorean theorem as $\text{leg}^2 + \text{leg}^2 = \text{hyp}^2$.

A right triangle is a triangle that contains a 90° angle. Furthermore, the sum of the squares of the two legs (the shorter sides) of a right triangle equals the square of the hypotenuse (the longest side). This important fact is known as the Pythagorean theorem. For the right triangle shown in Figure 4-6, the Pythagorean theorem is stated as

$$a^2 + b^2 = c^2$$

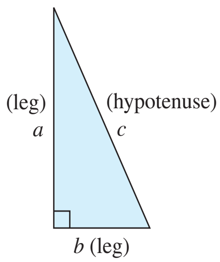


Figure 4-6

In this formula, a and b are the legs and c is the hypotenuse. Notice that the hypotenuse is the longest side and is opposite the right angle.

The triangle given in Figure 4-7 is a right triangle. We have

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (5\text{ ft})^2 + (12\text{ ft})^2 &= (13\text{ ft})^2 \\ 25\text{ ft}^2 + 144\text{ ft}^2 &= 169\text{ ft}^2 \\ 169\text{ ft}^2 &= 169\text{ ft}^2\ \checkmark \end{aligned}$$

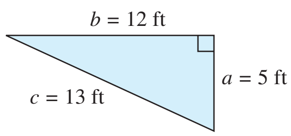


Figure 4-7

Answer

8. The width is 1 in., and the length is 2 in.

Example 9 Application of a Quadratic Equation

A region of coastline off Biscayne Bay is approximately in the shape of a right angle. The corresponding triangular area has sandbars and is marked off on navigational charts as being shallow water. If one leg of the triangle is 0.5 mi shorter than the other leg, and the hypotenuse is 2.5 mi, find the lengths of the legs of the triangle (Figure 4-8).

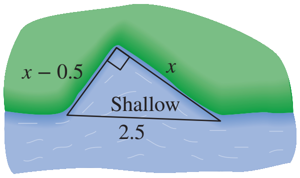


Figure 4-8

Solution:

Let x represent the longer leg.

Then $x - 0.5$ represents the shorter leg.

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean theorem}$$

$$x^2 + (x - 0.5)^2 = (2.5)^2$$

$$x^2 + \overbrace{(x)^2 - 2(x)(0.5) + (0.5)^2} = 6.25$$

$$x^2 + x^2 - x + 0.25 = 6.25$$

TIP: Recall that the square of a binomial results in a perfect square trinomial.

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(x - 0.5)^2 = (x)^2 - 2(x)(0.5) + (0.5)^2$$

$$= x^2 - x + 0.25$$

$$2x^2 - x - 6 = 0$$ Write the equation in the form $ax^2 + bx + c = 0$.

$$(2x + 3)(x - 2) = 0$$ Factor.

$$2x + 3 = 0 \quad \text{or} \quad x - 2 = 0$$ Set both factors to zero.

$$\cancel{x = -\frac{3}{2}} \quad \text{or} \quad x = 2$$ Solve both equations for x.

The side of a triangle cannot be negative, so we reject the solution $x = -\frac{3}{2}$.

Therefore, one leg of the triangle is 2 mi.

The other leg is $x - 0.5 = 2 - 0.5 = 1.5$ mi.

Skill Practice

9. The longer leg of a right triangle measures 7 ft more than the shorter leg. The hypotenuse is 8 ft longer than the shorter leg. Find the lengths of the sides of the triangle.

3. Definition of a Quadratic Function

We have already graphed several basic functions by plotting points, including $f(x) = x^2$. This function is called a quadratic function, and its graph is in the shape of a **parabola**. In general, any second-degree polynomial function is a quadratic function.

Definition of a Quadratic Function

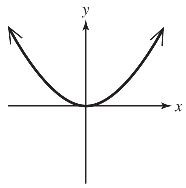
Let a, b, and c represent real numbers such that $a \neq 0$. Then a function defined by $f(x) = ax^2 + bx + c$ is called a **quadratic function**.

Answer

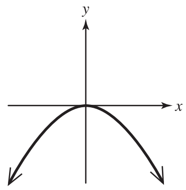
9. The sides are 5 ft, 12 ft, and 13 ft.

The graph of a quadratic function is a parabola that opens upward or downward. The leading coefficient a determines the direction of the parabola. For the quadratic function defined by $f(x) = ax^2 + bx + c$:

If $a > 0$, the parabola opens upward. For example, $f(x) = x^2$



If $a < 0$, the parabola opens downward. For example, $g(x) = -x^2$



Recall that the x-intercepts of a function $y = f(x)$ are the real solutions to the equation $f(x) = 0$. The y-intercept is found by evaluating $f(0)$.

Example 10 Finding the *x*- and *y*-Intercepts of a Quadratic Function

Find the x- and y-intercepts.

$$f(x) = x^2 - x - 12$$

Solution:

To find the x-intercept, substitute $f(x) = 0$.

$f(x) = x^2 - x - 12$	
$0 = x^2 - x - 12$	Substitute 0 for $f(x)$. The result is a quadratic equation.
$0 = (x - 4)(x + 3)$	Factor.
$x - 4 = 0$ or $x + 3 = 0$	Set each factor equal to zero.
$x = 4$ or $x = -3$	Solve each equation.

The x-intercepts are (4, 0) and (−3, 0).

To find the y-intercept, find $f(0)$.

$f(x) = x^2 - x - 12$	
$f(0) = (0)^2 - (0) - 12$	Substitute $x = 0$.
$= -12$	

The y-intercept is (0, −12).

Skill Practice

10. Find the x- and y-intercepts of the function defined by $f(x) = x^2 + 8x + 12$.

Answer

10. x-intercepts: (−6, 0) and (−2, 0); y-intercept: (0, 12)

4. Applications of Quadratic Functions

Example 11 Application of a Quadratic Function

A model rocket is shot vertically upward with an initial velocity of 288 ft/sec. The function given by $h(t) = -16t^2 + 288t$ relates the rocket's height $h(t)$ (in feet) to the time t after launch (in seconds).

a. Find $h(0)$, $h(5)$, $h(10)$, and $h(15)$, and interpret the meaning of these function values in the context of the rocket's height and time after launch.

b. Find the t-intercepts of the function, and interpret their meaning in the context of the rocket's height and time after launch.

c. Find the time(s) at which the rocket is at a height of 1152 ft.

Solution:

a. $h(t) = -16t^2 + 288t$

$h(0) = -16(0)^2 + 288(0) = 0$

$h(5) = -16(5)^2 + 288(5) = 1040$

$h(10) = -16(10)^2 + 288(10) = 1280$

$h(15) = -16(15)^2 + 288(15) = 720$

$h(0) = 0$ means that at $t = 0$ sec, the height of the rocket is 0 ft.

$h(5) = 1040$ means that 5 sec after launch, the height is 1040 ft.

$h(10) = 1280$ means that 10 sec after launch, the height is 1280 ft.

$h(15) = 720$ means that 15 sec after launch, the height is 720 ft.

b. The t-intercepts of the function are represented by the real solutions of the equation $h(t) = 0$.

$-16t^2 + 288t = 0$ Set $h(t) = 0$.

$-16t(t - 18) = 0$ Factor.

$-16t = 0$ or $t - 18 = 0$ Apply the zero product rule.

$t = 0$ or $t = 18$

The rocket is at ground level initially (at $t = 0$ sec) and then again after 18 sec when it hits the ground.

c. Set $h(t) = 1152$ and solve for t.

$h(t) = -16t^2 + 288t$

$1152 = -16t^2 + 288t$ Substitute 1152 for $h(t)$.

$16t^2 - 288t + 1152 = 0$ Set the equation equal to zero.

$16(t^2 - 18t + 72) = 0$ Factor out the GCF.

$16(t - 6)(t - 12) = 0$ Factor.

$t = 6$ or $t = 12$

The rocket will reach a height of 1152 ft after 6 sec (on the way up) and after 12 sec (on the way down). (See Figure 4-9.)

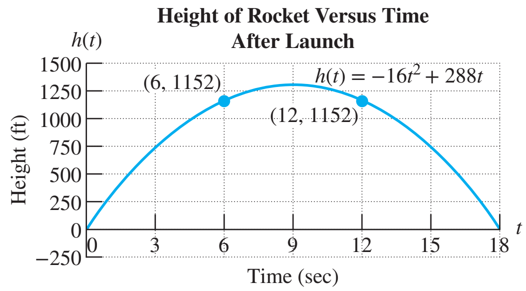


Figure 4-9

Skill Practice An object is dropped from the top of a building that is 144 ft high. The function given by $h(t) = -16t^2 + 144$ relates the height $h(t)$ of the object (in feet) to the time t in seconds after it is dropped.

11. a. Find $h(0)$ and interpret the meaning of the function value in the context of this problem.

b. Find the t-intercept(s) and interpret the meaning in the context of this problem.

c. When will the object be 128 ft above ground level?

Answer

11a. $h(0) = 144$, which is the initial height of the object (after 0 sec).

b. The t-intercept is (3, 0), which means the object is at ground level (0 ft high) after 3 sec. The intercept (−3, 0) does not make sense for this problem since time cannot be negative.

c. One second after release, the object will be 128 ft above ground level.

Section 4.8 Activity

A.1. a. The equation $5x - 20 = 0$ is (choose one: linear or quadratic) because the term of highest degree is (choose one: 1 or 2).

b. The equation $3x^2 + 11x - 4 = 0$ is (choose one: linear or quadratic) because the term of highest degree is (choose one: 1 or 2).

A.2. a. Given the equation $ab = 0$, what do you know about a or b?

b. Given the equation $(2x + 1)(x - 6) = 0$, then either ________ = 0 or ________ = 0.

c. Write the solution set to the equation $(2x + 1)(x - 6) = 0$.

A.3. Solve the equation $3x^2 + 11x - 4 = 0$ by following these steps.

a. Write the equation with one side equal to zero and the other side factored.

b. Set each factor equal to 0.

c. Solve the individual equations from part (b) and write the solution set for the original equation.

d. Do the solutions check in the original equation?

A.4. Solve the equation $4x^2 = 4(10x - 16)$ by following these steps.

a. Write the equation with one side equal to zero and the other side factored.

b. Set each factor equal to 0.

c. Solve the individual equations from part (b) and write the solution set for the original equation.

d. Do the solutions check in the original equation?

For Exercises A.5–A.10,

a. Identify the equations as linear, quadratic, or a higher-order polynomial equation.

b. Solve the equation.

A.5. $x(x + 10) = -9$

A.6. $4x - 15 = 3(x + 2)$

A.7. $x^2 = 10x - 25$

A.8. $2(x - 3)(2x + 1)(5 - x) = 0$

A.9. $50w^3 - 2w = 0$

A.10. $y^3 - 3y^2 = 16y - 48$

A.11. The product of two consecutive even integers is 14 more than their sum. Find the two integers by following these steps.

a. Let x represent the first integer. Write an expression representing the second integer.

b. Write an expression that represents the product of the two integers.

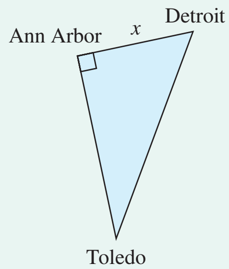
c. Write an expression that represents 14 more than the sum of the two integers.

d. Write an equation that indicates that the product of the two integers is 14 more than their sum.

e. Solve the equation.

f. Interpret your solution(s) and verify that the solutions meet the criteria of the problem.

A.12. On a map, Detroit, Michigan, Ann Arbor, Michigan, and Toledo, Ohio, form the vertices of a right triangle. The distance between Ann Arbor and Toledo is 11 miles more than the distance between Ann Arbor and Detroit. The distance between Toledo and Detroit is 22 miles more than the distance between Ann Arbor and Detroit. To find the distances between each pair of cities, follow these steps.



a. Let x represent the distance from Ann Arbor and Detroit. Write an expression in terms of x for the distance between Ann Arbor and Toledo.

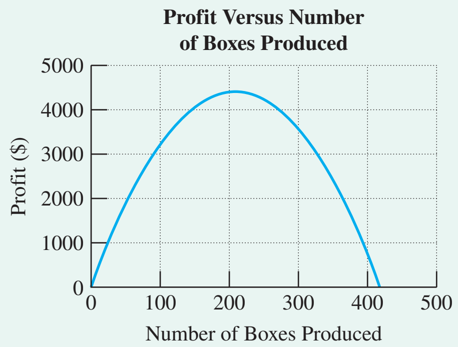
b. Write an expression in terms of x for the distance between Toledo and Detroit.

c. Use the Pythagorean theorem to write an equation that relates the lengths of the three sides.

d. Solve the equation from part (c). Which solution does not make sense in the context of this problem?

e. Find the distances between each pair of cities.

A.13. Cassandra makes decorative flower boxes that she sells at craft shows. Her profit is a function of the number of boxes she produces. Her profit $P(x)$ (in dollars) is given by $P(x) = -\frac{1}{10}x(x - 420)$, where x is the number of boxes produced.



a. Categorize the function as one of the following: constant, linear, quadratic, other.

b. Evaluate $P(100)$ and interpret its meaning.

c. Evaluate $P(200)$, $P(300)$, and $P(400)$.

d. The profit increases to a point and then drops as more boxes are produced. Explain why this might happen.

e. Find the x-intercepts of the function and interpret their meaning.

Practice Exercises

Section 4.8

Prerequisite Review

For Exercises R.1–R.4, solve the equation.

R.1. $d + 11 = 0$

R.2. $w - 21 = 0$

R.3. $5x - 9 = 0$

R.4. $6x + 5 = 0$

For Exercises R.5–R.10, factor completely.

R.5. $x^2 + x - 72$

R.6. $y^2 + 16y + 48$

R.7. $4t^2 + 5t - 6$

R.8. $5w^2 - 8w - 21$

R.9. $4h^3 - 36h$

R.10. $3x^3 - 75x$

R.11. Given $f(x) = x^2 - x - 20$, find

a. $f(0)$ **b.** $f(5)$ **c.** $f(-4)$

R.12. Given $g(x) = x^2 + 2x - 35$, find

a. $g(0)$ **b.** $g(5)$ **c.** $g(-7)$

For Exercises R.13–R.14, a statement for $y = f(x)$ is given. Determine the x- and y-intercepts.

R.13. $f(x) = -2x - 8$

R.14. $g(x) = 5x - 10$

Vocabulary and Key Concepts

1. An equation that can be written in the form $ax^2 + bx + c = 0$, $a \neq 0$, is called a ____________ equation.
2. The zero product rule states that if $ab = 0$, then $a =$ ____________ or $b =$ ____________.
3. The ____________ theorem states that given a right triangle with legs a and b and hypotenuse c, then $a^2 + b^2 =$ ____________.
4. A function defined by $f(x) = ax^2 + bx + c$, $a \neq 0$, is called a ____________ function.
5. Given a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, find the x-intercept(s) by solving the equation ____________. Find the ____________ intercept by evaluating $f(0)$.
6. If x is an integer, then ____________ represents the next greater integer. If x is an odd integer, then ____________ represents the next greater odd integer. Likewise if x is an even integer, then ____________ represents the next greater even integer.
7. The area of a rectangle of length l and width w is given by $A =$ ____________.
8. The area of a triangle with base b and height h is given by the formula $A =$ ____________.

Concept 1: Solving Equations by Using the Zero Product Rule

9. What conditions are necessary to solve an equation by using the zero product rule?
10. State the zero product rule.

For Exercises 11–16, determine which of the equations are written in the correct form to apply the zero product rule directly. If an equation is not in the correct form, explain what is wrong.

11. $2x(x - 3) = 0$

12. $(u + 1)(u - 3) = 10$

13. $3p^2 - 7p + 4 = 0$

14. $t^2 - t - 12 = 0$

15. $a(a + 3)^2 = 5$

16. $\left(\frac{2}{3}x - 5\right)\left(x + \frac{1}{2}\right) = 0$

For Exercises 17–20, factor the polynomial or solve the equation as indicated.

17. **a.** Factor. $w^2 - 81$
b. Solve. $w^2 - 81 = 0$

18. **a.** Factor. $p^2 - 25$
b. Solve. $p^2 - 25 = 0$

19. **a.** Factor. $3x^2 + 14x - 5$
b. Solve. $3x^2 + 14x - 5 = 0$

20. **a.** Factor. $2y^2 - y - 3$
b. Solve. $2y^2 - y - 3 = 0$

For Exercises 21–56, solve the equation. **(See Examples 1–6.)**

21. $(x + 3)(x + 5) = 0$

22. $(x + 7)(x - 4) = 0$

23. $(2w + 9)(5w - 1) = 0$

24. $(3a + 1)(4a - 5) = 0$

25. $x(x + 4)(10x - 3) = 0$

26. $t(t - 6)(3t - 11) = 0$

27. $0 = 5(y - 0.4)(y + 2.1)$

28. $0 = -4(z - 7.5)(z - 9.3)$

29. $x^2 + 6x - 27 = 0$

30. $2x^2 + x - 15 = 0$

31. $2x^2 + 5x = 3$

32. $-11x = 3x^2 - 4$

33. $10x^2 = 15x$

34. $5x^2 = 7x$

35. $6(y - 2) - 3(y + 1) = 8$

36. $4x + 3(x - 9) = 6x + 1$

37. $-9 = y(y + 6)$

38. $-62 = t(t - 16) + 2$

39. $9p^2 - 15p - 6 = 0$

40. $6y^2 + 2y = 48$

41. $(x + 1)(2x - 1)(x - 3) = 0$

42. $2x(x - 4)^2(4x + 3) = 0$

43. $(y - 3)(y + 4) = 8$

44. $(t + 10)(t + 5) = 6$

45. $(2a - 1)(a - 1) = 6$

46. $w(6w + 1) = 2$

47. $p^2 + (p + 7)^2 = 169$

48. $x^2 + (x + 2)^2 = 100$

49. $3t(t + 5) - t^2 = 2t^2 + 4t - 1$

50. $a^2 - 4a - 2 = (a + 3)(a - 5)$

51. $2x^3 - 8x^2 - 24x = 0$

52. $2p^3 + 20p^2 + 42p = 0$

53. $w^3 = 16w$

54. $12x^3 = 27x$

55. $0 = 2x^3 + 5x^2 - 18x - 45$

56. $0 = 3y^3 + y^2 - 48y - 16$

Concept 2: Applications of Quadratic Equations

57. If 5 is added to the square of a number, the result is 30. Find all such numbers.

58. Four less than the square of a number is 77. Find all such numbers.

59. The square of a number is equal to 12 more than the number. Find all such numbers.

60. The square of a number is equal to 20 more than the number. Find all such numbers.

61. The product of two consecutive integers is 42. Find the integers.

62. The product of two consecutive integers is 110. Find the integers.

63. The product of two consecutive odd integers is 63. Find the integers. **(See Example 7.)**

64. The product of two consecutive even integers is 120. Find the integers.

65. A rectangular pen has an area of 35 ft^2. If the width is 2 ft less than the length, find the dimensions of the pen. **(See Example 8.)**

66. The length of a rectangular photograph is 7 in. more than the width. If the area is 78 in.^2, what are the dimensions of the photograph?

67. The length of a rectangular room is 5 yd more than the width. If the area is 300 yd^2, find the length and the width of the room.

68. The top of a rectangular dining room table is twice as long as it is wide. Find the dimensions of the table if the area is 18 ft^2.

69. The height of a triangle is 1 in. more than the base. If the height is increased by 2 in. while the base remains the same, the new area becomes 20 in.^2

a. Find the base and height of the original triangle.

b. Find the area of the original triangle.

70. The base of a triangle is 2 cm more than the height. If the base is increased by 4 cm while the height remains the same, the new area is 56 cm^2.

a. Find the base and height of the original triangle.

b. Find the area of the original triangle.

71. The area of a triangular garden is 25 ft^2. The base is twice the height. Find the base and the height of the triangle.

72. The height of a triangle is 1 in. more than twice the base. If the area is 18 in.^2, find the base and height of the triangle.

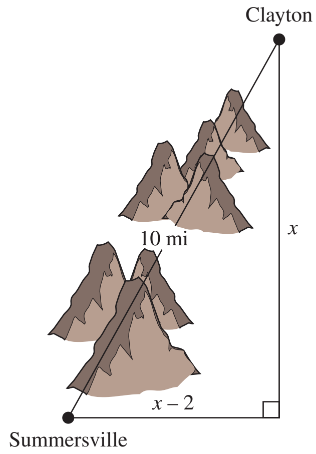
73. The sum of the squares of two consecutive positive integers is 41. Find the integers.

74. The sum of the squares of two consecutive, positive even integers is 164. Find the integers.

75. Justin must travel from Summersville to Clayton. He can drive 10 mi through the mountains at 40 mph, or he can drive east and then north on superhighways at 60 mph. The alternative route forms a right angle as shown in the diagram. The eastern leg is 2 mi less than the northern leg. **(See Example 9.)**

a. Find the total distance Justin would travel in going the alternative route.

b. If Justin wants to minimize the time of the trip, which route should he take?



76. A 17-ft ladder is standing up against a wall. The distance between the bottom of the ladder and the wall is 7 ft less than the distance between the top of the ladder and the base of the wall. Find the distance between the bottom of the ladder and the wall.

77. A right triangle has side lengths represented by three consecutive even integers. Find the lengths of the three sides, measured in meters.

78. The hypotenuse of a right triangle is 3 m more than twice the short leg. The longer leg is 2 m more than twice the shorter leg. Find the lengths of the sides.

79. Determine the length of the radius of a circle whose area is numerically equal to its circumference.

80. Determine the length of the radius of a circle whose area is numerically twice its circumference.

Concept 3: Definition of a Quadratic Function

For Exercises 81–84, **a.** Find the values of x for which $f(x) = 0$. **b.** Find $f(0)$.

81. $f(x) = x^2 - 3x$

82. $f(x) = 4x^2 + 2x$

83. $f(x) = x^2 - 6x - 7$

84. $f(x) = 2x^2 + 11x + 5$

For Exercises 85–88, find the x- and y-intercepts for the functions defined by $y = f(x)$. **(See Example 10.)**

85. $f(x) = \frac{1}{2}(x - 2)(x + 1)(2x)$

86. $f(x) = (x + 1)(x - 2)(x + 3)^2$

87. $f(x) = x^2 - 2x + 1$

88. $f(x) = x^2 + 4x + 4$

For Exercises 89–92, find the x-intercepts of each function, and use that information to match the function with its graph.

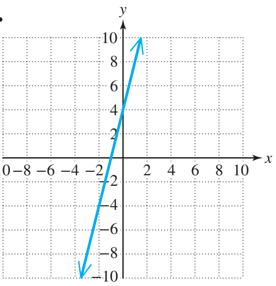
89. $g(x) = x^2 - 9$

90. $h(x) = x(x - 2)(x + 4)$

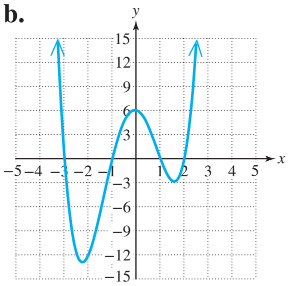
91. $f(x) = 4(x + 1)$

92. $k(x) = (x + 1)(x + 3)(x - 2)(x - 1)$

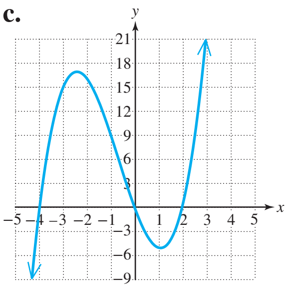
a.



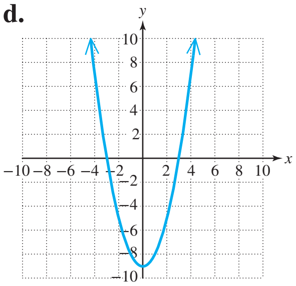
b.



c.



d.



Concept 4: Applications of Quadratic Functions

93. A rocket is fired upward from ground level with an initial velocity of 490 m/sec. The height of the rocket $s(t)$ in meters is a function of the time t in seconds after launch. **(See Example 11.)**

$$s(t) = -4.9t^2 + 490t$$

a. What characteristics of s indicate that it is a quadratic function?

b. Find the t-intercepts of the function.

c. What do the t-intercepts mean in the context of this problem?

d. At what times is the rocket at a height of 485.1 m?

94. A company makes water purification systems. The factory can produce x water systems per year. The profit $P(x)$ the company makes is a function of the number of systems x it produces.

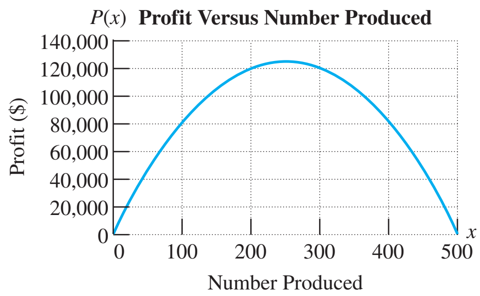
$$P(x) = -2x^2 + 1000x$$

a. Is this function linear or quadratic?

b. Find the number of water systems x that would produce a zero profit.

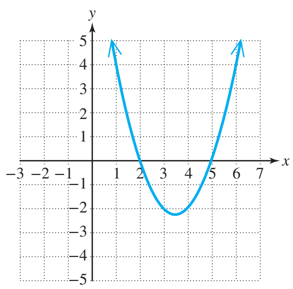
c. What points on the graph do the answers in part (b) represent?

d. Find the number of systems for which the profit is \$80,000.

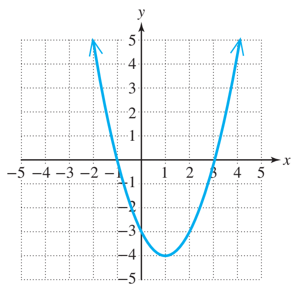


For Exercises 95–100, factor the expressions represented by $f(x)$. Explain how the factored form relates to the graph of the function.

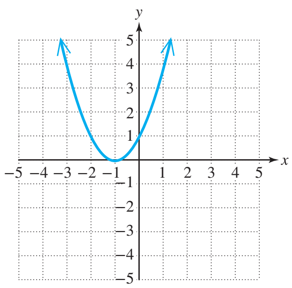
95. $f(x) = x^2 - 7x + 10$



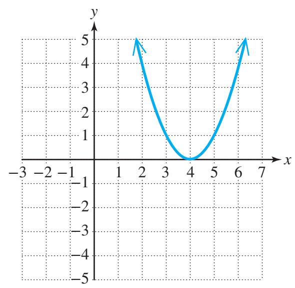
96. $f(x) = x^2 - 2x - 3$



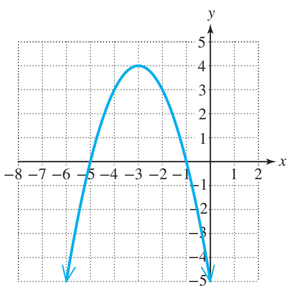
97. $f(x) = x^2 + 2x + 1$



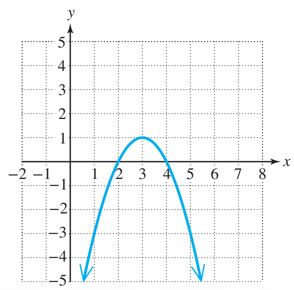
98. $f(x) = x^2 - 8x + 16$



99. $f(x) = -x^2 - 6x - 5$



100. $f(x) = -x^2 + 6x - 8$



Technology Connections

For Exercises 101–104, graph Y_1. Use the *Zero* feature to approximate the x-intercepts. Then solve $Y_1 = 0$ and compare the solutions to the x-intercepts.

101. $Y_1 = -x^2 + x + 2$

102. $Y_1 = -x^2 - x + 20$

103. $Y_1 = x^2 - 6x + 9$

104. $Y_1 = x^2 + 4x + 4$

Expanding Your Skills

105. The surface area of a right circular cylinder is represented by $2\pi r^2 + 2\pi rh$. If the surface area is 156π ft^2 and the height is 7 ft, determine the radius of the cylinder.

106. Determine the length and width of a rectangle with a perimeter of 20 yd and an area of 16 yd^2.

107. Determine the length and width of a rectangle with a perimeter of 28 ft and an area of 48 ft^2.

For Exercises 108–111, find an equation that has the given solutions. For example, 2 and -1 are solutions to $(x - 2)(x + 1) = 0$ or $x^2 - x - 2 = 0$. In general, x_1 and x_2 are solutions to the equation $a(x - x_1)(x - x_2) = 0$, where a can be any nonzero real number. For each exercise there is more than one correct answer depending on your choice of a.

108. $x = -3$ and $x = 1$

109. $x = 2$ and $x = -2$

110. $x = 0$ and $x = -5$

111. $x = 0$ and $x = -3$

Chapter 4 Summary

Section 4.1 Properties of Integer Exponents and Scientific Notation

Key Concepts

Let a and b ($b \neq 0$) represent real numbers and m and n represent positive integers.

$b^m \cdot b^n = b^{m+n}$ $\qquad \dfrac{b^m}{b^n} = b^{m-n}$

$(b^m)^n = b^{mn}$ $\qquad (ab)^m = a^m b^m$

$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ $\qquad b^0 = 1$

$b^{-n} = \left(\dfrac{1}{b}\right)^n = \dfrac{1}{b^n}$

A number expressed in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer, is written in **scientific notation**.

Examples

Example 1

$$\left(\frac{2x^2y}{z^{-1}}\right)^{-3} \cdot (x^{-4}y^0)$$

$$= \left(\frac{2^{-3}x^{-6}y^{-3}}{z^3}\right)(x^{-4} \cdot 1)$$

$$= \frac{2^{-3}x^{-10}y^{-3}}{z^3}$$

$$= \frac{1}{2^3x^{10}y^3z^3} \quad \text{or} \quad \frac{1}{8x^{10}y^3z^3}$$

Example 2

$0.0000002 \times 35{,}000$

$= (2 \times 10^{-7})(3.5 \times 10^4)$

$= 7 \times 10^{-3}$ or 0.007

Section 4.2 Addition and Subtraction of Polynomials and Polynomial Functions

Key Concepts

A **polynomial** in x is defined by a sum of terms of the form ax^n, where a is a real number and n is a whole number.

- a is the **coefficient** of the term.
- n is the **degree of the term**.

The **degree of a polynomial** is the greatest degree of its terms. The term of a polynomial with the greatest degree is the **leading term**. Its coefficient is the **leading coefficient**.

A one-term polynomial is a **monomial**.
A two-term polynomial is a **binomial**.
A three-term polynomial is a **trinomial**.

To add or subtract polynomials, add or subtract *like* terms.

Examples

Example 1

$7y^4 - 2y^2 + 3y + 8$

is a polynomial with leading coefficient 7 and degree 4.

Example 2

$f(x) = 4x^3 - 6x - 11$

f is a polynomial function with leading term $4x^3$ and leading coefficient 4. The degree of f is 3.

Example 3

For $f(x) = 4x^3 - 6x - 11$, find $f(-1)$.

$f(-1) = 4(-1)^3 - 6(-1) - 11$

$= -9$

Example 4

$(-4x^3y + 3x^2y^2) - (7x^3y - 5x^2y^2)$

$= -4x^3y + 3x^2y^2 - 7x^3y + 5x^2y^2$

$= -11x^3y + 8x^2y^2$

Section 4.3 Multiplication of Polynomials

Key Concepts

To multiply polynomials, multiply each term in the first polynomial by each term in the second polynomial.

Special Products

1. Multiplication of **conjugates**

   $(x + y)(x - y) = x^2 - y^2$

   The product is called a **difference of squares**.

2. Square of a binomial

   $(x + y)^2 = x^2 + 2xy + y^2$

   $(x - y)^2 = x^2 - 2xy + y^2$

   The product is called a **perfect square trinomial**.

Examples

Example 1

$$\begin{aligned}&(x - 2)(3x^2 - 4x + 11)\\ &\quad = 3x^3 - 4x^2 + 11x - 6x^2 + 8x - 22\\ &\quad = 3x^3 - 10x^2 + 19x - 22\end{aligned}$$

Example 2

$$\begin{aligned}&(3x + 5)(3x - 5)\\ &\quad = (3x)^2 - (5)^2\\ &\quad = 9x^2 - 25\end{aligned}$$

Example 3

$$\begin{aligned}&(4y + 3)^2\\ &\quad = (4y)^2 + (2)(4y)(3) + (3)^2\\ &\quad = 16y^2 + 24y + 9\end{aligned}$$

Section 4.4 Division of Polynomials

Key Concepts

Division of polynomials:

1. For division by a monomial, use the properties

   $$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad \text{and} \quad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

   for $c \neq 0$.

2. If the divisor has more than one term, use long division.
   - First write each polynomial in descending order.
   - Insert placeholders for missing powers.

Examples

Example 1

$$\begin{aligned}&\frac{-12a^2 - 6a + 9}{-3a}\\ &\quad = \frac{-12a^2}{-3a} - \frac{6a}{-3a} + \frac{9}{-3a}\\ &\quad = 4a + 2 - \frac{3}{a}\end{aligned}$$

Example 2

$(3x^2 - 5x + 1) \div (x + 2)$

$$\begin{array}{r} 3x - 11 \\ x + 2\,\overline{)\,3x^2 - 5x + 1} \\ -(3x^2 + 6x) \quad\;\; \\ -11x + 1 \\ -(-11x - 22) \\ 23 \end{array}$$

Answer: $3x - 11 + \dfrac{23}{x + 2}$

3. **Synthetic division** may be used to divide a polynomial by a binomial in the form $x - r$, where r is a constant.

Example 3

$(3x^2 - 5x + 1) \div (x + 2)$

$$\begin{array}{r|rrr} -2 & 3 & -5 & 1 \\ & & -6 & 22 \\ \hline & 3 & -11 & \boxed{23} \end{array}$$

Answer: $3x - 11 + \dfrac{23}{x+2}$

Section 4.5 Greatest Common Factor and Factoring by Grouping

Key Concepts

The **greatest common factor (GCF)** is the largest factor common to all terms of a polynomial. To factor out the GCF from a polynomial, use the distributive property.

A four-term polynomial may be **factored by grouping**.

Steps to Factor by Grouping

1. Identify and factor out the GCF from all four terms.
2. Factor out the GCF from the first pair of terms. Factor out the GCF from the second pair of terms. (Sometimes it is necessary to factor out the *opposite* of the GCF.)
3. If the two pairs of terms share a common binomial factor, factor out the binomial factor.

Examples

Example 1

$3x^2(a + b) - 6x(a + b)$

$= 3x(a + b)x - 3x(a + b)(2)$

$= 3x(a + b)(x - 2)$

Example 2

$60xa - 30xb - 80ya + 40yb$

$= 10[6xa - 3xb \;\vdots\; - 8ya + 4yb]$

$= 10[3x(2a - b) - 4y(2a - b)]$

$= 10(2a - b)(3x - 4y)$

Section 4.6 Factoring Trinomials

Key Concepts

AC-Method

To factor trinomials of the form $ax^2 + bx + c$:

1. Factor out the GCF. Find the product ac.
2. Find two integers whose product is ac and whose sum is b. (If no pair of numbers can be found, then the trinomial is prime.)
3. Rewrite the middle term bx as the sum of two terms whose coefficients are the numbers found in step 2.
4. Factor the polynomial by grouping.

Examples

Example 1

$10y^2 + 35y - 20 = 5(2y^2 + 7y - 4)$

$ac = (2)(-4) = -8$

Find two integers whose product is -8 and whose sum is 7. The numbers are 8 and -1.

$5[2y^2 + 8y \;\vdots\; - 1y - 4]$

$= 5[2y(y + 4) - 1(y + 4)]$

$= 5(y + 4)(2y - 1)$

Trial-and-Error Method

To factor trinomials in the form $ax^2 + bx + c$:

1. Factor out the GCF.
2. List the pairs of factors of a and the pairs of factors of c. Consider the reverse order in either list.
3. Construct two binomials of the form

Factors of a

$$(\square x \quad \square)(\square x \quad \square)$$

Factors of c

4. Test each combination of factors until the product of the outer terms and the product of inner terms add to the middle term.
5. If no combination of factors works, the polynomial is prime.

Example 2

$10y^2 + 35y - 20 = 5(2y^2 + 7y - 4)$

The pairs of factors of 2 are $2 \cdot 1$.
The pairs of factors of -4 are

$-1 \cdot 4$	$1 \cdot (-4)$
$-2 \cdot 2$	$2 \cdot (-2)$
$-4 \cdot 1$	$4 \cdot (-1)$

$(2y - 2)(y + 2) = 2y^2 + 2y - 4$	No
$(2y - 4)(y + 1) = 2y^2 - 2y - 4$	No
$(2y + 1)(y - 4) = 2y^2 - 7y - 4$	No
$(2y + 2)(y - 2) = 2y^2 - 2y - 4$	No
$(2y + 4)(y - 1) = 2y^2 + 2y - 4$	No
$(2y - 1)(y + 4) = 2y^2 + 7y - 4$	Yes

Therefore, $10y^2 + 35y - 20$ factors as $5(2y - 1)(y + 4)$.

The factored form of a **perfect square trinomial** is the square of a binomial:

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

Example 3

$$9w^2 - 30wz + 25z^2$$
$$= (3w)^2 - 2(3w)(5z) + (5z)^2$$
$$= (3w - 5z)^2$$

Sometimes it is easier to factor a polynomial after making a substitution.

Example 4

$(7v^2 - 1)^2 - (7v^2 - 1) - 12$	Let $u = (7v^2 - 1)$.
$= u^2 - u - 12$	Substitute.
$= (u + 3)(u - 4)$	Factor.
$= (7v^2 - 1 + 3)(7v^2 - 1 - 4)$	Back substitute.
$= (7v^2 + 2)(7v^2 - 5)$	Simplify.

Section 4.7 Factoring Binomials

Key Concepts

Factoring Binomials: Summary

Difference of squares:

$a^2 - b^2 = (a + b)(a - b)$

Sum of squares:

If a and b share no common factors, then $a^2 + b^2$ is prime.

Examples

Example 1

$$25u^2 - 9v^4 = (5u)^2 - (3v^2)^2$$
$$= (5u + 3v^2)(5u - 3v^2)$$

Example 2

$32 + 2w^2 = 2(16 + w^2)$ cannot be factored further because $16 + w^2$ is a sum of squares.

Difference of cubes:

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Sum of cubes:

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Sometimes it is necessary to group three terms with one term.

Example 3

$8c^3 - d^6 = (2c - d^2)(4c^2 + 2cd^2 + d^4)$

Example 4

$27w^9 + 64x^3$

$= (3w^3 + 4x)(9w^6 - 12w^3x + 16x^2)$

Example 5

$4a^2 - 12ab + 9b^2 - c^2$

$= 4a^2 - 12ab + 9b^2 \mid - c^2$ Group 3 by 1.

$= (2a - 3b)^2 - c^2$ Perfect square trinomial.

$= (2a - 3b - c)(2a - 3b + c)$ Difference of squares.

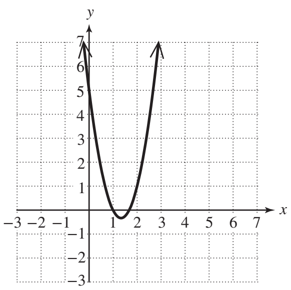
Section 4.8 Solving Equations by Using the Zero Product Rule

Key Concepts

An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, is a **quadratic equation**.

The **zero product rule** states that if $ab = 0$, then $a = 0$ or $b = 0$. The zero product rule can be used to solve a quadratic equation or higher-degree polynomial equation that is factored and equal to zero.

$f(x) = ax^2 + bx + c$ $(a \neq 0)$ defines a **quadratic function**. The x-intercepts of a function defined by $y = f(x)$ are determined by finding the real solutions to the equation $f(x) = 0$. The y-intercept of $y = f(x)$ is at $f(0)$.



Examples

Example 1

$0 = x(2x - 3)(x + 4)$

$x = 0$ or $2x - 3 = 0$ or $x + 4 = 0$

$x = \frac{3}{2}$ or $x = -4$

The solution set is $\left\{0, \frac{3}{2}, -4\right\}$.

Example 2

Find the x-intercepts.

$f(x) = 3x^2 - 8x + 5$

$0 = 3x^2 - 8x + 5$

$0 = (3x - 5)(x - 1)$

$3x - 5 = 0$ or $x - 1 = 0$

$x = \frac{5}{3}$ or $x = 1$

The x-intercepts are $\left(\frac{5}{3}, 0\right)$ and $(1, 0)$.

Find the y-intercept.

$f(x) = 3x^2 - 8x + 5$

$f(0) = 3(0)^2 - 8(0) + 5$

$f(0) = 5$

The y-intercept is $(0, 5)$.

Chapter 4 Review Exercises

Section 4.1

For Exercises 1–8, simplify the expression and write the answer with positive exponents.

1. $(3x)^3(3x)^2$

2. $(-6x^{-4})(3x^{-8})$

3. $\dfrac{24x^5y^3}{-8x^4y}$

4. $\dfrac{-18x^{-2}y^3}{-12x^{-5}y^5}$

5. $(-2a^2b^{-5})^{-3}$

6. $(-4a^{-2}b^3)^{-2}$

7. $\left(\dfrac{-4x^4y^{-2}}{5x^{-1}y^4}\right)^{-4}$

8. $\left(\dfrac{25x^2y^{-3}}{5x^4y^{-2}}\right)^{-5}$

9. Write the numbers in scientific notation.

a. For a recent year, the population of Asia was 3,686,600,000.

b. A nanometer is 0.000001 of a millimeter.

10. Write the numbers in scientific notation.

a. A millimeter is 0.001 of a meter.

b. The population of Asia is predicted to be 5,155,700,000 by 2040.

11. Write the numbers in standard form.

a. A micrometer is 1×10^{-3} of a millimeter.

b. A nanometer is 1×10^{-9} of a meter.

12. Write the numbers in standard form.

a. The total square footage of shopping centers in the United States is approximately 5.23×10^9 ft^2.

b. The total yearly sales of those shopping centers is $\$1.091 \times 10^{12}$. (*Source:* International Council of Shopping Centers)

For Exercises 13–16, perform the indicated operations. Write the answer in scientific notation.

13. $\dfrac{2{,}500{,}000}{0.0004}$

14. $\dfrac{0.0005}{25{,}000}$

15. $(3.6 \times 10^8)(9 \times 10^{-2})$

16. $(7 \times 10^{-12})(5.2 \times 10^3)$

Section 4.2

For Exercises 17–18, identify the polynomial as a monomial, binomial, or trinomial; then give the degree of the polynomial.

17. $6x^4 + 10x - 1$

18. 18

19. Given the polynomial function defined by $g(x) = 4x - 7$, find the function values.

a. $g(0)$ **b.** $g(-4)$ **c.** $g(3)$

20. Given the polynomial function defined by $p(x) = -x^4 - x + 12$, find the function values.

a. $p(0)$ **b.** $p(1)$ **c.** $p(-2)$

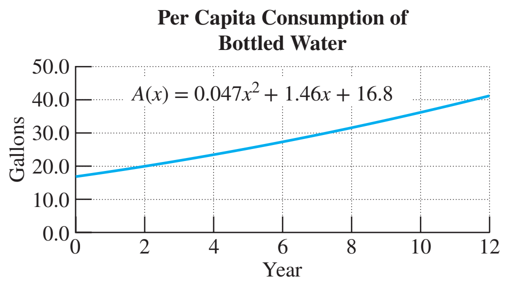
21. The amount $A(x)$ of bottled water consumed per capita in the United States can be approximated by

$$A(x) = 0.047x^2 + 1.46x + 16.8$$

where x is the number of years since the study began, and $A(x)$ is measured in gallons.

a. Evaluate $A(5)$ and interpret the meaning in the context of the problem.

b. Interpret the meaning of $A(15)$.



For Exercises 22–33, add or subtract the polynomials as indicated.

22. $(x^2 - 2x - 3xy - 7) + (-3x^2 - x + 2xy + 6)$

23.
$$\begin{array}{r} 7xy - 3xz + 5yz \\ \underline{+13xy - 15xz - 8yz} \end{array}$$

24.
$$\begin{array}{r} -4a^3 + 8a^2 - 3a \\ \underline{-(-7a^3 + 3a^2 - 9a)} \end{array}$$

25. $(3a^2 - 2a - a^3) - (5a^2 - a^3 - 8a)$

26. $\left(\frac{5}{8}x^4 - \frac{1}{4}x^2 - \frac{1}{2}\right) - \left(-\frac{3}{8}x^4 + \frac{3}{4}x^2 + \frac{1}{2}\right)$

27. $\left(\frac{5}{6}x^4 + \frac{1}{2}x^2 - \frac{1}{3}\right) - \left(-\frac{1}{6}x^4 - \frac{1}{4}x^2 - \frac{1}{3}\right)$

28. $(7x - y) - [-(2x + y) - (-3x - 6y)]$

29. $-(4x - 4y) - [(4x + 2y) - (3x + 7y)]$

30. Add $-4x + 6$ to $-7x - 5$.

31. Add $2x^2 - 4x$ to $2x^2 - 7x$.

32. Subtract $-4x + 6$ from $-7x - 5$.

33. Subtract $2x^2 - 4x$ from $2x^2 - 7x$.

Section 4.3

For Exercises 34–51, multiply the polynomials.

34. $2x(x^2 - 7x - 4)$

35. $-3x(6x^2 - 5x + 4)$

36. $(x + 6)(x - 7)$

37. $(x - 2)(x - 9)$

38. $\left(\frac{1}{2}x + 1\right)\left(\frac{1}{2}x - 5\right)$

39. $\left(-\frac{1}{5} + 2y\right)\left(\frac{1}{5} + y\right)$

40. $(3x + 5)(9x^2 - 15x + 25)$

41. $(x - y)(x^2 + xy + y^2)$

42. $(2x - 5)^2$

43. $\left(\frac{1}{2}x + 4\right)^2$

44. $(3y - 11)(3y + 11)$

45. $(6w - 1)(6w + 1)$

46. $\left(\frac{2}{3}t + 4\right)\left(\frac{2}{3}t - 4\right)$

47. $\left(z + \frac{1}{4}\right)\left(z - \frac{1}{4}\right)$

48. $[(x + 2) - b][(x + 2) + b]$

49. $[c - (w + 3)][c + (w + 3)]$

50. $(2x + 1)^3$

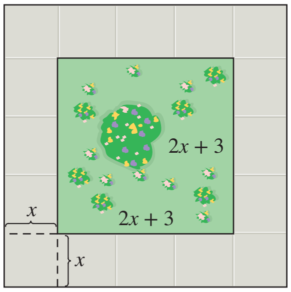
51. $(y^2 - 3)^3$

52. A square garden is surrounded by a walkway of uniform width x. If the sides of the garden are given by the expression $2x + 3$, find and simplify a polynomial for the following.

a. The area of the garden.

b. The area of the walkway and garden.

c. The area of the walkway only.



53. The length of a rectangle is 2 ft more than 3 times the width. Let x represent the width of the rectangle.

a. Write a function P that represents the perimeter of the rectangle.

b. Write a function A that represents the area of the rectangle.

54. In parts (a) and (b), one of the statements is true and the other is false. Identify the true statement and explain why the false statement is incorrect.

a. $2x^2 + 5x = 7x^3$ $\qquad (2x^2)(5x) = 10x^3$

b. $4x - 7x = -3x$ $\qquad 4x - 7x = -3$

Section 4.4

For Exercises 55–56, divide the polynomials.

55. $(6x^3y + 12x^2y^2 - 9xy^3) \div (3xy)$

56. $(10x^4 + 15x^3 - 20x^2) \div (-5x^2)$

57. **a.** Divide $(9y^4 + 14y^2 - 8) \div (3y + 2)$.

b. Identify the quotient and the remainder.

c. Explain how you can check your answer.

For Exercises 58–61, divide the polynomials by using long division.

58. $(x^2 + 7x + 10) \div (x + 5)$

59. $(x^2 + 8x - 16) \div (x + 4)$

60. $(2x^5 - 4x^4 + 2x^3 - 4) \div (x^2 - 3x)$

61. $(2x^5 + 3x^3 + x^2 - 4) \div (x^2 + x)$

62. Explain the conditions under which you may use synthetic division.

63. The following table is the result of a synthetic division.

3	2	5	−2	6	1
		6	33	93	297
	2	11	31	99	298

Use x as the variable.

a. Identify the divisor.

b. Identify the quotient.

c. Identify the remainder.

For Exercises 64–68, divide the polynomials by using synthetic division.

64. $(t^3 - 3t^2 + 8t - 12) \div (t - 2)$

65. $(x^2 + 7x + 14) \div (x + 5)$

66. $(x^2 + 8x + 20) \div (x + 4)$

67. $(w^3 - 6w^2 + 8) \div (w - 3)$

68. $(p^4 - 16) \div (p - 2)$

Section 4.5

For Exercises 69–72, factor by removing the greatest common factor.

69. $-x^3 - 4x^2 + 11x$

70. $21w^3 - 7w + 14$

71. $5x(x - 7) - 2(x - 7)$

72. $3t(t + 4) + 5(t + 4)$

For Exercises 73–76, factor by grouping (remember to take out the GCF first).

73. $m^3 - 8m^2 + m - 8$

74. $24x^3 - 36x^2 + 72x - 108$

75. $4ax^2 + 2bx^2 - 6ax - 3xb$

76. $y^3 - 6y^2 + y - 6$

Section 4.6

77. What characteristics determine a perfect square trinomial?

For Exercises 78–87, factor the polynomials by using any method.

78. $18x^2 + 27xy + 10y^2$

79. $3m^2 + mt - 10t^2$

80. $60a^2 + 65a^3 - 20a^4$

81. $2k^2 + 7k^3 + 6k^4$

82. $49x^2 + 36 - 84x$

83. $80z + 32 + 50z^2$

84. $(9w + 2)^2 + 4(9w + 2) - 5$

85. $(4x + 3)^2 - 12(4x + 3) + 36$

86. $18a^4 + 39a^2 - 15$

87. $3w^4 - 2w^2 - 5$

Section 4.7

For Exercises 88–94, factor the binomials.

88. $25 - y^2$

89. $x^3 - \dfrac{1}{27}$

90. $b^2 + 64$

91. $h^3 + 9h$

92. $a^3 + 64$

93. $k^4 - 16$

94. $9y^3 - 4y$

For Exercises 95–98, factor by grouping and by using the difference of squares.

95. $x^2 - 8xy + 16y^2 - 9$ (*Hint:* Group three terms that make up a perfect square trinomial, then factor as a difference of squares.)

96. $a^2 + 12a + 36 - b^2$

97. $t^2 + 16t + 64 - 25c^2$

98. $y^2 - 6y + 9 - 16x^2$

Section 4.8

99. How do you determine if an equation in the variables x and y is quadratic?

100. What shape is the graph of a quadratic function?

For Exercises 101–104, label the equation as quadratic or linear.

101. $x^2 + 6x = 7$

102. $(x - 3)(x + 4) = 9$

103. $2x - 5 = 3$

104. $x + 3 = 5x^2$

105. **a.** Factor. $5x^2 + 6x - 8$

b. Solve. $5x^2 + 6x - 8 = 0$

106. **a.** Factor. $3x^2 - 19x + 28$

b. Solve. $3x^2 - 19x + 28 = 0$

For Exercises 107–110, use the zero product rule to solve the equations.

107. $x^2 - 2x - 15 = 0$

108. $8x^2 = 59x - 21$

109. $2t(t + 5) + 1 = 3t - 3 - t^2$

110. $3(x - 1)(x + 5)(2x - 9) = 0$

For Exercises 111–114, find the x- and y-intercepts of the function. Then match the function with its graph.

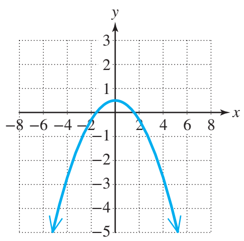
111. $f(x) = -4x^2 + 4$

112. $g(x) = 2x^2 - 2$

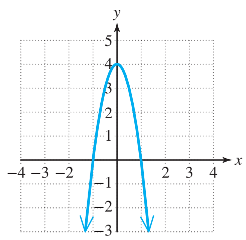
113. $h(x) = 5x^3 - 10x^2 - 20x + 40$

114. $k(x) = -\frac{1}{8}x^2 + \frac{1}{2}$

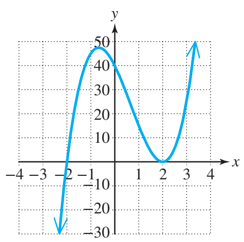
a.



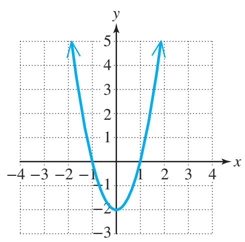
b.



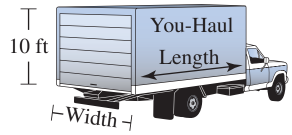
c.



d.



115. A moving van has the capacity to hold 1200 ft^3 in volume. If the van is 10 ft high and the length is 1 ft less than twice the width, find the dimensions of the van.



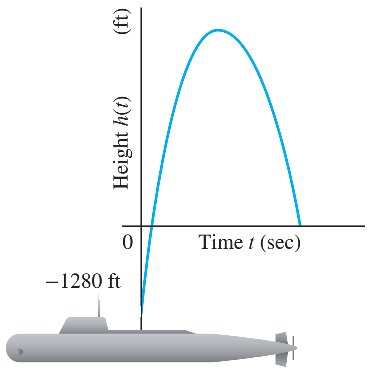
116. A missile is shot upward from a submarine 1280 ft below sea level. The initial velocity of the missile is 672 ft/sec. A function that approximates the height of the missile (relative to sea level) is given by

$$h(t) = -16t^2 + 672t - 1280$$

where $h(t)$ is the height in feet and t is the time in seconds.

a. Complete the table to determine the height of the missile for the given values of t.

Time t (sec)	Height $h(t)$ (ft)
0	
1	
3	
10	
20	
30	
42	



b. Interpret the meaning of a negative value of $h(t)$.

c. Factor the function to find the time required for the missile to emerge from the water and the time required for the missile to reenter the water. (*Hint:* The height of the missile will be zero at sea level.)

Chapter 4 Test

For Exercises 1–4, simplify the expression, and write the answer with positive exponents only.

1. $\dfrac{20a^7}{4a^{-6}}$

2. $\dfrac{x^6x^3}{x^{-2}}$

3. $\left(\dfrac{-3x^6}{5y^7}\right)^2$

4. $\dfrac{(2^{-1}xy^{-2})^{-3}(x^{-4}y)}{(x^0y^5)^{-1}}$

5. Multiply. $(8 \times 10^{-6})(7.1 \times 10^5)$

6. Divide. (Write the answer in scientific notation.) $(9{,}200{,}000) \div (0.004)$

7. For the function defined by $F(x) = 5x^3 - 2x^2 + 8$, find the function values $F(-1)$, $F(2)$, and $F(0)$.

8. Perform the indicated operations. Write the answer in descending order.

$$(5x^2 - 7x + 3) - (x^2 + 5x - 25) + (4x^2 + 4x - 20)$$

For Exercises 9–11, multiply the polynomials.

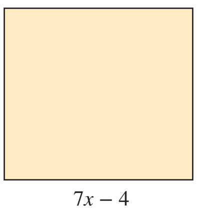
9. $(2a - 5)(a^2 - 4a - 9)$

10. $\left(\frac{1}{3}x - \frac{3}{2}\right)(6x + 4)$

11. $(5x - 4y^2)(5x + 4y^2)$

12. Explain why $(5x + 7)^2 \neq 25x^2 + 49$.

13. Write and simplify an expression that describes the area of the square.



14. Divide the polynomials.

$$(2x^3y^4 + 5x^2y^2 - 6xy^3 - xy) \div (2xy)$$

15. Divide the polynomials.

$$(10p^3 + 13p^2 - p + 3) \div (2p + 3)$$

16. Divide the polynomials by using synthetic division. $(y^4 - 2y + 5) \div (y - 2)$

For Exercises 17–32, factor completely.

17. $17y + 3y^2 - 28$

18. $x^2 - 5x - 4$

19. $3a^2 + 27ab + 54b^2$

20. $c^4 - 1$

21. $xy - 7x + 3y - 21$

22. $49 + p^2$

23. $-10u^2 + 30u - 20$

24. $12t^2 - 75$

25. $5y^2 - 50y + 125$

26. $21q^2 + 14q$

27. $2x^3 + x^2 - 8x - 4$

28. $y^3 - 125$

29. $x^2 + 8x + 16 - y^2$

30. $r^6 - 256r^2$

31. $(x^2 + 1)^2 + 3(x^2 + 1) + 2$

32. $12a - 6ac + 2b - bc$

For Exercises 33–38, solve the equation.

33. $(2x - 3)(x + 5) = 0$

34. $w^2 - 7w = 0$

35. $y^2 - 6y = 16$

36. $x(5x + 4) = 1$

37. $4p - 64p^3 = 0$

38. $t^2 + \frac{1}{2}t + \frac{1}{16} = 0$

39. A child launches a toy rocket from the ground. The height, $h(x)$, of the rocket can be determined by its horizontal distance, x, from the launch pad by

$$h(x) = -\frac{x^2}{256} + x$$

where x and h are in feet and $x \geq 0$ and $h \geq 0$.

How many feet from the launch pad will the rocket hit the ground?

40. The population of Japan, $P(t)$ (in millions), can be approximated by

$$P(t) = -0.01t^2 - 0.062t + 127.7,$$

where $t = 0$ represents the number of years since the study began.

a. Evaluate $P(4)$ and interpret in the context of the problem.

b. Approximate the number of people in Japan in year 6.

c. If the trend continues, predict the population of Japan in year 15.

For Exercises 41–44, find the x- and y-intercepts of the function. Then match the function with its graph.

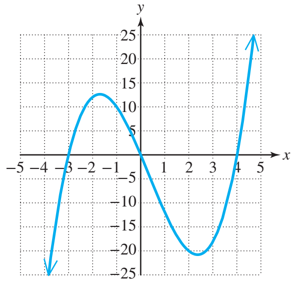
41. $f(x) = x^2 - 6x + 8$

42. $k(x) = x^3 + 4x^2 - 9x - 36$

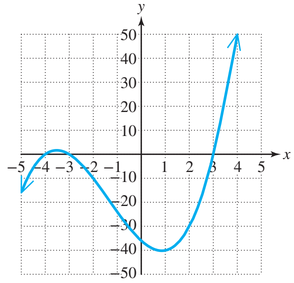
43. $p(x) = -2x^2 - 8x - 6$

44. $q(x) = x^3 - x^2 - 12x$

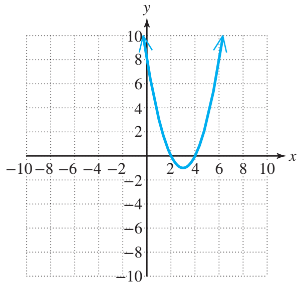
a.



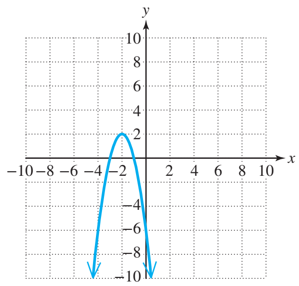
b.



c.



d.



Rational Expressions and Rational Equations

5

CHAPTER OUTLINE

Mathematics in Biology

"Mark and recapture" is a commonly used method to estimate an animal population when it is not practical to count the members of the population individually. For example, imagine trying to count all of the red shouldered hawks in Everglades National Park, an area of 2357 square miles consisting of mostly wetland and swamp! It would be impossible to find and count hawks individually and to avoid redundancy (counting the same hawk more than once).

Steve Byland/iStockphoto/Getty Images

Instead, biologists might capture a small sample of hawks and place an identifying tag around a leg of each hawk. Then the tagged hawks are released back into the environment. After waiting an optimal period of time, another sample of hawks is captured. The ratio of the original number of tagged hawks to the total population is approximately equal to the ratio of the number of recaptured tagged hawks to the number of recaptured hawks.

For example, suppose that 40 hawks are initially tagged in a total population of x hawks. A second sample of 100 hawks taken 3 months later includes 2 tagged hawks from the original group. This is represented by the following equation.

$$\begin{array}{rcl} \text{Number of hawks originally tagged} \longrightarrow & \dfrac{40}{x} = \dfrac{2}{100} & \longleftarrow \text{Number of tagged hawks recaptured} \\ \text{Total population} \longrightarrow & & \longleftarrow \text{Number of hawks recaptured} \end{array}$$

This equation is called a **proportion** and the solution is 2000. Thus, there are approximately 2000 red shouldered hawks in Everglades National Park. In this chapter we will solve proportions and other applications involving algebraic fractions called **rational expressions**.

Section 5.1 Rational Expressions and Rational Functions

Concepts

1. Rational Functions
2. Simplifying Rational Expressions
3. Simplifying Ratios of −1

1. Rational Functions

We have previously introduced polynomials and polynomial functions. The ratio of two polynomials defines a **rational expression**. This leads to the following definition of a rational function.

Definition of a Rational Function

A function is a **rational function** if it can be written in the form $f(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomial functions and $q(x) \neq 0$.

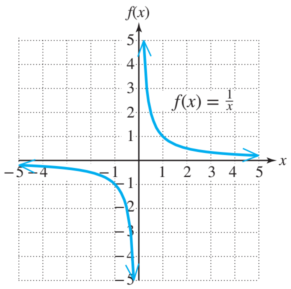


Figure 5-1

For example, the functions f, g, h, and k are rational functions.

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{2}{x-3}, \qquad h(a) = \frac{a+7}{a^2-5}, \qquad k(t) = \frac{t+4}{2t^2-11t+5}$$

We have already introduced the rational function defined by $f(x) = \frac{1}{x}$. Recall that $f(x) = \frac{1}{x}$ has a restriction on its domain that $x \neq 0$ and the graph of $f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$ (Figure 5-1).

In this course, we restrict our discussion of functions defined by $y = f(x)$ to those whose domain and range are subsets of the set of real numbers. Therefore, the domain of a function is the set of real numbers that when substituted into the function produce a real number. For a rational function, we must exclude values that make the denominator zero. To find the domain of a rational function, we offer these guidelines.

Finding the Domain of a Rational Function

Step 1 Set the denominator equal to zero and solve the resulting equation.

Step 2 The domain is the set of all real numbers *excluding* the values found in step 1.

Example 1 Evaluating a Rational Function

Given $g(x) = \dfrac{2}{x-3}$

a. Determine the function values if possible.
$g(0)$, $g(1)$, $g(2)$, $g(2.5)$, $g(2.9)$, $g(3)$, $g(3.1)$, $g(3.5)$, $g(4)$, and $g(5)$

b. Write the domain of the function.

FOR REVIEW

The expression $\frac{2}{0}$ is undefined because there is no real number that when multiplied by 0 equals 2.

Solution:

a.

$$g(0) = \frac{2}{(0)-3} = -\frac{2}{3} \qquad g(3) = \frac{2}{(3)-3} = \frac{2}{0} \quad \text{(undefined)}$$

$$g(1) = \frac{2}{(1)-3} = -1 \qquad g(3.1) = \frac{2}{(3.1)-3} = \frac{2}{0.1} = 20$$

$$g(2) = \frac{2}{(2)-3} = -2 \qquad g(3.5) = \frac{2}{(3.5)-3} = \frac{2}{0.5} = 4$$

$$g(2.5) = \frac{2}{(2.5)-3} = \frac{2}{-0.5} = -4 \qquad g(4) = \frac{2}{(4)-3} = 2$$

$$g(2.9) = \frac{2}{(2.9)-3} = \frac{2}{-0.1} = -20 \qquad g(5) = \frac{2}{(5)-3} = 1$$

b. $g(x) = \dfrac{2}{x-3}$ The value of the function is undefined when the denominator equals zero.

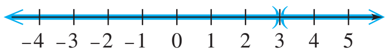
$x - 3 = 0$ Set the denominator equal to zero and solve for x.

$x = 3$ The value $x = 3$ must be excluded from the domain.

The domain can be written in set-builder notation or in interval notation.

Set-builder notation: $\{x \mid x \text{ is a real number}, x \neq 3\}$

Interval notation: $(-\infty, 3) \cup (3, \infty)$



Skill Practice Determine the function values if possible.

$$h(x) = \frac{x+2}{x+6}$$

1. $h(0)$ **2.** $h(2)$ **3.** $h(-2)$ **4.** $h(-6)$

5. Write the domain in set-builder notation.

6. Write the domain in interval notation.

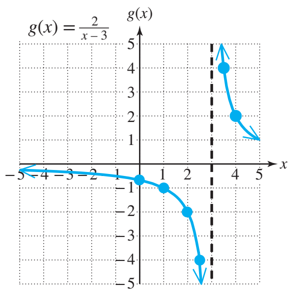


Figure 5-2

The graph of $g(x) = \frac{2}{x-3}$ is shown in Figure 5-2. Notice that the function has a vertical asymptote at $x = 3$ where the function is undefined.

Example 2 Finding the Domain of a Rational Function

Write the domain in set-builder notation and in interval notation.

a. $f(x) = \dfrac{x+4}{2x^2 - 11x + 5}$ **b.** $g(x) = \dfrac{x}{x^2 + 4}$

Solution:

a. $f(x) = \dfrac{x+4}{2x^2 - 11x + 5}$

$2x^2 - 11x + 5 = 0$ Set the denominator equal to zero and solve for x.

$(2x - 1)(x - 5) = 0$ This is a factorable quadratic equation.

$2x - 1 = 0$ or $x - 5 = 0$

$x = \dfrac{1}{2}$ or $x = 5$

The domain written in set-builder notation is $\{x \mid x \text{ is a real number and } x \neq \frac{1}{2}, x \neq 5\}$.

The domain written in interval notation is $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 5\right) \cup (5, \infty)$.

TIP: The domain of a rational function excludes values for which the denominator is zero.

Answers

1. $\frac{1}{3}$ **2.** $\frac{1}{2}$

3. 0 **4.** Undefined

5. $\{x \mid x \text{ is a real number}, x \neq -6\}$

6. $(-\infty, -6) \cup (-6, \infty)$

FOR REVIEW

Recall that the sum of squares is not factorable. The expression $x^2 + 4$ is a prime polynomial.

b. $g(x) = \dfrac{x}{x^2 + 4}$

Because the quantity x^2 is nonnegative for any real number x, the denominator $x^2 + 4$ cannot equal zero; therefore, no real numbers are excluded from the domain.

The domain written in set-builder notation is $\{x \mid x \text{ is a real number}\}$.

The domain written in interval notation is $(-\infty, \infty)$.

Skill Practice Write the domain in set-builder notation and in interval notation.

7. $h(x) = \dfrac{x + 10}{2x^2 - x - 1}$ **8.** $g(t) = \dfrac{t}{t^2 + 1}$

2. Simplifying Rational Expressions

A rational expression is an expression in the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$. As with fractions, it is often advantageous to simplify rational expressions to lowest terms.

The method for simplifying rational expressions mirrors the process to simplify fractions. In each case, factor the numerator and denominator. Common factors in the numerator and denominator form a ratio of 1 and can be simplified.

Simplifying a fraction: $\dfrac{15}{35} \xrightarrow{\text{factor}} \dfrac{3 \cdot 5}{7 \cdot 5} = \dfrac{3}{7} \cdot \dfrac{5}{5} = \dfrac{3}{7} \cdot 1 = \dfrac{3}{7}$

Simplifying a rational expression:

$$\frac{x^2 - x - 12}{x^2 - 16} \xrightarrow{\text{factor}} \frac{(x + 3)(x - 4)}{(x + 4)(x - 4)} = \frac{(x + 3)}{(x + 4)} \cdot \frac{(x - 4)}{(x - 4)}$$

$$= \frac{(x + 3)}{(x + 4)} \cdot 1$$

$$= \frac{(x + 3)}{(x + 4)}$$

This process is stated formally as the fundamental principle of rational expressions.

Fundamental Principle of Rational Expressions

Let p, q, and r represent polynomials. Then

$$\frac{pr}{qr} = \frac{p}{q} \cdot \frac{r}{r} = \frac{p}{q} \cdot 1 = \frac{p}{q} \qquad \text{for } q \neq 0 \text{ and } r \neq 0$$

A rational expression is not defined for the values of the variable that make the denominator equal to zero. We refer to these values as *restrictions* on the variable.

Answers

7. $\{x \mid x$ is a real number and $x \neq -\frac{1}{2}, x \neq 1\}$; $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 1) \cup (1, \infty)$

8. $\{t \mid t$ is a real number$\}$; $(-\infty, \infty)$

Example 3 Simplifying a Rational Expression

Given the expression $\dfrac{2x^3 + 12x^2 + 16x}{6x + 24}$

a. Factor the numerator and denominator.

b. Determine the restrictions on x.

c. Simplify the expression.

Solution:

a. $\dfrac{2x^3 + 12x^2 + 16x}{6x + 24}$

$= \dfrac{2x(x^2 + 6x + 8)}{6(x + 4)}$ Factor the numerator and denominator.

$= \dfrac{2x(x + 4)(x + 2)}{6(x + 4)}$

b. The expression is not defined for all values of x for which the denominator is equal to zero.

$6(x + 4) = 0$ Solve the equation.

$x = -4$

The restriction on x is that $x \neq -4$.

c. $\dfrac{2x(x + 4)(x + 2)}{2 \cdot 3(x + 4)} = \dfrac{x(x + 2)}{3} \cdot \overset{1}{\dfrac{\cancel{2(x + 4)}}{\cancel{2(x + 4)}}}$ Simplify the ratio of common factors.

$= \dfrac{x(x + 2)}{3}$ (provided $x \neq -4$)

Avoiding Mistakes

Always determine the restrictions on the variable *before* simplifying an expression.

Skill Practice Given $\dfrac{x^2 + 3x - 28}{2x + 14}$

9. a. Factor the numerator and denominator.

b. Determine the restrictions on x.

c. Simplify the expression.

From Example 3, it is important to note that the expressions

$$\frac{2x^3 + 12x^2 + 16x}{6x + 24} \quad \text{and} \quad \frac{x(x + 2)}{3}$$

are equal for all values of x that make each expression a real number. Therefore,

$$\frac{2x^3 + 12x^2 + 16x}{6x + 24} = \frac{x(x + 2)}{3}$$

for all values of x *except* $x = -4$. (At $x = -4$, the original expression is undefined.) From this point forward, when we simplify rational expressions we will not explicitly write the restrictions by the simplified form. These will be implied from the original expression.

Example 4 involves simplifying a quotient of two monomials.

Answer

9. a. $\dfrac{(x + 7)(x - 4)}{2(x + 7)}$ **b.** $x \neq -7$

c. $\dfrac{x - 4}{2}$ provided $x \neq -7$

FOR REVIEW

The expression in Example 4 can be simplified using the properties of exponents. Recall:

$\frac{b^m}{b^n} = b^{m-n} \quad b^{-n} = \frac{1}{b^n}$

Thus, $\frac{2x^2y^8}{8x^4y^3} = \frac{2^1x^2y^8}{2^3x^4y^3}$

$= 2^{1-3}x^{2-4}y^{8-3}$

$= 2^{-2}x^{-2}y^5 = \frac{y^5}{2^2x^2}$

Example 4 Simplifying a Rational Expression

Simplify. $\frac{2x^2y^8}{8x^4y^3}$

Solution:

$\frac{2x^2y^8}{8x^4y^3}$ This expression has the restriction that $x \neq 0$ and $y \neq 0$.

$= \frac{2x^2y^8}{2^3x^4y^3}$ Factor the denominator.

$= \frac{y^5}{2^2x^2} \cdot \frac{\overset{1}{\cancel{2x^2y^3}}}{\cancel{2x^2y^3}}$ Simplify common factors whose ratio is 1.

$= \frac{y^5}{4x^2}$

Skill Practice Simplify.

10. $\frac{9a^5b^3}{18a^8b}$

Example 5 Simplifying a Rational Expression

Simplify. $\frac{t^3 + 8}{t^2 + 4t + 4}$

Solution:

$\frac{t^3 + 8}{t^2 + 4t + 4} = \frac{(t + 2)(t^2 - 2t + 4)}{(t + 2)^2}$ Factor the numerator and denominator. The numerator is a sum of cubes. The denominator is a perfect square trinomial. The restriction on t is $t \neq -2$.

$= \frac{(t^2 - 2t + 4)}{(t + 2)} \cdot \frac{\overset{1}{\cancel{(t + 2)}}}{\cancel{(t + 2)}}$ Simplify common factors whose ratio is 1.

$= \frac{t^2 - 2t + 4}{t + 2}$

TIP: Recall the formula to factor a sum of cubes.

$a^3 + b^3$
$= (a + b)(a^2 - ab + b^2)$

Recall the formula to factor a perfect square trinomial.

$a^2 + 2ab + b^2 = (a + b)^2$

Skill Practice Simplify.

11. $\frac{p^3 - 27}{p^2 - 6p + 9}$

Answers

10. $\frac{b^2}{2a^3}$

11. $\frac{p^2 + 3p + 9}{p - 3}$

Avoiding Mistakes

Because the fundamental property of rational expressions is based on the identity property of multiplication, reducing applies only to factors (remember that factors are multiplied). Therefore, terms that are added or subtracted cannot be reduced. For example:

$$\frac{3x}{3y} = \frac{\overset{1}{\cancel{3}}}{\cancel{3}} \cdot \frac{x}{y} = (1) \cdot \frac{x}{y} = \frac{x}{y}$$

Reduce common factor.

However, $\frac{x+3}{y+3}$ cannot be simplified.

Cannot reduce common terms.

3. Simplifying Ratios of −1

When two factors are identical in the numerator and denominator, they form a ratio of 1 and can be simplified. Sometimes we encounter two factors that are *opposites* and form a ratio equal to −1. For example:

Simplified Form	Details/Notes
$\frac{-5}{5} = -1$	The ratio of a number and its opposite is −1.
$\frac{100}{-100} = -1$	The ratio of a number and its opposite is −1.
$\frac{x+7}{-x-7} = -1$	$\frac{x+7}{-x-7} = \frac{x+7}{-1(x+7)} = \frac{\overset{1}{\cancel{x+7}}}{-1(\cancel{x+7})} = \frac{1}{-1} = -1$
	Factor out −1.
$\frac{2-x}{x-2} = -1$	$\frac{2-x}{x-2} = \frac{-1(-2+x)}{x-2} = \frac{-1(\overset{1}{\cancel{x-2}})}{\cancel{x-2}} = \frac{-1}{1} = -1$

Recognizing factors that are opposites is useful when simplifying rational expressions. For example, $a - b$ and $b - a$ are opposites because the opposite of $a - b$ can be written $-(a - b) = -a + b = b - a$. Therefore, in general, $\frac{a-b}{b-a} = -1$.

Example 6 Simplifying a Rational Expression

Simplify the rational expression to lowest terms. $\frac{x-5}{25-x^2}$

Solution:

$$\frac{x-5}{25-x^2}$$

$$= \frac{x-5}{(5-x)(5+x)}$$ Factor.

Notice that $x - 5$ and $5 - x$ are opposites and form a ratio of −1.

$$= \frac{1}{(5+x)} \cdot \frac{\overset{-1}{\cancel{(x-5)}}}{\cancel{(5-x)}}$$ In general, $\frac{a-b}{b-a} = -1$.

$$= \frac{1}{5+x}(-1)$$

$$= -\frac{1}{5+x} \quad \text{or} \quad -\frac{1}{x+5}$$

TIP: The factor of −1 may be applied in front of the rational expression, or it may be applied to the numerator or to the denominator. Therefore, the final answer may be written in different forms.

$$-\frac{1}{x+5} \quad \text{or} \quad \frac{-1}{x+5} \quad \text{or} \quad \frac{1}{-(x+5)}$$

Skill Practice Simplify the expression.

12. $\dfrac{20-5x}{x^2-x-12}$

Answer

12. $-\dfrac{5}{x+3}$

Section 5.1 Activity

A.1. Given $f(x) = \dfrac{6}{x+2}$, find

a. $f(0)$ **b.** $f(1)$

c. $f(-1)$ **d.** $f(-2)$

A.2. Refer to $f(x) = \dfrac{6}{x+2}$.

a. Why is f undefined for $x = -2$?

b. Write the domain of f in interval notation.

A.3. Given $g(x) = \dfrac{2x^2+6x-20}{3x^2-5x-2}$,

a. Write the function with the numerator and denominator in factored form.

b. What are the restrictions on x?

c. Write the domain of g in interval notation.

d. Write the function in simplified form.

A.4. **a.** Write the opposite of $y - 5$. **b.** Simplify $\dfrac{y-5}{5-y}$.

A.5. **a.** Write the opposite of $-4a - 2b$. **b.** Simplify $\dfrac{4a+2b}{-4a-2b}$.

A.6. Simplify the expression and identify the restricted values of the variable.

$$\frac{72t-2t^3}{t^3-5t^2-6t}$$

Practice Exercises

Section 5.1

Study Skills Exercise

As the content grows in complexity, it is important to remind yourself of your long term goals. Times of challenge are great opportunities to strengthen your character. Strategies that will help you persist include:

- Self-awareness of your math skills.
- Not fearing mistakes, but rather learning from mistakes.
- Advocating for yourself when you need help.
- Not procrastinating on your math homework or in seeking help.

With persistence you will find success.

Prerequisite Review

For Exercises R.1–R.2, factor the number into a product of prime factors.

R.1. 84 **R.2.** 1350

For Exercises R.3–R.10, simplify the expression.

R.3. $-\dfrac{48}{84}$ **R.4.** $\dfrac{120}{600}$ **R.5.** $\dfrac{-34}{34}$ **R.6.** $\dfrac{18}{-18}$

R.7. $\dfrac{x^9}{x^7}$ **R.8.** $\dfrac{y^{15}}{y^{10}}$ **R.9.** $\dfrac{a^6 \cdot a^5}{a^{12}}$ **R.10.** $\dfrac{b^2 \cdot b^8}{b^{16}}$

For Exercises R.11–R.20, factor completely.

R.11. $24x^4 - 60x^3$ **R.12.** $120y^2 + 28y$ **R.13.** $36c^2 - 25d^2$ **R.14.** $9y^2 - 64z^2$

R.15. $14w^2 - 29w + 12$ **R.16.** $10x^2 - 13x - 3$ **R.17.** $8ac + 12a - 2bc - 3b$ **R.18.** $14x^2 + 63x + 2xy + 9y$

R.19. $16x^2 - 72xy + 81y^2$ **R.20.** $121c^2 + 22cd + d^2$

For Exercises R.21–R.22, graph the set and write the set in interval notation.

R.21. $\{x \mid x \text{ is a real number and } x \neq -3\}$

R.22. $\{x \mid x \text{ is a real number and } x \neq 2\}$

Vocabulary and Key Concepts

1. a. A ____________ function is a function that can be written as $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

b. The domain of a rational function is all real numbers excluding the values of the variable that make the ____________ equal to zero.

c. For polynomials p, q, and r, where ($q \neq 0$ and $r \neq 0$), $\dfrac{pr}{qr}$ simplifies to ____________.

d. The ratio $\dfrac{a-b}{a-b} =$ ____________, whereas the ratio $\dfrac{a-b}{b-a} =$ ____________ provided that $a \neq b$.

Concept 1: Rational Functions

For Exercises 2–6, determine the function values, if possible. **(See Example 1.)**

2. $f(x) = \dfrac{5}{x+1}$; $f(0), f(2), f(-1), f(-6)$

3. $k(x) = \dfrac{-3}{x+4}$; $k(0), k(-1), k(2), k(-4)$

4. $m(x) = \dfrac{x-4}{x+6}$; $m(-6), m(-4), m(0), m(4)$

5. $n(a) = \dfrac{3a+1}{a^2+1}$; $n(1), n(0), n\left(-\dfrac{1}{3}\right), n(-1)$

6. $f(t) = \dfrac{2t-8}{t^2+9}$; $f(4), f(-4), f(3), f(-3)$

For Exercises 7–20,

a. Write the domain in set-builder notation.

b. Write the domain in interval notation. **(See Example 2.)**

7. $f(x) = \dfrac{9}{x}$

8. $g(a) = -\dfrac{10}{a}$

9. $h(v) = \dfrac{v+1}{v-7}$

10. $p(t) = \dfrac{t+9}{t+3}$

11. $k(x) = \dfrac{3x-1}{2x-5}$

12. $n(t) = \dfrac{6t+5}{3t+8}$

13. $f(q) = \dfrac{q+1}{q^2+6q-27}$

14. $k(a) = \dfrac{a^2}{2a^2+3a-5}$

15. $h(c) = \dfrac{c}{c^2+25}$

16. $m(x) = \dfrac{x}{x^2+16}$

17. $f(x) = \dfrac{x+5}{x^2-25}$

18. $g(t) = \dfrac{t+4}{t^2-16}$

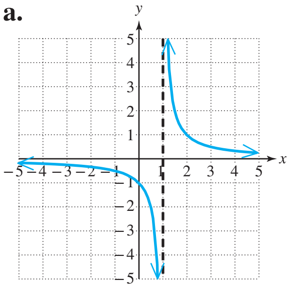
19. $p(x) = \dfrac{x-5}{3}$

20. $r(x) = \dfrac{x+2}{8}$

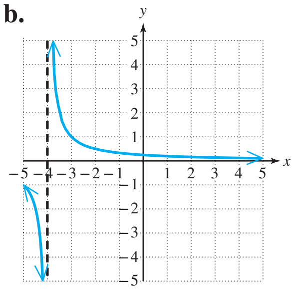
For Exercises 21–24, write the domain of each function in interval notation and use that information to match the function with its graph.

21. $m(x) = \dfrac{1}{x+4}$ **22.** $n(x) = \dfrac{1}{x+1}$ **23.** $q(x) = \dfrac{1}{x-4}$ **24.** $p(x) = \dfrac{1}{x-1}$

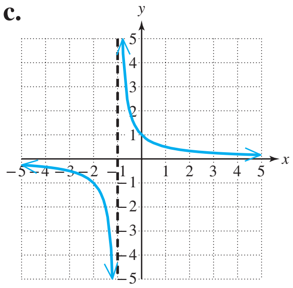
a.



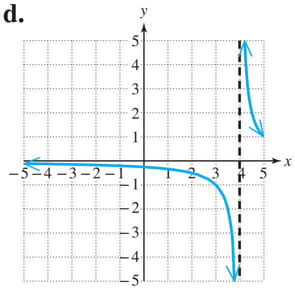
b.



c.



d.



Concept 2: Simplifying Rational Expressions

For Exercises 25–26, simplify the expression if possible.

25. **a.** $\dfrac{8x}{4y}$ **b.** $\dfrac{8+x}{4+y}$

26. **a.** $\dfrac{a-21}{14+b}$ **b.** $\dfrac{-21a}{14b}$

For Exercises 27–30,

a. Factor the numerator and denominator.

b. Determine the restrictions on x.

c. Simplify the expression. **(See Example 3.)**

27. $\dfrac{x^2+6x+8}{x^2+3x-4}$ **28.** $\dfrac{x^2-6x}{2x^2-11x-6}$

29. $\dfrac{x^2-18x+81}{x^2-81}$ **30.** $\dfrac{x^2+14x+49}{x^2-49}$

For Exercises 31–52, simplify the rational expression. **(See Examples 4–5.)**

31. $\dfrac{100x^3y^5}{36xy^8}$ **32.** $\dfrac{48ab^3c^2}{6a^7bc^0}$ **33.** $\dfrac{7w^{11}z^6}{14w^3z^3}$ **34.** $\dfrac{12r^9s^3}{24r^8s^4}$

35. $\dfrac{-3m^4n}{12m^6n^4}$ **36.** $\dfrac{-5x^3y^2}{20x^4y^2}$ **37.** $\dfrac{6a+18}{9a+27}$ **38.** $\dfrac{5y-15}{3y-9}$

39. $\dfrac{x-5}{x^2-25}$ **40.** $\dfrac{3z-6}{3z^2-12}$ **41.** $\dfrac{-7c}{21c^2-35c}$ **42.** $\dfrac{2p+3}{2p^2+7p+6}$

43. $\dfrac{2t^2+7t-4}{-2t^2-5t+3}$ **44.** $\dfrac{y^2+8y-9}{y^2-5y+4}$ **45.** $\dfrac{(p+1)(2p-1)^4}{(p+1)^2(2p-1)^2}$ **46.** $\dfrac{r(r-3)^5}{r^3(r-3)^2}$

47. $\dfrac{9 - z^2}{2z^2 + z - 15}$ **48.** $\dfrac{2c^2 + 2c - 12}{-8 + 2c + c^2}$ **49.** $\dfrac{2z^3 + 128}{16 + 8z + z^2}$ **50.** $\dfrac{p^3 - 1}{5 - 10p + 5p^2}$

51. $\dfrac{10x^3 - 25x^2 + 4x - 10}{-4 - 10x^2}$ **52.** $\dfrac{8x^3 - 12x^2 + 6x - 9}{16x^4 - 9}$

Concept 3: Simplifying Ratios of −1

For Exercises 53–64, simplify the rational expression. **(See Example 6.)**

53. $\dfrac{r + 6}{6 + r}$ **54.** $\dfrac{a + 2}{2 + a}$ **55.** $\dfrac{b + 8}{-b - 8}$ **56.** $\dfrac{7 + w}{-7 - w}$

57. $\dfrac{10 - x}{x - 10}$ **58.** $\dfrac{y - 14}{14 - y}$ **59.** $\dfrac{2t - 2}{1 - t}$ **60.** $\dfrac{5p - 10}{2 - p}$

61. $\dfrac{c + 4}{c - 4}$ **62.** $\dfrac{b + 2}{b - 2}$ **63.** $\dfrac{y - x}{12x^2 - 12y^2}$ **64.** $\dfrac{4w^2 - 49z^2}{14z - 4w}$

Mixed Exercises

For Exercises 65–84, simplify the rational expression.

65. $\dfrac{t^2 - 1}{t^2 + 7t + 6}$ **66.** $\dfrac{x^2 + 4x + 4}{x^2 - 4}$ **67.** $\dfrac{8p + 8}{2p^2 - 4p - 6}$ **68.** $\dfrac{15y - 15}{3y^2 + 9y - 12}$

69. $\dfrac{-16a^2bc^4}{8ab^2c^4}$ **70.** $\dfrac{-9x^3yz^2}{27x^4yz}$ **71.** $\dfrac{x^2 - y^2}{8y - 8x}$ **72.** $\dfrac{p^2 - 49}{14 - 2p}$

73. $\dfrac{b + 4}{2b^2 + 5b - 12}$ **74.** $\dfrac{c - 6}{3c^2 - 17c - 6}$ **75.** $\dfrac{-2x + 34}{-4x + 6}$ **76.** $\dfrac{-9w - 3}{3w + 12}$

77. $\dfrac{(a - 2)^2(a - 5)^3}{(a - 2)^3(a - 5)}$ **78.** $\dfrac{t^2(t - 11)^4}{t^5(t - 11)^2}$ **79.** $\dfrac{4x - 2x^2}{5x - 10}$ **80.** $\dfrac{2y - 6}{3y^2 - y^3}$

81. $\dfrac{x^3 - 2x^2 - 25x + 50}{x^3 + 5x^2 - 4x - 20}$ **82.** $\dfrac{4y^3 + 12y^2 - y - 3}{2y^3 + y^2 - 18y - 9}$ **83.** $\dfrac{t^3 + 8}{3t^2 + t - 10}$ **84.** $\dfrac{w^3 - 27}{4w^2 - 5w - 21}$

Expanding Your Skills

85. Write a rational expression whose domain is $(-\infty, 2) \cup (2, \infty)$. (Answers may vary.)

86. Write a rational expression whose domain is $(-\infty, 3) \cup (3, \infty)$. (Answers may vary.)

87. Write a rational function whose domain is $(-\infty, -5) \cup (-5, \infty)$. (Answers may vary.)

88. Write a rational function whose domain is $(-\infty, -6) \cup (-6, \infty)$. (Answers may vary.)

Multiplication and Division of Rational Expressions

Section 5.2

Concepts

1. Multiplication of Rational Expressions

Recall that to multiply fractions, we multiply the numerators and multiply the denominators. The same is true for multiplying rational expressions.

Multiplication Property of Rational Expressions

Let p, q, r, and s represent polynomials, such that $q \neq 0$ and $s \neq 0$. Then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

For example:

Multiply the Fractions	**Multiply the Rational Expressions**
$\frac{2}{3} \cdot \frac{5}{7} = \frac{10}{21}$	$\frac{2x}{3y} \cdot \frac{5z}{7} = \frac{10xz}{21y}$

Sometimes it is possible to simplify a ratio of common factors to 1 *before* multiplying. To do so, we must first factor the numerators and denominators of each fraction.

$$\frac{7}{10} \cdot \frac{15}{21} \xrightarrow{\text{Factor.}} \frac{7}{2 \cdot 5} \cdot \frac{3 \cdot 5}{3 \cdot 7} = \frac{\overset{1}{\cancel{7}} \cdot \overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{5}}}{2 \cdot \cancel{5} \cdot \cancel{3} \cdot \cancel{7}} = \frac{1}{2}$$

The same process is also used to multiply rational expressions.

Multiplying Rational Expressions

Step 1 Factor the numerator and denominator of each expression.
Step 2 Multiply the numerators, and multiply the denominators.
Step 3 Reduce the ratios of common factors to 1 or -1 and simplify.

Example 1 Multiplying Rational Expressions

Multiply. $\dfrac{5a - 5b}{10} \cdot \dfrac{2}{a^2 - b^2}$

Solution:

$$\frac{5a - 5b}{10} \cdot \frac{2}{a^2 - b^2}$$

$$= \frac{5(a - b)}{5 \cdot 2} \cdot \frac{2}{(a - b)(a + b)}$$ Factor numerator and denominator.

$$= \frac{5(a - b) \cdot 2}{5 \cdot 2 \cdot (a - b)(a + b)}$$ Multiply.

$$= \frac{\overset{1}{\cancel{5}}\overset{1}{\cancel{(a - b)}} \cdot \overset{1}{\cancel{2}}}{\cancel{5} \cdot \cancel{2} \cdot \cancel{(a - b)}(a + b)} = \frac{1}{a + b}$$ Reduce common factors and simplify.

Avoiding Mistakes

If all factors in the numerator simplify to 1, do not forget to write the factor of 1 in the numerator.

Skill Practice Multiply.

1. $\dfrac{3y-6}{6y} \cdot \dfrac{y^2+3y+2}{y^2-4}$

Example 2 Multiplying Rational Expressions

Multiply. $\dfrac{4w-20p}{2w^2-50p^2} \cdot \dfrac{2w^2+7wp-15p^2}{3w+9p}$

Solution:

$$\frac{4w-20p}{2w^2-50p^2} \cdot \frac{2w^2+7wp-15p^2}{3w+9p}$$

$$= \frac{4(w-5p)}{2(w^2-25p^2)} \cdot \frac{(2w-3p)(w+5p)}{3(w+3p)}$$ Factor numerator and denominator.

$$= \frac{2 \cdot 2(w-5p)}{2(w-5p)(w+5p)} \cdot \frac{(2w-3p)(w+5p)}{3(w+3p)}$$ Factor further.

$$= \frac{2 \cdot 2(w-5p)(2w-3p)(w+5p)}{2(w-5p)(w+5p) \cdot 3(w+3p)}$$ Multiply.

$$= \frac{\overset{1}{\cancel{2}} \cdot 2\overset{1}{\cancel{(w-5p)}}(2w-3p)\overset{1}{\cancel{(w+5p)}}}{\cancel{2}\cancel{(w-5p)}\cancel{(w+5p)} \cdot 3(w+3p)}$$ Simplify common factors.

$$= \frac{2(2w-3p)}{3(w+3p)}$$

Notice that the expression is left in factored form to show that it has been simplified to lowest terms.

Skill Practice Multiply.

2. $\dfrac{p^2+8p+16}{10p+10} \cdot \dfrac{2p+6}{p^2+7p+12}$

2. Division of Rational Expressions

Recall that to divide fractions, multiply the first fraction by the reciprocal of the second fraction.

Divide: $\dfrac{15}{14} \div \dfrac{10}{49} \xrightarrow[\text{of the second fraction.}]{\text{Multiply by the reciprocal}} \dfrac{15}{14} \cdot \dfrac{49}{10} = \dfrac{3 \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{7}} \cdot 7}{2 \cdot \cancel{7} \cdot 2 \cdot \cancel{5}} = \dfrac{21}{4}$

The same process is used for dividing rational expressions.

Division Property of Rational Expressions

Let p, q, r, and s represent polynomials, such that $q \neq 0$, $r \neq 0$, $s \neq 0$. Then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r} = \frac{ps}{qr}$$

Answers

1. $\dfrac{y+1}{2y}$ **2.** $\dfrac{p+4}{5(p+1)}$

Example 3 Dividing Rational Expressions

Divide. $\dfrac{8t^3+27}{9-4t^2} \div \dfrac{4t^2-6t+9}{2t^2-t-3}$

Solution:

$$\frac{8t^3+27}{9-4t^2} \div \frac{4t^2-6t+9}{2t^2-t-3}$$

$$= \frac{8t^3+27}{9-4t^2} \cdot \frac{2t^2-t-3}{4t^2-6t+9}$$

Multiply the first fraction by the reciprocal of the second.

$$= \frac{(2t+3)(4t^2-6t+9)}{(3-2t)(3+2t)} \cdot \frac{(2t-3)(t+1)}{4t^2-6t+9}$$

Factor numerator and denominator. Notice that $8t^3+27$ is a sum of cubes. Furthermore, $4t^2-6t+9$ does not factor over the real numbers.

$$= \frac{\overset{1}{\cancel{(2t+3)}}\overset{1}{\cancel{(4t^2-6t+9)}}\overset{-1}{\cancel{(2t-3)}}(t+1)}{\cancel{(3-2t)}\cancel{(3+2t)}\cancel{(4t^2-6t+9)}}$$

Simplify to lowest terms.

$$= (-1)\frac{(t+1)}{1}$$

$$= -(t+1) \quad \text{or} \quad -t-1$$

Avoiding Mistakes

When dividing rational expressions, your first step should be to take the reciprocal of the second fraction (divisor). Do this first so that you do not forget.

Skill Practice Divide.

3. $\dfrac{x^2+x}{5x^3-x^2} \div \dfrac{10x^2+12x+2}{25x^2-1}$

TIP: In Example 3, the factors $(2t-3)$ and $(3-2t)$ are opposites and form a ratio of -1. The factors $(2t+3)$ and $(3+2t)$ are equal and form a ratio of 1.

$$\frac{2t-3}{3-2t} = -1 \quad \text{whereas} \quad \frac{2t+3}{3+2t} = 1$$

Answer

3. $\dfrac{1}{2x}$

Section 5.2 Activity

A.1. Multiply the fractions. $\dfrac{8}{35} \cdot \dfrac{21}{4}$

A.2. Multiply the rational expressions by simplifying common factors whose ratio is 1 or -1.

$$\frac{x^2(3x-1)}{(x-4)(2x+5)} \cdot \frac{(4-x)(6+x)}{5x(x+6)(3x-1)}$$

A.3. Multiply the rational expressions $\dfrac{2y^3-20y^2+50y}{-2y^4+3y^3} \cdot \dfrac{2y^2+7y-15}{4y^2-100}$ by following these steps.

a. Factor the numerator and denominator of both rational expressions.

b. Multiply and write the simplified expression.

A.4. Divide the fractions. $\dfrac{10}{21} \div \dfrac{15}{14}$

A.5. Divide the rational expressions $\dfrac{4ac^2d}{5cd^2} \div \dfrac{8a^2c}{25ab}$ by following these steps.

a. Write the division expression as a related multiplication expression. That is, multiply the first fraction (the dividend) by the reciprocal of the second fraction (the divisor).

b. Simplify common factors whose ratio is 1 or -1.

A.6. Divide the rational expressions $\dfrac{1 - 2x}{x^3 + 8} \div \dfrac{4x^2 - 1}{2x^2 + 5x + 2}$ by following these steps.

a. Multiply the first expression by the reciprocal of the second expression.

b. Factor the numerator and denominator of each expression.

c. Simplify the expression.

Section 5.2 Practice Exercises

Study Skills Exercise

Talking out loud to yourself and/or teaching the material to another person is also a great way to retain information. As you are speaking and relating a concept to another individual, you are increasing your own understanding of the content.

- Pick a key concept that was presented in this section. Develop a lesson for teaching this concept to another student. Include important definitions or rules and worked out example problems.

Prerequisite Review

For Exercises R.1–R.4, identify the reciprocal of the given expressions. Assume all variable expressions represent nonzero real numbers.

R.1. **a.** $-\dfrac{1}{7}$ **b.** $\dfrac{3}{4}$ **c.** 9 **d.** y

R.2. **a.** $\dfrac{4}{11}$ **b.** -5 **c.** $\dfrac{1}{4}$ **d.** $\dfrac{3}{y}$

R.3. $\dfrac{x - 4}{x^2 + 15x + 14}$

R.4. $\dfrac{2x^2 + 17x + 21}{x + 9}$

For Exercises R.5–R.12, perform the indicated operation.

R.5. $\dfrac{4}{7} \cdot \dfrac{21}{2}$

R.6. $\dfrac{5}{6} \cdot \dfrac{8}{3}$

R.7. $-\dfrac{4}{3} \div \dfrac{16}{9}$

R.8. $\dfrac{24}{5} \div \left(-\dfrac{18}{5}\right)$

R.9. $-8 \cdot \left(-\dfrac{7}{16}\right)$

R.10. $\left(-\dfrac{5}{18}\right) \cdot (-6)$

R.11. $\dfrac{\frac{15}{4}}{\frac{25}{2}}$

R.12. $\dfrac{\frac{5}{3}}{\frac{10}{9}}$

Vocabulary and Key Concepts

1. **a.** Given polynomials p, q, r, and s such that $q \neq 0$ and $s \neq 0$, $\frac{p}{q} \cdot \frac{r}{s} = \frac{\square}{\square}$.

b. Given polynomials p, q, r, and s such that $q \neq 0$, $r \neq 0$, and $s \neq 0$, $\frac{p}{q} \div \frac{r}{s} = \frac{\square}{\square}$.

Concept 1: Multiplication of Rational Expressions

For Exercises 2–17, multiply the rational expression. **(See Examples 1–2.)**

2. $\frac{5}{12t} \cdot \frac{3t}{10}$

3. $\frac{9y}{14} \cdot \frac{7}{15y}$

4. $-6ab \cdot \left(\frac{5}{3ab}\right)$

5. $\left(-\frac{11}{6m^2}\right) \cdot 4m^2$

6. $\frac{8w^2}{9} \cdot \frac{3}{2w^4}$

7. $\frac{16}{z^7} \cdot \frac{z^4}{8}$

8. $\frac{5p^2q^4}{12pq^3} \cdot \frac{6p^2}{20q^2}$

9. $\frac{27r^5}{7s} \cdot \frac{28rs^3}{9r^3s^2}$

10. $\frac{3z+12}{8z^3} \cdot \frac{16z^3}{9z+36}$

11. $\frac{x^2y}{x^2-4x-5} \cdot \frac{2x^2-13x+15}{xy^3}$

12. $\frac{3y^2+18y+15}{6y+6} \cdot \frac{y-5}{y^2-25}$

13. $\frac{10w-8}{w+2} \cdot \frac{3w^2-w-14}{25w^2-16}$

14. $\frac{x-5y}{x^2+xy} \cdot \frac{y^2-x^2}{10y-2x}$

15. $\frac{3x-15}{4x^2-2x} \cdot \frac{10x-20x^2}{5-x}$

16. $x(x+5)^2 \cdot \frac{2}{x^2-25}$

17. $y(y^2-4) \cdot \frac{y}{y+2}$

Concept 2: Division of Rational Expressions

For Exercises 18–35, divide the rational expressions. **(See Example 3.)**

18. $\frac{10cd}{21d} \div \frac{15c}{14}$

19. $\frac{8v}{25} \div \frac{4vw}{35w}$

20. $-18p^2 \div \frac{45p^2}{2}$

21. $-\frac{12k^3}{5} \div 6k^3$

22. $\frac{5x}{7} \div \frac{10x^2}{21}$

23. $\frac{2a}{7b^3} \div \frac{10a^5}{77}$

24. $\frac{6x^2y^2}{(x-2)} \div \frac{3xy^2}{(x-2)^2}$

25. $\frac{(r+3)^2}{4r^3s} \div \frac{r+3}{rs}$

26. $\frac{t^2+5t}{t+1} \div (t+5)$

27. $\frac{6p+7}{p+2} \div (36p^2-49)$

28. $\frac{a}{a-10} \div \frac{a^3+6a^2-40a}{a^2-100}$

29. $\frac{b^2-6b+9}{b^2-b-6} \div \frac{b^2-9}{4}$

30. $\frac{2x^2+5xy+2y^2}{4x^2-y^2} \div \frac{x^2+xy-2y^2}{2x^2+xy-y^2}$

31. $\frac{6s^2+st-2t^2}{6s^2-5st+t^2} \div \frac{3s^2+17st+10t^2}{6s^2+13st-5t^2}$

32. $\frac{x^4-x^3+x^2-x}{2x^3+2x^2+x+1} \div \frac{x^3-4x^2+x-4}{2x^3-8x^2+x-4}$

33. $\frac{a^3+a+a^2+1}{a^3+a^2+ab^2+b^2} \div \frac{a^3+a+a^2b+b}{2a^2+2ab+ab^2+b^3}$

34. $\frac{3y-y^2}{y^3-27} \div \frac{y}{y^2+3y+9}$

35. $\frac{8x-4x^2}{xy-2y+3x-6} \div \frac{3x+6}{y+3}$

Mixed Exercises

For Exercises 36–55, perform the indicated operations.

36. $\dfrac{8a^4b^3}{3c} \div \dfrac{a^7b^2}{9c}$

37. $\dfrac{3x^5}{2x^2y^7} \div \dfrac{4x^3y}{6y^6}$

38. $\dfrac{2}{25x^2} \cdot \dfrac{5x}{12} \div \dfrac{2}{15x}$

39. $\dfrac{4y}{7} \div \dfrac{y^2}{14} \cdot \dfrac{3}{y}$

40. $\dfrac{10x^2 - 13xy - 3y^2}{8x^2 - 10xy - 3y^2} \cdot \dfrac{2y + 8x}{2x^2 + 2y^2}$

41. $\dfrac{6a^2 + ab - b^2}{10a^2 + 5ab} \cdot \dfrac{2a^3 + 4a^2b}{3a^2 + 5ab - 2b^2}$

42. $(3m^2 - 12m) \div \dfrac{m^2 - 4m}{m^2 - 6m + 8}$

43. $(2x^2 + 8) \div \dfrac{x^4 - 16}{x^2 + x - 6}$

44. $\dfrac{(a + b)^2}{a - b} \cdot \dfrac{a^3 - b^3}{a^2 - b^2} \div \dfrac{a^2 + ab + b^2}{(a - b)^2}$

45. $\dfrac{m^2 - n^2}{(m - n)^2} \div \dfrac{m^2 - 2mn + n^2}{m^2 - mn + n^2} \cdot \dfrac{(m - n)^4}{m^3 + n^3}$

46. $\dfrac{x^2 - 4y^2}{x + 2y} \div (x + 2y) \cdot \dfrac{2y}{x - 2y}$

47. $\dfrac{x^2 - 6xy + 9y^2}{x^2 - 4y^2} \cdot \dfrac{x^2 - 5xy + 6y^2}{3y - x} \div \dfrac{x^2 - 9y^2}{x + 2y}$

48. $\dfrac{8x^3 - 27y^3}{4x^2 - 9y^2} \div \dfrac{8x^2 + 12xy + 18y^2}{2x + 3y}$

49. $\dfrac{25m^2 - 1}{125m^3 - 1} \div \dfrac{5m + 1}{25m^2 + 5m + 1}$

50. $\dfrac{m^3 + 2m^2 - mn^2 - 2n^2}{m^3 - m^2 - 20m} \cdot \dfrac{m^3 - 25m}{m^3 + m^2n - 4m - 4n}$

51. $\dfrac{2a^2 + ab - 8a - 4b}{2a^2 - 6a + ab - 3b} \cdot \dfrac{a^2 - 6a + 9}{a^2 - 16}$

52. $\dfrac{7}{3x + 15} \cdot (x + 5) \div \dfrac{14}{9x - 27}$

53. $\dfrac{45}{2x + 1} \cdot (8x + 4) \div \dfrac{27}{4x + 2}$

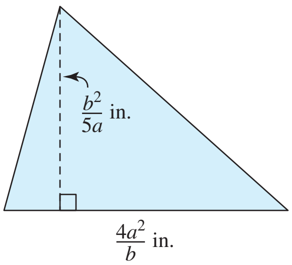
54. $\dfrac{12y + 3}{6y^2 - y - 12} \div \dfrac{4y^2 - 19y - 5}{2y^2 - y - 3}$

55. $\dfrac{2x^2 - 11x - 6}{3x - 2} \div \dfrac{2x^2 - 5x - 3}{3x^2 - 7x - 6}$

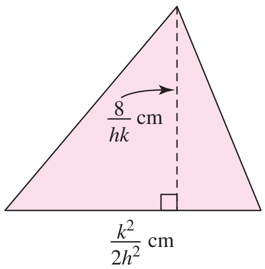
Expanding Your Skills

For Exercises 56–59, write an expression for the area of the figure and simplify.

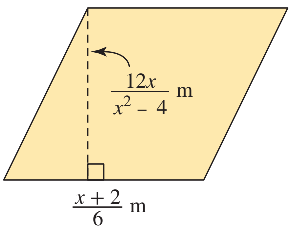
56.



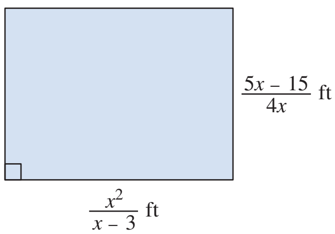
57.



58.



59.



Addition and Subtraction of Rational Expressions

Section 5.3

Concepts

1. Addition and Subtraction of Rational Expressions with Like Denominators

To add or subtract rational expressions, the expressions must have the same denominator. As with fractions, we add or subtract rational expressions with the same denominator by combining the terms in the numerator and then writing the result over the common denominator. Then, if possible, we simplify the expression to lowest terms.

Addition and Subtraction Properties of Rational Expressions

Let p, q, and r represent polynomials where $q \neq 0$. Then

1. $\dfrac{p}{q} + \dfrac{r}{q} = \dfrac{p + r}{q}$ **2.** $\dfrac{p}{q} - \dfrac{r}{q} = \dfrac{p - r}{q}$

Example 1 Adding and Subtracting Rational Expressions with Like Denominators

Add or subtract as indicated.

a. $\dfrac{1}{8} + \dfrac{3}{8}$ **b.** $\dfrac{5x}{2x - 1} + \dfrac{3}{2x - 1}$ **c.** $\dfrac{x^2}{x - 4} - \dfrac{x + 12}{x - 4}$

Solution:

a. $\dfrac{1}{8} + \dfrac{3}{8} = \dfrac{1 + 3}{8}$ Add the terms in the numerator.

$= \dfrac{4}{8}$

$= \dfrac{1}{2}$ Simplify the fraction.

b. $\dfrac{5x}{2x - 1} + \dfrac{3}{2x - 1} = \dfrac{5x + 3}{2x - 1}$ Add the terms in the numerator. The answer is already in lowest terms.

c. $\dfrac{x^2}{x - 4} - \dfrac{x + 12}{x - 4}$ Combine the terms in the numerator. Use parentheses to group the terms in the numerator that follow the subtraction sign. This will help you remember to apply the distributive property.

$= \dfrac{x^2 - (x + 12)}{x - 4}$

$= \dfrac{x^2 - x - 12}{x - 4}$ Apply the distributive property.

$= \dfrac{(x - 4)(x + 3)}{(x - 4)}$ Factor the numerator and denominator.

$= \dfrac{\overset{1}{\cancel{(x - 4)}}(x + 3)}{\cancel{(x - 4)}}$ Simplify the rational expression.

$= x + 3$

Skill Practice Add or subtract as indicated.

1. $\dfrac{5}{12} - \dfrac{1}{12}$ **2.** $\dfrac{4c-3}{c-2} + \dfrac{8}{c-2}$ **3.** $\dfrac{t^2}{t-7} - \dfrac{5t+14}{t-7}$

2. Least Common Denominator

If two rational expressions have different denominators, each expression must be rewritten with a common denominator before adding or subtracting the expressions. The **least common denominator (LCD)** of two or more rational expressions is defined as the least common multiple of the denominators.

For example, consider the fractions $\frac{1}{20}$ and $\frac{1}{8}$. By inspection, the least common denominator is 40. To understand why, find the prime factorization of both denominators.

$$20 = 2^2 \cdot 5 \qquad \text{and} \qquad 8 = 2^3$$

A common multiple of 20 and 8 must be a multiple of 5, a multiple of 2^2, and a multiple of 2^3. However, any number that is a multiple of $2^3 = 8$ is automatically a multiple of $2^2 = 4$. Therefore, it is sufficient to construct the least common denominator as the product of unique prime factors, where each factor is raised to its highest power.

$$\text{The LCD of } \frac{1}{2^2 \cdot 5} \text{ and } \frac{1}{2^3} \text{ is } 2^3 \cdot 5 = 40.$$

Finding the LCD of Two or More Rational Expressions

Step 1 Factor all denominators completely.

Step 2 The LCD is the product of unique prime factors from the denominators, where each factor is raised to the highest power to which it appears in any denominator.

Example 2 Finding the LCD of Rational Expressions

Find the LCD of the rational expressions.

a. $\dfrac{1}{12}, \dfrac{5}{18}, \dfrac{7}{30}$ **b.** $\dfrac{1}{2x^3y}, \dfrac{5}{16xy^2z}$

Solution:

a. $\dfrac{1}{12}, \dfrac{5}{18}, \dfrac{7}{30}$

$\dfrac{1}{2^2 \cdot 3}, \dfrac{5}{2 \cdot 3^2}, \dfrac{7}{2 \cdot 3 \cdot 5}$ Factor the denominators completely.

$12 = 2^2 \cdot 3$ 2^2 is the greatest power of 2 that appears.

$18 = 2^1 3^2$ 3^2 is the greatest power of 3 that appears.

$30 = 2^1 \cdot 3^1 5^1$ 5^1 is the greatest power of 5 that appears.

$\text{LCD} = 2^2 \cdot 3^2 \cdot 5 = 180$ The LCD is the product of the factors 2, 3, and 5, where each factor is raised to its highest power.

Answers

1. $\dfrac{1}{3}$ **2.** $\dfrac{4c+5}{c-2}$ **3.** $t+2$

b. $\dfrac{1}{2x^3y}, \dfrac{5}{16xy^2z}$

$\dfrac{1}{2x^3y}, \dfrac{5}{2^4xy^2z}$ Factor the denominators completely.

$\text{LCD} = 2^4x^3y^2z$ The LCD is the product of the factors 2, x, y, and z, where each factor is raised to its highest power.

$= 16x^3y^2z$

Skill Practice Find the LCD of the rational expressions.

4. $\dfrac{7}{40}, \dfrac{1}{15}, \dfrac{5}{6}$ **5.** $\dfrac{1}{9a^3b^2}, \dfrac{5}{18a^4b}$

Example 3 Finding the LCD of Rational Expressions

Find the LCD of the rational expressions.

a. $\dfrac{x^2+3}{x^2+9x+20}, \dfrac{6}{x^2+8x+16}$ **b.** $\dfrac{x+4}{x-3}, \dfrac{1}{3-x}$

Solution:

a. $\dfrac{x^2+3}{x^2+9x+20}, \dfrac{6}{x^2+8x+16}$

$\dfrac{x^2+3}{(x+4)(x+5)}, \dfrac{6}{(x+4)^2}$ Factor the denominators completely.

$\text{LCD} = (x+5)(x+4)^2$ The LCD is the product of the factors $(x+5)$ and $(x+4)$, where each factor is raised to its highest power.

b. $\dfrac{x+4}{x-3}, \dfrac{1}{3-x}$ The denominators are already factored.

Notice that $x-3$ and $3-x$ are opposite factors. If -1 is factored from either expression, the binomial factors will be the same.

$\dfrac{x+4}{x-3}, \dfrac{1}{-1(-3+x)}$ $\qquad$ $\dfrac{x+4}{-1(-x+3)}, \dfrac{1}{3-x}$

Factor out -1. $\qquad$ Factor out -1.

same binomial factors $\qquad$ same binomial factors

$\text{LCD} = (x-3)(-1)$ $\qquad$ $\text{LCD} = (-1)(3-x)$

$= -x+3$ $\qquad$ $= -3+x$

$= 3-x$ $\qquad$ $= x-3$

The LCD is either $(3-x)$ or $(x-3)$.

Skill Practice Find the LCD of the rational expressions.

6. $\dfrac{5}{x^2+4x+4}, \dfrac{x+1}{x^2-x-6}$ **7.** $\dfrac{6}{z-7}, \dfrac{1}{7-z}$

Answers

4. 120 **5.** $18a^4b^2$
6. $(x+2)^2(x-3)$
7. $z-7$ or $7-z$

3. Equivalent Rational Expressions

Rational expressions can be added if they have common denominators. Once the LCD has been determined, each rational expression must be converted to an **equivalent rational expression** with the indicated denominator.

Using the identity property of multiplication, we know that for $q \neq 0$ and $r \neq 0$,

$$\frac{p}{q} = \frac{p}{q} \cdot 1 = \frac{p}{q} \cdot \frac{r}{r} = \frac{pr}{qr}$$

This principle is used to convert a rational expression to an equivalent expression with a different denominator. For example, $\frac{1}{2}$ can be converted to an equivalent expression with a denominator of 12 as follows:

$$\frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2} \cdot \frac{6}{6} = \frac{1 \cdot 6}{2 \cdot 6} = \frac{6}{12}$$

In this example, we multiplied $\frac{1}{2}$ by a convenient form of 1. The ratio $\frac{6}{6}$ was chosen so that the product produced a new denominator of 12. Notice that multiplying $\frac{1}{2}$ by $\frac{6}{6}$ is equivalent to multiplying the numerator and denominator of the original expression by 6.

Example 4 Creating Equivalent Rational Expressions

Convert each expression to an equivalent rational expression with the indicated denominator.

a. $\frac{7}{5p^2} = \frac{\quad}{20p^6}$ **b.** $\frac{w}{w+5} = \frac{\quad}{w^2 + 3w - 10}$

Solution:

a. $\frac{7}{5p^2} = \frac{\quad}{20p^6}$

$\frac{7}{5p^2} \cdot \frac{4p^4}{4p^4} = \frac{28p^4}{20p^6}$ $5p^2$ must be multiplied by $4p^4$ to create $20p^6$. Multiply numerator and denominator by $4p^4$.

b. $\frac{w}{w+5} = \frac{\quad}{w^2 + 3w - 10}$

$= \frac{\quad}{(w+5)(w-2)}$ Factor the denominator.

$\frac{w}{w+5} = \frac{w}{w+5} \cdot \frac{w-2}{w-2}$ Multiply the numerator and denominator by the missing factor $(w - 2)$.

$= \frac{w^2 - 2w}{(w+5)(w-2)}$

TIP: Notice that in Example 4 we multiplied the polynomials in the numerator but left the denominator in factored form. This convention is followed because when we add and subtract rational expressions, the terms in the numerators must be combined.

Skill Practice Convert each expression to an equivalent rational expression with the indicated denominator.

8. $\frac{1}{8xy} = \frac{\quad}{16x^3y^2}$ **9.** $\frac{5b}{b-3} = \frac{\quad}{b^2 - 9}$

Answers

8. $\frac{2x^2y}{16x^3y^2}$ **9.** $\frac{5b^2 + 15b}{b^2 - 9}$

4. Addition and Subtraction of Rational Expressions with Unlike Denominators

To add or subtract rational expressions with unlike denominators, we must convert each expression to an equivalent expression with the same denominator. For example, consider adding the expressions $\frac{3}{x-2}+\frac{5}{x+1}$. The LCD is $(x-2)(x+1)$. For each expression, identify the factors from the LCD that are missing in the denominator. Then multiply the numerator and denominator of the expression by the missing factor(s):

$$\frac{(3)}{(x-2)}\cdot\frac{(x+1)}{(x+1)}+\frac{(5)}{(x+1)}\cdot\frac{(x-2)}{(x-2)}$$ The rational expressions now have the same denominator and can be added.

$$=\frac{3(x+1)+5(x-2)}{(x-2)(x+1)}$$ Combine terms in the numerator.

$$=\frac{3x+3+5x-10}{(x-2)(x+1)}$$ Clear parentheses and simplify.

$$=\frac{8x-7}{(x-2)(x+1)}$$

Adding and Subtracting Rational Expressions

Step 1 Factor the denominator of each rational expression.

Step 2 Identify the LCD.

Step 3 Rewrite each rational expression as an equivalent expression with the LCD as its denominator.

Step 4 Add or subtract the numerators, and write the result over the common denominator.

Step 5 Simplify, if possible.

Example 5 Adding Rational Expressions with Unlike Denominators

Add. $\frac{3}{7b}+\frac{4}{b^2}$

Solution:

$$\frac{3}{7b}+\frac{4}{b^2}$$ **Step 1:** The denominators are already factored.

Step 2: The LCD is $7b^2$.

$$=\frac{3}{7b}\cdot\frac{b}{b}+\frac{4}{b^2}\cdot\frac{7}{7}$$ **Step 3:** Write each expression with the LCD.

$$=\frac{3b}{7b^2}+\frac{28}{7b^2}$$ **Step 4:** Add the numerators, and write the result over the LCD.

$$=\frac{3b+28}{7b^2}$$ **Step 5:** Simplify.

Skill Practice Add.

10. $\frac{4}{5y}+\frac{1}{3y^3}$

Answer

10. $\frac{12y^2+5}{15y^3}$

Example 6 **Subtracting Rational Expressions with Unlike Denominators**

Subtract. $\dfrac{3t-2}{t^2+4t-12} - \dfrac{5}{2t+12}$

Solution:

$$\frac{3t-2}{t^2+4t-12} - \frac{5}{2t+12}$$

$$= \frac{3t-2}{(t+6)(t-2)} - \frac{5}{2(t+6)}$$

Step 1: Factor the denominators.

Step 2: The LCD is $2(t+6)(t-2)$.

$$= \frac{(2)}{(2)} \cdot \frac{(3t-2)}{(t+6)(t-2)} - \frac{5}{2(t+6)} \cdot \frac{(t-2)}{(t-2)}$$

Step 3: Write each expression with the LCD.

$$= \frac{2(3t-2) - 5(t-2)}{2(t+6)(t-2)}$$

Step 4: Add the numerators and write the result over the LCD.

$$= \frac{6t-4-5t+10}{2(t+6)(t-2)}$$

Step 5: Simplify.

$$= \frac{t+6}{2(t+6)(t-2)}$$

Combine *like* terms.

$$= \frac{\overset{1}{\cancel{t+6}}}{2(\cancel{t+6})(t-2)}$$

Simplify.

$$= \frac{1}{2(t-2)}$$

Skill Practice Subtract.

11. $\dfrac{2x+3}{x^2+x-2} - \dfrac{5}{3x-3}$

Example 7 **Adding and Subtracting Rational Expressions with Unlike Denominators**

Add and subtract as indicated. $\dfrac{2}{x} + \dfrac{x}{x+3} - \dfrac{3x+18}{x^2+3x}$

Solution:

$$\frac{2}{x} + \frac{x}{x+3} - \frac{3x+18}{x^2+3x}$$

$$= \frac{2}{x} + \frac{x}{x+3} - \frac{3x+18}{x(x+3)}$$

Step 1: Factor the denominators.

Step 2: The LCD is $x(x+3)$.

$$= \frac{2}{x} \cdot \frac{(x+3)}{(x+3)} + \frac{x}{(x+3)} \cdot \frac{x}{x} - \frac{3x+18}{x(x+3)}$$

Step 3: Write each expression with the LCD.

Answer

11. $\dfrac{1}{3(x+2)}$

$= \dfrac{2(x+3) + x^2 - (3x+18)}{x(x+3)}$ **Step 4:** Add the numerators, and write the result over the LCD.

$= \dfrac{2x + 6 + x^2 - 3x - 18}{x(x+3)}$ **Step 5:** Simplify.

$= \dfrac{x^2 - x - 12}{x(x+3)}$ Combine *like* terms.

$= \dfrac{(x-4)(x+3)}{x(x+3)}$ Factor the numerator.

$= \dfrac{(x-4)\overset{1}{\cancel{(x+3)}}}{x\cancel{(x+3)}}$ Simplify.

$= \dfrac{x-4}{x}$

Avoiding Mistakes

It is important to insert parentheses around the quantity being subtracted.

Skill Practice Add.

12. $\dfrac{a^2 + a + 24}{a^2 - 9} + \dfrac{5}{a+3}$

Example 8 Subtracting Rational Expressions with Unlike Denominators

Subtract. $\dfrac{6}{w} - \dfrac{4}{-w}$

Solution:

$\dfrac{6}{w} - \dfrac{4}{-w}$ **Step 1:** The denominators are already factored.

Step 2: The denominators are opposites and differ by a factor of -1. The LCD can either be taken as w or $-w$. We will use an LCD of w.

$= \dfrac{6}{w} - \dfrac{4}{-w} \cdot \dfrac{(-1)}{(-1)}$ **Step 3:** Write each expression with the LCD. Note that $(-w)(-1) = w$.

$= \dfrac{6}{w} - \dfrac{-4}{w}$

$= \dfrac{6 - (-4)}{w}$ **Step 4:** Subtract the numerators, and write the result over the LCD.

$= \dfrac{10}{w}$ **Step 5:** Simplify.

Skill Practice Subtract.

13. $\dfrac{3}{-y} - \dfrac{5}{y}$

Answers

12. $\dfrac{a+3}{a-3}$ **13.** $-\dfrac{8}{y}$

Example 9 Adding Rational Expressions with Unlike Denominators

Add. $\dfrac{x^2}{x-y} + \dfrac{y^2}{y-x}$

Solution:

$\dfrac{x^2}{x-y} + \dfrac{y^2}{y-x}$

Step 1: The denominators are already factored.

Step 2: The denominators are opposites and differ by a factor of -1. The LCD can be taken as either $(x-y)$ or $(y-x)$. We will use an LCD of $(x-y)$.

$= \dfrac{x^2}{(x-y)} + \dfrac{y^2}{(y-x)} \cdot \dfrac{(-1)}{(-1)}$

Step 3: Write each expression with the LCD. Note that $(y-x)(-1) = -y + x = x - y$.

$= \dfrac{x^2}{x-y} + \dfrac{-y^2}{x-y}$

$= \dfrac{x^2 - y^2}{x-y}$

Step 4: Combine the numerators, and write the result over the LCD.

$= \dfrac{(x+y)\overset{1}{\cancel{(x-y)}}}{\cancel{x-y}}$

Step 5: Factor and simplify to lowest terms.

$= x + y$

Skill Practice Add.

14. $\dfrac{3a}{a-5} + \dfrac{15}{5-a}$

Answer

14. 3

Section 5.3 Activity

For Exercises A.1–A.2, perform the indicated operations and simplify the result.

A.1. $\dfrac{5}{12} - \dfrac{1}{12}$ **A.2.** $\dfrac{x^2 + 2x - 7}{x-5} - \dfrac{2x+18}{x-5}$

A.3. To add or subtract expressions with different denominators, we must first convert the expressions to equivalent expressions with a common denominator. Consider the sum $\dfrac{5}{4xy^4} + \dfrac{1}{8x^2}$.

a. Factor the denominators of each expression.

b. To build the least common denominator of the two expressions, follow these steps. Identify each unique prime factor from the two denominators and the highest power to which it appears in either denominator. Then take the product of these.

Write the greatest power to which 2 appears: $2^{\square}$

Write the greatest power to which x appears: $x^{\square}$

Write the greatest power to which y appears: $y^{\square}$

The product is ______.

c. Multiply each term by an appropriate ratio of 1 so that the denominator of each term is the LCD $8x^2y^4$. (*Hint:* Determine the missing factors from each denominator that are needed so that the product equals the LCD.)

$$\frac{5}{4xy^4}\cdot\frac{\square}{\square}+\frac{1}{8x^2}\cdot\frac{\square}{\square}$$

d. Add the fractions from part (c).

For Exercises A.4–A.6, add or subtract the expressions by following these steps.

a. Factor the denominators.

b. Identify the LCD.

c. Write each term as an equivalent expression with the LCD as the denominator.

d. Add or subtract the numerators and write the result over the common denominator.

e. Simplify the result and write the expression in lowest terms.

A.4. $\frac{3}{5x}+\frac{7}{15}$

A.5. $\frac{2}{3x-18}-\frac{8}{x^2-36}$

A.6. $\frac{y}{y+4}+\frac{1}{y-7}-\frac{11}{y^2-3y-28}$

A.7. Consider the expression $\frac{4}{x-3}+\frac{9}{3-x}$.

a. Explain why the LCD can be taken as $x-3$ or as $3-x$.

b. Add the fractions using $x-3$ as the LCD.

c. Add the fractions using $3-x$ as the LCD.

d. Show that the expressions $\frac{-5}{x-3}$ and $\frac{5}{3-x}$ are equivalent.

Practice Exercises — Section 5.3

Study Skills Exercise

Math anxiety can stem from negative past experiences, fear of failure in a current course, or experiencing failure in a specific content area. However, advocating for yourself will help you build confidence. Ask questions in class or privately with your instructor. Learning new concepts requires productive struggle, so don't be afraid to make mistakes, take risks, and seek assistance when needed.

- Review the key concepts of this section and identify one to three concepts or problems that you do not understand. Continue to seek help until you are satisfied that your questions have been answered and that you are confident in your understanding.

Prerequisite Review

For Exercises R.1–R.2, find the least common multiple of the pair of numbers.

R.1. 18 and 24

R.2. 12 and 15

For Exercises R.3–R.4, identify the least common denominator of the given expressions.

R.3. $\frac{7}{12}$ and $\frac{5}{18}$

R.4. $\frac{3}{14}$ and $\frac{13}{63}$

For Exercises R.5–R.10, simplify the expression and identify restricted values of the variables.

R.5. $\dfrac{3x^4}{15x^3y}$

R.6. $\dfrac{36c^2d^2}{3c^5d}$

R.7. $\dfrac{6y^2 - 21y - 12}{6y - 24}$

R.8. $\dfrac{4x^2 - 100}{4x^2 + 20x}$

R.9. $\dfrac{30 - 10x}{x - 3}$

R.10. $\dfrac{7t - 14}{2 - t}$

For Exercises R.11–R.14, multiply.

R.11. $-5t(t - 6)$

R.12. $-7w(w - 10)$

R.13. $(3x + 1)(5x - 7)$

R.14. $(2y + 9)(y - 4)$

R.15. Which of the expressions are equivalent to $\dfrac{2 - x}{6}$? Circle all that apply.

a. $\dfrac{x - 2}{-6}$ **b.** $\dfrac{-(x - 2)}{6}$ **c.** $-\dfrac{x - 2}{6}$ **d.** $-\dfrac{2 - x}{6}$

R.16. Which of the expressions are equivalent to $\dfrac{-5}{y - 3}$? Circle all that apply.

a. $-\dfrac{5}{y - 3}$ **b.** $\dfrac{5}{3 - y}$ **c.** $-\dfrac{5}{3 - y}$ **d.** $\dfrac{-5}{3 - y}$

Vocabulary and Key Concepts

1. a. Given polynomials p, q, and r such that $q \neq 0$, $\dfrac{p}{q} + \dfrac{r}{q} = \dfrac{\square}{\square}$ and $\dfrac{p}{q} - \dfrac{r}{q} = \dfrac{\square}{\square}$

b. The __________ __________ __________ (LCD) of two rational expressions is defined as the least common multiple of their denominators.

Concept 1: Addition and Subtraction of Rational Expressions with Like Denominators

For Exercises 2–14, add or subtract as indicated and simplify if possible. **(See Example 1.)**

2. $\dfrac{14}{3} + \left(-\dfrac{2}{3}\right)$

3. $-\dfrac{3}{5} - \dfrac{12}{5}$

4. $-\dfrac{3}{x} + \dfrac{4}{x}$

5. $-\dfrac{2}{y} - \dfrac{3}{y}$

6. $\dfrac{3}{x^2} + \dfrac{1}{x^2}$

7. $\dfrac{3}{5x} + \dfrac{7}{5x}$

8. $\dfrac{1}{2x^2} - \dfrac{5}{2x^2}$

9. $\dfrac{x}{x^2 - 2x - 3} - \dfrac{3}{x^2 - 2x - 3}$

10. $\dfrac{x}{x^2 + 4x - 12} + \dfrac{6}{x^2 + 4x - 12}$

11. $\dfrac{5x - 1}{(2x + 9)(x - 6)} - \dfrac{3x - 6}{(2x + 9)(x - 6)}$

12. $\dfrac{4 - x}{8x + 1} - \dfrac{5x - 6}{8x + 1}$

13. $\dfrac{x + 2}{x - 5} + \dfrac{x - 12}{x - 5}$

14. $\dfrac{2x - 1}{x - 2} + \dfrac{x - 5}{x - 2}$

Concept 2: Least Common Denominator

For Exercises 15–26, find the least common denominator (LCD). **(See Examples 2–3.)**

15. $\dfrac{5}{8};\ \dfrac{3}{20x}$

16. $\dfrac{y}{15a};\ \dfrac{y^2}{35}$

17. $\dfrac{-5}{6m^4};\ \dfrac{1}{15mn^7}$

18. $\dfrac{13}{12cd^5};\ \dfrac{9}{8c^3}$

19. $\dfrac{6}{(x-4)(x+2)};\ \dfrac{-8}{(x-4)(x-6)}$

20. $\dfrac{x}{(2x-1)(x-7)};\ \dfrac{2}{(2x-1)(x+1)}$

21. $\dfrac{3}{x(x-1)(x+7)^2};\ \dfrac{-1}{x^2(x+7)}$

22. $\dfrac{14}{(x-2)^2(x+9)};\ \dfrac{41}{x(x-2)(x+9)}$

23. $\dfrac{5}{x-6};\ \dfrac{x-5}{x^2-8x+12}$

24. $\dfrac{7a}{a+4};\ \dfrac{a+12}{a^2-16}$

25. $\dfrac{3a}{a-4};\ \dfrac{5}{4-a}$

26. $\dfrac{10}{x-6};\ \dfrac{x+1}{6-x}$

Concept 3: Equivalent Rational Expressions

For Exercises 27–32, fill in the blank to make an equivalent fraction with the given denominator. **(See Example 4.)**

27. $\dfrac{5}{3x} = \dfrac{\quad}{9x^2y}$

28. $\dfrac{-5}{xy} = \dfrac{\quad}{4x^2y^3}$

29. $\dfrac{2x}{x-1} = \dfrac{\quad}{x(x-1)(x+2)}$

30. $\dfrac{5x}{2x-5} = \dfrac{\quad}{(2x-5)(x+8)}$

31. $\dfrac{y}{y+6} = \dfrac{\quad}{y^2+5y-6}$

32. $\dfrac{t^2}{t-8} = \dfrac{\quad}{t^2-6t-16}$

Concept 4: Addition and Subtraction of Rational Expressions with Unlike Denominators

For Exercises 33–58, add or subtract as indicated. **(See Examples 5–9.)**

33. $\dfrac{4}{3p} - \dfrac{5}{2p^2}$

34. $\dfrac{6}{5a^2b} - \dfrac{1}{10ab}$

35. $\dfrac{s-1}{s} - \dfrac{t+1}{t}$

36. $\dfrac{x+2}{x} - \dfrac{y-2}{y}$

37. $\dfrac{4a-2}{3a+12} - \dfrac{a-2}{a+4}$

38. $\dfrac{6y+5}{5y-25} - \dfrac{y+2}{y-5}$

39. $\dfrac{10}{b(b+5)} + \dfrac{2}{b}$

40. $\dfrac{6}{w(w-2)} + \dfrac{3}{w}$

41. $\dfrac{x-2}{x-6} - \dfrac{x+2}{6-x}$

42. $\dfrac{x-10}{x-8} - \dfrac{x+10}{8-x}$

43. $\dfrac{6b}{b-4} - \dfrac{1}{b+1}$

44. $\dfrac{a}{a-3} - \dfrac{5}{a+6}$

45. $\dfrac{2}{2x+1} + \dfrac{4}{x-2}$

46. $\dfrac{3}{y+6} + \dfrac{1}{3y+1}$

47. $\dfrac{y-2}{y-4} + \dfrac{2y^2-15y+12}{y^2-16}$

48. $\dfrac{x^2+13x+18}{x^2-9} + \dfrac{x+1}{x+3}$

49. $\dfrac{x+2}{x^2-36} - \dfrac{x}{x^2+9x+18}$

50. $\dfrac{7}{x^2-x-2} + \dfrac{x}{x^2+4x+3}$

51. $\dfrac{5}{w} + \dfrac{8}{-w}$

52. $\dfrac{4}{y} + \dfrac{5}{-y}$

53. $\dfrac{n}{5-n} + \dfrac{2n-5}{n-5}$

54. $\dfrac{c}{7-c}+\dfrac{2c-7}{c-7}$

55. $\dfrac{2}{3x-15}+\dfrac{x}{25-x^2}$

56. $\dfrac{5}{9-x^2}-\dfrac{4}{x^2+4x+3}$

57. $\dfrac{m}{20+9m+m^2}-\dfrac{4}{12+7m+m^2}$

58. $\dfrac{t}{6+5t+t^2}-\dfrac{2}{2+3t+t^2}$

Mixed Exercises

For Exercises 59–80, simplify.

59. $\dfrac{x+3}{x^2}+\dfrac{x+5}{2x}$

60. $\dfrac{x+2}{5x^2}+\dfrac{x+4}{15x}$

61. $w+2+\dfrac{1}{w-2}$

62. $h-3+\dfrac{1}{h+3}$

63. $\dfrac{9}{x^2-2x+1}-\dfrac{x-3}{x^2-x}$

64. $\dfrac{2}{4z^2-12z+9}-\dfrac{z+1}{2z^2-3z}$

65. $\dfrac{t+1}{t+3}-\dfrac{t-2}{t-3}+\dfrac{6}{t^2-9}$

66. $\dfrac{y-3}{y-2}-\dfrac{y+1}{2y-5}+\dfrac{-4y+7}{2y^2-9y+10}$

67. $(x-1)\cdot\left[\dfrac{3}{x^2-1}+\dfrac{x}{2x-2}\right]$

68. $(3x-2)\cdot\left[\dfrac{x}{3x^2+x-2}+\dfrac{2}{x+1}\right]$

69. $\dfrac{3z}{z-3}-\dfrac{z}{z+4}$

70. $\dfrac{2p}{p-5}-\dfrac{p}{p+6}$

71. $\dfrac{2x}{x^2-y^2}-\dfrac{1}{x-y}+\dfrac{1}{y-x}$

72. $\dfrac{3w-1}{2w^2+w-3}-\dfrac{2-w}{w-1}-\dfrac{w}{1-w}$

73. $(2p+1)\cdot\left[\dfrac{2p}{6p+3}-\dfrac{1}{p+4}\right]$

74. $(y+8)\cdot\left[\dfrac{4}{2y+1}-\dfrac{y}{2y^2+17y+8}\right]$

75. $\dfrac{1}{x+5}+\dfrac{3}{(x+5)^2}-\dfrac{2}{(x+5)^3}$

76. $\dfrac{1}{x-2}+\dfrac{4}{(x-2)^2}-\dfrac{3}{(x-2)^3}$

77. $\dfrac{-10}{z^2-6z+5}+\dfrac{15}{z^2-4z-5}$

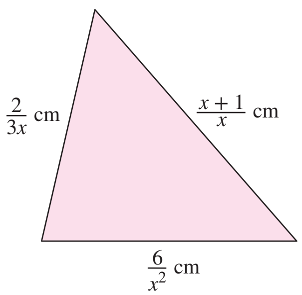
78. $\dfrac{-4}{n^2+6n+5}+\dfrac{3}{n^2+7n+10}$

79. $\dfrac{5}{x^2-4}+\dfrac{2}{x^3-8}$

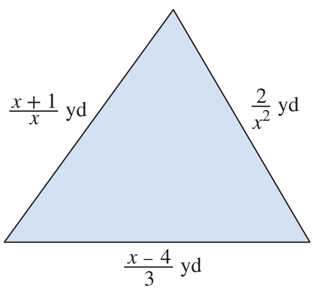
80. $\dfrac{-2}{x^2-9}+\dfrac{3}{x^3-27}$

For Exercises 81–84, write an expression that represents the perimeter of the figure and simplify.

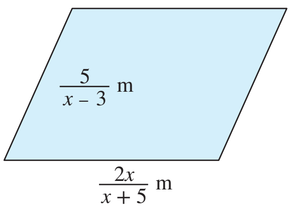
81.



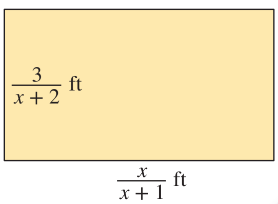
82.



83.



84.



Complex Fractions

Section 5.4

Concepts

1. Simplifying Complex Fractions by Method I

A **complex fraction** is an expression containing one or more fractional expressions in the numerator, denominator, or both. For example:

$$\frac{\frac{5x^2}{y}}{\frac{10x}{y^2}} \quad \text{and} \quad \frac{2+\frac{1}{2}-\frac{1}{3}}{\frac{3}{4}+\frac{1}{6}}$$

are complex fractions.

Two methods will be presented to simplify complex fractions. The first method (Method I) follows the order of operations to simplify the numerator and denominator separately before dividing. The process is summarized as follows.

Simplifying a Complex Fraction—Method I

Step 1 Add or subtract expressions in the numerator to form a single fraction. Add or subtract expressions in the denominator to form a single fraction.

Step 2 Divide the rational expressions from step 1.

Step 3 Simplify to lowest terms, if possible.

Example 1 Simplifying a Complex Fraction by Method I

Simplify the expression. $\dfrac{\frac{5x^2}{y}}{\frac{10x}{y^2}}$

Solution:

$$\frac{\frac{5x^2}{y}}{\frac{10x}{y^2}}$$

Step 1: The numerator and denominator of the complex fraction are already single fractions.

$$= \frac{5x^2}{y} \div \frac{10x}{y^2}$$

Step 2: Divide the fractions.

$$= \frac{5x^2}{y} \cdot \frac{y^2}{10x}$$

Multiply the numerator of the complex fraction by the reciprocal of the denominator.

$$= \frac{5x^2y^2}{10xy}$$

$$= \frac{1}{2}x^{2-1}y^{2-1}$$

Step 3: Simplify.

$$= \frac{1}{2}xy \quad \text{or} \quad \frac{xy}{2}$$

Skill Practice Simplify the expression.

1. $\dfrac{\frac{18a^3}{b^2}}{\frac{6a^2}{b}}$

FOR REVIEW

The expression in Example 1 is a complex fraction with variables. Before attempting Example 1, consider a similar looking numerical fraction.

$$\frac{\frac{5}{3}}{\frac{10}{9}} = \frac{5}{3} \div \frac{10}{9}$$

$$= \frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{3}}} \cdot \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{10}}} = \frac{3}{2}$$

Answer

1. $\dfrac{3a}{b}$

Sometimes it is necessary to simplify the numerator and denominator of a complex fraction before the division is performed. This is illustrated in Example 2.

Example 2 Simplifying a Complex Fraction by Method I

Simplify the expression. $\dfrac{\dfrac{1}{4x}-\dfrac{3}{2}}{3-\dfrac{1}{2x}}$

Solution:

$$\dfrac{\dfrac{1}{4x}-\dfrac{3}{2}}{3-\dfrac{1}{2x}}$$

Step 1: Combine fractions in the numerator and denominator separately.

$$=\dfrac{\dfrac{1}{4x}-\dfrac{2x}{2x}\cdot\dfrac{3}{2}}{\dfrac{2x}{2x}\cdot\dfrac{3}{1}-\dfrac{1}{2x}}$$

The LCD in the numerator is $4x$.
The LCD in the denominator is $2x$.

$$=\dfrac{\dfrac{1-6x}{4x}}{\dfrac{6x-1}{2x}}$$

Step 2: Divide the expression in the numerator of the complex fraction by the expression in the denominator.

$$=\dfrac{\overset{-1}{\cancel{1-6x}}}{\underset{2}{\cancel{4x}}}\cdot\dfrac{\overset{1}{\cancel{2x}}}{\cancel{6x-1}}$$

Multiply by the reciprocal of the divisor.

$$=-\dfrac{1}{2}$$

Step 3: Simplify to lowest terms.

Skill Practice Simplify the expression.

2. $\dfrac{\dfrac{1}{9m}-\dfrac{4}{3}}{4-\dfrac{1}{3m}}$

2. Simplifying Complex Fractions by Method II

We will now use a second method to simplify complex fractions—Method II. Recall that multiplying the numerator and denominator of a rational expression by the same quantity does not change the value of the expression. This is the basis for Method II.

Simplifying a Complex Fraction—Method II

Step 1 Multiply the numerator and denominator of the complex fraction by the LCD of *all* individual fractions within the expression.

Step 2 Apply the distributive property, and simplify the numerator and denominator.

Step 3 Simplify to lowest terms, if possible.

Answer

2. $-\dfrac{1}{3}$

Example 3 Simplifying a Complex Fraction by Method II

Simplify by using Method II. $\dfrac{4-\dfrac{6}{x}}{\dfrac{2}{x}-\dfrac{3}{x^2}}$

Solution:

$\dfrac{4-\dfrac{6}{x}}{\dfrac{2}{x}-\dfrac{3}{x^2}}$ The LCD of all individual terms is x^2.

$= \dfrac{x^2\cdot\left(4-\dfrac{6}{x}\right)}{x^2\cdot\left(\dfrac{2}{x}-\dfrac{3}{x^2}\right)}$ **Step 1:** Multiply the numerator and denominator of the complex fraction by the LCD of x^2.

$= \dfrac{x^2\cdot(4)-x^2\cdot\left(\dfrac{6}{x}\right)}{x^2\cdot\left(\dfrac{2}{x}\right)-x^2\cdot\left(\dfrac{3}{x^2}\right)}$ **Step 2:** Apply the distributive property.

$= \dfrac{4x^2-6x}{2x-3}$

$= \dfrac{2x\cancel{(2x-3)}^{1}}{\cancel{2x-3}}$ **Step 3:** Factor and simplify.

$= 2x$

Skill Practice Simplify by using Method II.

3. $\dfrac{y-\dfrac{1}{y}}{1-\dfrac{1}{y^2}}$

Example 4 Simplifying a Complex Fraction by Method II

Simplify by using Method II. $\dfrac{x^{-1}-x^{-2}}{1+2x^{-1}-3x^{-2}}$

Solution:

$\dfrac{x^{-1}-x^{-2}}{1+2x^{-1}-3x^{-2}}$

$= \dfrac{\dfrac{1}{x}-\dfrac{1}{x^2}}{1+\dfrac{2}{x}-\dfrac{3}{x^2}}$ Rewrite the expression with positive exponents. The LCD of all individual terms is x^2.

TIP: When writing $2x^{-1}$ with positive exponents recall that

$2x^{-1} = 2\cdot\dfrac{1}{x}$

$= \dfrac{2}{1}\cdot\dfrac{1}{x} = \dfrac{2}{x}$

Answer

3. y

$$= \frac{x^2 \cdot \left(\frac{1}{x} - \frac{1}{x^2}\right)}{x^2 \cdot \left(1 + \frac{2}{x} - \frac{3}{x^2}\right)}$$

Step 1: Multiply the numerator and denominator of the complex fraction by the LCD x^2.

$$= \frac{x^2\left(\frac{1}{x}\right) - x^2\left(\frac{1}{x^2}\right)}{x^2(1) + x^2\left(\frac{2}{x}\right) - x^2\left(\frac{3}{x^2}\right)}$$

Step 2: Apply the distributive property.

$$= \frac{x - 1}{x^2 + 2x - 3}$$

$$= \frac{x - 1}{(x + 3)(x - 1)}$$

Step 3: Factor and simplify to lowest terms.

$$= \frac{\overset{1}{\cancel{x - 1}}}{(x + 3)\cancel{(x - 1)}}$$

$$= \frac{1}{x + 3}$$

Skill Practice Simplify by using Method II.

4. $\dfrac{c^{-1}b - b^{-1}c}{b^{-1} + c^{-1}}$

Example 5 Simplifying a Complex Fraction by Method II

Simplify the expression by Method II. $\dfrac{\frac{1}{w + 3} - \frac{1}{w - 3}}{1 + \frac{9}{w^2 - 9}}$

Solution:

$$\frac{\frac{1}{w + 3} - \frac{1}{w - 3}}{1 + \frac{9}{w^2 - 9}}$$

$$= \frac{\frac{1}{w + 3} - \frac{1}{w - 3}}{1 + \frac{9}{(w + 3)(w - 3)}}$$

Factor all denominators to find the LCD.

The LCD of $\frac{1}{1}$, $\frac{1}{w + 3}$, $\frac{1}{w - 3}$, and $\frac{9}{(w + 3)(w - 3)}$ is $(w + 3)(w - 3)$.

$$= \frac{(w + 3)(w - 3)\left(\frac{1}{w + 3} - \frac{1}{w - 3}\right)}{(w + 3)(w - 3)\left[1 + \frac{9}{(w + 3)(w - 3)}\right]}$$

Step 1: Multiply the numerator and denominator of the complex fraction by $(w + 3)(w - 3)$.

Answer

4. $b - c$

$$= \frac{\cancel{(w+3)}(w-3)\left(\frac{1}{\cancel{w+3}}\right) - (w+3)\cancel{(w-3)}\left(\frac{1}{\cancel{w-3}}\right)}{(w+3)(w-3)1 + \cancel{(w+3)}\cancel{(w-3)}\left[\frac{9}{\cancel{(w+3)}\cancel{(w-3)}}\right]}$$

Step 2: Distributive property.

$$= \frac{(w-3)-(w+3)}{(w+3)(w-3)+9}$$

Step 3: Simplify.

$$= \frac{w-3-w-3}{w^2-9+9}$$

Apply the distributive property.

$$= \frac{-6}{w^2}$$

$$= -\frac{6}{w^2}$$

Skill Practice Simplify by using Method II.

5. $\dfrac{\dfrac{2}{x+1} - \dfrac{1}{x-1}}{\dfrac{x}{x-1} - \dfrac{1}{x+1}}$

Answer

5. $\dfrac{x-3}{x^2+1}$

Section 5.4 Activity

A.1. **a.** Add the fractions. $\frac{5}{6} + \frac{2}{9}$

b. Subtract the fractions. $\frac{2}{3} - \frac{1}{6}$

c. Simplify the complex fraction using the order of operations (Method I). That is, simplify the numerator, simplify the denominator, and then divide the results.

$$\frac{\frac{5}{6} + \frac{2}{9}}{\frac{2}{3} - \frac{1}{6}}$$

A.2. **a.** Subtract the fractions. $\frac{1}{x^2} - \frac{4}{y^2}$

b. Add the fractions. $\frac{1}{x} + \frac{2}{y}$

c. Use Method I to simplify the complex fraction. Remember to write the expression in lowest terms.

$$\frac{\frac{1}{x^2} - \frac{4}{y^2}}{\frac{1}{x} - \frac{2}{y}}$$

A.3. **a.** Given the fractions $\frac{5}{6}, \frac{2}{9}, \frac{2}{3}$, and $\frac{1}{6}$, identify the LCD.

b. Multiply. $18 \cdot \left(\frac{5}{6} + \frac{2}{9}\right)$

c. Multiply. $18 \cdot \left(\frac{2}{3} - \frac{1}{6}\right)$

d. Simplify the complex fraction by $\dfrac{\frac{5}{6}+\frac{2}{9}}{\frac{2}{3}-\frac{1}{6}}$ by using Method II. To begin, multiply the numerator and denominator by the LCD of all terms in the fraction. Compare your answer to the answer to Exercise A.1(c).

$$\frac{18\cdot\left(\frac{5}{6}+\frac{2}{9}\right)}{18\cdot\left(\frac{2}{3}-\frac{1}{6}\right)}=$$

A.4. **a.** Given the expressions $\frac{1}{x^2}$, $\frac{4}{y^2}$, $\frac{1}{x}$, and $\frac{2}{y}$, identify the LCD.

b. Multiply. $x^2y^2\cdot\left(\frac{1}{x^2}-\frac{4}{y^2}\right)$

c. Multiply. $x^2y^2\cdot\left(\frac{1}{x}+\frac{2}{y}\right)$

d. Simplify the complex fraction by using Method II. That is, multiply numerator and denominator by the LCD of all individual terms in the complex fraction.

$$\frac{x^2y^2\cdot\left(\frac{1}{x^2}-\frac{4}{y^2}\right)}{x^2y^2\cdot\left(\frac{1}{x}-\frac{2}{y}\right)}$$

A.5. Consider the expression $\dfrac{x^{-1}-4x^{-2}}{1-3x^{-1}-4x^{-2}}$.

a. Write each term in the numerator and denominator with positive exponents.
b. What is the LCD of all individual terms in the expression from part (a)?
c. Simplify the complex fraction by using Method II.

Section 5.4 Practice Exercises

Prerequisite Review

For Exercises R.1–R.6, apply the distributive property.

R.1. $15\left(\frac{3}{5}x-\frac{1}{5}\right)$ **R.2.** $18\left(\frac{7}{6}y+\frac{4}{3}\right)$ **R.3.** $y^2\left(1+\frac{10}{y}+\frac{9}{y^2}\right)$

R.4. $x^2\left(1-\frac{1}{x}-\frac{12}{x^2}\right)$ **R.5.** $cd\left(\frac{1}{c}-\frac{1}{d}\right)$ **R.6.** $x^2y^2\left(\frac{1}{x^2}-\frac{1}{y^2}\right)$

For Exercises R.7–R.10, perform the indicated operation.

R.7. **a.** $\frac{2}{3}-\frac{1}{6}$ **b.** $\frac{2}{3}+\frac{1}{6}$ **c.** $\frac{2}{3}\div\frac{1}{6}$

R.8. **a.** $\frac{3}{10}+\frac{3}{5}$ **b.** $\frac{3}{10}-\frac{3}{5}$ **c.** $\frac{3}{10}\div\left(-\frac{3}{5}\right)$

R.9. $\frac{5}{x^2}+\frac{3}{2x}$ **R.10.** $\frac{6}{y}-\frac{5}{4y^3}$

For Exercises R.11–R.12, write the expression with positive exponents only.

R.11. $7x^{-3}$ **R.12.** $-2w^{-1}$

Vocabulary and Key Concepts

1. A __________ fraction is an expression containing one or more fractional expressions in the numerator, denominator, or both.

2. Divide the fractions.

a. $\dfrac{4x}{9} \div \dfrac{10x}{3}$ **b.** $\dfrac{\dfrac{4x}{9}}{\dfrac{10x}{3}}$

Concept 1: Simplifying Complex Fractions by Method I

For Exercises 3–10, simplify the complex fractions by using Method I. **(See Examples 1–2.)**

3. $\dfrac{\dfrac{5x^2}{9y^2}}{\dfrac{3x}{y^2x}}$ **4.** $\dfrac{\dfrac{3w^2}{4rs}}{\dfrac{15wr}{s^2}}$ **5.** $\dfrac{\dfrac{x-6}{3x}}{\dfrac{3x-18}{9}}$ **6.** $\dfrac{\dfrac{a+4}{6}}{\dfrac{16-a^2}{3}}$

7. $\dfrac{\dfrac{2}{3}+\dfrac{1}{6}}{\dfrac{1}{2}-\dfrac{1}{4}}$ **8.** $\dfrac{\dfrac{7}{8}+\dfrac{3}{4}}{\dfrac{1}{3}-\dfrac{5}{6}}$ **9.** $\dfrac{8-\dfrac{5}{2x}}{\dfrac{5}{8x}-2}$ **10.** $\dfrac{10-\dfrac{3}{5x}}{\dfrac{3}{10x}-5}$

Concept 2: Simplifying Complex Fractions by Method II

For Exercises 11–42, simplify the complex fractions by using Method II. **(See Examples 3–5.)**

11. $\dfrac{\dfrac{7y}{y+3}}{\dfrac{1}{4y+12}}$ **12.** $\dfrac{\dfrac{6x}{x-5}}{\dfrac{1}{4x-20}}$ **13.** $\dfrac{1+\dfrac{1}{3}}{\dfrac{5}{6}-1}$ **14.** $\dfrac{2+\dfrac{4}{5}}{-1+\dfrac{3}{10}}$

15. $\dfrac{\dfrac{3q}{p}-q}{q-\dfrac{q}{p}}$ **16.** $\dfrac{\dfrac{b}{a}+3b}{b+\dfrac{2b}{a}}$ **17.** $\dfrac{\dfrac{2}{a}+\dfrac{3}{a^2}}{\dfrac{4}{a^2}-\dfrac{9}{a}}$ **18.** $\dfrac{\dfrac{2}{y^2}+\dfrac{1}{y}}{\dfrac{4}{y^2}-\dfrac{1}{y}}$

19. $\dfrac{t^{-1}-1}{1-t^{-2}}$ **20.** $\dfrac{d^{-2}-c^{-2}}{c^{-1}-d^{-1}}$ **21.** $\dfrac{-8}{\dfrac{6w}{w-1}-4}$ **22.** $\dfrac{6}{2z-\dfrac{10}{z-4}}$

23. $\dfrac{\dfrac{y}{y+3}}{\dfrac{y}{y+3}+y}$ **24.** $\dfrac{\dfrac{4}{w-4}}{\dfrac{4}{w-4}-1}$ **25.** $\dfrac{1-\dfrac{1}{x}-\dfrac{6}{x^2}}{1-\dfrac{4}{x}+\dfrac{3}{x^2}}$ **26.** $\dfrac{1+\dfrac{1}{x}-\dfrac{12}{x^2}}{\dfrac{9}{x^2}+\dfrac{3}{x}-2}$

27. $\dfrac{2-\dfrac{2}{t+1}}{2+\dfrac{2}{t}}$ **28.** $\dfrac{3+\dfrac{3}{p-1}}{3-\dfrac{3}{p}}$ **29.** $\dfrac{\dfrac{2}{a}-\dfrac{3}{a+1}}{\dfrac{2}{a+1}-\dfrac{3}{a}}$ **30.** $\dfrac{\dfrac{5}{b}+\dfrac{4}{b+1}}{\dfrac{4}{b}-\dfrac{5}{b+1}}$

31. $\dfrac{\dfrac{1}{y+2}+\dfrac{4}{y-3}}{\dfrac{2}{y-3}-\dfrac{7}{y+2}}$

32. $\dfrac{\dfrac{1}{t-4}+\dfrac{1}{t+5}}{\dfrac{6}{t+5}+\dfrac{2}{t-4}}$

33. $\dfrac{\dfrac{2}{x+h}-\dfrac{2}{x}}{h}$

34. $\dfrac{\dfrac{1}{2x+2h}-\dfrac{1}{2x}}{h}$

35. $\dfrac{x^{-2}}{x+3x^{-1}}$

36. $\dfrac{x^{-1}+x^{-2}}{5x^{-2}}$

37. $\dfrac{2a^{-1}+3b^{-2}}{a^{-1}-b^{-1}}$

38. $\dfrac{2m^{-1}+n^{-1}}{m^{-2}-4n^{-1}}$

39. $\dfrac{\dfrac{1}{4+h}-\dfrac{1}{4}}{h}$

40. $\dfrac{\dfrac{1}{3+3h}-\dfrac{1}{3}}{h}$

41. $\dfrac{\dfrac{6}{x+h}-\dfrac{6}{x}}{h}$

42. $\dfrac{\dfrac{-3}{x+h}+\dfrac{3}{x}}{h}$

Expanding Your Skills

43. The slope formula is used to find the slope of the line passing through the points (x_1, y_1) and (x_2, y_2). Write the slope formula from memory.

For Exercises 44–47, find the slope of the line that passes through the given points.

44. $\left(1\frac{1}{2}, \frac{2}{5}\right), \left(\frac{1}{4}, -2\right)$

45. $\left(-\frac{3}{7}, \frac{3}{5}\right), (-1, -3)$

46. $\left(\frac{5}{8}, \frac{9}{10}\right), \left(-\frac{1}{16}, -\frac{1}{5}\right)$

47. $\left(\frac{1}{4}, \frac{1}{3}\right), \left(\frac{1}{8}, \frac{1}{6}\right)$

48. Show that $(x+x^{-1})^{-1} = \dfrac{x}{x^2+1}$ by writing the expression on the left without negative exponents and simplifying.

49. Show that $(x^{-1}+y^{-1})^{-1} = \dfrac{xy}{x+y}$ by writing the expression on the left without negative exponents and simplifying.

50. Simplify. $\dfrac{x}{1-\left(1+\dfrac{1}{x}\right)^{-1}}$

51. Simplify. $\dfrac{x}{1-\left(1-\dfrac{1}{x}\right)^{-1}}$

Problem Recognition Exercises

Operations on Rational Expressions

For Exercises 1–24, identify the operation (addition, subtraction, multiplication, or division), then simplify the expression. In each case, be sure to ask yourself if you need a common denominator.

1. $\dfrac{2}{2y-3}-\dfrac{3}{2y}+1$

2. $(x+5)+\left(\dfrac{7}{x-4}\right)$

3. $\dfrac{5x^2-6x+1}{x^2-1}\div\dfrac{16x^2-9}{4x^2+7x+3}-\dfrac{x}{4x-3}$

4. $\dfrac{a^2-25}{3a^2+3ab}\cdot\dfrac{a^2+4a+ab+4b}{a^2+9a+20}$

5. $\dfrac{4}{y+1}+\dfrac{y+2}{y^2-1}-\dfrac{3}{y-1}$

6. $\dfrac{8w^2}{w^3-16w}-\dfrac{4w}{w^2-4w}$

7. $\dfrac{a^2-16}{2x+6}\cdot\dfrac{x+3}{a-4}$

8. $\dfrac{t^2-9}{t}\div\dfrac{t+3}{t+2}$

9. $\dfrac{2+\dfrac{1}{a}}{4-\dfrac{1}{a^2}}$

10. $\dfrac{\dfrac{6x^2y}{5}}{\dfrac{3x}{y}}$

11. $\dfrac{6xy}{x^2-y^2}+\dfrac{x+y}{y-x}$

12. $(x^2-6x+8)\cdot\left(\dfrac{3}{x-2}\right)$

13. $\dfrac{3}{x-2}-\dfrac{x-2}{6}$

14. $\dfrac{5}{x+7}+\dfrac{x+7}{10}$

15. $\dfrac{1}{w-1}-\dfrac{w+2}{3w-3}$

16. $\dfrac{3y+6}{y^2-3y-10}\div\dfrac{27}{y-5}$

17. $\dfrac{y+\dfrac{2}{y}-3}{1-\dfrac{2}{y}}$

18. $\dfrac{2}{t-3}-\dfrac{3}{t+2}+5$

19. $\dfrac{4x^2+22x+24}{4x+4}\cdot\dfrac{6x+6}{4x^2-9}$

20. $\dfrac{12x^3y^5z}{5x^4}\div\dfrac{16xy^7}{10z^2}$

21. $\dfrac{3x-1}{4}+\dfrac{7}{6x-2}$

22. $\dfrac{2x^{-1}+3x^{-2}}{x^{-2}-5x^{-1}}$

23. $(y+2)\cdot\dfrac{2y+1}{y^2-4}-\dfrac{y-2}{y+3}$

24. $\dfrac{a^2}{a-10}-\dfrac{100-20a}{10-a}$

Solving Rational Equations

Section 5.5

Concepts

1. Solving Rational Equations
2. Formulas Involving Rational Equations

1. Solving Rational Equations

Thus far, we have studied two types of equations in one variable: linear equations and quadratic equations. In this section, we will study another type of equation called a rational equation.

Definition of a Rational Equation

An equation with one or more rational expressions is called a **rational equation**.

The following equations are rational equations:

$$\frac{3}{5}+\frac{1}{x}=\frac{2}{3} \qquad 3-\frac{6w}{w+1}=\frac{6}{w+1}$$

To understand the process of solving a rational equation, first review the procedure of clearing fractions.

Example 1 Solving an Equation with Fractions

Solve the equation. $\frac{1}{2}x + \frac{1}{3} = \frac{1}{4}x$

Solution:

$\frac{1}{2}x + \frac{1}{3} = \frac{1}{4}x$ The LCD of all terms in the equation is 12.

$12\left(\frac{1}{2}x + \frac{1}{3}\right) = 12\left(\frac{1}{4}x\right)$ Multiply both sides by 12 to clear fractions.

$12 \cdot \frac{1}{2}x + 12 \cdot \frac{1}{3} = 12 \cdot \frac{1}{4}x$ Apply the distributive property.

$6x + 4 = 3x$ Solve the resulting equation.

$3x = -4$

$x = -\frac{4}{3}$

Check: $\frac{1}{2}x + \frac{1}{3} = \frac{1}{4}x$

$\frac{1}{2}\left(-\frac{4}{3}\right) + \frac{1}{3} \stackrel{?}{=} \frac{1}{4}\left(-\frac{4}{3}\right)$

$-\frac{2}{3} + \frac{1}{3} \stackrel{?}{=} -\frac{1}{3}$

$-\frac{1}{3} \stackrel{?}{=} -\frac{1}{3}$ ✓

The solution set is $\left\{-\frac{4}{3}\right\}$.

FOR REVIEW

The technique of clearing fractions utilizes the multiplication property of equality. Multiplying both sides of an equation by the same nonzero real number results in an equivalent equation.

Skill Practice Solve the equation.

1. $\frac{1}{2}x + \frac{21}{20} = \frac{1}{5}x$

The same process of clearing fractions is used to solve rational equations when variables are present in the denominator.

Example 2 Solving a Rational Equation

Solve the equation. $\frac{3}{5} + \frac{1}{x} = \frac{2}{3}$

Solution:

$\frac{3}{5} + \frac{1}{x} = \frac{2}{3}$ The LCD of all terms in the equation is $15x$. Note that in this equation there is a restriction that $x \neq 0$.

$15x\left(\frac{3}{5} + \frac{1}{x}\right) = 15x\left(\frac{2}{3}\right)$ Multiply by $15x$ to clear fractions.

Answer

1. $\left\{-\frac{7}{2}\right\}$

$$15x \cdot \frac{3}{5} + 15x \cdot \frac{1}{x} = 15x \cdot \frac{2}{3}$$ Apply the distributive property.

$$9x + 15 = 10x$$ Solve the resulting equation.

$$15 = x$$

Check: $x = 15$

$$\frac{3}{5} + \frac{1}{x} = \frac{2}{3}$$

$$\frac{3}{5} + \frac{1}{(15)} \stackrel{?}{=} \frac{2}{3}$$

$$\frac{9}{15} + \frac{1}{15} \stackrel{?}{=} \frac{2}{3}$$

$$\frac{10}{15} \stackrel{?}{=} \frac{2}{3} \checkmark$$

The solution set is $\{15\}$.

Skill Practice Solve the equation.

2. $\dfrac{3}{y} + \dfrac{4}{3} = -1$

Example 3 Solving a Rational Equation

Solve the equation. $3 - \dfrac{6w}{w+1} = \dfrac{6}{w+1}$

Solution:

$$3 - \frac{6w}{w+1} = \frac{6}{w+1}$$

The LCD of all terms in the equation is $w + 1$. Note that in this equation there is a restriction that $w \neq -1$.

$$(w+1)(3) - (w+1)\left(\frac{6w}{w+1}\right) = (w+1)\left(\frac{6}{w+1}\right)$$

Multiply by $(w + 1)$ on both sides to clear fractions.

$$(w+1)(3) - \cancel{(w+1)}\left(\frac{6w}{\cancel{w+1}}\right) = \cancel{(w+1)}\left(\frac{6}{\cancel{w+1}}\right)$$

Apply the distributive property.

$$3w + 3 - 6w = 6$$ Solve the resulting equation.

$$-3w = 3$$

$$w = -1$$

Check:

$$3 - \frac{6w}{w+1} = \frac{6}{w+1}$$

$$3 - \frac{6(-1)}{(-1)+1} \stackrel{?}{=} \frac{6}{(-1)+1}$$

The denominator is 0 for the value of $w = -1$.

Answer

2. $\left\{-\frac{9}{7}\right\}$

The value -1 is one of the restrictions on w found in the first step. As expected, the value $w = -1$ does not check. Since no other potential solution exists, the equation has no solution.

The solution set is the empty set, { }.

Skill Practice Solve the equation.

3. $5 - \dfrac{8}{x+2} = \dfrac{4x}{x+2}$

Examples 1–3 show that the steps to solve a rational equation mirror the process of clearing fractions. However, we must check whether the potential solutions are defined in each expression in the original equation. A potential solution that does not check is called an **extraneous solution**.

The steps for solving a rational equation are summarized as follows.

Solving a Rational Equation

Step 1 Factor the denominators of all rational expressions. Identify any values of the variable for which any expression is undefined.

Step 2 Identify the LCD of all terms in the equation.

Step 3 Multiply both sides of the equation by the LCD.

Step 4 Solve the resulting equation.

Step 5 Check the potential solutions in the original equation. Note that any value from step 1 for which the equation is undefined cannot be a solution to the equation.

Example 4 Solving a Rational Equation

Solve the equation. $1 + \dfrac{3}{x} = \dfrac{28}{x^2}$

Solution:

$$1 + \frac{3}{x} = \frac{28}{x^2}$$ The LCD of all terms in the equation is x^2. Expressions will be undefined for $x = 0$.

$$x^2\left(1 + \frac{3}{x}\right) = x^2\left(\frac{28}{x^2}\right)$$ Multiply both sides by x^2 to clear fractions.

$$x^2 \cdot 1 + x^2 \cdot \frac{3}{x} = x^2 \cdot \frac{28}{x^2}$$ Apply the distributive property.

$$x^2 + 3x = 28$$ The resulting equation is quadratic.

$$x^2 + 3x - 28 = 0$$ Set the equation equal to zero and factor.

$$(x+7)(x-4) = 0$$

$$x = -7 \quad \text{or} \quad x = 4$$

FOR REVIEW

Recall that to solve a quadratic equation, set one side equal to zero and factor the other side. If the product of factors equals zero, then one or both factors must be zero.

Thus, if $(x+7)(x-4) = 0$, then $x + 7 = 0$ or $x - 4 = 0$.

Answer

3. { } (The value -2 does not check.)

Check: $x = -7$

$$1 + \frac{3}{x} = \frac{28}{x^2}$$

$$1 + \frac{3}{-7} \stackrel{?}{=} \frac{28}{(-7)^2}$$

$$\frac{7}{7} - \frac{3}{7} \stackrel{?}{=} \frac{28}{49}$$

$$\frac{4}{7} \stackrel{?}{=} \frac{4}{7} \checkmark$$

Check: $x = 4$

$$1 + \frac{3}{x} = \frac{28}{x^2}$$

$$1 + \frac{3}{4} \stackrel{?}{=} \frac{28}{(4)^2}$$

$$\frac{4}{4} + \frac{3}{4} \stackrel{?}{=} \frac{28}{16}$$

$$\frac{7}{4} \stackrel{?}{=} \frac{7}{4} \checkmark$$

The solution set is $\{-7, 4\}$.

Skill Practice Solve the equation.

4. $1 + \dfrac{6}{x} = \dfrac{16}{x^2}$

Example 5 Solving a Rational Equation

Solve. $\dfrac{36}{p^2 - 9} = \dfrac{2p}{p + 3} - 1$

Solution:

$$\frac{36}{p^2 - 9} = \frac{2p}{p + 3} - 1$$

$$\frac{36}{(p + 3)(p - 3)} = \frac{2p}{p + 3} - 1$$

The LCD is $(p + 3)(p - 3)$. Expressions will be undefined for $p = 3$ and $p = -3$.

Multiply both sides by the LCD to clear fractions.

$$(p + 3)(p - 3)\left[\frac{36}{(p + 3)(p - 3)}\right] = (p + 3)(p - 3)\left(\frac{2p}{p + 3}\right) - (p + 3)(p - 3)1$$

$$\cancel{(p + 3)(p - 3)}\left[\frac{36}{\cancel{(p + 3)(p - 3)}}\right] = \cancel{(p + 3)}(p - 3)\left(\frac{2p}{\cancel{p + 3}}\right) - (p + 3)(p - 3)1$$

$36 = 2p(p - 3) - (p + 3)(p - 3)$ Solve the resulting equation.

$36 = 2p^2 - 6p - (p^2 - 9)$ The equation is quadratic.

$36 = 2p^2 - 6p - p^2 + 9$

$36 = p^2 - 6p + 9$

$0 = p^2 - 6p - 27$ Set the equation equal to zero and factor.

$0 = (p - 9)(p + 3)$

$p = 9$ or $p = -3$

Answer

4. $\{-8, 2\}$

Check: $p = 9$

$$\frac{36}{p^2 - 9} = \frac{2p}{p+3} - 1$$

$$\frac{36}{(9)^2 - 9} \stackrel{?}{=} \frac{2(9)}{(9)+3} - 1$$

$$\frac{36}{72} \stackrel{?}{=} \frac{18}{12} - 1$$

$$\frac{1}{2} \stackrel{?}{=} \frac{3}{2} - 1$$

$$\frac{1}{2} \stackrel{?}{=} \frac{1}{2} \checkmark$$

Check: $p = -3$

$$\frac{36}{p^2 - 9} = \frac{2p}{p+3} - 1$$

$$\frac{36}{(-3)^2 - 9} \stackrel{?}{=} \frac{2(-3)}{(-3)+3} - 1$$

Denominator is zero.

Here the value -3 is *not* a solution to the original equation because it is restricted in the original equation. However, 9 checks in the original equation.

The solution set is $\{9\}$.

Skill Practice Solve.

5. $\dfrac{6}{x+2} - \dfrac{20x}{x^2 - x - 6} = \dfrac{x}{x+2}$

2. Formulas Involving Rational Equations

Example 6 Solving a Literal Equation Involving Rational Expressions

Solve for the indicated variable. $V = \dfrac{mv}{m+M}$ for m

Avoiding Mistakes

Variables in algebra are case-sensitive. For example, M and m are different variables.

Solution:

$$V = \frac{mv}{m+M} \quad \text{for } m$$

$$V(m+M) = \left(\frac{mv}{\cancel{m+M}}\right)\cancel{(m+M)}$$ Multiply by the LCD and clear fractions.

$$V(m+M) = mv$$

$$Vm + VM = mv$$ Use the distributive property to clear parentheses.

$$Vm - mv = -VM$$ Collect all m terms on one side.

$$m(V - v) = -VM$$ Factor out m.

$$\frac{m\cancel{(V-v)}}{\cancel{(V-v)}} = \frac{-VM}{(V-v)}$$ Divide by $(V - v)$.

$$m = \frac{-VM}{V - v}$$

TIP: The factor of –1 that appears in the numerator may be written in the denominator or out in front of the expression. The following expressions are equivalent:

$$m = \frac{-VM}{V-v} \quad \text{or} \quad m = \frac{VM}{-(V-v)} = \frac{VM}{-V+v} = \frac{VM}{v-V} \quad \text{or} \quad m = -\frac{VM}{V-v}$$

Answer

5. $\{-9\}$ (The value -2 does not check.)

Skill Practice Solve the equation for x.

6. $y = \dfrac{ax + b}{x + d}$

Answer

6. $x = \dfrac{b - yd}{y - a}$ or $x = \dfrac{yd - b}{a - y}$

Section 5.5 Activity

A.1. **a.** What is the LCD of the expressions $\frac{5}{3}t$, $\frac{2}{5}$, and $\frac{7}{15}$?

b. Multiply. $15 \cdot \left(\frac{5}{3}t - \frac{2}{5}\right)$

c. Multiply. $15 \cdot \left(t - \frac{7}{15}\right)$

d. Solve the equation by first multiplying both sides by the LCD of all terms in the equation.

$$\frac{5}{3}t - \frac{2}{5} = t - \frac{7}{15}$$

A.2. **a.** Factor the denominators of the terms in the equation

$$\frac{1}{x+4} + \frac{x}{x-4} = \frac{-8}{x^2 - 16}.$$

b. What are the restrictions on the variable?

c. What is the LCD of all terms in the equation

$$\frac{1}{x+4} + \frac{x}{x-4} = \frac{-8}{(x+4)(x-4)}?$$

d. Solve the equation by first multiplying both sides by the LCD of all terms in the equation.

$$\frac{1}{x+4} + \frac{x}{x-4} = \frac{-8}{x^2 - 16}.$$

A.3. **a.** What is the smallest power of x that can be used to clear the fractions in the equation $2 - \frac{17}{x} + \frac{21}{x^2} = 0$.

b. Clear the fractions in the equation part (a).

c. Solve the equation.

A.4. The equation $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ is used to study electromagnetism in physics. Solve the equation for a by following these steps.

a. To "move" a out of the denominator, clear the fractions in the equation.

b. The variable a appears in two terms in the equation $bc = ac + ab$. To isolate a, we want only one occurrence of a in the equation. To accomplish this, factor out a as the greatest common factor on the left side of the equation.

c. Use the resulting equation from part (b) to complete the process to solve for a.

Practice Exercises Section 5.5

Prerequisite Review

For Exercises R.1–R.6, determine the least common denominator.

R.1. $\dfrac{7}{2x}$ and $\dfrac{5}{x-4}$

R.2. $\dfrac{-3}{y+7}$ and $\dfrac{8}{3y}$

R.3. $\dfrac{2}{w+5}$, $\dfrac{5}{4w+20}$, and $-\dfrac{7}{8}$

R.4. $\dfrac{2}{3n-21}$, $-\dfrac{11}{6}$, and $-\dfrac{4}{n-7}$

R.5. $-\dfrac{4}{p^2}$ and $\dfrac{8}{p^3}$

R.6. $\dfrac{7}{x^2}$ and $\dfrac{1}{x}$

For Exercises R.7–R.10, apply the distributive property and simplify.

R.7. $2x(x-4)\left(\dfrac{7}{2x}+\dfrac{5}{x-4}\right)$

R.8. $3y(y+7)\left(-\dfrac{3}{y+7}+\dfrac{8}{3y}\right)$

R.9. $p^3\left(-\dfrac{4}{p^2}-\dfrac{8}{p^3}\right)$

R.10. $x^2\left(\dfrac{7}{x^2}-\dfrac{1}{x}\right)$

For Exercises R.11–R.12, identify the restricted values of the expression.

R.11. $\dfrac{7}{2x}+\dfrac{5}{x-4}$

R.12. $-\dfrac{3}{y+7}+\dfrac{8}{3y}$

For Exercises R.13–R.16, solve the equation.

R.13. $x^2-12x=-32$

R.14. $x^2+8x=-12$

R.15. $4x(x+2)=21$

R.16. $3x^2=2-5x$

For Exercises R.17–R.20, solve for the indicated variable.

R.17. $A=lw$ for w

R.18. $A=2\pi rh$ for r

R.19. $2x-7y=14$ for y

R.20. $-3x-10y=30$ for y

Vocabulary and Key Concepts

1. a. The equation $\dfrac{5}{x+2}+\dfrac{1}{2}=\dfrac{4}{5}$ is an example of a __________ equation.

b. After solving a rational equation, check each potential solution to determine if it makes the __________ equal to zero in one or more of the rational expressions. If so, that potential solution is not part of the solution set.

c. Given $\dfrac{3}{2x+1}+\dfrac{36}{2x^2-7x-4}=\dfrac{4}{x-4}$, is it possible for 4 to be a solution to the equation?

Concept 1: Solving Rational Equations

2. Why is it important to check your answer when solving a rational equation?

For Exercises 3–6, identify the restricted values of the variable that would make any expression in the equation undefined.

3. $2-\dfrac{3}{y}=\dfrac{5}{y^2}$

4. $7+\dfrac{20}{z}=\dfrac{3}{z^2}$

5. $\dfrac{4c}{c-5}-\dfrac{1}{c+1}=\dfrac{3c^2+3}{c^2-4c-5}$

6. $\dfrac{3x}{x+2}-\dfrac{5}{x-4}=\dfrac{2x^2-14x}{x^2-2x-8}$

For Exercises 7–42, solve the rational equation. **(See Examples 1–5.)**

7. $\dfrac{1}{2}x-\dfrac{2}{3}=\dfrac{5}{12}$

8. $\dfrac{8}{9}+\dfrac{5}{6}m=\dfrac{1}{2}$

9. $\dfrac{x+2}{3}-\dfrac{x-4}{4}=\dfrac{1}{2}$

10. $\dfrac{x+6}{3}-\dfrac{x+8}{5}=0$

11. $\dfrac{3y}{4}-2=\dfrac{5y}{6}$

12. $\dfrac{2w}{5}-8=\dfrac{4w}{2}$

13. $\dfrac{5}{4p}-\dfrac{7}{6}+3=0$

14. $\dfrac{7}{15w}-\dfrac{3}{10}-2=0$

15. $\dfrac{1}{2}-\dfrac{3}{2x}=\dfrac{4}{x}-\dfrac{5}{12}$

16. $\dfrac{2}{3x}+\dfrac{1}{4}=\dfrac{11}{6x}-\dfrac{1}{3}$

17. $\dfrac{3}{x-4}+2=\dfrac{5}{x-4}$

18. $\dfrac{5}{x+3}-2=\dfrac{7}{x+3}$

19. $\dfrac{1}{3}+\dfrac{2}{w-3}=1$

20. $\dfrac{3}{5}+\dfrac{7}{p+2}=2$

21. $\dfrac{12}{x}-\dfrac{12}{x-5}=\dfrac{2}{x}$

22. $\dfrac{25}{y}-\dfrac{25}{y-2}=\dfrac{2}{y}$

23. $\dfrac{3}{a^2} - \dfrac{4}{a} = -1$

24. $\dfrac{3}{w^2} = 2 + \dfrac{1}{w}$

25. $\dfrac{1}{4}a - 4a^{-1} = 0$

26. $\dfrac{1}{3}t - 12t^{-1} = 0$

27. $\dfrac{y}{y+3} + \dfrac{2}{y^2+3y} = \dfrac{6}{y}$

28. $\dfrac{-8}{t^2-6t} + \dfrac{t}{t-6} = \dfrac{1}{t}$

29. $\dfrac{4}{t-2} - \dfrac{8}{t^2-2t} = -2$

30. $\dfrac{x}{x+6} = \dfrac{72}{x^2-36} + 4$

31. $\dfrac{6}{5y+10} - \dfrac{1}{y-5} = \dfrac{4}{y^2-3y-10}$

32. $\dfrac{-3}{x^2-7x+12} - \dfrac{2}{x^2+x-12} = \dfrac{10}{x^2-16}$

33. $\dfrac{x}{x-5} + \dfrac{1}{5} = \dfrac{5}{x-5}$

34. $\dfrac{x}{x-2} + \dfrac{2}{3} = \dfrac{2}{x-2}$

35. $\dfrac{6}{x^2-4x+3} - \dfrac{1}{x-3} = \dfrac{1}{4x-4}$

36. $\dfrac{1}{4x^2-36} - \dfrac{5}{x+3} + \dfrac{2}{x-3} = 0$

37. $\dfrac{1}{k+2} - \dfrac{4}{k-2} - \dfrac{k^2}{4-k^2} = 0$

38. $\dfrac{h}{2} - \dfrac{h}{h-4} = \dfrac{4}{4-h}$

39. $\dfrac{5}{x^2-7x+12} = \dfrac{2}{x-3} + \dfrac{5}{x-4}$

40. $\dfrac{9}{x^2+7x+10} = \dfrac{5}{x+2} - \dfrac{3}{x+5}$

41. $\dfrac{4}{x^2+7x+12} - \dfrac{7}{x^2+8x+15} = \dfrac{1}{x^2+9x+20}$

42. $\dfrac{5}{x^2-6x+8} - \dfrac{2}{x^2+3x-10} = \dfrac{8}{x^2+x-20}$

Concept 2: Formulas Involving Rational Equations

For Exercises 43–60, solve the formula for the indicated variable. **(See Example 6.)**

43. $K = \dfrac{ma}{F}$ for m

44. $K = \dfrac{ma}{F}$ for a

45. $K = \dfrac{IR}{E}$ for E

46. $K = \dfrac{IR}{E}$ for R

47. $I = \dfrac{E}{R+r}$ for R

48. $I = \dfrac{E}{R+r}$ for r

49. $h = \dfrac{2A}{B+b}$ for B

50. $\dfrac{V}{\pi h} = r^2$ for h

51. $x = \dfrac{at+b}{t}$ for t

52. $\dfrac{T+mf}{m} = g$ for m

53. $\dfrac{x-y}{xy} = z$ for x

54. $\dfrac{w-n}{wn} = P$ for w

55. $a+b = \dfrac{2A}{h}$ for h

56. $1 + rt = \dfrac{A}{P}$ for P

57. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ for R

58. $\dfrac{b+a}{ab} = \dfrac{1}{f}$ for b

59. $v = \dfrac{s_2 - s_1}{t_2 - t_1}$ for t_2

60. $a = \dfrac{v_2 - v_1}{t_2 - t_1}$ for v_1

Mixed Exercises

For Exercises 61–74, solve the equation.

61. $\dfrac{3}{x+2}+\dfrac{2}{x}=\dfrac{-4}{x^2+2x}$

62. $\dfrac{1}{y^2-3y}+\dfrac{8}{y}=\dfrac{2}{y-3}$

63. $4c(c+1)=3(c^2+4)$

64. $3t(2t-2)=5(t^2-1)$

65. $\dfrac{2}{v-1}-\dfrac{4}{v+5}=\dfrac{3}{v^2+4v-5}$

66. $\dfrac{-2}{a+4}-\dfrac{3}{a-5}=\dfrac{6}{a^2-a-20}$

67. $5(x-9)=3(x+4)-2(4x+1)$

68. $4z-3(5z-3)=z-12$

69. $\dfrac{3y}{10}-\dfrac{5}{2y}=\dfrac{y}{5}$

70. $\dfrac{2h}{3}-\dfrac{8}{3h}=\dfrac{h}{2}$

71. $\dfrac{1}{2}(4d-1)+\dfrac{2}{3}(2d+2)=\dfrac{5}{6}(4d+1)$

72. $\dfrac{2}{5}(2b+5)+\dfrac{1}{10}(7b-10)=\dfrac{1}{2}(3b+2)$

73. $8t^{-1}+2=3t^{-1}$

74. $6z^{-2}-5z^{-1}=0$

Expanding Your Skills

75. Find the value of y so that the slope of the line between the points $(3, 1)$ and $(11, y)$ is $\frac{1}{2}$.

76. Find the value of x so that the slope of the line between the points $(-2, -5)$ and $(x, 10)$ is 3.

77. Find the value of x so that the slope of the line between the points $(4, -2)$ and $(x, 2)$ is 4.

78. Find the value of y so that the slope of the line between the points $(3, 2)$ and $(-1, y)$ is $-\frac{3}{4}$.

Problem Recognition Exercises

Rational Equations vs. Expressions

1. **a.** Simplify. $\dfrac{3}{w-5}+\dfrac{10}{w^2-25}-\dfrac{1}{w+5}$

b. Solve. $\dfrac{3}{w-5}+\dfrac{10}{w^2-25}-\dfrac{1}{w+5}=0$

c. Identify each problem in parts (a) and (b) as either an equation or an expression.

2. **a.** Simplify. $\dfrac{x}{2x+4}+\dfrac{2}{3x+6}-1$

b. Solve. $\dfrac{x}{2x+4}+\dfrac{2}{3x+6}=1$

c. Identify each problem in parts (a) and (b) as either an equation or an expression.

For Exercises 3–20, first ask yourself whether the problem is an expression to simplify or an equation to solve. Then simplify or solve as indicated.

3. $\dfrac{2}{a^2+4a+3}+\dfrac{1}{a+3}$

4. $\dfrac{1}{c+6}+\dfrac{4}{c^2+8c+12}$

5. $\dfrac{7}{y^2-y-2}+\dfrac{1}{y+1}-\dfrac{3}{y-2}=0$

6. $\dfrac{3}{b+2}-\dfrac{1}{b-1}-\dfrac{5}{b^2+b-2}=0$

7. $\dfrac{x}{x-1}-\dfrac{12}{x^2-x}$

8. $\dfrac{3}{5t-20}+\dfrac{4}{t-4}$

9. $\dfrac{3}{w} - 5 = \dfrac{7}{w} - 1$

10. $\dfrac{-3}{y^2} - \dfrac{1}{y} = -2$

11. $\dfrac{4p+1}{8p-12} + \dfrac{p-3}{2p-3}$

12. $\dfrac{x+1}{2x+4} - \dfrac{x^2}{x+2}$

13. $\dfrac{1}{2x^2} + \dfrac{1}{6x}$

14. $\dfrac{5}{4a} + \dfrac{1}{6a^2}$

15. $\dfrac{3}{2t} + \dfrac{2}{3t^2} = \dfrac{-1}{t}$

16. $\dfrac{-3}{b^2} + \dfrac{1}{5b} = \dfrac{1}{2b}$

17. $\dfrac{3}{c^2+4c+3} - \dfrac{2}{c^2+6c+9}$

18. $\dfrac{1}{y^2-10y+25} - \dfrac{3}{y^2-7y+10}$

19. $\dfrac{4}{w-4} - \dfrac{36}{2w^2-7w-4} = \dfrac{3}{2w+1}$

20. $\dfrac{2}{x-3} - \dfrac{5}{x+2} = \dfrac{25}{x^2-x-6}$

Applications of Rational Equations and Proportions

Section 5.6

Concepts

1. Solving Proportions
2. Applications of Proportions
3. Similar Triangles
4. Applications of Rational Equations

1. Solving Proportions

A proportion is a rational equation that equates two ratios.

Definition of Ratio and Proportion

1. The **ratio** of a to b is $\dfrac{a}{b}$ $(b \neq 0)$ and can also be expressed as $a:b$ or $a \div b$.
2. An equation that equates two ratios or rates is called a **proportion**.
   Therefore, if $b \neq 0$ and $d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ is a proportion.

The process for solving rational equations can be used to solve proportions.

Example 1 Solving a Proportion

Solve the proportion. $\dfrac{5}{19} = \dfrac{95}{y}$

Solution:

$\dfrac{5}{19} = \dfrac{95}{y}$ The LCD is $19y$. Note that $y \neq 0$.

$19y\left(\dfrac{5}{19}\right) = 19y\left(\dfrac{95}{y}\right)$ Multiply both sides by the LCD.

$\cancel{19}y\left(\dfrac{5}{\cancel{19}}\right) = 19\cancel{y}\left(\dfrac{95}{\cancel{y}}\right)$ Clear fractions.

$5y = 1805$ Solve the resulting equation.

$\dfrac{5y}{5} = \dfrac{1805}{5}$

$y = 361$ The solution 361 checks in the original equation.

The solution set is $\{361\}$.

Skill Practice Solve the proportion.

1. $\dfrac{8}{5} = \dfrac{12}{x}$

2. Applications of Proportions

Example 2 Solving a Proportion

The recommended ratio of total cholesterol to HDL cholesterol is 7 to 2. If Rich's blood test revealed that he has a total cholesterol level of 210 mg/dL (milligrams per deciliter), what should his HDL level be to fit within the recommendations?

Solution:

One method of solving this problem is to set up a proportion. Write two equivalent ratios depicting the amount of total cholesterol to HDL cholesterol. Let x represent the unknown amount of HDL cholesterol.

$$\boxed{\text{Given ratio}} \longrightarrow \frac{7}{2} = \frac{210}{x} \longleftarrow \boxed{\dfrac{\text{Amount of total cholesterol}}{\text{Amount of HDL cholesterol}}}$$

$$2x\left(\frac{7}{2}\right) = 2\cancel{x}\left(\frac{210}{\cancel{x}}\right) \qquad \text{Multiply both sides by the LCD } 2x.$$

$$7x = 420 \qquad \text{Clear fractions.}$$

$$x = 60$$

Rich's HDL cholesterol level should be 60 mg/dL to fit within the recommended level.

Skill Practice

2. The ratio of cats to dogs at an animal rescue facility is 8 to 5. How many dogs are in the facility if there are 400 cats?

Example 3 Solving a Proportion

The ratio of male to female police officers in a certain town is 11:3. If the total number of officers is 112, how many are men and how many are women?

Sean Locke Photography/ Shutterstock

Solution:

Let x represent the number of male police officers.

Then $112 - x$ represents the number of female police officers.

Answers

1. $\left\{\dfrac{15}{2}\right\}$ **2.** There are 250 dogs.

$$\boxed{\frac{\text{Male}}{\text{Female}}} \rightarrow \frac{11}{3} = \frac{x}{112 - x} \leftarrow \boxed{\frac{\text{Number of males}}{\text{Number of females}}}$$

$$\cancel{3}(112 - x)\left(\frac{11}{\cancel{3}}\right) = 3\cancel{(112 - x)}\left(\frac{x}{\cancel{112 - x}}\right)$$ Multiply both sides by $3(112 - x)$.

$$11(112 - x) = 3x$$ The resulting equation is linear.

$$1232 - 11x = 3x$$

$$1232 = 14x$$

$$\frac{1232}{14} = \frac{14x}{14}$$

$$x = 88$$

Then $112 - x = 112 - 88 = 24$

There are 88 male police officers and 24 female officers.

Skill Practice

3. Professor Wolfe has a ratio of passing students to failing students of 5 to 4. One semester he had a total of 207 students. How many students passed and how many failed?

3. Similar Triangles

Proportions are used in geometry with **similar triangles**. Two triangles are similar if their corresponding angles are equal. In such a case, the lengths of the corresponding sides are proportional. In Figure 5-3, triangle *ABC* is similar to triangle *XYZ*. Therefore, the following ratios are equivalent.

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$$

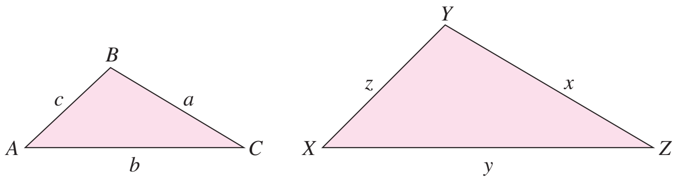


Figure 5-3

Example 4 Using Similar Triangles in an Application

The shadow cast by a yardstick is 2 ft long. The shadow cast by a tree is 11 ft long. Find the height of the tree.

Solution:

Let x represent the height of the tree.

We will assume that the measurements were taken at the same time of day. Therefore, the angle of the sun is the same on both objects, and we can set up similar triangles (Figure 5-4).

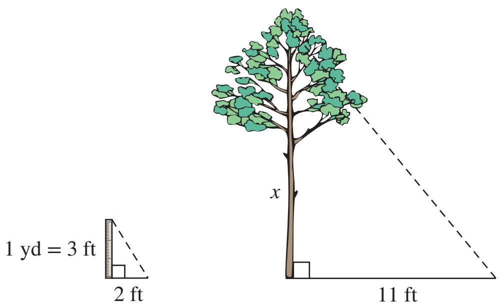


Figure 5-4

Answer

3. 115 passed and 92 failed.

$$\frac{\text{Height of yardstick}}{\text{Height of tree}} \rightarrow \frac{3 \text{ ft}}{x \text{ ft}} = \frac{2 \text{ ft}}{11 \text{ ft}} \leftarrow \frac{\text{Length of yardstick's shadow}}{\text{Length of tree's shadow}}$$

$$\frac{3}{x} = \frac{2}{11} \qquad \text{Write an equation.}$$

$$11\cancel{x} \cdot \left(\frac{3}{\cancel{x}}\right) = \cancel{11}x \cdot \left(\frac{2}{\cancel{11}}\right) \qquad \text{Multiply by the LCD.}$$

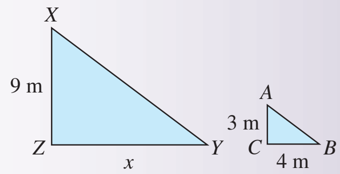
$$33 = 2x \qquad \text{Solve the equation.}$$

$$16.5 = x \qquad \text{Interpret the results.}$$

The tree is 16.5 ft high.

Skill Practice

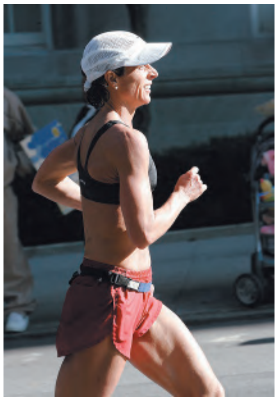
4. Triangle XYZ is similar to triangle ABC. Solve for x.



4. Applications of Rational Equations

Example 5 Solving an Application Involving Distance, Rate, and Time

An athlete's average speed on her bike is 14 mph faster than her average speed running. She can bike 31.5 mi in the same time that it takes her to run 10.5 mi. Find her speed running and her speed biking.



Lars A. Niki

Solution:

Because the speed biking is given in terms of the speed running, let x represent the running speed.

Let x represent the speed running.

Then $x + 14$ represents the speed biking.

Organize the given information in a chart.

	Distance	Rate	Time
Running	10.5	x	$\frac{10.5}{x}$
Biking	31.5	$x + 14$	$\frac{31.5}{x + 14}$

Because $d = rt$, then $t = \frac{d}{r}$

Answer

4. $x = 12$ m

The time required to run 10.5 mi is the same as the time required to bike 31.5 mi, so we can equate the two expressions for time:

$$\frac{10.5}{x} = \frac{31.5}{x + 14}$$ The LCD is $x(x + 14)$.

$$\cancel{x}(x + 14)\left(\frac{10.5}{\cancel{x}}\right) = x\cancel{(x + 14)}\left(\frac{31.5}{\cancel{x + 14}}\right)$$ Multiply by $x(x + 14)$ to clear fractions.

$$10.5(x + 14) = 31.5x$$ The resulting equation is linear.

$$10.5x + 147 = 31.5x$$ Solve for x.

$$-21x = -147$$

$$x = 7$$

Then $x + 14 = 7 + 14 = 21$.

The athlete runs 7 mph and bikes 21 mph.

Skill Practice

5. Devon can cross-country ski 5 km/hr faster than his sister, Shanelle. Devon skis 45 km in the same amount of time that Shanelle skis 30 km. Find their speeds.

Example 6 Solving an Application Involving Distance, Rate, and Time

Valentina travels 70 km to Rome by train, and then takes a bus 30 km to the Coliseum. The bus travels 24 km/hr slower than the train. If the total time traveling on the bus and train is 2 hr, find the average speed of the train and the average speed of the bus.

Digital Stock/Corbis

Solution:

Because the speed of the bus is given in terms of the speed of the train, let x represent the speed of the train.

Let x represent the speed of the train.

Let $x - 24$ represent the speed of the bus.

Organize the given information in a chart.

	Distance	Rate	Time
Train	70	x	$\frac{70}{x}$
Bus	30	$x - 24$	$\frac{30}{x - 24}$

Fill in the last column with $t = \frac{d}{r}$.

Answer

5. Shanelle skis 10 km/hr and Devon skis 15 km/hr.

In this problem, we are given that the total time is 2 hr. So we add the two times to equal 2.

$$\frac{70}{x} + \frac{30}{x-24} = 2$$ The LCD is $x(x-24)$.

Avoiding Mistakes

Remember to multiply all terms on both sides of the equation by the LCD.

$$\cancel{x}(x-24)\left(\frac{70}{\cancel{x}}\right) + x\cancel{(x-24)}\frac{30}{\cancel{x-24}} = x(x-24)(2)$$ Multiply by the LCD to clear the fractions.

$$70(x-24) + 30x = 2x(x-24)$$

$$70x - 1680 + 30x = 2x^2 - 48x$$ The resulting equation is quadratic.

$$0 = 2x^2 - 148x + 1680$$ Set the equation equal to 0.

$$0 = 2(x^2 - 74x + 840)$$ Factor.

$$0 = 2(x-60)(x-14)$$ Solve for x.

$$x - 60 = 0 \quad \text{or} \quad x - 14 = 0$$ Set each factor equal to 0.

$$x = 60 \quad \text{or} \quad x = 14$$

If $x = 14$, then the rate of the bus would be $14 - 24 = -10$. Because a negative rate of -10 km/hr is not reasonable, we reject $x = 14$ as a solution. Therefore, the solution is $x = 60$. That is, the average rate of the train is 60 km/hr and the average rate of the bus is $60 - 24$ or 36 km/hr.

Skill Practice

6. Jason drives 50 mi to a train station and then continues his trip with a 210-mi train ride. The car travels 20 mph slower than the train. If the total travel time is 4 hr, find the average speed of the car and the average speed of the train.

Example 7 Solving an Application Involving "Work"

JoAn can wallpaper a bathroom in 3 hr. Bonnie can wallpaper the same bathroom in 5 hr. How long would it take them if they worked together?

Tom Grill/Fuse/Getty Images

Solution:

Let x represent the amount of time required for both people working together to complete the job.

Answer

6. The car's average speed is 50 mph and the train's average speed is 70 mph.

One method to approach this problem is to add the rates of speed at which each person works.

$$\begin{pmatrix}\text{JoAn's}\\ \text{speed}\end{pmatrix} + \begin{pmatrix}\text{Bonnie's}\\ \text{speed}\end{pmatrix} = \begin{pmatrix}\text{speed working}\\ \text{together}\end{pmatrix}$$

$$\frac{1 \text{ job}}{3 \text{ hr}} + \frac{1 \text{ job}}{5 \text{ hr}} = \frac{1 \text{ job}}{x \text{ hr}}$$

$$\frac{1}{3} + \frac{1}{5} = \frac{1}{x}$$ Set up an equation.

$$15x \cdot \left(\frac{1}{3} + \frac{1}{5}\right) = 15x \cdot \left(\frac{1}{x}\right)$$ Multiply by the LCD, $15x$ to clear fractions.

$$5x + 3x = 15$$ Solve the resulting equation.

$$8x = 15$$

$$x = \frac{15}{8} \quad \text{or} \quad x = 1\frac{7}{8}$$

JoAn and Bonnie can wallpaper the bathroom in $1\frac{7}{8}$ hr.

Skill Practice

7. Antonio can install a new roof in 4 days. Bob can install the same size roof in 6 days. How long will it take them to install a roof if they work together?

TIP: An alternative approach to solving a "work" problem is to determine the portion of the job that each person can complete in 1 hr. Let x represent the amount of time required to complete the job working together. Then

- JoAn completes $\frac{1}{3}$ of the job in 1 hr, and $\frac{1}{3}x$ jobs in x hours.
- Bonnie completes $\frac{1}{5}$ of the job in 1 hr and $\frac{1}{5}x$ jobs in x hours.

$$\begin{pmatrix}\text{Portion of the job}\\ \text{completed by JoAn}\end{pmatrix} + \begin{pmatrix}\text{portion of the job}\\ \text{completed by Bonnie}\end{pmatrix} = \begin{pmatrix}\text{1 whole}\\ \text{job}\end{pmatrix}$$

$$\frac{1}{3}x + \frac{1}{5}x = 1$$

$$15 \cdot \left(\frac{1}{3}x + \frac{1}{5}x\right) = 15 \cdot (1)$$

$$5x + 3x = 15$$

$$8x = 15$$

$$x = \frac{15}{8} \quad \text{or} \quad x = 1\frac{7}{8}$$

The time working together is $1\frac{7}{8}$ hr.

Answer

7. It will take them $\frac{12}{5}$ days or $2\frac{2}{5}$ days.

Section 5.6 Activity

A.1. **a.** Three cookies have 150 calories. Therefore ________ cookies have 450 calories.

b. Written as an equation, the statement in part (a) is:

$$\frac{3 \text{ cookies}}{150 \text{ calories}} = \frac{\boxed{}}{450 \text{ calories}}$$

A.2. In a certain state, the sales tax is \$11.70 on an item that costs \$180. Suppose that another item costs \$1250. To use a proportion to find the sales tax follow these steps.

a. In this scenario, what is the unknown quantity. Let x be this value.

b. Write a proportion that represents this scenario.

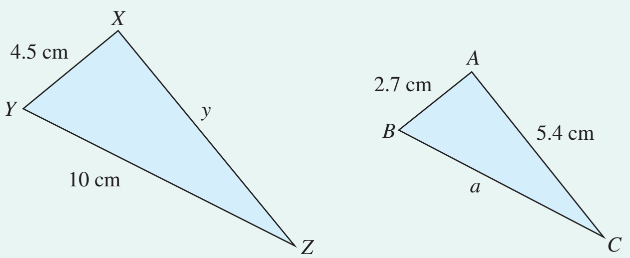
c. Solve the proportion and interpret your answer.

A.3. At a small engineering college, the ratio of men to women is 5:4. Suppose that the total number of students is 1890.

a. Let x represent the number of men. Then the number of women is represented by ________.

b. Set up and solve a proportion to determine the number of women at the school and the number of men at the school.

A.4. Triangle ABC is similar to triangle XYZ. Solve for a and y.



A.5. Mike's average speed riding a new bicycle is 4 mph more than his speed riding his old bicycle. If he can ride 28 miles on his new bike in the same amount of time that he can ride 21 miles on his old bike, what is his average speed on his new bike and his average speed on his old bike?

a. The unknowns to be found are Mike's rate of speed on his new bicycle and his rate of speed on his old bicycle. Let x represent one of these two unknowns.

b. Complete the table.

	Distance (mi)	Rate (mph)	Time (hr)
Riding old bike			
Riding new bike			

c. Write an equation that indicates that the time to ride the old bike 21 mi is the same as the time to ride the new bike 28 mi.

d. Solve the equation from part (c).

e. Interpret the solution to the equation and verify that the solution makes sense in the context of this problem.

A.6. Two hoses of different flow rates are used to fill a man-made pond in an apartment complex. When both hoses are working together, it takes 4 hours to fill the pond. If the hoses work separately, it takes the smaller hose 6 hours longer than the larger hose to fill the pond. How long would it take the larger hose to fill the pond alone?

a. Let x represent the time required for the larger hose to fill the pond. Write an expression that represents the time required for the smaller hose.

b. This scenario can be modeled by adding the rates of speed at which the smaller hose and the larger hose fill the pond. Fill in the blanks.

$$\left(\begin{array}{c}\text{The rate at which the}\\ \text{the smaller hose works}\end{array}\right) + \left(\begin{array}{c}\text{The rate at which the}\\ \text{larger hose works}\end{array}\right) = \left(\begin{array}{c}\text{The rate at which the}\\ \text{hoses work together}\end{array}\right)$$

$$\frac{1 \text{ pond}}{\square \text{ hr}} + \frac{1 \text{ pond}}{\square \text{ hr}} = \frac{1 \text{ pond}}{4 \text{ hr}} \quad \text{or simply} \quad \frac{1}{\square} + \frac{1}{\square} = \frac{1}{4}$$

c. Solve the equation from part (b).

d. Interpret the solution to the equation and verify that the solution makes sense in the context of this problem.

Practice Exercises — Section 5.6

Prerequisite Review

For Exercises R.1–R.8, solve the equation or simplify the expression.

R.1. $3 - \dfrac{6}{x} = x + 8$

R.2. $2 + \dfrac{6}{x} = x + 7$

R.3. $\dfrac{5}{3x-6} - \dfrac{3}{4x-8}$

R.4. $\dfrac{4}{5t-1} + \dfrac{1}{10t-2}$

R.5. $\dfrac{2}{y-1} - \dfrac{5}{4} = \dfrac{-1}{y+1}$

R.6. $\dfrac{5}{w-2} = 7 - \dfrac{10}{w+2}$

R.7. $\dfrac{ab}{6} \div \dfrac{a^2}{12} \cdot \dfrac{a+1}{ab+b}$

R.8. $\dfrac{8p^2-32}{p^2-4p+4} \cdot \dfrac{3p^2-3p-6}{2p^2+20p+32}$

Vocabulary and Key Concepts

1. An equation that equates two rates or ratios is called a ______________.

2. Given similar triangles, the lengths of corresponding sides are ______________.

Concept 1: Solving Proportions

For Exercises 3–18, solve the proportion. **(See Example 1.)**

3. $\dfrac{y}{6} = \dfrac{20}{15}$

4. $\dfrac{12}{18} = \dfrac{14}{x}$

5. $\dfrac{9}{75} = \dfrac{m}{50}$

6. $\dfrac{n}{15} = \dfrac{12}{45}$

7. $\dfrac{p-1}{4} = \dfrac{p+3}{3}$

8. $\dfrac{q-5}{2} = \dfrac{q+2}{3}$

9. $\dfrac{x+1}{5} = \dfrac{4}{15}$

10. $\dfrac{t-1}{7} = \dfrac{2}{21}$

11. $\dfrac{5-2x}{x} = \dfrac{1}{4}$

12. $\dfrac{2y+3}{y} = \dfrac{3}{2}$

13. $\dfrac{2}{y-1} = \dfrac{y-3}{4}$

14. $\dfrac{1}{x-5} = \dfrac{x-3}{3}$

15. $\dfrac{1}{49w} = \dfrac{w}{9}$

16. $\dfrac{1}{4z} = \dfrac{z}{25}$

17. $\dfrac{x+3}{5x+26} = \dfrac{2}{x+4}$

18. $\dfrac{-2}{x-2} = \dfrac{x-3}{8x+11}$

Concept 2: Applications of Proportions

19. A preschool advertises that it has a 3-to-1 ratio of children to adults. If 18 children are enrolled, how many adults must be on the staff? **(See Example 2.)**

20. An after-school care facility tries to maintain a 4-to-1 ratio of children to adults. If the facility hired five adults, what is the maximum number of children that can enroll?

21. A 3.5-oz box of candy has a total of 21.0 g of fat. How many grams of fat would a 14-oz box of candy contain?

22. A 6-oz box of candy has 350 calories. How many calories would a 10-oz box contain?

23. A fisherman in the North Atlantic catches eight swordfish for a total of 1840 lb. How many swordfish were caught if a commercial fishing boat arrives in port with 230,000 lb of swordfish?

24. If a 64-oz bottle of laundry detergent costs $12.00, how much would an 80-oz bottle cost?

25. Pam drives her hybrid 243 mi in city driving on 4.5 gal of gas. At this rate how many gallons of gas are required to drive 621 mi?

26. On a map, the distance from Sacramento, California, to San Francisco, California, is 8 cm. The legend gives the actual distance as 96 mi. On the same map, Fatima measured 7 cm from Sacramento to Modesto, California. What is the actual distance?

27. Yellowstone National Park in Wyoming has the largest population of free-roaming bison. To approximate the number of bison, 200 are captured and tagged and then left free to roam. Later, a sample of 120 bison is observed and 6 have tags. Approximate the population of bison in the park.

28. Laws have been instituted in Florida to help save the manatee. To establish the number of manatees in Florida, 150 manatees were tagged. A new sample was taken later, and among the 40 manatees in the sample, 3 were tagged. Approximate the number of manatees in Florida.

Jeff Vanuga/Corbis

29. The ratio of men to women enrolled in a math course to train elementary school teachers is 1 to 5. If the total enrollment in these classes is approximately 186 students per semester, how many men are enrolled? **(See Example 3.)**

30. The ratio of Hank's income spent on rent to his income spent on car payments is 3 to 1. If he spends a total of $1640 per month on the rent and car payment, how much does he spend on each item?

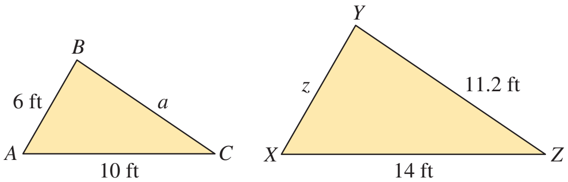
31. The ratio of single men in their 20s to single women in their 20s is 119 to 100 (*Source:* U.S. Census). In a random group of 1095 single college students in their 20s, approximately how many are men and how many are women?

32. A chemist mixes water and alcohol in a 7 to 8 ratio. If she makes a 450-L solution, how much is water and how much is alcohol?

Concept 3: Similar Triangles

For Exercises 33–36, triangle *ABC* is similar to triangle *XYZ*. Find the lengths of the missing sides. **(See Example 4.)**

33.



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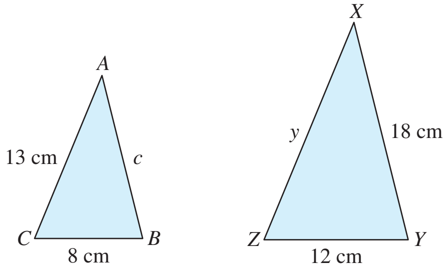

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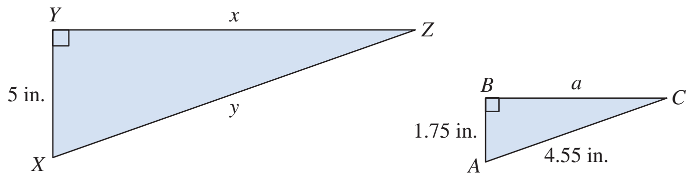

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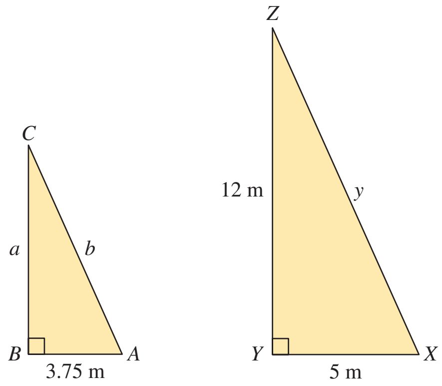

Concept 4: Applications of Rational Equations

37. If 5 is added to the reciprocal of a number, the result is $\frac{16}{3}$. Find the number.

38. If $\frac{2}{3}$ is added to the reciprocal of a number, the result is $\frac{17}{3}$. Find the number.

39. If 7 is decreased by the reciprocal of a number, the result is $\frac{9}{2}$. Find the number.

40. If a number is added to its reciprocal, the result is $\frac{13}{6}$. Find the number.

For Exercises 41–42, use the fact that distance = (rate)(time).

41. A truck travels 7 mph faster than a car. Let x represent the speed of the car.

a. Write an expression for the speed of the truck.

b. Write an expression for the time it takes the car to travel 48 mi.

c. Write an expression for the time it takes the truck to travel 83 mi.

42. A car travels 4 mph slower than a motorcycle. Let x represent the speed of the motorcycle.

a. Write an expression for the speed of the car.

b. Write an expression for the time it takes the motorcycle to travel 50 mi.

c. Write an expression for the time it takes the car to travel 145 mi.

43. A motorist travels 80 mi while driving in a bad rainstorm. In sunny weather, the motorist drives 20 mph faster and covers 120 mi in the same amount of time. Find the speed of the motorist in the rainstorm and the speed in sunny weather. **(See Example 5.)**

44. Brooke walks 2 km/hr slower than her older sister Adrianna. If Brooke can walk 12 km in the same amount of time that Adrianna can walk 18 km, find their speeds.

45. Two out-of-town firefighting crews have been called to a wildfire in the mountains. The Wescott Fire Station is 96 mi from the fire, and the Broadmoor Fire Station is 88 mi from the fire. The fire truck from the Wescott Fire Station travels 6.4 mph faster than the Broadmoor fire truck. If it takes the trucks the same amount of time to reach the fire, what is the average speed of each truck?

46. Kathy can run 3 mi to the beach in the same amount of time Dennis can ride his bike 7 mi to work. Kathy runs 8 mph slower than Dennis rides his bike. Find their speeds.

47. A bicyclist rides 30 mi against a wind and returns 30 mi with the wind. His average speed for the return trip is 5 mph faster. How fast did the cyclist ride against the wind if the total time of the trip was 5 hr? **(See Example 6.)**

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48. A boat travels 60 mi to an island and 60 mi back again. Changes in the wind and tide made the average speed on the return trip 3 mph slower than the speed on the way out. If the total time of the trip took 9 hr, find the speed going to the island and the speed of the return trip.

49. Celeste walked 140 ft on a moving walkway at the airport. Then she walked on the ground for 100 ft. She travels 2 ft/sec faster on the walkway than she does on the ground. If the time it takes her to travel the total distance of 240 ft is 40 sec, how fast does Celeste travel on and off the moving walkway?

50. Julio rides his bike 6 mi and gets a flat tire. Then he has to walk with the bike for another mile. His speed walking is 6 mph less than his speed riding the bike. If the total time is 1 hr, find his speed riding the bike and his speed walking.

51. Beatrice participates in professional triathlons. She runs 2 mph faster than her friend Joe, a weekend athlete. If they each run 12 mi, Beatrice finishes 30 min $\left(\frac{1}{2}\text{ hr}\right)$ ahead of Joe. Determine how fast each person runs.

52. A bus leaves a terminal at 9:00. A car leaves 1 hr later and averages 10 mph faster than the bus. If the car overtakes the bus after 200 mi, find the average speed of the bus and the average speed of the car.

53. One painter can paint a room in 6 hr. Another painter can paint the same room in 8 hr. How long would it take them working together? **(See Example 7.)**

54. Karen can wax her SUV in 2 hr. Clarann can wax the same SUV in 3 hr. If they work together, how long will it take them to wax the SUV?

55. A new housing development offers fenced-in yards that all have the same dimensions. Joel can fence a yard in 12 hr, and Michael can fence a yard in 15 hr. How long will it take if they work together?

David Papazian/Corbis

56. Ted can change an advertisement on a billboard in 4 hr. Marie can do the same job in 5 hr. How long would it take them if they worked together?

57. A swimming pool takes 30 hr to fill using an old pump. When a new pump was installed, it took only 12 hr to fill the pool with both pumps. However, the old pump had to be repaired.

a. Determine how long it would take for the new pump to fill the pool alone.

b. If the new pump begins filling the empty pool at 4 P.M. on Thursday, when should the technician return to stop the pump?

58. One carpenter can complete a kitchen in 8 days. With the help of another carpenter, they can do the job together in 4 days. How long would it take the second carpenter if he worked alone?

59. Gus works twice as fast as Sid. Together they can dig a garden in 4 hr. How long would it take each person working alone?

60. It takes a child 3 times longer to vacuum a house than an adult. If it takes 1 hr for one adult and one child working together to vacuum a house, how long would it take each person working alone?

Randy Faris/Corbis/Getty Images

Variation

Section 5.7

Concepts

1. Definition of Direct and Inverse Variation

In this section, we introduce the concept of variation. Direct and inverse variation models can show how one quantity varies in proportion to another.

Definition of Direct and Inverse Variation

Let k be a nonzero constant real number. Then,

1. y varies **directly** as x.
   y is directly proportional to x. } $y = kx$

2. y varies **inversely** as x.
   y is inversely proportional to x. } $y = \frac{k}{x}$

Note: The value of k is called the constant of variation.

For a car traveling 30 mph, the equation $d = 30t$ indicates that the distance traveled is *directly proportional* to the time of travel. For positive values of k, when two variables are directly related, as one variable increases, the other variable will also increase. Likewise, if one variable decreases, the other will decrease. In the equation $d = 30t$, the longer the time of the trip, the greater the distance traveled. The shorter the time of the trip, the shorter the distance traveled.

For positive values of k, when two positive variables are *inversely related,* as one variable increases, the other will decrease, and vice versa. Consider a car traveling between Toronto and Montreal, a distance of 500 km. The time required to make the trip is inversely proportional to the speed of travel: $t = 500/r$. As the rate of speed, r, increases, the quotient $500/r$ will decrease. Thus, the time will decrease. Similarly, as the rate of speed decreases, the trip will take longer.

2. Translations Involving Variation

The first step in using a variation model is to write an English phrase as an equivalent mathematical equation.

Example 1 Translating to a Variation Model

Write each expression as an equivalent mathematical model.

a. The circumference of a circle varies directly as the radius.

b. At a constant temperature, the volume of a gas varies inversely as the pressure.

c. The length of time of a meeting is directly proportional to the *square* of the number of people present.

Solution:

a. Let C represent circumference and r represent radius. The variables are directly related, so use the model $C = kr$.

b. Let V represent volume and P represent pressure. Because the variables are inversely related, use the model $V = \frac{k}{P}$.

c. Let t represent time, and let N be the number of people present at a meeting. Because t is directly related to N^2, use the model $t = kN^2$.

Skill Practice Write each expression as an equivalent mathematical model.

1. The distance, d, driven in a particular time varies directly with the speed of the car, s.

2. The weight of an individual kitten, w, varies inversely with the number of kittens in the litter, n.

3. The value of v varies inversely as the square root of b.

Sometimes a variable varies directly as the product of two or more other variables. In this case, we have joint variation.

Definition of Joint Variation

Let k be a nonzero constant real number. Then the following statements are equivalent:

$$\left.\begin{array}{l} y \text{ varies \textbf{jointly} as } w \text{ and } z. \\ y \text{ is jointly proportional to } w \text{ and } z. \end{array}\right\} \quad y = kwz$$

Example 2 Translating to a Variation Model

Write each expression as an equivalent mathematical model.

a. y varies jointly as u and the square root of v.

b. The gravitational force of attraction between two planets varies jointly as the product of their masses and inversely as the square of the distance between them.

Solution:

a. $y = ku\sqrt{v}$

b. Let m_1 and m_2 represent the masses of the two planets. Let F represent the gravitational force of attraction and d represent the distance between the planets.

The variation model is: $F = \dfrac{km_1m_2}{d^2}$

Skill Practice Write each expression as an equivalent mathematical model.

4. The value of q varies jointly as u and v.

5. The value of x varies directly as the square of y and inversely as z.

Answers

1. $d = ks$ **2.** $w = \dfrac{k}{n}$

3. $v = \dfrac{k}{\sqrt{b}}$

4. $q = kuv$ **5.** $x = \dfrac{ky^2}{z}$

3. Applications of Variation

Consider the variation models $y = kx$ and $y = \frac{k}{x}$. In either case, if values for x and y are known, we can solve for k. Once k is known, we can use the variation equation to find y if x is known, or to find x if y is known. This concept is the basis for solving many applications involving variation.

Finding a Variation Model

Step 1 Write a general variation model that relates the variables given in the problem. Let k represent the constant of variation.

Step 2 Solve for k by substituting known values of the variables into the model from step 1.

Step 3 Substitute the value of k into the original variation model from step 1.

Example 3 Solving an Application Involving Direct Variation

The variable z varies directly as w. When w is 16, z is 56.

a. Write a variation model for this situation. Use k as the constant of variation.

b. Solve for the constant of variation.

c. Find the value of z when w is 84.

Solution:

a. $z = kw$

b. $z = kw$

$56 = k(16)$ Substitute known values for z and w. Then solve for the unknown value of k.

$\frac{56}{16} = \frac{k(16)}{16}$ To isolate k, divide both sides by 16.

$\frac{7}{2} = k$ Simplify $\frac{56}{16}$ to $\frac{7}{2}$.

c. With the value of k known, the variation model can now be written as:

$z = \frac{7}{2}w$

$z = \frac{7}{2}(84)$ To find z when $w = 84$, substitute $w = 84$ into the equation.

$z = 294$

Skill Practice The variable t varies directly as the square of v. When v is 8, t is 32.

6. Write a variation model for this relationship.

7. Solve for the constant of variation.

8. Find t when $v = 10$.

Answers

6. $t = kv^2$ **7.** $\frac{1}{2}$ **8.** 50

Example 4 Solving an Application Involving Direct Variation

The speed of a racing canoe in still water varies directly as the square root of the length of the canoe.

a. If a 16-ft canoe can travel 6.2 mph in still water, find a variation model that relates the speed of a canoe to its length.

b. Find the speed of a 25-ft canoe.

Solution:

a. Let s represent the speed of the canoe and L represent the length. The general variation model is $s = k\sqrt{L}$. To solve for k, substitute the known values for s and L.

$$s = k\sqrt{L}$$

$$6.2 = k\sqrt{16} \qquad \text{Substitute } s = 6.2 \text{ mph and } L = 16 \text{ ft.}$$

$$6.2 = k \cdot 4$$

$$\frac{6.2}{4} = \frac{4k}{4} \qquad \text{Solve for } k.$$

$$k = 1.55$$

$$s = 1.55\sqrt{L} \qquad \text{Substitute } k = 1.55 \text{ into the model } s = k\sqrt{L}.$$

b. $s = 1.55\sqrt{L}$

$$= 1.55\sqrt{25} \qquad \text{Find the speed when } L = 25 \text{ ft.}$$

$$= 7.75 \text{ mph} \qquad \text{The speed is 7.75 mph.}$$

Skill Practice

9. The amount of water needed by a mountain hiker varies directly as the time spent hiking. The hiker needs 2.4 L for a 4-hr hike. How much water will be needed for a 5-hr hike?

Example 5 Solving an Application Involving Inverse Variation

The loudness of sound measured in decibels (dB) varies inversely as the square of the distance between the listener and the source of the sound. If the loudness of sound is 17.92 dB at a distance of 10 ft from a home theater speaker, what is the decibel level 20 ft from the speaker?

Solution:

Let L represent the loudness of sound in decibels and d represent the distance in feet. The inverse relationship between decibel level and the square of the distance is modeled by

$$L = \frac{k}{d^2}$$

$$17.92 = \frac{k}{(10)^2} \qquad \text{Substitute } L = 17.92 \text{ dB and } d = 10 \text{ ft.}$$

Answer

9. 3 L

$$17.92 = \frac{k}{100}$$

$$(17.92)100 = \frac{k}{100} \cdot 100$$ Solve for k (clear fractions).

$$k = 1792$$

$$L = \frac{1792}{d^2}$$ Substitute $k = 1792$ into the original model $L = \frac{k}{d^2}$.

With the value of k known, we can find L for any value of d.

$$L = \frac{1792}{(20)^2}$$ Find the loudness when $d = 20$ ft.

$$= 4.48 \text{ dB}$$ The loudness is 4.48 dB.

Notice that the loudness of sound is 17.92 dB at a distance 10 ft from the speaker. When the distance from the speaker is increased to 20 ft, the decibel level decreases to 4.48 dB. This is consistent with an inverse relationship. For $k > 0$, as one variable is increased, the other is decreased. It also seems reasonable that the farther one moves away from the source of a sound, the softer the sound becomes.

Skill Practice

10. The yield on a bond varies inversely as the price. The yield on a particular bond is 5% when the price is \$100. Find the yield when the price is \$125.

Example 6 Solving an Application Involving Joint Variation

The kinetic energy of an object varies jointly as the weight of the object at sea level and as the square of its velocity. During a hurricane, a 0.5-lb stone traveling at 60 mph has 81 J (joules) of kinetic energy. Suppose the wind speed doubles to 120 mph. Find the kinetic energy.

Solution:

Let E represent the kinetic energy, let w represent the weight, and let v represent the velocity of the stone. The variation model is

$$E = kwv^2$$

$$81 = k(0.5)(60)^2$$ Substitute $E = 81$ J, $w = 0.5$ lb, and $v = 60$ mph.

$$81 = k(0.5)(3600)$$ Simplify exponents.

$$81 = k(1800)$$

$$\frac{81}{1800} = \frac{k(1800)}{1800}$$ Divide by 1800.

$$0.045 = k$$ Solve for k.

Answer

10. 4%

With the value of k known, the model $E = kwv^2$ can now be written as $E = 0.045wv^2$. We now find the kinetic energy of a 0.5-lb stone traveling at 120 mph.

$$E = 0.045(0.5)(120)^2$$
$$= 324$$

The kinetic energy of a 0.5-lb stone traveling 120 mph is 324 J.

Skill Practice

11. The amount of simple interest earned in an account varies jointly as the interest rate and time of the investment. An account earns \$72 in 4 years at 2% interest. How much interest would be earned in 3 years at a rate of 5%?

Answer

11. \$135

In Example 6, when the velocity increased by 2 times, the kinetic energy increased by 4 times (note that $324 \text{ J} = 4 \cdot 81 \text{ J}$). This factor of 4 occurs because the kinetic energy is proportional to the *square* of the velocity. When the velocity increased by 2 times, the kinetic energy increased by 2^2 times.

Section 5.7 Activity

A.1. Which equation represents a direct relationship between z and p and which represents an inverse relationship?

$$z = kp \qquad \text{and} \qquad z = \frac{k}{p}$$

A.2. **a.** Consider the equation $z = 6p$, where z and p are positive real numbers. If p increases, then z will (choose one: increase or decrease) proportionally.

b. Consider the equation $z = \frac{6}{p}$, where z and p are positive real numbers. If p increases, then z will (choose one: increase or decrease) proportionally.

A.3. The amount of medicine that a physician prescribes for a patient varies directly as the weight of the patient. A physician prescribes 3 g (grams) of a medicine for a 150-lb person.

a. Write a variation equation using A for the amount of medicine, w for the weight of the patient, and k as the constant of variation.

b. Solve for k by using the fact that 3 g of medicine is given to a 150-lb person. Write the resulting variation model.

c. Based on the variation model, as the weight of the patient increases, will more or less medicine be prescribed?

d. How many grams should the physician prescribe for a 180-lb person?

e. How many grams should the physician prescribe for a 225-lb person?

f. How many grams should the physician prescribe for a 120-lb person?

A.4. The average cost of producing DVDs is inversely proportional to the number of DVDs produced. If 5000 DVDs are produced, the average cost per DVD is \$0.48.

a. Write a variation equation using C for the average cost, n for the number of DVDs produced, and k as the constant of variation.

b. Solve for k by using the known average cost of \$0.48 for 5000 DVDs. Write the resulting variation model.

c. Based on the variation model, as the number of DVDs produced increases, will the average cost per DVD go up or down?

d. What would the average cost be if 6000 DVDs are produced?

e. What would the average cost be if 8000 DVDs are produced?

f. What would the average cost be if 2400 DVDs are produced?

A.5. The variable y varies jointly as b and the square root of c.

a. Write a variation model using k as the constant of variation.

b. If y is 150 when b is 10 and c is 25, find the value of k.

c. Write the variation model using the numerical value of k.

d. Determine the value of y when b is 7 and c is 36.

Practice Exercises

Section 5.7

Prerequisite Review

For Exercises R.1–R.4, solve the equation.

R.1. $40 = k \cdot 8$

R.2. $75 = k \cdot 12$

R.3. $0.16 = \dfrac{k}{4}$

R.4. $1.25 = \dfrac{k}{5}$

R.5. Given $x = \dfrac{ky^2}{z}$ and $k = 4$, $y = 6$, and $z = 3$, solve for x.

R.6. Given $y = \dfrac{ka}{b^3}$ and $k = 10$, $a = 12$, and $b = 2$, solve for y.

R.7. Given $p = ka\sqrt{c}$ and $k = 12$, $c = 9$, and $p = 72$, solve for a.

R.8. Given $t = kxy^3$ and $k = 5$, $y = 3$, and $t = 54$, solve for x.

Vocabulary and Key Concepts

1. **a.** Let k be a nonzero constant. If y varies directly as x, then $y =$ __________, where k is the constant of variation.

b. Let k be a nonzero constant. If y varies inversely as x, then $y =$ __________, where k is the constant of variation.

c. Let k be a nonzero constant. If y varies jointly as x and w, then $y =$ __________, where k is the constant of variation.

Concept 1: Definition of Direct and Inverse Variation

2. In the equation $r = kt$, does r vary directly or inversely as t?

3. In the equation $w = \dfrac{k}{v}$, does w vary directly or inversely as v?

4. In the equation $P = \dfrac{k \cdot c}{v}$, does P vary directly or inversely as v?

Concept 2: Translations Involving Variation

For Exercises 5–16, write a variation model. Use k as the constant of variation. **(See Examples 1–2.)**

5. T varies directly as q.

6. W varies directly as z.

7. b varies inversely as c.

8. m varies inversely as t.

9. Q is directly proportional to x and inversely proportional to y.

10. d is directly proportional to p and inversely proportional to n.

11. c varies jointly as s and t.

12. w varies jointly as p and f.

13. L varies jointly as w and the square root of v.

14. q varies jointly as v and the cube root of w.

15. x varies directly as the square of y and inversely as z.

16. a varies directly as n and inversely as the square of d.

Concept 3: Applications of Variation

For Exercises 17–22, find the constant of variation, k. **(See Example 3.)**

17. y varies directly as x and when x is 4, y is 18.

18. m varies directly as x and when x is 8, m is 22.

19. p varies inversely as q and when q is 16, p is 32.

20. T varies inversely as x and when x is 40, T is 200.

21. y varies jointly as w and v. When w is 50 and v is 0.1, y is 8.75.

22. N varies jointly as t and p. When t is 1 and p is 7.5, N is 330.

For Exercises 23–34, solve for the indicated variable. **(See Example 3.)**

23. x varies directly as p. If $x = 50$ when $p = 10$, find x when p is 14.

24. y is directly proportional to z. If $y = 12$ when $z = 36$, find y when z is 21.

25. b is inversely proportional to c. If b is 4 when c is 3, find b when $c = 2$.

26. q varies inversely as w. If q is 8 when w is 50, find q when w is 125.

27. Z varies directly as the square of w. If $Z = 14$ when $w = 4$, find Z when $w = 8$.

28. m varies directly as the square of x. If $m = 200$ when $x = 20$, find m when x is 32.

29. Q varies inversely as the square of p. If $Q = 4$ when $p = 3$, find Q when $p = 2$.

30. z is inversely proportional to the square of t. If $z = 15$ when $t = 4$, find z when $t = 10$.

31. L varies jointly as a and the square root of b. If $L = 72$ when $a = 8$ and $b = 9$, find L when $a = \frac{1}{2}$ and $b = 36$.

32. Y varies jointly as the cube of x and the square root of w. $Y = 128$ when $x = 2$ and $w = 16$. Find Y when $x = \frac{1}{2}$ and $w = 64$.

33. B varies directly as m and inversely as n. $B = 20$ when $m = 10$ and $n = 3$. Find B when $m = 15$ and $n = 12$.

34. R varies directly as s and inversely as t. $R = 14$ when $s = 2$ and $t = 9$. Find R when $s = 4$ and $t = 3$.

For Exercises 35–50, use a variation model to solve for the unknown value. **(See Examples 4–6.)**

35. The weight of a person's heart varies directly as the person's actual weight. For a 150-lb man, his heart would weigh 0.75 lb.

a. Approximate the weight of a 184-lb man's heart.

b. How much does your heart weigh?

36. The number of calories, C, in beer varies directly with the number of ounces, n. If 12 oz of beer contains 153 calories, how many calories are in 40 oz of beer?

37. The number of turkeys needed for a banquet is directly proportional to the number of guests that must be fed. Master Chef Rico knows that he needs to cook 3 turkeys to feed 42 guests.

a. How many turkeys should he cook to feed 70 guests?

b. How many turkeys should he cook to feed 140 guests?

c. How many turkeys should be cooked to feed 700 guests at an inaugural ball?

d. How many turkeys should be cooked for a wedding with 100 guests?

38. An author self-publishes a book and finds that the number of books she can sell per month varies inversely as the price of the book. The author can sell 1500 books per month when the price is set at \$8 per book.

a. How many books would she expect to sell if the price were \$12?

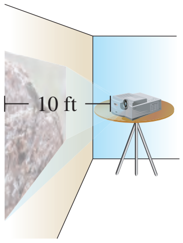
b. How many books would she expect to sell if the price were \$15?

c. How many books would she expect to sell if the price were \$6?

39. The amount of pollution entering the atmosphere over a given time varies directly as the number of people living in an area. If 80,000 people create 56,800 tons of pollutants, how many tons enter the atmosphere in a city with a population of 500,000?

Patrick Clark/Photodisc/Getty Images

40. The area of a picture projected on a wall varies directly as the square of the distance from the projector to the wall. If a 10-ft distance produces a 16-ft^2 picture, what is the area of a picture produced when the projection unit is moved to a distance 20 ft from the wall?



41. The stopping distance of a car varies directly as the square of the speed of the car. If a car traveling 40 mph has a stopping distance of 109 ft, find the stopping distance of a car that travels 25 mph. (Round the answer to one decimal place.)

42. The intensity of a light source varies inversely as the square of the distance from the source. If the intensity of a light bulb is 400 lumen/m^2 (lux) at a distance of 5 m, determine the intensity at 8 m.

43. The power in an electric circuit varies jointly as the current and the square of the resistance. If the power is 144 W (watts) when the current is 4 A (amperes) and the resistance is 6 Ω (ohms), find the power when the current is 3 A and the resistance is 10 Ω.

44. Some bodybuilders claim that, within safe limits, the number of repetitions that a person can complete on a given weight-lifting exercise is inversely proportional to the amount of weight lifted. Roxanne can bench press 45 lb for 15 repetitions.

a. How many repetitions can Roxanne bench with 60 lb of weight?

b. How many repetitions can Roxanne bench with 75 lb of weight?

c. How many repetitions can Roxanne bench with 100 lb of weight?

45. The current in a wire varies directly as the voltage and inversely as the resistance. If the current is 9 A when the voltage is 90 V (volts) and the resistance is 10 Ω, find the current when the voltage is 185 V and the resistance is 10 Ω.

46. The resistance of a wire varies directly as its length and inversely as the square of its diameter. A 40-ft wire 0.1 in. in diameter has a resistance of 4 Ω. What is the resistance of a 50-ft wire with a diameter of 0.2 in.?

47. The weight of a medicine ball varies directly as the cube of its radius. A ball with a radius of 3 in. weighs 4.32 lb. How much does a medicine ball weigh if its radius is 5 in.?

48. The area of an equilateral triangle varies directly as the square of the length of the sides. For an equilateral triangle with 7-cm sides, the area is 21.22 cm^2. What is the area of an equilateral triangle with 17-cm sides? Round to the nearest whole unit.

49. The amount of simple interest earned in an account varies jointly as the amount of principal invested and the amount of time the money is invested. If \$2500 in principal earns \$500 in interest after 4 years, then how much interest will be earned on \$7000 invested for 10 years?

50. The amount of simple interest earned in an account varies jointly as the amount of principal invested and the amount of time the money is invested. If \$6000 in principal earns \$840 in interest after 2 years, then how much interest will be earned on \$4500 invested for 8 years?

Chapter 5 Summary

Section 5.1 Rational Expressions and Rational Functions

Key Concepts

A **rational expression** is in the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

The *domain* of a rational function excludes the values for which the denominator is zero.

To simplify a rational expression to lowest terms, factor the numerator and denominator completely. Then simplify factors whose ratio is 1 or -1. A rational expression written in lowest terms will still have the same restrictions on the domain as the original expression.

Examples

Example 1

Find the domain of the function.

$$f(x) = \frac{x-3}{(x+4)(2x-1)}$$

Domain: $\left\{x \,\middle|\, x \text{ is a real number and } x \neq -4, x \neq \frac{1}{2}\right\}$

or $\quad (-\infty, -4) \cup (-4, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

Example 2

Simplify to lowest terms.

$$\frac{t^2 - 6t - 16}{5t + 10}$$

$$= \frac{(t-8)(t+2)}{5(t+2)}$$

$$= \frac{(t-8)}{5} \cdot \frac{\overset{1}{\cancel{(t+2)}}}{\cancel{(t+2)}}$$

$$= \frac{(t-8)}{5} \cdot (1) = \frac{t-8}{5}$$

Section 5.2 Multiplication and Division of Rational Expressions

Key Concepts

To multiply rational expressions, factor the numerators and denominators completely. Then simplify factors whose ratio is 1 or −1.

To divide rational expressions, multiply by the reciprocal of the divisor.

Examples

Example 1

$$\frac{b^2 - a^2}{a^2 - 2ab + b^2} \cdot \frac{a^2 - 3ab + 2b^2}{2a + 2b}$$

$$= \frac{\overset{-1}{\cancel{(b-a)}}\overset{1}{\cancel{(b+a)}} \cdot (a-2b)\overset{1}{\cancel{(a-b)}}}{\cancel{(a-b)}\cancel{(a-b)} \cdot 2\cancel{(a+b)}} \qquad \text{Factor.}$$

$$= -\frac{a-2b}{2} \quad \text{or} \quad \frac{2b-a}{2} \qquad \text{Simplify.}$$

Example 2

$$\frac{9x+3}{x^2-4} \div \frac{3x+1}{4x+8}$$

$$= \frac{9x+3}{x^2-4} \cdot \frac{4x+8}{3x+1} \qquad \text{Multiply by the reciprocal.}$$

$$= \frac{3\overset{1}{\cancel{(3x+1)}}}{(x-2)\cancel{(x+2)}} \cdot \frac{4\overset{1}{\cancel{(x+2)}}}{\cancel{3x+1}} \qquad \text{Factor.}$$

$$= \frac{12}{x-2} \qquad \text{Simplify.}$$

Section 5.3 Addition and Subtraction of Rational Expressions

Key Concepts

To add or subtract rational expressions, the expressions must have the same denominator.

The **least common denominator (LCD)** is the product of unique factors from the denominators, in which each factor is raised to its highest power.

Steps to Add or Subtract Rational Expressions

1. Factor the denominator of each rational expression.
2. Identify the LCD.
3. Rewrite each rational expression as an equivalent expression with the LCD as its denominator. (This is accomplished by multiplying the numerator and denominator of each rational expression by the missing factor(s) from the LCD.)
4. Add or subtract the numerators, and write the result over the common denominator.
5. Simplify, if possible.

Examples

Example 1

For $\dfrac{1}{3(x-1)^3(x+2)}$ and $\dfrac{-5}{6(x-1)(x+7)^2}$

LCD $= 6(x-1)^3(x+2)(x+7)^2$

Example 2

$$\frac{c}{c^2-c-12}-\frac{1}{2c-8}$$

$$=\frac{c}{(c-4)(c+3)}-\frac{1}{2(c-4)} \quad \text{Factor the denominators.}$$

The LCD is $2(c-4)(c+3)$

$$\frac{2}{2}\cdot\frac{c}{(c-4)(c+3)}-\frac{1}{2(c-4)}\cdot\frac{(c+3)}{(c+3)} \quad \text{Write equivalent fractions with LCD.}$$

$$=\frac{2c-(c+3)}{2(c-4)(c+3)} \quad \text{Subtract.}$$

$$=\frac{2c-c-3}{2(c-4)(c+3)} \quad \text{Simplify.}$$

$$=\frac{c-3}{2(c-4)(c+3)}$$

Section 5.4 Complex Fractions

Key Concepts

Complex fractions can be simplified by using Method I or Method II.

Method I uses the order of operations to simplify the numerator and denominator separately before multiplying by the reciprocal of the denominator of the complex fraction.

To use Method II, multiply the numerator and denominator of the complex fraction by the LCD of all the individual fractions. Then simplify the result.

Examples

Example 1

Simplify by using Method I.

$$\frac{\dfrac{5x+15}{7}}{\dfrac{x+3}{2}}$$

$$= \frac{5x+15}{7} \div \frac{x+3}{2}$$

$$= \frac{5x+15}{7} \cdot \frac{2}{x+3}$$

$$= \frac{5\overset{1}{\cancel{(x+3)}}}{7} \cdot \frac{2}{\cancel{x+3}}$$

$$= \frac{10}{7}$$

Example 2

Simplify by using Method II.

$$\frac{1-\dfrac{4}{w^2}}{1-\dfrac{1}{w}-\dfrac{6}{w^2}} \qquad \text{The LCD is } w^2.$$

$$= \frac{w^2\left(1-\dfrac{4}{w^2}\right)}{w^2\left(1-\dfrac{1}{w}-\dfrac{6}{w^2}\right)}$$

$$= \frac{w^2-4}{w^2-w-6}$$

$$= \frac{(w-2)\overset{1}{\cancel{(w+2)}}}{(w-3)\cancel{(w+2)}}$$

$$= \frac{w-2}{w-3}$$

Section 5.5 Solving Rational Equations

Key Concepts

Steps to Solve a Rational Equation

1. Factor the denominators of all rational expressions. Identify any restrictions on the variable.
2. Identify the LCD of all expressions in the equation.
3. Multiply both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check each potential solution.

Examples

Example 1

$$\frac{1}{w} - \frac{1}{2w-1} = \frac{-2w}{2w-1}$$

The LCD is $w(2w - 1)$.

$$\cancel{w}(2w-1)\frac{1}{\cancel{w}} - w(\cancel{2w-1})\cdot\frac{1}{\cancel{2w-1}}$$

$$= w(\cancel{2w-1})\cdot\frac{-2w}{\cancel{2w-1}}$$

$$(2w-1)1 - w(1) = w(-2w)$$

$$2w - 1 - w = -2w^2 \qquad \text{(quadratic equation)}$$

$$2w^2 + w - 1 = 0$$

$$(2w-1)(w+1) = 0$$

$$\cancel{w = \frac{1}{2}} \quad \text{or} \quad w = -1$$

The solution set is $\{-1\}$. (The value $\frac{1}{2}$ does not check.)

Section 5.6 Applications of Rational Equations and Proportions

Key Concepts

An equation that equates two **ratios** is called a **proportion**.

$$\frac{a}{b} = \frac{c}{d} \quad \text{provided } b \neq 0,\ d \neq 0$$

Example 1

A sample of 85 g of a particular ice cream contains 17 g of fat. How much fat does 324 g of the same ice cream contain?

$$\begin{array}{rcl} \text{fat (g)} \longrightarrow & \dfrac{17}{85} = \dfrac{x}{324} & \longleftarrow \text{fat (g)} \\ \text{ice cream (g)} \longrightarrow & & \longleftarrow \text{ice cream (g)} \end{array}$$

$$(\cancelto{1}{85}\cdot 324)\cdot\frac{17}{\cancel{85}} = (85\cdot\cancelto{1}{324})\cdot\frac{x}{\cancel{324}} \qquad \text{Multiply by the LCD.}$$

$$5508 = 85x$$

$$x = 64.8 \text{ g}$$

There would be 64.8 g of fat in 324 g of ice cream.

Examples

Example 2

An old water pump can fill a tank in 6 hr, and a new pump can fill the tank in 4 hr. How long will it take to fill the tank if both pumps are working?

Let x represent the time working together.

$$\begin{pmatrix}\text{Speed of}\\ \text{old pump}\end{pmatrix} + \begin{pmatrix}\text{speed of}\\ \text{new pump}\end{pmatrix} = \begin{pmatrix}\text{speed working}\\ \text{together}\end{pmatrix}$$

$$\frac{1}{6} + \frac{1}{4} = \frac{1}{x}$$

$$12x\cdot\left(\frac{1}{6}+\frac{1}{4}\right) = 12x\cdot\left(\frac{1}{x}\right)$$

$$2x + 3x = 12$$

$$5x = 12$$

$$x = \frac{12}{5} \qquad \text{It will take } \frac{12}{5} \text{ hr.}$$

Section 5.7 Variation

Key Concepts

Direct Variation

For a constant k,
y varies **directly** as x.
y is directly proportional to x.
$\bigg\}\ y = kx$

Inverse Variation

For a constant k,
y varies **inversely** as x.
y is inversely proportional to x.
$\bigg\}\ y = \dfrac{k}{x}$

Joint Variation

For a constant k,
y varies **jointly** as w and z.
y is jointly proportional to w and z.
$\bigg\}\ y = kwz$

Steps to Find a Variation Model

1. Write a general variation model that relates the variables given in the problem. Let k represent the constant of variation.
2. Solve for k by substituting known values of the variables into the model from step 1.
3. Substitute the value of k into the original variation model from step 1.

Examples

Example 1

t varies directly as the square root of x.

$t = k\sqrt{x}$

Example 2

W is inversely proportional to the cube of x.

$W = \dfrac{k}{x^3}$

Example 3

y is jointly proportional to x and to the square of z.

$y = kxz^2$

Example 4

C varies directly as the square root of d and inversely as t. If $C = 12$ when d is 9 and t is 6, find C if d is 16 and t is 12.

Step 1: $C = \dfrac{k\sqrt{d}}{t}$

Step 2: $12 = \dfrac{k\sqrt{9}}{6} \Rightarrow 12 = \dfrac{k \cdot 3}{6} \Rightarrow k = 24$

Step 3: $C = \dfrac{24\sqrt{d}}{t} \Rightarrow C = \dfrac{24\sqrt{16}}{12} \Rightarrow C = 8$

Chapter 5 Review Exercises

Section 5.1

1. Let $k(y) = \dfrac{y}{y^2 - 1}$.

a. Find the function values (if they exist): $k(2), k(0), k(1), k(-1), k\left(\dfrac{1}{2}\right)$.

b. Identify the domain for k Write the answer in interval notation.

2. Let $h(x) = \dfrac{x}{x^2 + 1}$.

a. Find the function values (if they exist): $h(1), h(0), h(-1), h(-3), h\left(\dfrac{1}{2}\right)$.

b. Identify the domain for h. Write the answer in interval notation.

For Exercises 3–10, simplify the rational expression.

3. $\dfrac{28a^3b^3}{14a^2b^3}$

4. $\dfrac{25x^2yz^3}{125xyz}$

5. $\dfrac{x^2 - 4x + 3}{x - 3}$

6. $\dfrac{k^2 + 3k - 10}{k^2 - 5k + 6}$

7. $\dfrac{x^3 - 27}{9 - x^2}$

8. $\dfrac{a^4 - 81}{3 - a}$

9. $\dfrac{2t^2 + 3t - 5}{7 - 6t - t^2}$

10. $\dfrac{y^3 - 4y}{y^2 - 5y + 6}$

For Exercises 11–14, write the domain of each function in interval notation. Use that information to match the function with its graph.

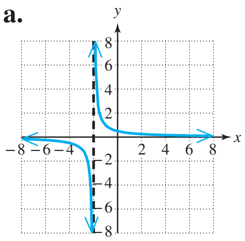
11. $f(x) = \dfrac{1}{x - 3}$

12. $m(x) = \dfrac{1}{x + 2}$

13. $k(x) = \dfrac{6}{x^2 - 3x}$

14. $p(x) = \dfrac{-2}{x^2 + 4}$

a.



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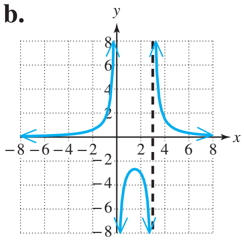
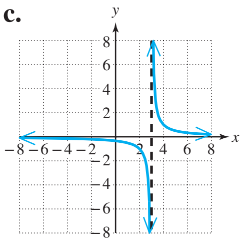
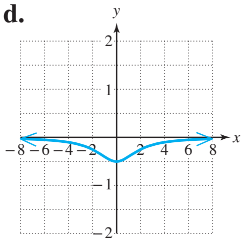


c.



d.



Section 5.2

For Exercises 15–26, multiply or divide as indicated.

15. $\dfrac{3a + 9}{a^2} \cdot \dfrac{a^3}{6a + 18}$

16. $\dfrac{4 - y}{5} \div \dfrac{2y - 8}{15}$

17. $\dfrac{x - 4y}{x^2 + xy} \div \dfrac{20y - 5x}{x^2 - y^2}$

18. $(x^2 + 5x - 24)\left(\dfrac{x + 8}{x - 3}\right)$

19. $\dfrac{7k + 28}{2k + 4} \cdot \dfrac{k^2 - 2k - 8}{k^2 + 2k - 8}$

20. $\dfrac{ab + 2a + b + 2}{ab - 3b + 2a - 6} \cdot \dfrac{ab - 3b + 4a - 12}{ab - b + 4a - 4}$

21. $\dfrac{x^2 + 8x - 20}{x^2 + 6x - 16} \div \dfrac{x^2 + 6x - 40}{x^2 + 3x - 40}$

22. $\dfrac{2b - b^2}{b^3 - 8} \cdot \dfrac{b^2 + 2b + 4}{b^2}$

23. $\dfrac{2w}{21} \div \dfrac{3w^2}{7} \cdot \dfrac{4}{w}$

24. $\dfrac{5y^2 - 20}{y^3 + 8} \div \dfrac{7y^2 - 14y}{y^3 + y}$

25. $\dfrac{x^2 + x - 20}{x^2 - 4x + 4} \cdot \dfrac{x^2 + x - 6}{12 + x - x^2} \div \dfrac{2x + 10}{10 - 5x}$

26. $(9k^2 - 25) \cdot \left(\dfrac{k + 5}{3k - 5}\right)$

Section 5.3

For Exercises 27–38, add or subtract as indicated.

27. $\dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{1}{x^3}$

28. $\dfrac{1}{x + 2} + \dfrac{5}{x - 2}$

29. $\dfrac{y}{2y - 1} + \dfrac{3}{1 - 2y}$

30. $\dfrac{a + 2}{2a + 6} - \dfrac{3}{a + 3}$

31. $\dfrac{4k}{k^2 + 2k + 1} + \dfrac{3}{k^2 - 1}$

32. $4x + 3 - \dfrac{2x + 1}{x + 4}$

33. $\dfrac{2}{a + 3} + \dfrac{2a^2 - 2a}{a^2 - 2a - 15}$

34. $\dfrac{6}{x^2 + 4x + 3} + \dfrac{7}{x^2 + 5x + 6}$

35. $\dfrac{2}{3x - 5} - 8$

36. $\dfrac{7}{4k^2 - k - 3} + \dfrac{1}{4k^2 - 7k + 3}$

37. $\dfrac{6a}{3a^2 - 7a + 2} + \dfrac{2}{1 - 3a} + \dfrac{3a}{a - 2}$

38. $4 + \dfrac{2y - 5}{y + 2} + \dfrac{y}{3 - y}$

Section 5.4

For Exercises 39–46, simplify the complex fraction.

39. $\dfrac{\dfrac{2x}{3x^2 - 3}}{\dfrac{4x}{6x - 6}}$

40. $\dfrac{\dfrac{k + 2}{3}}{\dfrac{5}{k - 2}}$

41. $\dfrac{\dfrac{2}{x} + \dfrac{1}{xy}}{\dfrac{4}{x^2}}$

42. $\dfrac{\dfrac{4}{y} - 1}{\dfrac{1}{y} - \dfrac{4}{y^2}}$

43. $\dfrac{\dfrac{1}{a - 1} + 1}{\dfrac{1}{a + 1} - 1}$

44. $\dfrac{\dfrac{3}{x - 1} - \dfrac{1}{1 - x}}{\dfrac{2}{x - 1} - \dfrac{2}{x}}$

45. $\dfrac{1 + xy^{-1}}{x^2y^{-2} - 1}$

46. $\dfrac{5a^{-1} + (ab)^{-1}}{3a^{-2}}$

For Exercises 47–48, find the slope of the line containing the two points.

47. $\left(\dfrac{2}{3}, -\dfrac{7}{4}\right)$ and $\left(\dfrac{13}{6}, -\dfrac{5}{3}\right)$

48. $\left(\dfrac{8}{15}, -\dfrac{1}{3}\right)$ and $\left(\dfrac{13}{10}, \dfrac{9}{5}\right)$

Section 5.5

For Exercises 49–54, solve the equation.

49. $\dfrac{x + 3}{x^2 - x} - \dfrac{8}{x^2 - 1} = 0$

50. $\dfrac{y}{y + 3} - \dfrac{3}{3 - y} = \dfrac{18}{y^2 - 9}$

51. $x - 9 = \dfrac{72}{x - 8}$

52. $\dfrac{3x + 1}{x + 5} = \dfrac{x - 1}{x + 1} + 2$

53. $5y^{-2} + 1 = 6y^{-1}$

54. $1 + \dfrac{7}{6}m^{-1} = \dfrac{13}{6}m^{-1}$

55. Solve for x. $\quad c = \dfrac{ax + b}{x}$

56. Solve for P. $\quad \dfrac{A}{rt} = P + \dfrac{P}{rt}$

Section 5.6

For Exercises 57–60, solve the proportion.

57. $\dfrac{5}{4} = \dfrac{x}{6}$

58. $\dfrac{x}{36} = \dfrac{6}{7}$

59. $\dfrac{x + 2}{3} = \dfrac{5(x + 1)}{4}$

60. $\dfrac{x}{x + 2} = \dfrac{-3}{5}$

61. In a football game, the quarterback completed 34 passes for 357 yd. At this rate how many yards would be gained for 22 passes?

62. Erik bought $108 Canadian with $100 American. At this rate, how many Canadian dollars can he buy with $235 American?

63. Tony rode 175 mi on a 2-day bicycle ride to benefit the Multiple Sclerosis Foundation. The second day he rode 5 mph slower than the first day because of a strong headwind. If Tony rode 100 mi on the first day and 75 mi on the second day in a total time of 10 hr, how fast did he ride each day?

64. Stephen drove his car 45 mi. He ran out of gas and had to walk 3 mi to a gas station. His speed driving is 15 times his speed walking. If the total time for the drive and walk was $1\frac{1}{2}$ hr, what was his speed driving?

65. Doug and Jean work as phone solicitors. They work in batches of 400 calls. Doug can finish a batch in an average of 8 hr, and Jean can finish a batch in 10 hr. How long would it take them to finish a batch if they worked together?

66. Two pipes can fill a tank in 6 hr. The larger pipe works twice as fast as the smaller pipe. How long would it take each pipe to fill the tank if they worked separately?

Section 5.7

67. The force applied to a spring varies directly with the distance that the spring is stretched.

a. Write a variation model using k as the constant of variation.

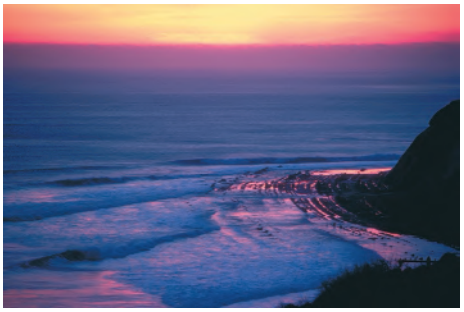
b. When 6 lb of force is applied, the spring stretches 2 ft. Find k.

c. How much force is required to stretch the spring 4.2 ft?

68. Suppose y varies inversely with x and $y = 32$ when $x = 2$. Find y when $x = 4$.

69. Suppose y varies jointly with x and the square root of z, and $y = 3$ when $x = 3$ and $z = 4$. Find y when $x = 8$ and $z = 9$.

70. The distance d that one can see to the horizon varies directly as the square root of the height above sea level. If a person 25 m above sea level can see 30 km, how far can a person see if she is 64 m above sea level?



Digital Stock/Corbis

Chapter 5 Test

1. For the function $h(x) = \dfrac{2x - 14}{x^2 - 49}$

a. Evaluate $h(0)$, $h(5)$, $h(7)$, and $h(-7)$, if possible.

b. Write the domain of h in interval notation.

2. Write the domain of $k(x) = \dfrac{5x - 3}{7}$ in interval notation.

3. For the function $f(x) = \dfrac{2x + 6}{x^2 - x - 12}$

a. Write the domain in set-builder notation.

b. Simplify to lowest terms.

For Exercises 4–5, simplify to lowest terms.

4. $\dfrac{12m^3n^7}{18mn^8}$

5. $\dfrac{9x^2 - 9}{3x^2 + 2x - 5}$

6. Find the slope of the line containing the points $\left(\dfrac{1}{12}, -\dfrac{3}{4}\right)$ and $\left(\dfrac{5}{6}, -\dfrac{8}{3}\right)$.

For Exercises 7–13, simplify.

7. $\dfrac{2x - 5}{25 - 4x^2} \cdot (2x^2 - x - 15)$

8. $\dfrac{x^2}{x - 4} - \dfrac{8x - 16}{x - 4}$

9. $\dfrac{4x}{x + 1} + x + \dfrac{2}{x + 1}$

10. $\dfrac{3 + \dfrac{3}{k}}{4 + \dfrac{4}{k}}$

11. $\dfrac{2u^{-1} + 2v^{-1}}{4u^{-3} + 4v^{-3}}$

12. $\dfrac{ax + bx + 2a + 2b}{ax - 3a + bx - 3b} \cdot \dfrac{x - 3}{5 - x} \div \dfrac{x + 2}{ax - 5a}$

13. $\dfrac{3}{x^2 + 8x + 15} - \dfrac{1}{x^2 + 7x + 12} - \dfrac{1}{x^2 + 9x + 20}$

For Exercises 14–16, solve the equation.

14. $\dfrac{7}{z + 1} - \dfrac{z - 5}{z^2 - 1} = \dfrac{6}{z}$

15. $\dfrac{3}{y^2 - 9} + \dfrac{4}{y + 3} = 1$

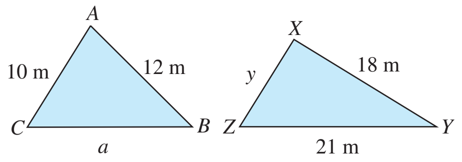
16. $\dfrac{4x}{x - 4} = 3 + \dfrac{16}{x - 4}$

17. Solve for T. $\quad \dfrac{1 + Tv}{T} = p$

18. Solve for m_1. $F = \dfrac{Gm_1m_2}{r^2}$

19. If the reciprocal of a number is added to 3 times the number, the result is $\frac{13}{2}$. Find the number.

20. Triangle ABC is similar to triangle XYZ. Find the lengths of the missing sides.



21. On a certain map, the distance between New York and Los Angeles is 8.2 in., and the actual distance is 2820 mi. What is the distance between two cities that are 5.7 in. apart on the same map? Round to the nearest mile.

22. Lance can ride 48 mi on his bike against the wind. With the wind at his back, he rides 4 mph faster and can ride 60 mi in the same amount of time. Find his speed riding against the wind and his speed riding with the wind.

23. Barbara can type a chapter in a book in 4 hr. Jack can type a chapter in a book in 10 hr. How long would it take them to type a chapter if they worked together?

24. Write a variation model using k as the constant of variation. The variable x varies directly as y and inversely as the square of t.

25. The period of a pendulum varies directly as the square root of the length of the pendulum. If the period of the pendulum is 2.2 sec when the length is 4 ft, find the period when the length is 9 ft.

Radicals and Complex Numbers

6

CHAPTER OUTLINE

Mathematics in Architecture

The area A of a triangle can be found if the length of one side (the base b) and the corresponding height h of the triangle are known. $A = \frac{1}{2}bh$

However, if the base and height are not known, but the lengths of the three sides a, b, and c are known, then we can use Heron's formula to find the area of the triangle. Let s represent the semi-perimeter of the triangle.

That is, $s = \frac{1}{2}(a + b + c)$.

Then Heron's formula gives the area A as

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

I. M. Pei (American architect, born 1917 in China)/ Photov.com/Pixtal/age fotostock

The Louvre pyramid, designed by architect I.M. Pei, is a glass structure that serves as the entrance to the Louvre Museum in Paris, France. Each triangular face is made of glass with sides of lengths 108.5 ft, 108.5 ft, and 116 ft. To determine the surface area of glass for each face, we can apply Heron's formula.

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(108.5 + 108.5 + 116) = 166.5$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{166.5(166.5 - 108.5)(166.5 - 108.5)(166.5 - 116)}$$

$$\approx 5318 \text{ ft}^2 \quad \text{Each face is approximately } 5318 \text{ ft}^2 \text{ of glass.}$$

Heron's formula is one application of radicals. In this chapter, we will simplify radical expressions and solve equations containing radicals.

Section 6.1 Definition of an *n*th Root

Concepts

1. Definition of a Square Root
2. Definition of an *n*th Root
3. Roots of Variable Expressions
4. Pythagorean Theorem
5. Radical Functions

1. Definition of a Square Root

The reverse operation to squaring a number is to find its square roots. For example, finding a square root of 36 is equivalent to asking, "when squared, what number equals 36?"

One obvious answer to this question is 6 because $(6)^2 = 36$, but -6 will also work, because $(-6)^2 = 36$.

Definition of a Square Root

b is a **square root** of a if $b^2 = a$.

Example 1 Identifying Square Roots

Identify the square roots of the real numbers.

a. 25 **b.** 49 **c.** 0 **d.** -9

Solution:

a. 5 is a square root of 25 because $(5)^2 = 25$.

-5 is a square root of 25 because $(-5)^2 = 25$.

b. 7 is a square root of 49 because $(7)^2 = 49$.

-7 is a square root of 49 because $(-7)^2 = 49$.

c. 0 is a square root of 0 because $(0)^2 = 0$.

d. There are no real numbers that when squared will equal a negative number; therefore, there are no real-valued square roots of -9.

Skill Practice Identify the square roots of the real numbers.

1. 64 **2.** 16 **3.** 1 **4.** -100

TIP:

- All positive real numbers have two real-valued square roots (one positive and one negative).
- Zero has only one square root, which is zero itself.
- A negative number has no real-valued square roots.

Recall that the positive square root of a real number can be denoted with a **radical sign** $\sqrt{}$.

Positive and Negative Square Roots

Let a represent a positive real number. Then

1. $\sqrt{a}$ is the *positive* square root of a. The positive square root is also called the **principal square root**.
2. $-\sqrt{a}$ is the *negative* square root of a.
3. $\sqrt{0} = 0$

Answers

1. -8 and 8
2. -4 and 4
3. -1 and 1
4. No real-valued square roots

Example 2 **Simplifying Square Roots**

Simplify the square roots.

a. $\sqrt{36}$ **b.** $\sqrt{\frac{4}{9}}$ **c.** $\sqrt{0.04}$

Solution:

a. $\sqrt{36}$ denotes the positive square root of 36.

$\sqrt{36} = 6$ because $(6)^2 = 36$

b. $\sqrt{\frac{4}{9}}$ denotes the positive square root of $\frac{4}{9}$.

$\sqrt{\frac{4}{9}} = \frac{2}{3}$ because $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$

c. $\sqrt{0.04}$ denotes the positive square root of 0.04.

$\sqrt{0.04} = 0.2$ because $(0.2)^2 = 0.04$

Skill Practice Simplify the square roots.

5. $\sqrt{81}$ **6.** $\sqrt{\frac{36}{49}}$ **7.** $\sqrt{0.09}$

The numbers 36, $\frac{4}{9}$, and 0.04 are **perfect squares** because their square roots are rational numbers.

Radicals that cannot be simplified to rational numbers are irrational numbers. Recall that an irrational number cannot be written as a terminating or repeating decimal. For example, the symbol $\sqrt{13}$ is used to represent the *exact* value of the square root of 13. The symbol $\sqrt{42}$ is used to represent the *exact* value of the square root of 42. These values can be approximated by a rational number by using a calculator.

$$\sqrt{13} \approx 3.605551275 \qquad \sqrt{42} \approx 6.480740698$$

TIP: Before using a calculator to evaluate a square root, try estimating the value first.
$\sqrt{13}$ must be a number between 3 and 4 because $\sqrt{9} < \sqrt{13} < \sqrt{16}$.
$\sqrt{42}$ must be a number between 6 and 7 because $\sqrt{36} < \sqrt{42} < \sqrt{49}$.

A negative number cannot have a real number as a square root because no real number when squared is negative. For example, $\sqrt{-25}$ is *not* a real number because there is no real number b for which $(b)^2 = -25$.

Answers

5. 9 **6.** $\frac{6}{7}$ **7.** 0.3

Example 3 Simplifying Square Roots

Simplify the square roots, if possible.

a. $\sqrt{-144}$ **b.** $-\sqrt{144}$ **c.** $\sqrt{-0.01}$ **d.** $-\sqrt{\frac{1}{9}}$

Solution:

a. $\sqrt{-144}$ is *not* a real number. No real number when squared equals -144.

b. $-\sqrt{144}$

$= -1 \cdot \sqrt{144}$

$= -1 \cdot 12$

$= -12$

TIP: For the expression $-\sqrt{144}$, the factor of -1 is *outside* the radical.

c. $\sqrt{-0.01}$ is *not* a real number. No real number when squared equals -0.01.

d. $-\sqrt{\frac{1}{9}}$

$= -1 \cdot \sqrt{\frac{1}{9}}$

$= -1 \cdot \frac{1}{3}$

$= -\frac{1}{3}$

Skill Practice Simplify the square roots, if possible.

8. $\sqrt{-81}$ **9.** $-\sqrt{64}$ **10.** $-\sqrt{0.25}$ **11.** $\sqrt{-\frac{1}{4}}$

2. Definition of an *n*th Root

Finding a square root of a number is the reverse process of squaring a number. This concept can be extended to finding a third root (called a cube root), a fourth root, and in general an ***n*th root**.

Definition of an *n*th Root

b is an nth root of a if $b^n = a$.

Example: 2 is a square root of 4 because 2^2 is 4.

Example: 2 is a third root of 8 because 2^3 is 8.

Example: 2 is a fourth root of 16 because 2^4 is 16.

The radical sign $\sqrt{}$ is used to denote the principal square root of a number. The symbol $\sqrt[n]{}$ is used to denote the principal nth root of a number. In the expression $\sqrt[n]{a}$, n is called the **index** of the radical, and a is called the **radicand**. For a square root, the index is 2, but it is usually not written ($\sqrt[2]{a}$ is denoted simply as $\sqrt{a}$). A radical with an index of 3 is called a **cube root**, denoted by $\sqrt[3]{a}$.

Answers

8. Not a real number **9.** -8

10. -0.5 **11.** Not a real number

Evaluating $\sqrt[n]{a}$

1. If $n > 1$ is an *even* integer and $a > 0$, then $\sqrt[n]{a}$ is the principal (positive) nth root of a. Example: $\sqrt[4]{625} = 5$
2. If $n > 1$ is an *odd* integer, then $\sqrt[n]{a}$ is the nth root of a.
   Example: $\sqrt[3]{8} = 2$, $\sqrt[3]{-8} = -2$
3. If $n > 1$ is an integer, then $\sqrt[n]{0} = 0$.

For the purpose of simplifying nth roots, it is helpful to know numbers that are perfect squares as well as the following common perfect cubes, fourth powers, and fifth powers.

Perfect Cubes	Perfect Fourth Powers	Perfect Fifth Powers
$1^3 = 1$	$1^4 = 1$	$1^5 = 1$
$2^3 = 8$	$2^4 = 16$	$2^5 = 32$
$3^3 = 27$	$3^4 = 81$	$3^5 = 243$
$4^3 = 64$	$4^4 = 256$	$4^5 = 1024$
$5^3 = 125$	$5^4 = 625$	$5^5 = 3125$

Example 4 Identifying the *n*th Root of a Real Number

Simplify the expressions, if possible.

a. $\sqrt{4}$ **b.** $\sqrt[3]{64}$ **c.** $\sqrt[5]{-32}$ **d.** $\sqrt[4]{81}$

e. $\sqrt[6]{1{,}000{,}000}$ **f.** $\sqrt{-100}$ **g.** $\sqrt[4]{-16}$

Solution:

a. $\sqrt{4} = 2$ because $(2)^2 = 4$

b. $\sqrt[3]{64} = 4$ because $(4)^3 = 64$

c. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

d. $\sqrt[4]{81} = 3$ because $(3)^4 = 81$

e. $\sqrt[6]{1{,}000{,}000} = 10$ because $(10)^6 = 1{,}000{,}000$

f. $\sqrt{-100}$ is not a real number. No real number when squared equals -100.

g. $\sqrt[4]{-16}$ is not a real number. No real number when raised to the fourth power equals -16.

Skill Practice Simplify if possible.

12. $\sqrt[4]{16}$ **13.** $\sqrt[3]{1000}$ **14.** $\sqrt[5]{-1}$ **15.** $\sqrt[5]{32}$

16. $\sqrt[5]{100{,}000}$ **17.** $\sqrt{-36}$ **18.** $\sqrt[3]{-27}$

Examples 4(f) and 4(g) illustrate that an nth root of a negative quantity is not a real number if the index is even. This is because no real number raised to an even power is negative.

3. Roots of Variable Expressions

Finding an nth root of a variable expression is similar to finding an nth root of a numerical expression. For roots with an even index, however, particular care must be taken to obtain a nonnegative result.

Answers

12. 2 **13.** 10 **14.** −1
15. 2 **16.** 10
17. Not a real number **18.** −3

Evaluating $\sqrt[n]{a^n}$

1. If n is a positive *odd* integer, then $\sqrt[n]{a^n} = a$.
2. If n is a positive *even* integer, then $\sqrt[n]{a^n} = |a|$.

The absolute value bars are necessary for roots with an even index because the variable a may represent a positive quantity or a negative quantity. By using absolute value bars, $\sqrt[n]{a^n} = |a|$ is nonnegative and represents the principal nth root of a^n.

Example 5 Simplifying Expressions of the Form $\sqrt[n]{a^n}$

Simplify the expressions.

a. $\sqrt[4]{(-3)^4}$ **b.** $\sqrt[5]{(-3)^5}$ **c.** $\sqrt{(x+2)^2}$ **d.** $\sqrt[3]{(a+b)^3}$ **e.** $\sqrt{y^4}$

Solution:

a. $\sqrt[4]{(-3)^4} = |-3| = 3$ Because this is an *even*-indexed root, absolute value bars are necessary to make the answer positive.

b. $\sqrt[5]{(-3)^5} = -3$ This is an *odd*-indexed root, so absolute value bars are not necessary.

c. $\sqrt{(x+2)^2} = |x+2|$ Because this is an *even*-indexed root, absolute value bars are necessary. The sign of the quantity $x + 2$ is unknown; however, $|x + 2| \geq 0$ regardless of the value of x.

d. $\sqrt[3]{(a+b)^3} = a + b$ This is an *odd*-indexed root, so absolute value bars are not necessary.

e. $\sqrt{y^4} = \sqrt{(y^2)^2}$

$= |y^2|$ Because this is an even-indexed root, use absolute value bars.

$= y^2$ However, because y^2 is nonnegative, the absolute value bars are not necessary.

Skill Practice Simplify the expressions.

19. $\sqrt{(-4)^2}$ **20.** $\sqrt[3]{(-4)^3}$ **21.** $\sqrt{(y+9)^2}$ **22.** $\sqrt[3]{(t+1)^3}$ **23.** $\sqrt[4]{v^8}$

If n is an even integer, then $\sqrt[n]{a^n} = |a|$; however, if the variable a is assumed to be *nonnegative*, then the absolute value bars may be dropped. That is, $\sqrt[n]{a^n} = a$ provided $a \geq 0$. In many examples and exercises, we will make the assumption that the variables within a radical expression are positive real numbers. In such a case, the absolute value bars are not needed to evaluate $\sqrt[n]{a^n}$.

Take a minute to examine the following patterns associated with perfect squares and perfect cubes. In general, any expression raised to an even power is a perfect square. An expression raised to a power that is a multiple of 3 is a perfect cube.

Perfect Squares	Perfect Cubes
$(x^1)^2 = x^2$	$(x^1)^3 = x^3$
$(x^2)^2 = x^4$	$(x^2)^3 = x^6$
$(x^3)^2 = x^6$	$(x^3)^3 = x^9$
$(x^4)^2 = x^8$	$(x^4)^3 = x^{12}$

These patterns may be extended to higher powers.

Answers

19. 4 **20.** -4 **21.** $|y+9|$
22. $t+1$ **23.** v^2

Example 6 **Simplifying *n*th Roots**

Simplify the expressions. Assume that all variables are positive real numbers.

a. $\sqrt{y^8}$ **b.** $\sqrt[3]{27a^3}$ **c.** $\sqrt[5]{\dfrac{a^5}{b^5}}$ **d.** $-\sqrt[4]{\dfrac{81x^4y^8}{16}}$

Solution:

a. $\sqrt{y^8} = \sqrt{(y^4)^2}$
$= y^4$

b. $\sqrt[3]{27a^3} = \sqrt[3]{(3a)^3}$
$= 3a$

c. $\sqrt[5]{\dfrac{a^5}{b^5}} = \sqrt[5]{\left(\dfrac{a}{b}\right)^5}$
$= \dfrac{a}{b}$

d. $-\sqrt[4]{\dfrac{81x^4y^8}{16}} = -\sqrt[4]{\left(\dfrac{3xy^2}{2}\right)^4}$
$= -\dfrac{3xy^2}{2}$

TIP: In Example 6, the variables are assumed to represent positive numbers. Therefore, absolute value bars are not necessary in the simplified form.

Skill Practice Simplify the expressions. Assume all variables represent positive real numbers.

24. $\sqrt{t^6}$ **25.** $\sqrt[3]{64y^{12}}$ **26.** $\sqrt[4]{\dfrac{x^4}{y^4}}$ **27.** $-\sqrt[5]{\dfrac{32a^5}{b^{10}}}$

4. Pythagorean Theorem

Given a right triangle with legs of lengths a and b and hypotenuse of length c, the **Pythagorean theorem** can be stated as $a^2 + b^2 = c^2$. See Figure 6-1.

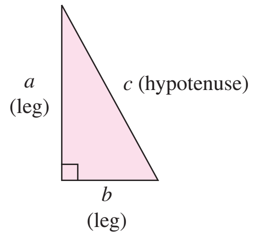
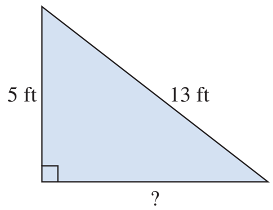


Figure 6-1

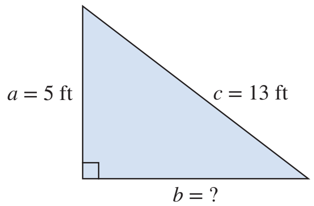
Example 7 **Applying the Pythagorean Theorem**

Use the Pythagorean theorem and the definition of the principal square root to find the length of the unknown side.



Solution:

Label the sides of the triangle.

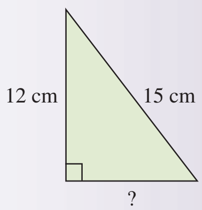


$a^2 + b^2 = c^2$	Apply the Pythagorean theorem.
$(5)^2 + b^2 = (13)^2$	
$25 + b^2 = 169$	Simplify.
$b^2 = 169 - 25$	Isolate b^2.
$b^2 = 144$	By definition, b must be one of the square roots of 144. Because b represents the length of a side of a triangle, choose the positive square root of 144.
$b = 12$	

The third side is 12 ft long.

Skill Practice

28. Use the Pythagorean theorem and the definition of the principal square root to find the length of the unknown side of the right triangle.



Answers

24. t^3 **25.** $4y^4$ **26.** $\dfrac{x}{y}$

27. $-\dfrac{2a}{b^2}$ **28.** 9 cm

Example 8 Applying the Pythagorean Theorem

Two boats leave a dock at 12:00 noon. One travels due north at 6 mph, and the other travels due east at 8 mph (Figure 6-2). How far apart are the two boats after 2 hr?

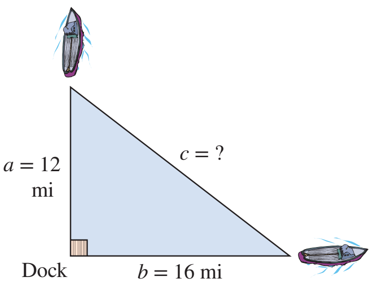


Figure 6-2

Solution:

The boat traveling north travels a distance of (6 mph)(2 hr) = 12 mi. The boat traveling east travels a distance of (8 mph)(2 hr) = 16 mi. The course of the boats forms a right triangle where the hypotenuse represents the distance between them.

$a^2 + b^2 = c^2$	
$(12)^2 + (16)^2 = c^2$	Apply the Pythagorean theorem.
$144 + 256 = c^2$	Simplify.
$400 = c^2$	
$\sqrt{400} = c$	By definition, c must be one of the square roots of 400. Choose the positive square root of 400 to
$20 = c$	represent the distance between the two boats.

The boats are 20 mi apart.

Skill Practice

29. Two cars leave from the same place at the same time. One travels west at 40 mph, and the other travels north at 30 mph. How far apart are they after 2 hr?

5. Radical Functions

If n is an integer greater than 1, then a function written in the form $f(x) = \sqrt[n]{x}$ is called a **radical function**. Note that if n is an even integer, then the function will be a real number only if the radicand is nonnegative. Therefore, the domain is restricted to nonnegative real numbers, or equivalently, $[0, \infty)$. If n is an odd integer, then the domain is all real numbers.

FOR REVIEW

Recall that the domain of a function is the set of real numbers that will produce a real number when input into the function.

Example 9 Determining the Domain of Radical Functions

For each function, write the domain in interval notation.

a. $g(t) = \sqrt[4]{t-2}$ **b.** $h(a) = \sqrt[3]{a-3}$ **c.** $k(x) = \sqrt{3-5x} + 2$

Solution:

a. $g(t) = \sqrt[4]{t-2}$	The index is even. The radicand must be nonnegative.
$t - 2 \geq 0$	Set the radicand greater than or equal to zero.
$t \geq 2$	Solve for t.

The domain is $[2, \infty)$.

b. $h(a) = \sqrt[3]{a-3}$	The index is odd; therefore, the domain is all real numbers.

The domain is $(-\infty, \infty)$.

Answer

29. 100 mi

c. $k(x) = \sqrt{3-5x} + 2$ The index is even; therefore, the radicand must be nonnegative.

$3 - 5x \geq 0$ Set the radicand greater than or equal to zero.

$-5x \geq -3$ Solve for x.

$\dfrac{-5x}{-5} \leq \dfrac{-3}{-5}$ Reverse the inequality sign.

$x \leq \dfrac{3}{5}$

The domain is $\left(-\infty, \frac{3}{5}\right]$.

Skill Practice For each function, write the domain in interval notation.

30. $f(x) = \sqrt{x+5}$ **31.** $g(t) = \sqrt[3]{t-9}$ **32.** $h(a) = \sqrt{1-2a}$

FOR REVIEW

Recall that the square root of a negative real number is not a real number. For example, $\sqrt{-64}$ is not a real number.

The cube root of a negative real number is a real number. For example, $\sqrt[3]{-64} = -4$.

Example 10 Graphing a Radical Function

Given $f(x) = \sqrt{3-x}$

a. Write the domain of f in interval notation.

b. Graph f by making a table of ordered pairs.

Solution:

a. $f(x) = \sqrt{3-x}$

$3 - x \geq 0$ The index is even. The radicand must be greater than or equal to zero.

$-x \geq -3$

$x \leq 3$ Reverse the inequality sign.

The domain is $(-\infty, 3]$.

b. Create a table of ordered pairs where x values are taken to be less than or equal to 3.

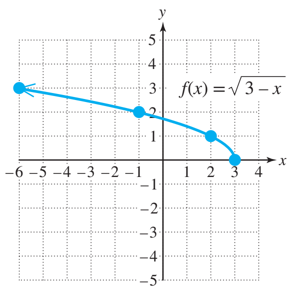
x	$f(x)$
3	0
2	1
−1	2
−6	3

$f(3) = \sqrt{3-3} = \sqrt{0} = 0$

$f(2) = \sqrt{3-2} = \sqrt{1} = 1$

$f(-1) = \sqrt{3-(-1)} = \sqrt{4} = 2$

$f(-6) = \sqrt{3-(-6)} = \sqrt{9} = 3$



FOR REVIEW

When using interval notation, a parenthesis, (or), indicates that an endpoint is not included in the set. A square bracket, [or], indicates that an endpoint is included in the set.

Skill Practice

33. Given $f(x) = \sqrt{x+4}$

a. Write the domain of f in interval notation.

b. Graph f by making a table of ordered pairs.

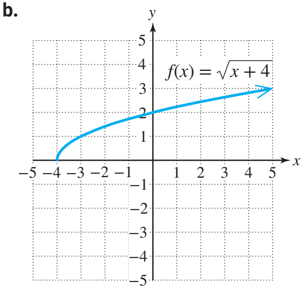
Answers

30. $[-5, \infty)$ **31.** $(-\infty, \infty)$

32. $\left(-\infty, \frac{1}{2}\right]$

33. a. $[-4, \infty)$

b.



Section 6.1 Activity

A.1. **a.** b is a **square root** of a if $b^{\square} =$ ____.

b. 3 is a square root of 9 because $3^{\square} =$ ____.

c. –3 is a square root of 9 because $(-3)^{\square} =$ ____.

d. Parts (b) and (c) illustrate that there are two square roots of a positive real number. One square root is positive and the other is negative. The **principal square root** of a positive real number a is denoted by ________. By definition, a principal square root is (choose one: nonnegative/negative).

e. Evaluate $\sqrt{9}$.

A.2. **a.** b is an ***n*th root** of a if $b^{\square} =$ ____.

b. –4 is a cube root of –64 because $(-4)^{\square} =$ ____.

c. 2 is a fifth root of 32 because $2^{\square} =$ ____.

d. Given $\sqrt[3]{125}$, the index is ____ and the radicand is ____.

e. Evaluate $\sqrt[3]{125}$.

A.3. **a.** Explain why $\sqrt{-4}$ is not a real number.

b. Explain why $\sqrt[4]{-16}$ is not a real number.

c. Evaluate $\sqrt[3]{-8}$ and explain why the answer *is* a real number.

A.4. **a.** The expression $\sqrt{a^2}$ simplifies as $|a|$. Explain why the absolute value bars are needed.

b. The expression $\sqrt[3]{a^3} = a$. Explain why no absolute value bars are needed.

A.5. **a.** The radicand of a root with an (choose one: odd/even) index must be nonnegative.

b. Given $f(x) = \sqrt{x}$, write the domain in interval notation.

c. Given $g(x) = \sqrt{x - 5}$, write the domain in interval notation.

d. Given $h(x) = \sqrt{5 - x}$, write the domain in interval notation.

A.6. **a.** The radicand of a root with an (choose one: odd/even) index can be positive, negative, or zero.

b. Given $f(x) = \sqrt[3]{x}$, write the domain in interval notation.

c. Given $g(x) = \sqrt[3]{x - 5}$, write the domain in interval notation.

d. Given $h(x) = \sqrt[3]{5 - x}$, write the domain in interval notation.

Section 6.1 Practice Exercises

Prerequisite Review

For Exercises R.1–R.12, simplify the expression.

R.1. 10^2

R.2. 8^2

R.3. $(0.4)^2$

R.4. $(0.13)^2$

R.5. $\left(-\frac{3}{5}\right)^3$

R.6. $\left(-\frac{1}{10}\right)^5$

R.7. $\left(-\frac{1}{2}\right)^6$

R.8. $\left(-\frac{1}{9}\right)^2$

R.9. $(x^7)^2$

R.10. $(t^{10})^2$

R.11. $(3t^4)^4$

R.12. $(3p^2)^3$

Vocabulary and Key Concepts

1. **a.** If $b^2 = a$, then ________ is a square root of ________.

b. Given $a > 0$, the symbol $\sqrt{a}$ denotes the positive or ________ square root of a.

c. b is an nth root of a if ________ = ________.

d. Given the symbol $\sqrt[n]{a}$, the value n is called the ________ and a is called the ________.

e. The symbol $\sqrt[3]{a}$ denotes the ________ root of a.

f. The expression $\sqrt{-4}$ (is/is not) a real number. The expression $-\sqrt{4}$ (is/is not) a real number.

g. The expression $\sqrt[n]{a^n} = |a|$ if n is (even/odd). The expression $\sqrt[n]{a^n} = a$ if n is (even/odd).

h. Given a right triangle with legs a and b and hypotenuse c, the ________ theorem is stated as $a^2 + b^2 =$ ________.

i. In interval notation, the domain of $f(x) = \sqrt{x}$ is ________, whereas the domain of $g(x) = \sqrt[3]{x}$ is ________.

j. Which of the following values of x are *not* in the domain of $h(x) = \sqrt{x+3}$: $x = -5, x = -4, x = -3, x = -2, x = -1, x = 0$?

Concept 1: Definition of a Square Root

2. Simplify the expression $\sqrt[3]{8}$. Explain how you can check your answer.

3. **a.** Find the square roots of 64. **(See Example 1.)**

b. Find $\sqrt{64}$.

c. Explain the difference between the answers in part (a) and part (b).

4. **a.** Find the square roots of 121.

b. Find $\sqrt{121}$.

c. Explain the difference between the answers in part (a) and part (b).

5. **a.** What is the principal square root of 81?

b. What is the negative square root of 81?

6. **a.** What is the principal square root of 100?

b. What is the negative square root of 100?

7. Using the definition of a square root, explain why $\sqrt{-36}$ is not a real number.

For Exercises 8–19, evaluate the roots without using a calculator. Identify those that are not real numbers. **(See Examples 2–3.)**

8. $\sqrt{25}$ **9.** $\sqrt{49}$ **10.** $-\sqrt{25}$ **11.** $-\sqrt{49}$

12. $\sqrt{-25}$ **13.** $\sqrt{-49}$ **14.** $\sqrt{\dfrac{100}{121}}$ **15.** $\sqrt{\dfrac{64}{9}}$

16. $\sqrt{0.64}$ **17.** $\sqrt{0.81}$ **18.** $-\sqrt{0.0144}$ **19.** $-\sqrt{0.16}$

Concept 2: Definition of an nth Root

20. Using the definition of an nth root, explain why $\sqrt[4]{-16}$ is not a real number.

For Exercises 21–38, evaluate the roots without using a calculator. Identify those that are not real numbers. **(See Example 4.)**

21. **a.** $\sqrt{64}$ **b.** $\sqrt[3]{64}$ **c.** $-\sqrt{64}$

d. $-\sqrt[3]{64}$ **e.** $\sqrt{-64}$ **f.** $\sqrt[3]{-64}$

22. **a.** $\sqrt{16}$ **b.** $\sqrt[4]{16}$ **c.** $-\sqrt{16}$
d. $-\sqrt[4]{16}$ **e.** $\sqrt{-16}$ **f.** $\sqrt[4]{-16}$

23. $\sqrt[3]{-27}$ **24.** $\sqrt[3]{-125}$ **25.** $\sqrt[3]{\frac{1}{8}}$ **26.** $\sqrt[5]{\frac{1}{32}}$

27. $\sqrt[5]{32}$ **28.** $\sqrt[4]{1}$ **29.** $\sqrt[3]{-\frac{125}{64}}$ **30.** $\sqrt[3]{-\frac{8}{27}}$

31. $\sqrt[4]{-1}$ **32.** $\sqrt[6]{-1}$ **33.** $\sqrt[6]{1{,}000{,}000}$ **34.** $\sqrt[4]{10{,}000}$

35. $-\sqrt[3]{0.008}$ **36.** $-\sqrt[4]{0.0016}$ **37.** $\sqrt[4]{0.0625}$ **38.** $\sqrt[3]{0.064}$

Concept 3: Roots of Variable Expressions

For Exercises 39–58, simplify the radical expressions. **(See Example 5.)**

39. $\sqrt{a^2}$ **40.** $\sqrt[4]{a^4}$ **41.** $\sqrt[3]{a^3}$ **42.** $\sqrt[5]{a^5}$

43. $\sqrt[6]{a^6}$ **44.** $\sqrt[7]{a^7}$ **45.** $\sqrt{(x+1)^2}$ **46.** $\sqrt[3]{(y+3)^3}$

47. $\sqrt{x^2y^4}$ **48.** $\sqrt[3]{(u+v)^3}$ **49.** $-\sqrt[3]{\frac{x^3}{y^3}}$, $y \neq 0$ **50.** $\sqrt[4]{\frac{a^4}{b^8}}$, $b \neq 0$

51. $\frac{2}{\sqrt[4]{x^4}}$, $x \neq 0$ **52.** $\sqrt{(-5)^2}$ **53.** $\sqrt[3]{(-92)^3}$ **54.** $\sqrt[6]{(50)^6}$

55. $\sqrt[10]{(-2)^{10}}$ **56.** $\sqrt[5]{(-2)^5}$ **57.** $\sqrt[7]{(-923)^7}$ **58.** $\sqrt[6]{(-417)^6}$

For Exercises 59–74, simplify the expressions. Assume all variables are positive real numbers. **(See Example 6.)**

59. $\sqrt{y^8}$ **60.** $\sqrt{x^4}$ **61.** $\sqrt{\frac{a^6}{b^2}}$ **62.** $\sqrt{\frac{w^2}{z^4}}$

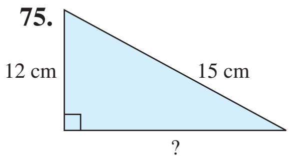
63. $-\sqrt{\frac{25}{q^2}}$ **64.** $-\sqrt{\frac{p^6}{81}}$ **65.** $\sqrt{9x^2y^4z^2}$ **66.** $\sqrt{4a^4b^2c^6}$

67. $\sqrt{\frac{h^2k^4}{16}}$ **68.** $\sqrt{\frac{4x^2}{y^8}}$ **69.** $-\sqrt[3]{\frac{t^3}{27}}$ **70.** $\sqrt[4]{\frac{16}{w^4}}$

71. $\sqrt[5]{32y^{10}}$ **72.** $\sqrt[3]{64x^6y^3}$ **73.** $\sqrt[6]{64p^{12}q^{18}}$ **74.** $\sqrt[4]{16r^{12}s^8}$

Concept 4: Pythagorean Theorem

For Exercises 75–78, find the length of the third side of each triangle by using the Pythagorean theorem. **(See Example 7.)**

75.


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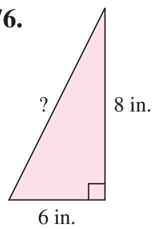

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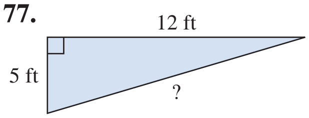

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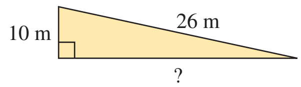


For Exercises 79–82, use the Pythagorean theorem.

79. Roberto and Sherona began running from the same place at the same time. They ran along two different paths that formed right angles with each other. Roberto ran 4 mi and stopped, while Sherona ran 3 mi and stopped. How far apart were they when they stopped? **(See Example 8.)**

80. Leine and Laura began hiking from their campground. Laura headed south while Leine headed east. Laura walked 12 mi and Leine walked 5 mi. How far apart were they when they stopped walking?

81. Two mountain bikers take off from the same place at the same time. One travels north at 4 mph, and the other travels east at 3 mph. How far apart are they after 5 hr?

82. Professor Ortiz leaves campus on her bike, heading west at 12 ft/sec. Professor Wilson leaves campus at the same time and walks south at 5 ft/sec. How far apart are they after 40 sec?

Concept 5: Radical Functions

For Exercises 83–86, evaluate the function for the given values of x. Then write the domain of the function in interval notation. **(See Example 9.)**

83. $h(x) = \sqrt{x-2}$
- **a.** $h(0)$
- **b.** $h(1)$
- **c.** $h(2)$
- **d.** $h(3)$
- **e.** $h(6)$

84. $k(x) = \sqrt{x+1}$
- **a.** $k(-3)$
- **b.** $k(-2)$
- **c.** $k(-1)$
- **d.** $k(0)$
- **e.** $k(3)$

85. $g(x) = \sqrt[3]{x-2}$
- **a.** $g(-6)$
- **b.** $g(1)$
- **c.** $g(2)$
- **d.** $g(3)$

86. $f(x) = \sqrt[3]{x+1}$
- **a.** $f(-9)$
- **b.** $f(-2)$
- **c.** $f(0)$
- **d.** $f(7)$

For each function defined in Exercises 87–94, write the domain in interval notation. **(See Example 9.)**

87. $f(x) = \sqrt{5-2x}$

88. $g(x) = \sqrt{3-4x}$

89. $k(x) = \sqrt[3]{4x-7}$

90. $R(x) = \sqrt[3]{x+1}$

91. $M(x) = \sqrt{x-5} + 3$

92. $N(x) = \sqrt{x+3} - 1$

93. $F(x) = \sqrt[3]{x+7} - 2$

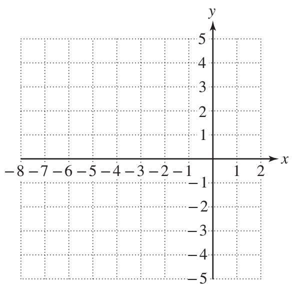
94. $G(x) = \sqrt[3]{x-10} + 4$

For Exercises 95–102,

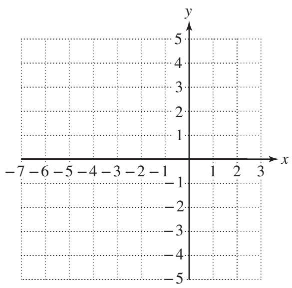
a. Write the domain of f in interval notation.

b. Graph f by making a table of ordered pairs. **(See Example 10.)**

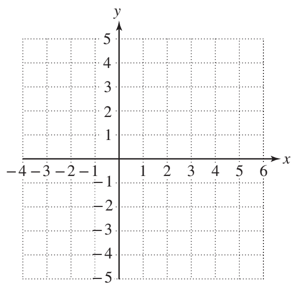
95. $f(x) = \sqrt{1-x}$



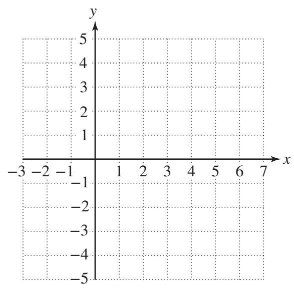
96. $f(x) = \sqrt{2-x}$



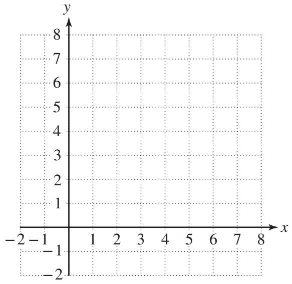
97. $f(x) = \sqrt{x+3}$



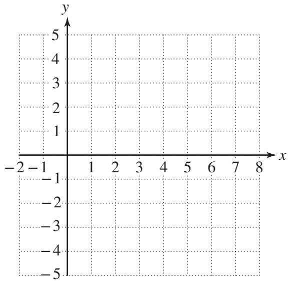
98. $f(x) = \sqrt{x+1}$



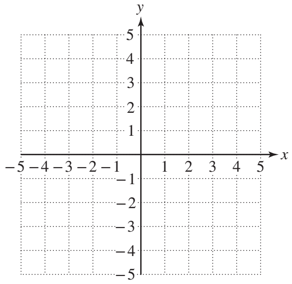
99. $f(x) = \sqrt{x} + 2$



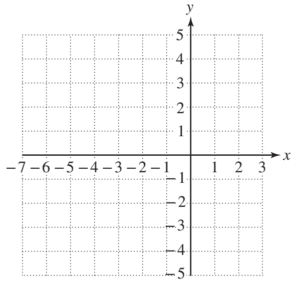
100. $f(x) = \sqrt{x} - 1$



101. $f(x) = \sqrt[3]{x - 1}$



102. $f(x) = \sqrt[3]{x + 2}$



Mixed Exercises

For Exercises 103–106, write the English phrase as an algebraic expression.

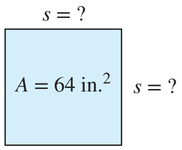
103. The sum of q and the square of p

104. The product of 11 and the cube root of x

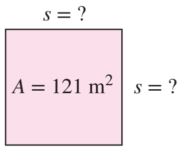
105. The quotient of 6 and the cube root of x

106. The difference of y and the principal square root of x

107. If a square has an area of 64 in.^{2}, then what are the lengths of the sides?



108. If a square has an area of 121 m^2, then what are the lengths of the sides?



Technology Connections

For Exercises 109–116, use a calculator to evaluate the expressions to four decimal places.

109. $\sqrt{69}$

110. $\sqrt{5798}$

111. $2 + \sqrt[3]{5}$

112. $3 - 2\sqrt[4]{10}$

113. $7\sqrt[4]{25}$

114. $-3\sqrt[3]{9}$

115. $\dfrac{3 - \sqrt{19}}{11}$

116. $\dfrac{5 + 2\sqrt{15}}{12}$

117. Graph $h(x) = \sqrt{x - 2}$. Use the graph to confirm the domain found in Exercise 83.

118. Graph $k(x) = \sqrt{x + 1}$. Use the graph to confirm the domain found in Exercise 84.

119. Graph $g(x) = \sqrt[3]{x - 2}$. Use the graph to confirm the domain found in Exercise 85.

120. Graph $f(x) = \sqrt[3]{x + 1}$. Use the graph to confirm the domain found in Exercise 86.

Rational Exponents

Section 6.2

Concepts

1. Definition of $a^{1/n}$ and $a^{m/n}$

In this section, the properties for simplifying expressions with integer exponents are expanded to include expressions with rational exponents. We begin by defining expressions of the form $a^{1/n}$.

Definition of $a^{1/n}$

Let a be a real number, and let n be an integer such that $n > 1$. If $\sqrt[n]{a}$ is a real number, then

$$a^{1/n} = \sqrt[n]{a}$$

Example 1 Evaluating Expressions of the Form $a^{1/n}$

Convert the expression to radical form and simplify, if possible.

a. $(-8)^{1/3}$ **b.** $81^{1/4}$ **c.** $-100^{1/2}$ **d.** $(-100)^{1/2}$ **e.** $16^{-1/2}$

Solution:

a. $(-8)^{1/3} = \sqrt[3]{-8} = -2$

b. $81^{1/4} = \sqrt[4]{81} = 3$

c. $-100^{1/2} = -1 \cdot 100^{1/2}$ The exponent applies only to the base of 100.

$= -1\sqrt{100}$

$= -10$

d. $(-100)^{1/2}$ is not a real number because $\sqrt{-100}$ is not a real number.

e. $16^{-1/2} = \dfrac{1}{16^{1/2}}$ Write the expression with a positive exponent. Recall that $b^{-n} = \dfrac{1}{b^n}$.

$= \dfrac{1}{\sqrt{16}}$

$= \dfrac{1}{4}$

Skill Practice Convert the expression to radical form and simplify, if possible.

1. $(-64)^{1/3}$ **2.** $16^{1/4}$ **3.** $-36^{1/2}$ **4.** $(-36)^{1/2}$ **5.** $64^{-1/3}$

If $\sqrt[n]{a}$ is a real number, then we can define an expression of the form $a^{m/n}$ in such a way that the multiplication property of exponents still holds true. For example:

$$16^{3/4} \nearrow (16^{1/4})^3 = (\sqrt[4]{16})^3 = (2)^3 = 8$$

$$16^{3/4} \searrow (16^3)^{1/4} = \sqrt[4]{16^3} = \sqrt[4]{4096} = 8$$

Answers

1. -4 **2.** 2 **3.** -6

4. Not a real number **5.** $\dfrac{1}{4}$

Definition of $a^{m/n}$

Let a be a real number, and let m and n be positive integers such that m and n share no common factors and $n > 1$. If $\sqrt[n]{a}$ is a real number, then

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \qquad \text{and} \qquad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

The rational exponent in the expression $a^{m/n}$ is essentially performing two operations. The numerator of the exponent raises the base to the mth power. The denominator takes the nth root.

Example 2 Evaluating Expressions of the Form $a^{m/n}$

Convert each expression to radical form and simplify.

a. $8^{2/3}$ **b.** $100^{5/2}$ **c.** $\left(\frac{1}{25}\right)^{3/2}$ **d.** $4^{-3/2}$ **e.** $(-81)^{3/4}$

Solution:

a. $8^{2/3} = (\sqrt[3]{8})^2$ Take the cube root of 8 and square the result.

$= (2)^2$ Simplify.

$= 4$

b. $100^{5/2} = (\sqrt{100})^5$ Take the square root of 100 and raise the result to the fifth power.

$= (10)^5$ Simplify.

$= 100{,}000$

c. $\left(\frac{1}{25}\right)^{3/2} = \left(\sqrt{\frac{1}{25}}\right)^3$ Take the square root of $\frac{1}{25}$ and cube the result.

$= \left(\frac{1}{5}\right)^3$ Simplify.

$= \frac{1}{125}$

d. $4^{-3/2} = \left(\frac{1}{4}\right)^{3/2} = \frac{1}{4^{3/2}}$ Write the expression with positive exponents.

$= \frac{1}{(\sqrt{4})^3}$ Take the square root of 4 and cube the result.

$= \frac{1}{2^3}$ Simplify.

$= \frac{1}{8}$

e. $(-81)^{3/4}$ is not a real number because $\sqrt[4]{-81}$ is not a real number.

Skill Practice Convert each expression to radical form and simplify.

6. $9^{3/2}$ **7.** $8^{5/3}$ **8.** $\left(\frac{1}{27}\right)^{4/3}$ **9.** $32^{-4/5}$ **10.** $(-4)^{3/2}$

2. Converting Between Rational Exponents and Radical Notation

Example 3 Using Radical Notation and Rational Exponents

Convert each expression to radical notation. Assume all variables represent positive real numbers.

a. $a^{3/5}$ **b.** $5^{1/3}x^{2/3}$ **c.** $3y^{1/4}$ **d.** $9z^{-3/4}$

Solution:

a. $a^{3/5} = \sqrt[5]{a^3}$ or $(\sqrt[5]{a})^3$

b. $5^{1/3}x^{2/3} = (5x^2)^{1/3} = \sqrt[3]{5x^2}$

c. $3y^{1/4} = 3\sqrt[4]{y}$ Note that the coefficient 3 is not raised to the $\frac{1}{4}$ power.

d. $9z^{-3/4} = 9 \cdot \dfrac{1}{z^{3/4}} = \dfrac{9}{\sqrt[4]{z^3}}$ Note that the coefficient 9 has an implied exponent of 1, not $-\frac{3}{4}$.

Skill Practice Convert each expression to radical notation. Assume all variables represent positive real numbers.

11. $t^{4/5}$ **12.** $2^{1/4}y^{3/4}$ **13.** $10p^{1/2}$ **14.** $11q^{-2/3}$

Example 4 Using Radical Notation and Rational Exponents

Convert each expression to an equivalent expression by using rational exponents. Assume that all variables represent positive real numbers.

a. $\sqrt[4]{b^3}$ **b.** $\sqrt{7a}$ **c.** $7\sqrt{a}$

Solution:

a. $\sqrt[4]{b^3} = b^{3/4}$ **b.** $\sqrt{7a} = (7a)^{1/2}$ **c.** $7\sqrt{a} = 7a^{1/2}$

Skill Practice Convert to an equivalent expression using rational exponents. Assume all variables represent positive real numbers.

15. $\sqrt[3]{x^2}$ **16.** $\sqrt{5y}$ **17.** $5\sqrt{y}$

3. Properties of Rational Exponents

The properties and definitions for simplifying expressions with integer exponents also apply to rational exponents.

Answers

6. 27 **7.** 32 **8.** $\frac{1}{81}$

9. $\frac{1}{16}$ **10.** Not a real number

11. $\sqrt[5]{t^4}$ **12.** $\sqrt[4]{2y^3}$ **13.** $10\sqrt{p}$

14. $\dfrac{11}{\sqrt[3]{q^2}}$ **15.** $x^{2/3}$ **16.** $(5y)^{1/2}$

17. $5y^{1/2}$

Definitions and Properties of Exponents

Let a and b be nonzero real numbers. Let m and n be rational numbers such that a^m, a^n, and b^m are real numbers.

Description	Property	Example
1. Multiplying like bases	$a^m a^n = a^{m+n}$	$x^{1/3}x^{4/3} = x^{5/3}$
2. Dividing like bases	$\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^{3/5}}{x^{1/5}} = x^{2/5}$
3. The power rule	$(a^m)^n = a^{mn}$	$(2^{1/3})^{1/2} = 2^{1/6}$
4. Power of a product	$(ab)^m = a^m b^m$	$(9y)^{1/2} = 9^{1/2}y^{1/2} = 3y^{1/2}$
5. Power of a quotient	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{4}{25}\right)^{1/2} = \dfrac{4^{1/2}}{25^{1/2}} = \dfrac{2}{5}$

Description	Definition	Example
1. Negative exponents	$a^{-m} = \left(\dfrac{1}{a}\right)^m = \dfrac{1}{a^m}$	$(8)^{-1/3} = \left(\dfrac{1}{8}\right)^{1/3} = \dfrac{1}{2}$
2. Zero exponent	$a^0 = 1$	$5^0 = 1$

Example 5 Simplifying Expressions With Rational Exponents

Use the properties of exponents to simplify the expressions. Assume all variables represent positive real numbers.

a. $y^{2/5}y^{3/5}$ **b.** $\dfrac{6a^{-1/2}}{a^{3/2}}$ **c.** $\left(\dfrac{s^{1/2}t^{1/3}}{w^{3/4}}\right)^4$

Solution:

a. $y^{2/5}y^{3/5} = y^{(2/5)+(3/5)}$ Multiply like bases by adding exponents.

$= y^{5/5}$ Simplify.

$= y$

b. $\dfrac{6a^{-1/2}}{a^{3/2}}$

$= 6a^{(-1/2)-(3/2)}$ Divide like bases by subtracting exponents.

$= 6a^{-2}$ Simplify: $-\dfrac{1}{2} - \left(\dfrac{3}{2}\right) = -\dfrac{4}{2} = -2$

$= \dfrac{6}{a^2}$ Simplify the negative exponent.

c. $\left(\dfrac{s^{1/2}t^{1/3}}{w^{3/4}}\right)^4 = \dfrac{s^{(1/2)\cdot 4}t^{(1/3)\cdot 4}}{w^{(3/4)\cdot 4}}$ Apply the power rule. Multiply exponents.

$= \dfrac{s^2t^{4/3}}{w^3}$ Simplify.

Skill Practice Use the properties of exponents to simplify the expressions. Assume all variables represent positive real numbers.

18. $x^{1/2} \cdot x^{3/4}$ **19.** $\dfrac{4k^{-2/3}}{k^{1/3}}$ **20.** $\left(\dfrac{a^{1/3}b^{1/2}}{c^{5/8}}\right)^6$

Answers

18. $x^{5/4}$ **19.** $\dfrac{4}{k}$ **20.** $\dfrac{a^2b^3}{c^{15/4}}$

4. Applications Involving Rational Exponents

Example 6 Applying Rational Exponents

Suppose P dollars in principal is invested in an account that earns interest annually. If after t years the investment grows to A dollars, then the annual rate of return r on the investment is given by

$$r = \left(\frac{A}{P}\right)^{1/t} - 1$$

Find the annual rate of return on \$5000 which grew to \$6894.21 after 6 years.

Solution:

$$r = \left(\frac{A}{P}\right)^{1/t} - 1$$

$$= \left(\frac{6894.21}{5000}\right)^{1/6} - 1 \qquad \text{where } A = \$6894.21,\ P = \$5000, \text{ and } t = 6$$

$$\approx 0.055 \text{ or } 5.5\%$$

The annual rate of return is 5.5%.

Skill Practice

21. The formula for the radius of a sphere is

$$r = \left(\frac{3V}{4\pi}\right)^{1/3}$$

where V is the volume. Find the radius of a sphere whose volume is 113.04 in.^3 (Use 3.14 for π.)

Answer

21. 3 in.

Section 6.2 Activity

A.1. **a.** Write the expression $a^{1/n}$ in radical notation.
b. Write $9^{1/2}$ as a radical and simplify.
c. Write $(-125)^{1/3}$ as a radical and simplify.
d. Explain why $(-16)^{1/2}$ is undefined.

A.2. **a.** Write the expression $a^{m/n}$ in radical notation.
b. Write $125^{2/3}$ as a radical and simplify.
c. Write $16^{3/4}$ as a radical and simplify.

The properties of integer exponents can be extended to rational exponents, provided the corresponding radicals are real numbers. For Exercises A.3–A.9, simplify the expressions. Apply your knowledge of the properties of integer exponents to similar expressions involving rational exponents. Assume that all variables represent positive real numbers.

A.3. **a.** $t^2 \cdot t^5$ **b.** $t^{1/2} \cdot t^{5/2}$

A.4. **a.** $\dfrac{y^9}{y^5}$ **b.** $\dfrac{y^{6/5}}{y^{1/3}}$

A.5. **a.** $(z^3)^5$ **b.** $(z^{2/3})^6$

A.6. **a.** n^{-6} **b.** $n^{-1/2}$

A.7. **a.** $\dfrac{2}{m^{-3}}$ **b.** $\dfrac{2}{m^{-1/3}}$

A.8. **a.** $(a^2b^4)^2$ **b.** $(a^2b^4)^{1/2}$

A.9. **a.** $\left(\dfrac{c^3}{d^4}\right)^3$ **b.** $\left(\dfrac{c^{1/3}}{d^{1/4}}\right)^{12}$

Section 6.2 Practice Exercises

Prerequisite Review

For Exercises R.1–R.10, simplify the expression.

R.1. $a^{-7} \cdot a^{12}$ **R.2.** $q^{-3} \cdot q^{10}$ **R.3.** $\dfrac{y^{17}}{y^{20}}$ **R.4.** $\dfrac{n^2}{n^{12}}$

R.5. $(n^4)^{-2}$ **R.6.** $(d^{-4})^3$ **R.7.** $\left(\dfrac{1}{5}a^2b^3\right)^3$ **R.8.** $\left(\dfrac{2}{3}x^4y\right)^2$

R.9. $w^4 \cdot w^{-4}$ **R.10.** $p^{-5} \cdot p^5$

For Exercises R.11–R.12,
a. Identify the index. **b.** Identify the radicand. **c.** Simplify.

R.11. $\sqrt[4]{16}$ **R.12.** $\sqrt[3]{125}$

For Exercises R.13–R.16, simplify the radicals

R.13. $(\sqrt[4]{81})^3$ **R.14.** $(\sqrt[4]{16})^3$ **R.15.** $\sqrt[3]{(t+2)^3}$ **R.16.** $\sqrt[5]{(a^2+b)^5}$

For the exercises in this set, assume that all variables represent positive real numbers unless otherwise stated.

Vocabulary and Key Concepts

1. **a.** If n is an integer greater than 1, then radical notation for $a^{1/n}$ is ___________.

   b. Assume that m and n are positive integers that share no common factors and $n > 1$. If $\sqrt[n]{a}$ exists, then in radical notation $a^{m/n} =$ ___________.

2. **a.** The radical notation for $x^{-1/2}$ is ___________.

   b. $8^{1/3} =$ ___________ and $8^{-1/3} =$ ___________.

Concept 1: Definition of $a^{1/n}$ and $a^{m/n}$

For Exercises 3–6, write the expression in radical form.

3. **a.** $49^{1/2}$ **b.** $-49^{1/2}$ **c.** $49^{-1/2}$

4. **a.** $121^{1/2}$ **b.** $-121^{1/2}$ **c.** $121^{-1/2}$

5. **a.** $(-64)^{1/3}$ **b.** $-64^{1/3}$ **c.** $64^{-1/3}$

6. **a.** $(-343)^{1/3}$ **b.** $-343^{1/3}$ **c.** $343^{-1/3}$

For Exercises 7–18, convert the expressions to radical form and simplify. **(See Example 1.)**

7. $144^{1/2}$ **8.** $16^{1/4}$ **9.** $-144^{1/2}$ **10.** $-16^{1/4}$

11. $(-144)^{1/2}$ **12.** $(-16)^{1/4}$ **13.** $(-64)^{1/3}$ **14.** $(-32)^{1/5}$

15. $25^{-1/2}$ **16.** $27^{-1/3}$ **17.** $-49^{-1/2}$ **18.** $-64^{-1/2}$

19. Explain how to interpret the expression $a^{m/n}$ as a radical.

20. Explain why $(\sqrt[3]{8})^4$ is easier to evaluate than $\sqrt[3]{8^4}$.

For Exercises 21–24, simplify the expression, if possible. **(See Example 2.)**

21. **a.** $16^{3/4}$ **b.** $-16^{3/4}$ **c.** $(-16)^{3/4}$ **d.** $16^{-3/4}$ **e.** $-16^{-3/4}$ **f.** $(-16)^{-3/4}$

22. **a.** $81^{3/4}$ **b.** $-81^{3/4}$ **c.** $(-81)^{3/4}$ **d.** $81^{-3/4}$ **e.** $-81^{-3/4}$ **f.** $(-81)^{-3/4}$

23. **a.** $25^{3/2}$ **b.** $-25^{3/2}$ **c.** $(-25)^{3/2}$ **d.** $25^{-3/2}$ **e.** $-25^{-3/2}$ **f.** $(-25)^{-3/2}$

24. **a.** $4^{3/2}$ **b.** $-4^{3/2}$ **c.** $(-4)^{3/2}$ **d.** $4^{-3/2}$ **e.** $-4^{-3/2}$ **f.** $(-4)^{-3/2}$

For Exercises 25–48, simplify the expression. **(See Example 2.)**

25. $64^{-3/2}$ **26.** $81^{-3/2}$ **27.** $243^{3/5}$ **28.** $1^{5/3}$

29. $-27^{-4/3}$ **30.** $-16^{-5/4}$ **31.** $\left(\frac{100}{9}\right)^{-3/2}$ **32.** $\left(\frac{49}{100}\right)^{-1/2}$

33. $(-4)^{-3/2}$ **34.** $(-49)^{-3/2}$ **35.** $(-8)^{1/3}$ **36.** $(-9)^{1/2}$

37. $-8^{1/3}$ **38.** $-9^{1/2}$ **39.** $\frac{1}{36^{-1/2}}$ **40.** $\frac{1}{16^{-1/2}}$

41. $\frac{1}{1000^{-1/3}}$ **42.** $\frac{1}{81^{-3/4}}$ **43.** $\left(\frac{1}{8}\right)^{2/3}+\left(\frac{1}{4}\right)^{1/2}$ **44.** $\left(\frac{1}{8}\right)^{-2/3}+\left(\frac{1}{4}\right)^{-1/2}$

45. $\left(\frac{1}{16}\right)^{-3/4}-\left(\frac{1}{49}\right)^{-1/2}$ **46.** $\left(\frac{1}{16}\right)^{1/4}-\left(\frac{1}{49}\right)^{1/2}$ **47.** $\left(\frac{1}{4}\right)^{1/2}+\left(\frac{1}{64}\right)^{-1/3}$ **48.** $\left(\frac{1}{36}\right)^{1/2}+\left(\frac{1}{64}\right)^{-5/6}$

Concept 2: Converting Between Rational Exponents and Radical Notation

For Exercises 49–56, convert each expression to radical notation. **(See Example 3.)**

49. $q^{2/3}$ **50.** $t^{3/5}$ **51.** $6y^{3/4}$ **52.** $8b^{4/9}$

53. $x^{2/3}y^{1/3}$ **54.** $c^{2/5}d^{3/5}$ **55.** $6r^{-2/5}$ **56.** $7x^{-3/4}$

For Exercises 57–64, write each expression by using rational exponents rather than radical notation. **(See Example 4.)**

57. $\sqrt[3]{x}$ **58.** $\sqrt[4]{a}$ **59.** $10\sqrt{b}$ **60.** $-2\sqrt[3]{t}$

61. $\sqrt[3]{y^2}$ **62.** $\sqrt[6]{z^5}$ **63.** $\sqrt[4]{a^2b^3}$ **64.** $\sqrt{abc}$

Concept 3: Properties of Rational Exponents

For Exercises 65–88, simplify the expressions by using the properties of rational exponents. Write the final answers using positive exponents only. **(See Example 5.)**

65. $x^{1/4}x^{-5/4}$

66. $2^{2/3}2^{-5/3}$

67. $\dfrac{p^{5/3}}{p^{2/3}}$

68. $\dfrac{q^{5/4}}{q^{1/4}}$

69. $(y^{1/5})^{10}$

70. $(x^{1/2})^8$

71. $6^{-1/5}6^{3/5}$

72. $a^{-1/3}a^{2/3}$

73. $\dfrac{4t^{-1/3}}{t^{4/3}}$

74. $\dfrac{5s^{-1/3}}{s^{5/3}}$

75. $(a^{1/3}a^{1/4})^{12}$

76. $(x^{2/3}x^{1/2})^6$

77. $(5a^2c^{-1/2}d^{1/2})^2$

78. $(2x^{-1/3}y^2z^{5/3})^3$

79. $\left(\dfrac{x^{-2/3}}{y^{-3/4}}\right)^{12}$

80. $\left(\dfrac{m^{-1/4}}{n^{-1/2}}\right)^{-4}$

81. $\left(\dfrac{16w^{-2}z}{2wz^{-8}}\right)^{1/3}$

82. $\left(\dfrac{50p^{-1}q}{2pq^{-3}}\right)^{1/2}$

83. $(25x^2y^4z^6)^{1/2}$

84. $(8a^6b^3c^9)^{2/3}$

85. $(x^2y^{-1/3})^6(x^{1/2}yz^{2/3})^2$

86. $(a^{-1/3}b^{1/2})^4(a^{-1/2}b^{3/5})^{10}$

87. $\left(\dfrac{x^{3m}y^{2m}}{z^{5m}}\right)^{1/m}$

88. $\left(\dfrac{a^{4n}b^{3n}}{c^n}\right)^{1/n}$

Concept 4: Applications Involving Rational Exponents

89. If P dollars in principal grows to A dollars after t years with annual interest, then the interest rate r is given by $r = \left(\dfrac{A}{P}\right)^{1/t} - 1$. **(See Example 6.)**

a. In one account, \$10,000 grows to \$16,802 after 5 years. Compute the interest rate. Round your answer to a tenth of a percent.

b. In another account, \$10,000 grows to \$18,000 after 7 years. Compute the interest rate. Round your answer to a tenth of a percent.

c. Which account produced a higher average yearly return?

90. If the area A of a square is known, then the length of its sides, s, can be computed by the formula $s = A^{1/2}$.

a. Compute the length of the sides of a square having an area of 100 in.^{2}

b. Compute the length of the sides of a square having an area of 72 in.^{2} Round your answer to the nearest 0.1 in.

91. The radius r of a sphere of volume V is given by $r = \left(\dfrac{3V}{4\pi}\right)^{1/3}$. Find the radius of a sphere having a volume of 85 in.^{3} Round your answer to the nearest 0.1 in.

92. Is $(a + b)^{1/2}$ the same as $a^{1/2} + b^{1/2}$? If not, give a counterexample.

Expanding Your Skills

For Exercises 93–104, write the expression using rational exponents. Then simplify and convert back to radical notation.

Example: $\sqrt[15]{x^{10}} \xrightarrow[\text{exponents}]{\text{Rational}} x^{10/15} \xrightarrow{\text{Simplify}} x^{2/3} \xrightarrow[\text{notation}]{\text{Radical}} \sqrt[3]{x^2}$

93. $\sqrt[6]{y^3}$ **94.** $\sqrt[4]{w^2}$ **95.** $\sqrt[12]{z^3}$ **96.** $\sqrt[18]{t^3}$

97. $\sqrt[9]{x^6}$ **98.** $\sqrt[12]{p^9}$ **99.** $\sqrt[6]{x^3y^6}$ **100.** $\sqrt[8]{m^2p^8}$

101. $\sqrt{16x^8y^6}$ **102.** $\sqrt{81a^{12}b^{20}}$ **103.** $\sqrt[3]{8x^3y^2z}$ **104.** $\sqrt[3]{64m^2n^3p}$

For Exercises 105–108, write the expression as a single radical.

105. $\sqrt{\sqrt[3]{x}}$ **106.** $\sqrt[3]{\sqrt{x}}$ **107.** $\sqrt[5]{\sqrt[3]{w}}$ **108.** $\sqrt[3]{\sqrt[4]{w}}$

For Exercises 109–116, use a calculator to approximate the expressions. Round to four decimal places, if necessary.

109. $9^{1/2}$ **110.** $125^{-1/3}$ **111.** $50^{-1/4}$ **112.** $(172)^{3/5}$

113. $\sqrt[3]{5^2}$ **114.** $\sqrt[4]{6^3}$ **115.** $\sqrt{10^3}$ **116.** $\sqrt[3]{16}$

Simplifying Radical Expressions

Section 6.3

Concepts

1. Multiplication Property of Radicals
2. Simplifying Radicals by Using the Multiplication Property of Radicals
3. Simplifying Radicals by Using the Order of Operations

1. Multiplication Property of Radicals

You may have already noticed certain properties of radicals involving a product or quotient.

Multiplication Property of Radicals

Let a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real. Then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

The multiplication property of radicals follows from the property of rational exponents.

$$\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n}b^{1/n} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

The multiplication property of radicals indicates that a product within a radicand can be written as a product of radicals, provided the roots are real numbers. For example:

$$\sqrt{144} = \sqrt{16} \cdot \sqrt{9}$$

The reverse process is also true. A product of radicals can be written as a single radical provided the roots are real numbers and they have the same indices.

$$\sqrt{3} \cdot \sqrt{12} = \sqrt{36}$$

2. Simplifying Radicals by Using the Multiplication Property of Radicals

In algebra, it is customary to simplify radical expressions.

Simplified Form of a Radical

Consider any radical expression where the radicand is written as a product of prime factors. The expression is in *simplified form* if all the following conditions are met:

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand does not contain a fraction.
3. There are no radicals in the denominator of a fraction.

For example, the following radicals are not simplified.

1. The expression $\sqrt[3]{x^5}$ fails condition 1.
2. The expression $\sqrt{\frac{1}{4}}$ fails condition 2.
3. The expression $\frac{1}{\sqrt[3]{8}}$ fails condition 3.

The expressions $\sqrt{x^2}$, $\sqrt{x^4}$, $\sqrt{x^6}$, and $\sqrt{x^8}$ are not simplified because they fail condition 1 (the exponents are not less than the index). However, each radicand is a perfect square and is easily simplified for $x \geq 0$.

$$\sqrt{x^2} = x$$
$$\sqrt{x^4} = x^2$$
$$\sqrt{x^6} = x^3$$
$$\sqrt{x^8} = x^4$$

However, how is an expression such as $\sqrt{x^9}$ simplified? This and many other radical expressions are simplified by using the multiplication property of radicals. We demonstrate the process in Examples 1–4.

Example 1 Using the Multiplication Property to Simplify a Radical Expression

Simplify the expression assuming that $x \geq 0$. $\sqrt{x^9}$

Solution:

The expression $\sqrt{x^9}$ is equivalent to $\sqrt{x^8 \cdot x}$.

$\sqrt{x^9} = \sqrt{x^8 \cdot x}$

$= \sqrt{x^8} \cdot \sqrt{x}$ Apply the multiplication property of radicals.

Note that x^8 is a perfect square because $x^8 = (x^4)^2$.

$= x^4\sqrt{x}$ Simplify.

Skill Practice Simplify the expression. Assume $a \geq 0$.

1. $\sqrt{a^{11}}$

Answer

1. $a^5\sqrt{a}$

In Example 1, the expression x^9 is not a perfect square. Therefore, to simplify $\sqrt{x^9}$, it was necessary to write the expression as the product of the largest perfect square and a remaining or "left-over" factor: $\sqrt{x^9} = \sqrt{x^8 \cdot x}$. This process also applies to simplifying nth roots, as shown in Example 2.

Example 2 **Using the Multiplication Property to Simplify Radical Expressions**

Simplify each expression. Assume all variables represent positive real numbers.

a. $\sqrt[4]{b^7}$ **b.** $\sqrt[3]{w^7z^9}$

Solution:

The goal is to rewrite each radicand as the product of the greatest perfect square (perfect cube, perfect fourth power, and so on) and a leftover factor.

a. $\sqrt[4]{b^7} = \sqrt[4]{b^4 \cdot b^3}$	b^4 is the greatest perfect fourth power in the radicand.
$= \sqrt[4]{b^4} \cdot \sqrt[4]{b^3}$	Apply the multiplication property of radicals.
$= b\sqrt[4]{b^3}$	Simplify.
b. $\sqrt[3]{w^7z^9} = \sqrt[3]{(w^6z^9) \cdot (w)}$	w^6z^9 is the greatest perfect cube in the radicand.
$= \sqrt[3]{w^6z^9} \cdot \sqrt[3]{w}$	Apply the multiplication property of radicals.
$= w^2z^3\sqrt[3]{w}$	Simplify.

Skill Practice Simplify the expressions. Assume all variables represent positive real numbers.

2. $\sqrt[4]{v^{25}}$ **3.** $\sqrt[3]{p^{17}q^{10}}$

Each expression in Example 2 involves a radicand that is a product of variable factors. If a numerical factor is present, sometimes it is necessary to factor the coefficient before simplifying the radical.

Example 3 **Using the Multiplication Property to Simplify a Radical**

Simplify the expression. $\sqrt{56}$

Solution:

$\sqrt{56} = \sqrt{2^3 \cdot 7}$	Factor the radicand.
$= \sqrt{(2^2) \cdot (2 \cdot 7)}$	2^2 is the greatest perfect square in the radicand.
$= \sqrt{2^2} \cdot \sqrt{2 \cdot 7}$	Apply the multiplication property of radicals.
$= 2\sqrt{14}$	Simplify.

$$\begin{array}{r|l} 2 & \underline{56} \\ 2 & \underline{28} \\ 2 & \underline{14} \\ & 7 \end{array}$$

TIP: It may be easier to visualize the greatest perfect square factor within the radicand as follows:

$$\sqrt{56} = \sqrt{4 \cdot 14}$$
$$= \sqrt{4} \cdot \sqrt{14}$$
$$= 2\sqrt{14}$$

Skill Practice Simplify.

4. $\sqrt{24}$

Answers

2. $v^6\sqrt[4]{v}$ **3.** $p^5q^3\sqrt[3]{p^2q}$ **4.** $2\sqrt{6}$

Example 4 Using the Multiplication Property to Simplify Radicals

Simplify. $6\sqrt{50}$

Solution:

$$6\sqrt{50} = 6\sqrt{5^2 \cdot 2} \quad \text{Factor the radicand.}$$
$$= 6 \cdot \sqrt{5^2} \cdot \sqrt{2} \quad \text{Apply the multiplication property of radicals.}$$
$$= 6 \cdot 5 \cdot \sqrt{2} \quad \text{Simplify.}$$
$$= 30\sqrt{2} \quad \text{Simplify.}$$

TIP: The radical can also be simplified as:

$$6\sqrt{50} = 6\sqrt{25 \cdot 2}$$
$$= 6\sqrt{25} \cdot \sqrt{2}$$
$$= 6 \cdot 5\sqrt{2}$$
$$= 30\sqrt{2}$$

Skill Practice Simplify.

5. $5\sqrt{18}$

Example 5 Using the Multiplication Property to Simplify Radicals

Simplify the expression. Assume that x, y, and z represent positive real numbers.

$\sqrt[3]{40x^3y^5z^7}$

Solution:

$$\sqrt[3]{40x^3y^5z^7}$$
$$= \sqrt[3]{2^3 5x^3y^5z^7} \quad \text{Factor the radicand.}$$
$$= \sqrt[3]{(2^3x^3y^3z^6) \cdot (5y^2z)} \quad 2^3x^3y^3z^6 \text{ is the greatest perfect cube.}$$
$$= \sqrt[3]{2^3x^3y^3z^6} \cdot \sqrt[3]{5y^2z} \quad \text{Apply the multiplication property of radicals.}$$
$$= 2xyz^2\sqrt[3]{5y^2z} \quad \text{Simplify.}$$

$$\begin{array}{r|l} 2 & 40 \\ 2 & 20 \\ 2 & 10 \\ & 5 \end{array}$$

TIP: In Example 5, the numerical coefficient within the radicand can be written $8 \cdot 5$ because 8 is the greatest perfect cube factor of 40:

$$\sqrt[3]{40x^3y^5z^7} = \sqrt[3]{8 \cdot 5x^3y^5z^7}$$
$$= \sqrt[3]{(8x^3y^3z^6)(5y^2z)}$$
$$= \sqrt[3]{8x^3y^3z^6} \cdot \sqrt[3]{5y^2z}$$
$$= 2xyz^2 \cdot \sqrt[3]{5y^2z}$$

Skill Practice Simplify. Assume that $a > 0$ and $b > 0$.

6. $\sqrt[4]{32a^{10}b^{19}}$

3. Simplifying Radicals by Using the Order of Operations

Often a radical can be simplified by applying the order of operations. In Example 6, the first step will be to simplify the expression within the radicand.

Example 6 Using the Order of Operations to Simplify Radicals

Use the order of operations to simplify the expressions. Assume $a > 0$.

a. $\sqrt{\dfrac{a^7}{a^3}}$ **b.** $\sqrt[3]{\dfrac{3}{81}}$

Answers

5. $15\sqrt{2}$ **6.** $2a^2b^4\sqrt[4]{2a^2b^3}$

Solution:

a. $\sqrt{\dfrac{a^7}{a^3}}$ The radicand contains a fraction. However, the fraction can be reduced to lowest terms.

$= \sqrt{a^4}$

$= a^2$ Simplify the radical.

b. $\sqrt[3]{\dfrac{3}{81}}$ The radical contains a fraction that can be simplified.

$= \sqrt[3]{\dfrac{1}{27}}$ Reduce to lowest terms.

$= \dfrac{1}{3}$ Simplify.

Skill Practice Use the order of operations to simplify the expressions. Assume $v > 0$.

7. $\sqrt{\dfrac{v^{21}}{v^5}}$ **8.** $\sqrt[5]{\dfrac{64}{2}}$

Example 7 Simplifying a Radical Expression

Simplify. $\dfrac{7\sqrt{50}}{10}$

Solution:

$\dfrac{7\sqrt{50}}{10} = \dfrac{7\sqrt{25 \cdot 2}}{10}$ 25 is the greatest perfect square in the radicand.

$= \dfrac{7 \cdot 5\sqrt{2}}{10}$ Simplify the radical.

$= \dfrac{7 \cdot \overset{1}{\cancel{5}}\sqrt{2}}{\underset{2}{\cancel{10}}}$ Simplify the fraction to lowest terms.

$= \dfrac{7\sqrt{2}}{2}$

Avoiding Mistakes

The expression $\frac{7\sqrt{2}}{2}$ cannot be simplified further because one factor of 2 is in the radicand and the other is outside the radical.

Skill Practice Simplify.

9. $\dfrac{2\sqrt{300}}{30}$

Answers

7. v^8 **8.** 2 **9.** $\dfrac{2\sqrt{3}}{3}$

Section 6.3 Activity

A.1. **a.** If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real numbers, then $\sqrt[n]{a} \cdot \sqrt[n]{b} =$ ________.

b. The multiplication property of radicals can be used in "reverse" to simplify radical expressions. For example, $\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} =$ ________.

c. Simplify $\sqrt{50}$ by factoring 50 such that one factor is a perfect square.)

$$\sqrt{50} = \sqrt{\square \cdot \square} = \sqrt{\square} \cdot \sqrt{\square} = \underline{\qquad\qquad}$$

d. Simplify $\sqrt[3]{x^7}$ by factoring x^7 as the product of the largest perfect cube and a "leftover" factor.)

$$\sqrt[3]{x^7} = \sqrt[3]{\square \cdot \square} = \sqrt[3]{\square} \cdot \sqrt[3]{\square} = \underline{\qquad\qquad}$$

A.2. Simplify $\sqrt{48x^5y^6}$ by following these steps.

a. Factor 48 into a product of prime factors.

b. Write the radical with the radicand factored as a product of prime factors.

c. Write the radicand as the product of the largest perfect square factor and a "leftover" factor.

$$\sqrt{48x^5y^6} = \sqrt{\underbrace{\left(\square \cdot x^{\square}y^{\square}\right)}_{\text{These factors should all be perfect squares.}} \cdot (\underline{\qquad})}$$

d. Simplify the radical.

A.3. Simplify $\sqrt{\dfrac{125x^5}{5x}}$ by first simplifying the radicand.

A.4. Simplify $\dfrac{5\sqrt{72}}{3}$. (*Hint:* Simplify the radical first.)

Section 6.3 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, find the prime factorization of the given number.

R.1. 882

R.2. 189

R.3. Which of the following are perfect squares?

4, 6, 9, 18, 25, 40, 100, 200, y^2, y^3, y^4, y^5, y^{10}, y^{15}

R.4. Which of the following are perfect cubes?

1, 4, 8, 16, 27, 30, 64, a^2, a^3, a^4, a^5, a^6, a^{10}, a^{15}

R.5. Which of the following are perfect fourth powers?

1, 4, 8, 16, 25, 40, 81, b^2, b^3, b^4, b^5, b^8, b^{10}

R.6. Which of the follow are perfect fifth powers?

1, 20, 25, 32, 100, 243, $10x$, x^5, x^{10}, x^{24}, x^{30}, x^{35}

For Exercises R.7–R.18, simplify the expression if possible. Assume that all variable expressions represent positive real numbers.

R.7. $\sqrt{\dfrac{81}{64}}$	**R.8.** $\sqrt{\dfrac{121}{16}}$	**R.9.** $\sqrt[3]{125}$	**R.10.** $\sqrt[4]{81}$
R.11. $-\sqrt{36}$	**R.12.** $\sqrt{-36}$	**R.13.** $-\sqrt[3]{64}$	**R.14.** $\sqrt[3]{-64}$
R.15. $\sqrt{b^{10}}$	**R.16.** $\sqrt[4]{t^{12}}$	**R.17.** $\sqrt{9c^4d^6}$	**R.18.** $\sqrt{16m^2n^8}$

For the exercises in this set, assume that all variables represent positive real numbers unless otherwise stated.

Vocabulary and Key Concepts

1. **a.** The multiplication property of radicals indicates that if $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{ab} =$ ________ $\cdot$ ________.

   b. Explain why the following radical is not in simplified form. $\sqrt{x^3}$

   c. The radical expression $\sqrt[3]{x^{10}}$ (is/is not) in simplified form.

   d. The radical expression $\sqrt{18}$ simplifies to $\_\_\sqrt{2}$.

   e. To simplify the radical expression $\sqrt[3]{t^{14}}$ the radicand is rewritten as $\sqrt[3]{\_\_ \cdot t^2}$.

2. On a calculator, $\sqrt{2}$ is given as 1.414213562. Is this decimal number the exact value of $\sqrt{2}$?

Concept 2: Simplifying Radicals by Using the Multiplication Property of Radicals

For Exercises 3–8, fill in the boxes inside the radicals. Then simplify the result.

3. $\sqrt{x^{11}} = \sqrt{x^{\square} \cdot x} = \sqrt{x^{\square}} \cdot \sqrt{x} =$ _____
4. $\sqrt{y^5} = \sqrt{y^{\square} \cdot y} = \sqrt{y^{\square}} \cdot \sqrt{y} =$ _____
5. $\sqrt[3]{x^{11}} = \sqrt[3]{x^{\square} \cdot x^2} = \sqrt[3]{x^{\square}} \cdot \sqrt[3]{x^2} =$ _____
6. $\sqrt[3]{y^5} = \sqrt[3]{y^{\square} \cdot y^2} = \sqrt[3]{y^{\square}} \cdot \sqrt[3]{y^2} =$ _____
7. $\sqrt[4]{x^{11}} = \sqrt[4]{x^{\square} \cdot x^3} = \sqrt[4]{x^{\square}} \cdot \sqrt[4]{x^3} =$ _____
8. $\sqrt[4]{y^5} = \sqrt[4]{y^{\square} \cdot y} = \sqrt[4]{y^{\square}} \cdot \sqrt[4]{y} =$ _____

For Exercises 9–32, simplify the radicals. **(See Examples 1–5.)**

9. $\sqrt{x^5}$	**10.** $\sqrt{p^{15}}$	**11.** $\sqrt[3]{q^7}$	**12.** $\sqrt[3]{r^{17}}$
13. $\sqrt{a^5b^4}$	**14.** $\sqrt{c^9d^6}$	**15.** $-\sqrt[4]{x^8y^{13}}$	**16.** $-\sqrt[4]{p^{16}q^{17}}$
17. $\sqrt{28}$	**18.** $\sqrt{63}$	**19.** $\sqrt{20}$	**20.** $\sqrt{50}$
21. $5\sqrt{18}$	**22.** $2\sqrt{24}$	**23.** $\sqrt[3]{54}$	**24.** $\sqrt[3]{250}$
25. $\sqrt{25ab^3}$	**26.** $\sqrt{64m^5n^{20}}$	**27.** $\sqrt[3]{40x^7}$	**28.** $\sqrt[3]{81y^{17}}$
29. $\sqrt[3]{-16x^6yz^3}$	**30.** $\sqrt[3]{-192a^6bc^2}$	**31.** $\sqrt[4]{80w^4z^7}$	**32.** $\sqrt[4]{32p^8qr^5}$

Concept 3: Simplifying Radicals by Using the Order of Operations

For Exercises 33–44, simplify the radical expressions. **(See Examples 6–7.)**

33. $\sqrt{\dfrac{x^3}{x}}$ **34.** $\sqrt{\dfrac{y^5}{y}}$ **35.** $\sqrt{\dfrac{p^7}{p^3}}$ **36.** $\sqrt{\dfrac{q^{11}}{q^5}}$

37. $\sqrt{\dfrac{50}{2}}$ **38.** $\sqrt{\dfrac{98}{2}}$ **39.** $\sqrt[3]{\dfrac{3}{24}}$ **40.** $\sqrt[3]{\dfrac{2}{250}}$

41. $\dfrac{5\sqrt[3]{16}}{6}$ **42.** $\dfrac{7\sqrt{18}}{9}$ **43.** $\dfrac{5\sqrt[3]{72}}{12}$ **44.** $\dfrac{3\sqrt[3]{250}}{10}$

Mixed Exercises

For Exercises 45–72, simplify the radical expressions.

45. $\sqrt{80}$ **46.** $\sqrt{108}$ **47.** $-6\sqrt{75}$ **48.** $-8\sqrt{8}$

49. $\sqrt{25x^4y^3}$ **50.** $\sqrt{125p^3q^2}$ **51.** $\sqrt[3]{27x^2y^3z^4}$ **52.** $\sqrt[3]{108a^3bc^2}$

53. $\sqrt{\dfrac{12w^5}{3w}}$ **54.** $\sqrt{\dfrac{64x^9}{4x^3}}$ **55.** $\sqrt{\dfrac{3y^3}{300y^{15}}}$ **56.** $\sqrt{\dfrac{4h}{100h^5}}$

57. $\sqrt[3]{\dfrac{16a^2b}{2a^2b^4}}$ **58.** $\sqrt[3]{\dfrac{-27a^4}{8a}}$ **59.** $\sqrt{2^3a^{14}b^8c^{31}d^{22}}$ **60.** $\sqrt{7^5u^{12}v^{20}w^{65}x^{80}}$

61. $\sqrt[3]{54a^6b^4}$ **62.** $\sqrt[3]{72m^5n^3}$ **63.** $-5a\sqrt{12a^3b^4c}$ **64.** $-7y\sqrt{75xy^5z^6}$

65. $\sqrt[4]{7x^5y}$ **66.** $\sqrt[4]{10cd^7}$ **67.** $\sqrt{54a^4b^2}$ **68.** $\sqrt{48r^6s^2}$

69. $\dfrac{2\sqrt{27}}{3}$ **70.** $\dfrac{7\sqrt{24}}{2}$ **71.** $\dfrac{3\sqrt{125}}{20}$ **72.** $\dfrac{10\sqrt{63}}{12}$

For Exercises 73–76, write a mathematical expression for the English phrase and simplify.

73. The quotient of 1 and the cube root of w^6

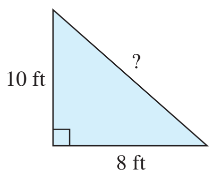
74. The principal square root of the quotient of h^2 and 49

75. The principal square root of the quantity k raised to the third power

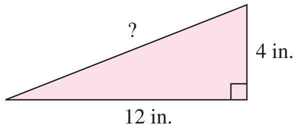
76. The cube root of $2x^4$

For Exercises 77–80, determine the length of the third side of the right triangle. Write the answer as a simplified radical.

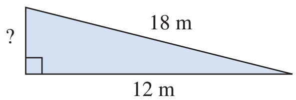
77.



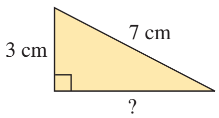
78.



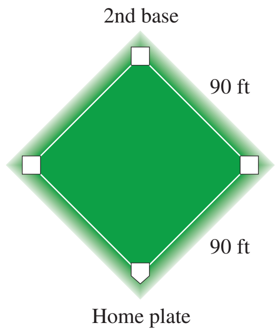
79.



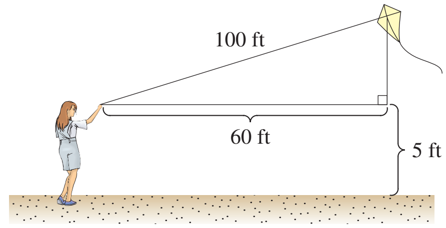
80.



81. On a baseball diamond, the bases are 90 ft apart. Find the exact distance from home plate to second base. Then round to the nearest tenth of a foot.

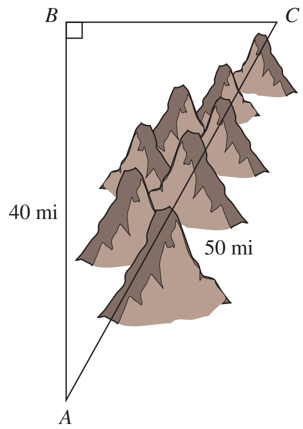


82. Linda is at the beach flying a kite. The kite is directly over a sand castle 60 ft away from Linda. If 100 ft of kite string is out (ignoring any sag in the string), how high is the kite? (Assume that Linda is 5 ft tall.) See figure.

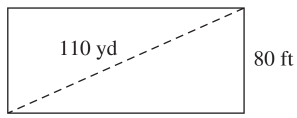


Expanding Your Skills

83. Tom has to travel from town *A* to town *C* across a small mountain range. He can travel one of two routes. He can travel on a four-lane highway from *A* to *B* and then from *B* to *C* at an average speed of 55 mph. Or he can travel on a two-lane road directly from town *A* to town *C*, but his average speed will be only 35 mph. If Tom is in a hurry, which route will take him to town *C* faster?



84. One side of a rectangular pasture is 80 ft in length. The diagonal distance is 110 yd. If fencing costs $3.29 per foot, how much will it cost to fence the pasture?



Addition and Subtraction of Radicals

Section 6.4

Concept

1. Addition and Subtraction of Radicals

1. Addition and Subtraction of Radicals

Definition of *Like* Radicals

Two radical terms are called ***like* radicals** if they have the same index and same radicand.

Like radicals can be added or subtracted by using the distributive property.

Same index

$$3\sqrt{x} + 7\sqrt{x} = \overbrace{(3+7)\sqrt{x}}^{\text{Distributive property}} = 10\sqrt{x}$$

Same radicand

Example 1 Adding and Subtracting Radicals

Add or subtract as indicated.

a. $6\sqrt{11} - 2\sqrt{11}$ **b.** $\sqrt{3} + \sqrt{3}$

Solution:

a. $6\sqrt{11} - 2\sqrt{11}$

$= (6 - 2)\sqrt{11}$ Apply the distributive property.

$= 4\sqrt{11}$ Simplify.

b. $\sqrt{3} + \sqrt{3}$

$= 1\sqrt{3} + 1\sqrt{3}$ Note that $\sqrt{3} = 1\sqrt{3}$.

$= (1 + 1)\sqrt{3}$ Apply the distributive property.

$= 2\sqrt{3}$ Simplify.

FOR REVIEW

When adding and subtracting *like* radicals, the distributive property is used to factor out the radical. Then the coefficients are added or subtracted as indicated.

$$6\sqrt{11} - 2\sqrt{11} = (6 - 2)\sqrt{11}$$
$$= 4\sqrt{11}$$

Avoiding Mistakes

The process of adding *like* radicals with the distributive property is similar to adding *like* terms. The end result is that the numerical coefficients are added and the radical factor is unchanged.

$$\sqrt{3} + \sqrt{3} = 1\sqrt{3} + 1\sqrt{3} = 2\sqrt{3}$$

Be careful: $\sqrt{3} + \sqrt{3} \neq \sqrt{6}$

In general: $\sqrt{x} + \sqrt{y} \neq \sqrt{x + y}$

Skill Practice Add or subtract as indicated.

1. $5\sqrt{6} - 8\sqrt{6}$ **2.** $\sqrt{10} + \sqrt{10}$

Example 2 Adding and Subtracting Radicals

Add or subtract as indicated.

a. $-2\sqrt[3]{ab} + 7\sqrt[3]{ab} - \sqrt[3]{ab}$ **b.** $\frac{1}{4}x\sqrt{3y} - \frac{3}{2}x\sqrt{3y}$

Solution:

a. $-2\sqrt[3]{ab} + 7\sqrt[3]{ab} - \sqrt[3]{ab}$

$= (-2 + 7 - 1)\sqrt[3]{ab}$ Apply the distributive property.

$= 4\sqrt[3]{ab}$ Simplify.

b. $\frac{1}{4}x\sqrt{3y} - \frac{3}{2}x\sqrt{3y}$

$= \left(\frac{1}{4} - \frac{3}{2}\right)x\sqrt{3y}$ Apply the distributive property.

$= \left(\frac{1}{4} - \frac{6}{4}\right)x\sqrt{3y}$ Get a common denominator.

$= -\frac{5}{4}x\sqrt{3y}$ Simplify.

Answers

1. $-3\sqrt{6}$ **2.** $2\sqrt{10}$

Skill Practice Add or subtract as indicated.

3. $5\sqrt[3]{xy} - 3\sqrt[3]{xy} + 7\sqrt[3]{xy}$

4. $\frac{5}{6}y\sqrt{2} + \frac{1}{4}y\sqrt{2}$

Example 3 shows that it is often necessary to simplify radicals before adding or subtracting.

Example 3 Adding and Subtracting Radicals

Simplify the radicals and add or subtract as indicated. Assume all variables represent positive real numbers.

a. $3\sqrt{8} + \sqrt{2}$ **b.** $8\sqrt{x^3y^2} - 3y\sqrt{x^3}$

c. $\sqrt{50x^2y^5} - 13y\sqrt{2x^2y^3} + xy\sqrt{98y^3}$

Solution:

a. $3\sqrt{8} + \sqrt{2}$ The radicals are different. Try simplifying the radicals first.

$= 3 \cdot 2\sqrt{2} + \sqrt{2}$ Simplify: $\sqrt{8} = \sqrt{2^3} = 2\sqrt{2}$

$= 6\sqrt{2} + \sqrt{2}$

$= (6 + 1)\sqrt{2}$ Apply the distributive property.

$= 7\sqrt{2}$ Simplify.

b. $8\sqrt{x^3y^2} - 3y\sqrt{x^3}$ The radicands are different. Simplify the radicals first.

$= 8xy\sqrt{x} - 3xy\sqrt{x}$ Simplify $\sqrt{x^3y^2} = xy\sqrt{x}$ and $\sqrt{x^3} = x\sqrt{x}$.

$= (8 - 3)xy\sqrt{x}$ Apply the distributive property.

$= 5xy\sqrt{x}$ Simplify.

c. $\sqrt{50x^2y^5} - 13y\sqrt{2x^2y^3} + xy\sqrt{98y^3}$ Simplify each radical.

$= 5xy^2\sqrt{2y} - 13xy^2\sqrt{2y} + 7xy^2\sqrt{2y}$

$$\left\{\begin{aligned} \sqrt{50x^2y^5} &= \sqrt{25 \cdot 2x^2y^5} \\ &= 5xy^2\sqrt{2y} \\ -13y\sqrt{2x^2y^3} &= -13xy^2\sqrt{2y} \\ xy\sqrt{98y^3} &= xy\sqrt{49 \cdot 2y^3} \\ &= 7xy^2\sqrt{2y} \end{aligned}\right.$$

$= (5 - 13 + 7)xy^2\sqrt{2y}$ Apply the distributive property.

$= -xy^2\sqrt{2y}$

Skill Practice Simplify the radicals and add or subtract as indicated. Assume all variables represent positive real numbers.

5. $\sqrt{75} + 2\sqrt{3}$

6. $4\sqrt{a^2b} - 6a\sqrt{b}$

7. $-3\sqrt{2y^3} + 5y\sqrt{18y} - 2\sqrt{50y^3}$

Answers

3. $9\sqrt[3]{xy}$ **4.** $\frac{13}{12}y\sqrt{2}$

5. $7\sqrt{3}$ **6.** $-2a\sqrt{b}$

7. $2y\sqrt{2y}$

In some cases, when two radicals are added, the resulting sum is written in factored form. This is demonstrated in Example 4.

Example 4 Adding Radical Expressions

Add the radicals. Assume that $x \geq 0$. $3\sqrt{2x^2} + \sqrt{8}$

Solution:

$3\sqrt{2x^2} + \sqrt{8}$

$= 3x\sqrt{2} + 2\sqrt{2}$ Simplify each radical. Notice that the radicands are the same, but the terms are not *like* terms. The first term has a factor of x and the second does not.

$= (3x + 2)\sqrt{2}$ Apply the distributive property. The expression cannot be simplified further because $3x$ and 2 are not *like* terms.

Skill Practice Add the radicals. Assume that $y \geq 0$.

8. $4\sqrt{45} - \sqrt{5y^4}$

Answer

8. $(12 - y^2)\sqrt{5}$

Section 6.4 Activity

A.1. Two radical expressions are *like* radicals if they have the same ________ and same ________.

A.2. Determine whether the radicals are *like* radicals. If they are not *like* radicals, explain why.

a. $\sqrt{7}$ and $\sqrt{3}$

b. $\sqrt[3]{xy}$ and $\sqrt{xy}$

c. $4\sqrt[3]{u}$ and $2\sqrt[3]{u}$

d. $8\sqrt{5x}$ and $\sqrt{5x}$

A.3. Simplify the expressions.

a. $4x + 7x - 2x$

b. $4\sqrt{x} + 7\sqrt{x} - 2\sqrt{x}$

c. $4\sqrt{2} + 7\sqrt{2} - 2\sqrt{2}$

d. Discuss the similarity in the process to simplify each expression in parts (a)–(c).

A.4. Given the expression $16\sqrt{3t} - 12\sqrt{3t} + \sqrt{3t}$,

a. Identify the coefficients of each term.

b. Simplify the expression.

A.5. Simplify the expression by following these steps.

$3y\sqrt{5x^5} - 2xy\sqrt{20x^3} + 2x^2\sqrt{45xy^2}$

a. Simplify $3y\sqrt{5x^5}$.

b. Simplify $-2xy\sqrt{20x^3}$.

c. Simplify $2x^2\sqrt{45xy^2}$.

d. Write the original expression with the radicals in simplified form.

e. Simplify the expression.

A.6. Consider the expression $3\sqrt{7x^2} + 5\sqrt{7}$.

a. Write the expression with the radicals simplified.

b. Add the radicals using the distributive property.

Practice Exercises

Section 6.4

Prerequisite Review

For Exercises R.1–R.2, determine whether the terms are *like* terms.

R.1. **a.** $3y$ and $\frac{1}{4}y$ **b.** $9a$ and $9b$ **c.** $2x^2y$ and $4x^2y$ **d.** $10x$ and 10

R.2. **a.** $-6p$ and -6 **b.** $\frac{3}{4}y^3$ and $6y^3$ **c.** $-11a^4b$ and $4ab^4$ **d.** $2.1x$ and $\frac{2}{5}x$

For Exercises R.3–R.6, combine *like* terms.

R.3. $4y - 6y + y$ **R.4.** $8p - p + 11p$

R.5. $\frac{3}{4}c^2d + \frac{1}{4}c^2d$ **R.6.** $\frac{2}{5}x^3y + \frac{8}{5}x^3y$

For Exercises R.7–R.10, simplify the expression. Assume that all variables represent positive real numbers.

R.7. $\sqrt{27}$ **R.8.** $\sqrt{84}$

R.9. $\sqrt[3]{250x^3y^4}$ **R.10.** $\sqrt[3]{16a^6b^8}$

For the exercises in this set, assume that all variables represent positive real numbers, unless otherwise stated.

Vocabulary and Key Concepts

1. **a.** Two radical terms are called *like* radicals if they have the same __________ and the same __________.

b. The expression $\sqrt{3x} + \sqrt{3x}$ simplifies to __________.

c. The expression $\sqrt{2} + \sqrt{3}$ (can/cannot) be simplified further, whereas the expression $\sqrt{2} \cdot \sqrt{3}$ (can/cannot) be simplified further.

2. **a.** The expression $\sqrt{18}$ simplifies to _____.

b. The expression $\sqrt{2} + \sqrt{18}$ simplifies to _____.

Concept 1: Addition and Subtraction of Radicals

For Exercises 3–8, determine if the radical terms are *like*.

3. $\sqrt{2}$ and $\sqrt[3]{2}$ **4.** $7\sqrt[3]{x}$ and $\sqrt[3]{x}$

5. $\sqrt{2}$ and $3\sqrt{2}$ **6.** $\sqrt[3]{x}$ and $\sqrt[4]{x}$

7. $\sqrt{2}$ and $\sqrt{5}$ **8.** $2\sqrt[4]{x}$ and $x\sqrt[4]{2}$

9. Explain the similarities between the pairs of expressions.

a. $7\sqrt{5} + 4\sqrt{5}$ and $7x + 4x$

b. $-2\sqrt{6} - 9\sqrt{3}$ and $-2x - 9y$

10. Explain the similarities between the pairs of expressions.

a. $-4\sqrt{3} + 5\sqrt{3}$ and $-4z + 5z$

b. $13\sqrt{7} - 18$ and $13a - 18$

For Exercises 11–14, simplify the expressions.

11. **a.** $3x + 5x$ **b.** $3\sqrt{x} + 5\sqrt{x}$

12. **a.** $-11m + 6m$ **b.** $-11\sqrt{m} + 6\sqrt{m}$

13. **a.** $-8t + t$ **b.** $-8\sqrt[3]{t} + \sqrt[3]{t}$

14. **a.** $-c + 7c$ **b.** $-\sqrt[4]{c} + 7\sqrt[4]{c}$

For Exercises 15–32, add or subtract the radical expressions, if possible. **(See Examples 1–2.)**

15. $3\sqrt{5} + 6\sqrt{5}$

16. $5\sqrt{a} + 3\sqrt{a}$

17. $3\sqrt[3]{tw} - 2\sqrt[3]{tw} + \sqrt[3]{tw}$

18. $6\sqrt[3]{7} - 2\sqrt[3]{7} + \sqrt[3]{7}$

19. $6\sqrt{10} - \sqrt{10}$

20. $13\sqrt{11} - \sqrt{11}$

21. $\sqrt[4]{3} + 7\sqrt[4]{3} - \sqrt[4]{14}$

22. $2\sqrt{11} + 3\sqrt{13} + 5\sqrt{11}$

23. $8\sqrt{x} + 2\sqrt{y} - 6\sqrt{x}$

24. $10\sqrt{10} - 8\sqrt{10} + \sqrt{2}$

25. $\sqrt[3]{ab} + a\sqrt[3]{b}$

26. $x\sqrt[4]{y} - y\sqrt[4]{x}$

27. $\sqrt{2t} + \sqrt[3]{2t}$

28. $\sqrt[4]{5c} + \sqrt[3]{5c}$

29. $\frac{5}{6}z\sqrt[3]{6} + \frac{7}{9}z\sqrt[3]{6}$

30. $\frac{3}{4}a\sqrt[4]{b} + \frac{1}{6}a\sqrt[4]{b}$

31. $0.81x\sqrt{y} - 0.11x\sqrt{y}$

32. $7.5\sqrt{pq} - 6.3\sqrt{pq}$

33. Explain the process for adding the two radicals. Then find the sum. $3\sqrt{2} + 7\sqrt{50}$

34. Explain the process for subtracting two radicals. Then find the difference. $\sqrt{12x} - \sqrt{75x}$

For Exercises 35–64, add or subtract the radical expressions as indicated. **(See Examples 3–4.)**

35. $\sqrt{36} + \sqrt{81}$

36. $3\sqrt{80} - 5\sqrt{45}$

37. $2\sqrt{12} + \sqrt{48}$

38. $5\sqrt{32} + 2\sqrt{50}$

39. $4\sqrt{7} + \sqrt{63} - 2\sqrt{28}$

40. $8\sqrt{3} - 2\sqrt{27} + \sqrt{75}$

41. $5\sqrt{18} + \sqrt{27} - 4\sqrt{50}$

42. $7\sqrt{40} - \sqrt{8} + 4\sqrt{50}$

43. $\sqrt[3]{81} - \sqrt[3]{24}$

44. $17\sqrt[3]{81} - 2\sqrt[3]{24}$

45. $3\sqrt{2a} - \sqrt{8a} - \sqrt{72a}$

46. $\sqrt{12t} - \sqrt{27t} + 5\sqrt{3t}$

47. $2s^2\sqrt[3]{s^2t^6} + 3t^2\sqrt[3]{8s^8}$

48. $4\sqrt[3]{x^4} - 2x\sqrt[3]{x}$

49. $7\sqrt[3]{x^4} - x\sqrt[3]{x}$

50. $6\sqrt[3]{y^{10}} - 3y^2\sqrt[3]{y^4}$

51. $5p\sqrt{20p^2} + p^2\sqrt{80}$

52. $2q\sqrt{48q^2} - \sqrt{27q^4}$

53. $\sqrt[3]{a^2b} - \sqrt[3]{8a^2b}$

54. $w\sqrt{80} - 3\sqrt{125w^2}$

55. $5x\sqrt{x} + 6\sqrt{x}$

56. $9y^2\sqrt{2} + 4\sqrt{2}$

57. $\sqrt{50x^2} - 3\sqrt{8}$

58. $\sqrt{9x^3} - \sqrt{25x}$

59. $11\sqrt[3]{54cd^3} - 2\sqrt[3]{2cd^3} + d\sqrt[3]{16c}$

60. $x\sqrt[3]{64x^5y^2} - x^2\sqrt[3]{x^2y^2} + 5\sqrt[3]{x^8y^2}$

61. $\frac{3}{2}ab\sqrt{24a^3} + \frac{4}{3}\sqrt{54a^5b^2} - a^2b\sqrt{150a}$

62. $mn\sqrt{72n} + \frac{2}{3}n\sqrt{8m^2n} - \frac{5}{6}\sqrt{50m^2n^3}$

63. $x\sqrt[3]{16} - 2\sqrt[3]{27x} + \sqrt[3]{54x^3}$

64. $5\sqrt[4]{y^5} - 2y\sqrt[4]{y} + \sqrt[4]{16y^7}$

Mixed Exercises

For Exercises 65–72, answer true or false. If an answer is false, explain why or give a counter example.

65. $\sqrt{x}+\sqrt{y}=\sqrt{x+y}$

66. $\sqrt{x}+\sqrt{x}=2\sqrt{x}$

67. $5\sqrt[3]{x}+2\sqrt[3]{x}=7\sqrt[3]{x}$

68. $6\sqrt{x}+5\sqrt[3]{x}=11\sqrt{x}$

69. $\sqrt{y}+\sqrt{y}=\sqrt{2y}$

70. $\sqrt{c^2+d^2}=c+d$

71. $2w\sqrt{5}+4w\sqrt{5}=6w^2\sqrt{5}$

72. $7x\sqrt{3}-2\sqrt{3}=(7x-2)\sqrt{3}$

For Exercises 73–76, write the English phrase as an algebraic expression. Simplify each expression.

73. The sum of the principal square root of 48 and the principal square root of 12

74. The sum of the cube root of 16 and the cube root of 2

75. The difference of 5 times the cube root of x^6 and the square of x

76. The sum of the cube of y and the principal fourth root of y^{12}

For Exercises 77–80, write an English phrase that translates the mathematical expression. (Answers may vary.)

77. $\sqrt{18}-5^2$

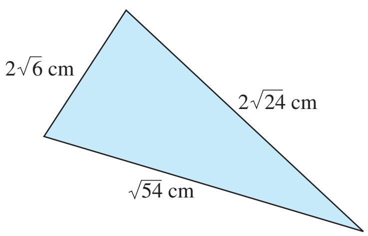
78. $4^3-\sqrt[3]{4}$

79. $\sqrt[4]{x}+y^3$

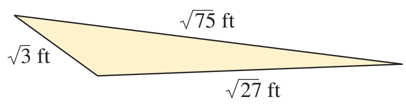
80. $a^4+\sqrt{a}$

For Exercises 81–82, find the exact value of the perimeter, and then approximate the value to one decimal place.

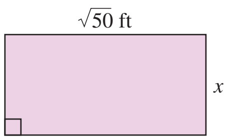
81.



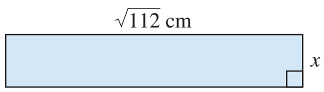
82.



83. The figure has perimeter $14\sqrt{2}$ ft. Find the value of x.



84. The figure has perimeter $12\sqrt{7}$ cm. Find the value of x.

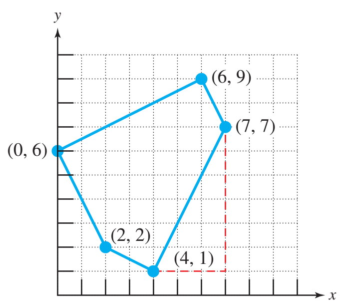


Expanding Your Skills

85. **a.** An irregularly shaped garden is shown in the figure. All distances are expressed in yards. Find the perimeter. (*Hint:* Use the Pythagorean theorem to find the length of each side.) Write the final answer in radical form.

b. Approximate your answer to two decimal places.

c. If edging costs \$1.49 per foot and sales tax is 6%, find the total cost of edging the garden.



Section 6.5 Multiplication of Radicals

Concepts

1. Multiplication Property of Radicals
2. Expressions of the Form $(\sqrt[n]{a})^n$
3. Special Case Products
4. Multiplying Radicals With Different Indices

1. Multiplication Property of Radicals

In this section, we will learn how to multiply radicals by using the multiplication property of radicals.

The Multiplication Property of Radicals

Let a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real. Then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

To multiply two radical expressions, we use the multiplication property of radicals along with the commutative and associative properties of multiplication.

Example 1 Multiplying Radical Expressions

Multiply each expression and simplify the result. Assume all variables represent positive real numbers.

a. $(3\sqrt{2})(5\sqrt{6})$ **b.** $(2x\sqrt{y})(-7\sqrt{xy})$ **c.** $(15c\sqrt[3]{cd})\left(\frac{1}{3}\sqrt[3]{cd^2}\right)$

Solution:

a. $(3\sqrt{2})(5\sqrt{6})$

$= (3 \cdot 5)(\sqrt{2} \cdot \sqrt{6})$	Commutative and associative properties of multiplication
$= 15\sqrt{12}$	Multiplication property of radicals
$= 15\sqrt{2^2 3}$	
$= 15 \cdot 2\sqrt{3}$	Simplify the radical.
$= 30\sqrt{3}$	

b. $(2x\sqrt{y})(-7\sqrt{xy})$

$= (2x)(-7)(\sqrt{y} \cdot \sqrt{xy})$	Commutative and associative properties of multiplication
$= -14x\sqrt{xy^2}$	Multiplication property of radicals
$= -14xy\sqrt{x}$	Simplify the radical.

c. $(15c\sqrt[3]{cd})\left(\frac{1}{3}\sqrt[3]{cd^2}\right)$

$= \left(15c \cdot \frac{1}{3}\right)(\sqrt[3]{cd} \cdot \sqrt[3]{cd^2})$	Commutative and associative properties of multiplication
$= 5c\sqrt[3]{c^2d^3}$	Multiplication property of radicals
$= 5cd\sqrt[3]{c^2}$	Simplify the radical.

Skill Practice Multiply the expressions and simplify the results. Assume all variables represent positive real numbers.

1. $\left(4\sqrt{6}\right)\left(-2\sqrt{3}\right)$
2. $\left(3ab\sqrt{b}\right)\left(-2\sqrt{ab}\right)$
3. $\left(2\sqrt[3]{4ab}\right)\left(5\sqrt[3]{2a^2b}\right)$

When multiplying radical expressions with more than one term, we use the distributive property.

Example 2 Multiplying Radical Expressions

Multiply. $3\sqrt{11}(2+\sqrt{11})$

Solution:

$3\sqrt{11}(2+\sqrt{11})$

$= 3\sqrt{11}\cdot(2) + 3\sqrt{11}\cdot\sqrt{11}$ Apply the distributive property.

$= 6\sqrt{11} + 3\sqrt{11^2}$ Multiplication property of radicals

$= 6\sqrt{11} + 3\cdot 11$ Simplify the radical.

$= 6\sqrt{11} + 33$

Skill Practice Multiply.

4. $5\sqrt{5}(2\sqrt{5}-2)$

Example 3 Multiplying Radical Expressions

Multiply.

a. $(\sqrt{5}+3\sqrt{2})(2\sqrt{5}-\sqrt{2})$ **b.** $(-10a\sqrt{b}+7b)(a\sqrt{b}+2b)$

Solution:

a. $(\sqrt{5}+3\sqrt{2})(2\sqrt{5}-\sqrt{2})$

$= 2\sqrt{5^2} - \sqrt{10} + 6\sqrt{10} - 3\sqrt{2^2}$ Apply the distributive property.

$= 2\cdot 5 + 5\sqrt{10} - 3\cdot 2$ Simplify radicals and combine *like* radicals.

$= 10 + 5\sqrt{10} - 6$

$= 4 + 5\sqrt{10}$ Combine *like* terms.

b. $(-10a\sqrt{b}+7b)(a\sqrt{b}+2b)$

$= -10a^2\sqrt{b^2} - 20ab\sqrt{b} + 7ab\sqrt{b} + 14b^2$ Apply the distributive property.

$= -10a^2b - 13ab\sqrt{b} + 14b^2$ Simplify and combine *like* terms.

FOR REVIEW

Compare the multiplication of radicals to the multiplication of polynomials.

$3(x+2) = 3\cdot x + 3\cdot 2$

$3\sqrt{11}(2+\sqrt{11})$
$= 3\sqrt{11}\cdot 2 + 3\sqrt{11}\cdot\sqrt{11}$

Answers

1. $-24\sqrt{2}$
2. $-6ab^2\sqrt{a}$
3. $20a\sqrt[3]{b^2}$
4. $50 - 10\sqrt{5}$

Skill Practice Multiply.

5. $(2\sqrt{3} - 3\sqrt{10})(\sqrt{3} + 2\sqrt{10})$ **6.** $(x\sqrt{y} + y)(3x\sqrt{y} - 2y)$

Example 4 Multiplying Radical Expressions

Multiply. $(2\sqrt{x} + \sqrt{y})(6 - \sqrt{x} + 8\sqrt{y})$

Solution:

$$(2\sqrt{x} + \sqrt{y})(6 - \sqrt{x} + 8\sqrt{y})$$

$= 12\sqrt{x} - 2\sqrt{x^2} + 16\sqrt{xy} + 6\sqrt{y} - \sqrt{xy} + 8\sqrt{y^2}$ Apply the distributive property.

$= 12\sqrt{x} - 2x + 16\sqrt{xy} + 6\sqrt{y} - \sqrt{xy} + 8y$ Simplify the radicals.

$= 12\sqrt{x} - 2x + 15\sqrt{xy} + 6\sqrt{y} + 8y$ Combine *like* terms.

Skill Practice Multiply.

7. $(\sqrt{t} + 3\sqrt{w})(4 - \sqrt{t} - \sqrt{w})$

2. Expressions of the Form $(\sqrt[n]{a})^n$

The multiplication property of radicals can be used to simplify an expression of the form $(\sqrt{a})^2$, where $a \geq 0$.

$$(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a \qquad \text{where } a \geq 0$$

This logic can be applied to nth roots.

- If $\sqrt[n]{a}$ is a real number, then $(\sqrt[n]{a})^n = a$.

Example 5 Simplifying Radical Expressions

Simplify the expressions. Assume all variables represent positive real numbers.

a. $(\sqrt{11})^2$ **b.** $(\sqrt[5]{z})^5$ **c.** $(\sqrt[3]{pq})^3$

Solution:

a. $(\sqrt{11})^2 = 11$ **b.** $(\sqrt[5]{z})^5 = z$ **c.** $(\sqrt[3]{pq})^3 = pq$

Skill Practice Simplify.

8. $(\sqrt{14})^2$ **9.** $(\sqrt[7]{q})^7$ **10.** $(\sqrt[5]{3z})^5$

Answers

5. $-54 + \sqrt{30}$
6. $3x^2y + xy\sqrt{y} - 2y^2$
7. $4\sqrt{t} - t - 4\sqrt{tw} + 12\sqrt{w} - 3w$
8. 14 **9.** q **10.** $3z$

3. Special Case Products

From Examples 2–4, you may have noticed a similarity between multiplying radical expressions and multiplying polynomials.

Recall that the square of a binomial results in a perfect square trinomial:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

The same patterns occur when squaring a radical expression with two terms.

Example 6 Squaring a Two-Term Radical Expression

Square the radical expressions. Assume all variables represent positive real numbers.

a. $(\sqrt{d}+3)^2$ **b.** $(5\sqrt{y}-\sqrt{2})^2$

Solution:

a. $(\sqrt{d}+3)^2$ — This expression is in the form $(a+b)^2$, where $a=\sqrt{d}$ and $b=3$.

$a^2 + 2ab + b^2$ — Apply the formula

$= (\sqrt{d})^2 + 2(\sqrt{d})(3) + (3)^2$ — $(a+b)^2 = a^2 + 2ab + b^2$.

$= d + 6\sqrt{d} + 9$ — Simplify.

b. $(5\sqrt{y}-\sqrt{2})^2$ — This expression is in the form $(a-b)^2$, where $a = 5\sqrt{y}$ and $b=\sqrt{2}$.

$a^2 - 2ab + b^2$ — Apply the formula

$= (5\sqrt{y})^2 - 2(5\sqrt{y})(\sqrt{2}) + (\sqrt{2})^2$ — $(a-b)^2 = a^2 - 2ab + b^2$.

$= 25y - 10\sqrt{2y} + 2$ — Simplify.

TIP: The product $(\sqrt{d}+3)^2$ can also be found by using the distributive property.

$$(\sqrt{d}+3)^2 = (\sqrt{d}+3)(\sqrt{d}+3)$$

Skill Practice Square the radical expressions. Assume all variables represent positive real numbers.

11. $(\sqrt{a}-5)^2$ **12.** $(4\sqrt{x}+\sqrt{3})^2$

Recall that the product of two conjugate binomials results in a difference of squares.

$$(a+b)(a-b) = a^2 - b^2$$

The same pattern occurs when multiplying two conjugate radical expressions.

Example 7 Multiplying Conjugate Radical Expressions

Multiply the radical expressions. Assume all variables represent positive real numbers.

a. $(\sqrt{3}+2)(\sqrt{3}-2)$ **b.** $\left(\frac{1}{3}\sqrt{s}-\frac{3}{4}\sqrt{t}\right)\left(\frac{1}{3}\sqrt{s}+\frac{3}{4}\sqrt{t}\right)$

Answers

11. $a - 10\sqrt{a} + 25$

12. $16x + 8\sqrt{3x} + 3$

TIP: The product $(\sqrt{3}+2)(\sqrt{3}-2)$ can also be found by using the distributive property.

$$(\sqrt{3}+2)(\sqrt{3}-2)$$

Solution:

a. $(\sqrt{3}+2)(\sqrt{3}-2)$ The expression is in the form $(a+b)(a-b)$, where $a=\sqrt{3}$ and $b=2$.

$$a^2 - b^2$$

$= (\sqrt{3})^2 - (2)^2$ Apply the formula $(a+b)(a-b)=a^2-b^2$.

$= 3 - 4$ Simplify.

$= -1$

b. $\left(\frac{1}{3}\sqrt{s}-\frac{3}{4}\sqrt{t}\right)\left(\frac{1}{3}\sqrt{s}+\frac{3}{4}\sqrt{t}\right)$ This expression is in the form $(a-b)(a+b)$, where $a=\frac{1}{3}\sqrt{s}$ and $b=\frac{3}{4}\sqrt{t}$.

$$a^2 - b^2$$

$= \left(\frac{1}{3}\sqrt{s}\right)^2 - \left(\frac{3}{4}\sqrt{t}\right)^2$ Apply the formula $(a+b)(a-b)=a^2-b^2$.

$= \frac{1}{9}s - \frac{9}{16}t$ Simplify.

Skill Practice Multiply the conjugates. Assume all variables represent positive real numbers.

13. $(\sqrt{5}+3)(\sqrt{5}-3)$ **14.** $\left(\frac{1}{2}\sqrt{a}+\frac{2}{5}\sqrt{b}\right)\cdot\left(\frac{1}{2}\sqrt{a}-\frac{2}{5}\sqrt{b}\right)$

4. Multiplying Radicals With Different Indices

The product of two radicals can be simplified provided the radicals have the same index. If the radicals have different indices, then we can use the properties of rational exponents to obtain a common index.

Example 8 Multiplying Radicals With Different Indices

Multiply the expressions. Write the answers in radical form.

a. $\sqrt[3]{5}\cdot\sqrt[4]{5}$ **b.** $\sqrt[3]{7}\cdot\sqrt{2}$

Solution:

a. $\sqrt[3]{5}\cdot\sqrt[4]{5}$

$= 5^{1/3}5^{1/4}$ Rewrite each expression with rational exponents.

$= 5^{(1/3)+(1/4)}$ Because the bases are equal, we can add the exponents.

$= 5^{(4/12)+(3/12)}$ Write the fractions with a common denominator.

$= 5^{7/12}$ Simplify the exponent.

$= \sqrt[12]{5^7}$ Rewrite the expression as a radical.

Answers

13. -4 **14.** $\frac{1}{4}a - \frac{4}{25}b$

b. $\sqrt[3]{7} \cdot \sqrt{2}$

$= 7^{1/3}2^{1/2}$	Rewrite each expression with rational exponents.
$= 7^{2/6}2^{3/6}$	Write the rational exponents with a common denominator.
$= (7^2 2^3)^{1/6}$	Apply the power rule of exponents.
$= \sqrt[6]{7^2 2^3}$	Rewrite the expression as a single radical.
$= \sqrt[6]{392}$	Simplify.

Skill Practice Multiply the expressions. Write the answers in radical form. Assume all variables represent positive real numbers.

15. $\sqrt{x} \cdot \sqrt[3]{x}$ **16.** $\sqrt[4]{a^3} \cdot \sqrt[3]{b}$

Answers

15. $\sqrt[6]{x^5}$ **16.** $\sqrt[12]{a^9b^4}$

Section 6.5 Activity

A.1. Recall that the multiplication property of radicals indicates that $\sqrt[n]{a} \cdot \sqrt[n]{b} =$ ________, provided that the two radicals represent real numbers.

A.2. **a.** Multiply. $(5x) \cdot (2x)$

b. Multiply. $(5\sqrt{x}) \cdot (2\sqrt{x})$

c. Multiply. $(5\sqrt[3]{x}) \cdot (2\sqrt[3]{x})$

d. Compare the process to multiply radicals in parts (b) and (c).

For Exercises A.3–A.6, multiply the expressions.

A.3. **a.** $\frac{1}{2}x(4x^2 + 8xy + 2y^2)$ **b.** $\frac{1}{2}\sqrt{x}(4\sqrt{x} + 8\sqrt{xy} + 2\sqrt{y})$

A.4. **a.** $(2x + 3)(5x - 6)$ **b.** $(2\sqrt{x} + 3)(5\sqrt{x} - 6)$

c. $(2\sqrt{x} + \sqrt{3})(5\sqrt{x} - \sqrt{6})$

A.5. **a.** $(5x - 3)^2$ **b.** $(5\sqrt{x} - 3)^2$

c. $(5\sqrt{x} - \sqrt{3})^2$

A.6. **a.** $(2y - 7)(2y + 7)$ **b.** $(2\sqrt{y} - 7)(2\sqrt{y} + 7)$

c. $(2\sqrt{y} - \sqrt{7})(2\sqrt{y} + \sqrt{7})$

A.7. To multiply two radical expressions with different indices, write the radicals with rational exponents. Then write the rational exponents with a common denominator. To illustrate this process, simplify $\sqrt[3]{7} \cdot \sqrt[5]{2}$ by following these steps.

a. Write $\sqrt[3]{7}$ using rational exponents.

b. Write $\sqrt[5]{2}$ using rational exponents.

c. Write the expression $\sqrt[3]{7} \cdot \sqrt[5]{2}$ using rational exponents.

d. Write the expression from part (c) with a common denominator in the exponents.

e. Convert the expression from part (d) as a product of radicals.

f. Multiply the radicals. Leave the radicand in factored form.

Section 6.5 Practice Exercises

Prerequisite Review

For Exercises R.1–R.10, simplify the expression.

R.1. $(5x^2)(-8xy)$

R.2. $(-6t^2w)(-3t^3)$

R.3. $-\frac{1}{5}a(15a^2 + 25ab - 10b^2)$

R.4. $\frac{1}{4}m(4m^2 - 8mn + 24n^2)$

R.5. $(3c + 4)(c^2 - 2c + 5)$

R.6. $(5d - 1)(2d^2 + 3d + 7)$

R.7. $(2a - 5b)^2$

R.8. $(3m + 9n)^2$

R.9. $\left(\frac{1}{2}m + 4n\right)\left(\frac{1}{2}m - 4n\right)$

R.10. $\left(\frac{3}{5}x - y\right)\left(\frac{3}{5}x + y\right)$

For Exercises R.11–R.12, add or subtract as indicated.

R.11. $-2y\sqrt{28} + 4\sqrt{63y^2}$

R.12. $4\sqrt{8x^3} - x\sqrt{50x}$

For the exercises in this set, assume that all variables represent positive real numbers, unless otherwise stated.

Vocabulary and Key Concepts

1. a. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a} \cdot \sqrt[n]{b} =$ ________.

b. If $x \geq 0$, then $\sqrt{x} \cdot \sqrt{x} =$ ________.

c. If $\sqrt[n]{a}$ is a real number, then $(\sqrt[n]{a})^n =$ ________.

d. Two binomials $(x + \sqrt{2})$ and $(x - \sqrt{2})$ are called ________ of each other, and their product is $(x)^2 - (\sqrt{2})^2$.

e. If $m \geq 0$ and $n \geq 0$, then $(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) =$ ________.

2. Which is the correct simplification of $(\sqrt{c} + 4)^2$?

$c + 16$ or $c + 8\sqrt{c} + 16$

Concept 1: Multiplication Property of Radicals

For Exercises 3–8, fill in the blanks and simplify the expressions. Assume that the variables represent positive real numbers.

3. a. $x \cdot x$ **b.** $\sqrt{x} \cdot \sqrt{x} = \sqrt{x^{\square}} =$ _____

4. a. $y^2 \cdot y$ **b.** $\sqrt[3]{y^2} \cdot \sqrt[3]{y} = \sqrt[3]{y^{\square}} =$ _____

5. a. $t \cdot t^5$ **b.** $\sqrt[3]{t} \cdot \sqrt[3]{t^5} = \sqrt[3]{\square} =$ _____

6. a. $m^3 \cdot m^5$ **b.** $\sqrt[4]{m^3} \cdot \sqrt[4]{m^5} = \sqrt[4]{\square} =$ _____

7. a. $(5x)(10x)$ **b.** $\sqrt{5x} \cdot \sqrt{10x} = \sqrt{\square} =$ _____

8. a. $(6y)(3y)$ **b.** $\sqrt{6y} \cdot \sqrt{3y} = \sqrt{\square} =$ _____

For Exercises 9–44, multiply the radical expressions. **(See Examples 1–4.)**

9. $\sqrt[3]{7} \cdot \sqrt[3]{3}$

10. $\sqrt[4]{6} \cdot \sqrt[4]{2}$

11. $\sqrt{2} \cdot \sqrt{10}$

12. $\sqrt[3]{4} \cdot \sqrt[3]{12}$

13. $\sqrt[4]{16} \cdot \sqrt[4]{64}$

14. $\sqrt{5x^3} \cdot \sqrt{10x^4}$

15. $(4\sqrt[3]{4})(2\sqrt[3]{5})$

16. $(2\sqrt{5})(3\sqrt{7})$

17. $(8a\sqrt{b})(-3\sqrt{ab})$

18. $(p\sqrt[4]{q^3})(\sqrt[4]{pq})$

19. $\sqrt{30} \cdot \sqrt{12}$

20. $\sqrt{20} \cdot \sqrt{54}$

21. $\sqrt{6x} \cdot \sqrt{12x}$

22. $(\sqrt{3ab^2})(\sqrt{21a^2b})$

23. $\sqrt{5a^3b^2} \cdot \sqrt{20a^3b^3}$

24. $\sqrt[3]{m^2n^2} \cdot \sqrt[3]{48m^4n^2}$

25. $(4\sqrt{3xy^3})(-2\sqrt{6x^3y^2})$

26. $(2\sqrt[4]{3x})(4\sqrt[4]{27x^6})$

27. $(\sqrt[3]{4a^2b})(\sqrt[3]{2ab^3})(\sqrt[3]{54a^2b})$

28. $(\sqrt[3]{9x^3y})(\sqrt[3]{6xy})(\sqrt[3]{8x^2y^5})$

29. $\sqrt{3}(4\sqrt{3} - 6)$

30. $3\sqrt{5}(2\sqrt{5} + 4)$

31. $\sqrt{2}(\sqrt{6} - \sqrt{3})$

32. $\sqrt{5}(\sqrt{3} + \sqrt{7})$

33. $-\frac{1}{3}\sqrt{x}(6\sqrt{x} + 7)$

34. $-\frac{1}{2}\sqrt{y}(8 - 3\sqrt{y})$

35. $(\sqrt{3} + 2\sqrt{10})(4\sqrt{3} - \sqrt{10})$

36. $(8\sqrt{7} - \sqrt{5})(\sqrt{7} + 3\sqrt{5})$

37. $(\sqrt{x} + 4)(\sqrt{x} - 9)$

38. $(\sqrt{w} - 2)(\sqrt{w} - 9)$

39. $(\sqrt[3]{y} + 2)(\sqrt[3]{y} - 3)$

40. $(4 + \sqrt[5]{p})(5 + \sqrt[5]{p})$

41. $(\sqrt{a} - 3\sqrt{b})(9\sqrt{a} - \sqrt{b})$

42. $(11\sqrt{m} + 4\sqrt{n})(\sqrt{m} + \sqrt{n})$

43. $(\sqrt{p} + 2\sqrt{q})(8 + 3\sqrt{p} - \sqrt{q})$

44. $(5\sqrt{s} - \sqrt{t})(\sqrt{s} + 5 + 6\sqrt{t})$

Concept 2: Expressions of the Form $(\sqrt[n]{a})^n$

For Exercises 45–52, simplify the expressions. Assume all variables represent positive real numbers. **(See Example 5.)**

45. $(\sqrt{15})^2$

46. $(\sqrt{58})^2$

47. $(\sqrt{3y})^2$

48. $(\sqrt{19yz})^2$

49. $(\sqrt[3]{6})^3$

50. $(\sqrt[5]{24})^5$

51. $\sqrt{709} \cdot \sqrt{709}$

52. $\sqrt{401} \cdot \sqrt{401}$

Concept 3: Special Case Products

53. **a.** Write the formula for the product of two conjugates. $(x + y)(x - y) =$

b. Multiply. $(x + 5)(x - 5)$

54. **a.** Write the formula for squaring a binomial. $(x + y)^2 =$

b. Multiply. $(x + 5)^2$

For Exercises 55–66, multiply the radical expressions. **(See Examples 6–7.)**

55. $(\sqrt{13} + 4)^2$

56. $(6 - \sqrt{11})^2$

57. $(\sqrt{p} - \sqrt{7})^2$

58. $(\sqrt{q} + \sqrt{2})^2$

59. $(\sqrt{2a} - 3\sqrt{b})^2$

60. $(\sqrt{3w} + 4\sqrt{z})^2$

61. $(\sqrt{3} + x)(\sqrt{3} - x)$

62. $(y + \sqrt{6})(y - \sqrt{6})$

63. $(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})$

64. $(\sqrt{15} + \sqrt{5})(\sqrt{15} - \sqrt{5})$

65. $\left(\frac{2}{3}\sqrt{x} + \frac{1}{2}\sqrt{y}\right)\left(\frac{2}{3}\sqrt{x} - \frac{1}{2}\sqrt{y}\right)$

66. $\left(\frac{1}{4}\sqrt{s} + \frac{1}{5}\sqrt{t}\right)\left(\frac{1}{4}\sqrt{s} - \frac{1}{5}\sqrt{t}\right)$

For Exercises 67–68, multiply the expressions.

67. **a.** $(\sqrt{3}+\sqrt{x})(\sqrt{3}-\sqrt{x})$
b. $(\sqrt{3}+\sqrt{x})(\sqrt{3}+\sqrt{x})$
c. $(\sqrt{3}-\sqrt{x})(\sqrt{3}-\sqrt{x})$

68. **a.** $(\sqrt{5}+\sqrt{y})(\sqrt{5}-\sqrt{y})$
b. $(\sqrt{5}+\sqrt{y})(\sqrt{5}+\sqrt{y})$
c. $(\sqrt{5}-\sqrt{y})(\sqrt{5}-\sqrt{y})$

Mixed Exercises

For Exercises 69–76, identify each statement as true or false. If an answer is false, explain why.

69. $\sqrt{3}\cdot\sqrt{2}=\sqrt{6}$

70. $\sqrt{5}\cdot\sqrt[3]{2}=\sqrt{10}$

71. $(x-\sqrt{5})^2=x-5$

72. $3(2\sqrt{5x})=6\sqrt{5x}$

73. $5(3\sqrt{4x})=15\sqrt{20x}$

74. $\dfrac{\sqrt{5x}}{5}=\sqrt{x}$

75. $\dfrac{3\sqrt{x}}{3}=\sqrt{x}$

76. $(\sqrt{t}-1)(\sqrt{t}+1)=t-1$

For Exercises 77–84, perform the indicated operations.

77. $(-\sqrt{6x})^2$

78. $(-\sqrt{8a})^2$

79. $(\sqrt{3x+1})^2$

80. $(\sqrt{x-1})^2$

81. $(\sqrt{x+3}-4)^2$

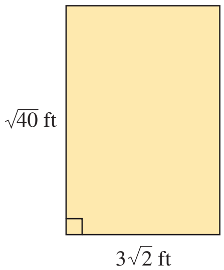
82. $(\sqrt{x+1}+3)^2$

83. $(\sqrt{2t-3}+5)^2$

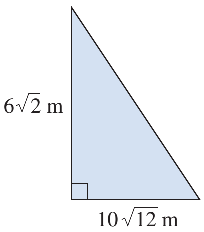
84. $(\sqrt{3w-2}-4)^2$

For Exercises 85–88, find the exact area.

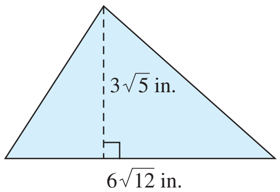
85.



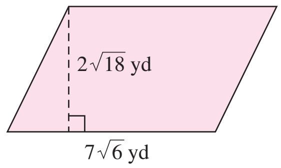
86.



87.



88.



Concept 4: Multiplying Radicals With Different Indices

For Exercises 89–100, multiply or divide the radicals with different indices. Write the answers in radical form and simplify. **(See Example 8.)**

89. $\sqrt{x}\cdot\sqrt[4]{x}$

90. $\sqrt[3]{y}\cdot\sqrt{y}$

91. $\sqrt[5]{2z}\cdot\sqrt[3]{2z}$

92. $\sqrt[3]{5w}\cdot\sqrt[4]{5w}$

93. $\sqrt[3]{p^2}\cdot\sqrt{p^3}$

94. $\sqrt[4]{q^3}\cdot\sqrt[3]{q^2}$

95. $\dfrac{\sqrt{u^3}}{\sqrt[3]{u}}$

96. $\dfrac{\sqrt{v^5}}{\sqrt[4]{v}}$

97. $\sqrt[3]{x}\cdot\sqrt[6]{y}$

98. $\sqrt{a}\cdot\sqrt[6]{b}$

99. $\sqrt[4]{8}\cdot\sqrt{3}$

100. $\sqrt{11}\cdot\sqrt[6]{2}$

Expanding Your Skills

For Exercises 101–106, multiply.

101. $\sqrt[3]{2xy}\cdot\sqrt[4]{5xy}$

102. $\sqrt{6ab}\cdot\sqrt[3]{7ab}$

103. $\sqrt[3]{4m^2n}\cdot\sqrt{6mn}$

104. $\sqrt[4]{5xy^3}\cdot\sqrt[3]{10x^2y}$

105. $(\sqrt[3]{a}+\sqrt[3]{b})(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2})$

106. $(\sqrt[3]{a}-\sqrt[3]{b})(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})$

Division of Radicals and Rationalization

Section 6.6

Concepts

1. Simplified Form of a Radical
2. Division Property of Radicals
3. Rationalizing the Denominator—One Term
4. Rationalizing the Denominator—Two Terms

1. Simplified Form of a Radical

Recall that for a radical expression to be in simplified form the following three conditions must be met.

Simplified Form of a Radical

Consider any radical expression in which the radicand is written as a product of prime factors. The expression is in simplified form if all the following conditions are met:

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand does not contain a fraction.
3. No radicals are in the denominator of a fraction.

In the previous sections, we have concentrated on the first condition in the simplification process. Next, we will demonstrate how to satisfy the second and third conditions involving radicals and fractions.

2. Division Property of Radicals

The multiplication property of radicals makes it possible to write a product within a radical to be separated as a product of radicals. We now state a similar property for radicals involving quotients.

Division Property of Radicals

Let a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real. Then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad b \neq 0$$

The division property of radicals indicates that a quotient within a radicand can be written as a quotient of radicals provided the roots are real numbers. For example:

$$\sqrt{\frac{25}{36}} = \frac{\sqrt{25}}{\sqrt{36}}$$

The reverse process is also true. A quotient of radicals can be written as a single radical provided that the roots are real numbers and that they have the same index.

$$\text{Same index} \quad \frac{\sqrt[3]{125}}{\sqrt[3]{8}} = \sqrt[3]{\frac{125}{8}}$$

In Examples 1 and 2, we will apply the division property of radicals to simplify radical expressions.

Example 1 Using the Division Property to Simplify Radicals

Use the division property of radicals to simplify the expressions. Assume the variables represent positive real numbers.

a. $\sqrt{\dfrac{a^6}{b^4}}$ **b.** $\sqrt[3]{\dfrac{81y^5}{x^3}}$

Solution:

a. $\sqrt{\dfrac{a^6}{b^4}}$ The radicand contains an irreducible fraction.

$= \dfrac{\sqrt{a^6}}{\sqrt{b^4}}$ Apply the division property to rewrite as a quotient of radicals.

$= \dfrac{a^3}{b^2}$ Simplify the radicals.

b. $\sqrt[3]{\dfrac{81y^5}{x^3}}$ The radicand contains an irreducible fraction.

$= \dfrac{\sqrt[3]{81y^5}}{\sqrt[3]{x^3}}$ Apply the division property to rewrite as a quotient of radicals.

$= \dfrac{\sqrt[3]{3^4 \cdot y^5}}{\sqrt[3]{x^3}}$ Factor the radicand in the numerator to simplify the radical.

$= \dfrac{3y\sqrt[3]{3y^2}}{x}$ Simplify the radicals in the numerator and the denominator. The expression is simplified because it now satisfies all conditions.

Skill Practice Simplify the expressions.

1. $\sqrt{\dfrac{x^4}{y^{10}}}$ **2.** $\sqrt[3]{\dfrac{w^7}{64}}$

Example 2 Using the Division Property to Simplify a Radical

Use the division property of radicals to simplify the expressions. Assume the variables represent positive real numbers.

$$\frac{\sqrt[4]{8p^7}}{\sqrt[4]{p^3}}$$

Solution:

$\dfrac{\sqrt[4]{8p^7}}{\sqrt[4]{p^3}}$ There is a radical in the denominator of the fraction.

$= \sqrt[4]{\dfrac{8p^7}{p^3}}$ Apply the division property to write the quotient under a single radical.

$= \sqrt[4]{8p^4}$ Simplify the fraction.

$= p\sqrt[4]{8}$ Simplify the radical.

Skill Practice Simplify the expression.

3. $\dfrac{\sqrt[3]{16y^4}}{\sqrt[3]{y}}$

Answers

1. $\dfrac{x^2}{y^5}$ **2.** $\dfrac{w^2\sqrt[3]{w}}{4}$ **3.** $2y\sqrt[3]{2}$

3. Rationalizing the Denominator—One Term

The third condition restricts radicals from the denominator of an expression. The process to remove a radical from the denominator is called **rationalizing the denominator**. In many cases, rationalizing the denominator creates an expression that is computationally simpler. For example,

$$\frac{6}{\sqrt{3}} = 2\sqrt{3} \qquad \text{and} \qquad \frac{-2}{2+\sqrt{6}} = 2-\sqrt{6}$$

We will demonstrate the process to rationalize the denominator as two separate cases:

- Rationalizing the denominator (one term)
- Rationalizing the denominator (two terms involving square roots)

To begin the first case, recall that the nth root of a perfect nth power simplifies completely.

$$\sqrt{x^2} = x \qquad x \geq 0$$
$$\sqrt[3]{x^3} = x$$
$$\sqrt[4]{x^4} = x \qquad x \geq 0$$
$$\sqrt[5]{x^5} = x$$
$$\vdots$$

Therefore, to rationalize a radical expression, use the multiplication property of radicals to create an nth root of an nth power.

Example 3 Rationalizing Radical Expressions

Fill in the missing radicand to rationalize the radical expressions. Assume all variables represent positive real numbers.

a. $\sqrt{a} \cdot \sqrt{?} = \sqrt{a^2} = a$ **b.** $\sqrt[3]{y} \cdot \sqrt[3]{?} = \sqrt[3]{y^3} = y$

c. $\sqrt[4]{2z^3} \cdot \sqrt[4]{?} = \sqrt[4]{2^4z^4} = 2z$

Solution:

a. $\sqrt{a} \cdot \sqrt{?} = \sqrt{a^2} = a$ What multiplied by $\sqrt{a}$ will equal $\sqrt{a^2}$?

$\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$

b. $\sqrt[3]{y} \cdot \sqrt[3]{?} = \sqrt[3]{y^3} = y$ What multiplied by $\sqrt[3]{y}$ will equal $\sqrt[3]{y^3}$?

$\sqrt[3]{y} \cdot \sqrt[3]{y^2} = \sqrt[3]{y^3} = y$

c. $\sqrt[4]{2z^3} \cdot \sqrt[4]{?} = \sqrt[4]{2^4z^4} = 2z$ What multiplied by $\sqrt[4]{2z^3}$ will equal $\sqrt[4]{2^4z^4}$?

$\sqrt[4]{2z^3} \cdot \sqrt[4]{2^3z} = \sqrt[4]{2^4z^4} = 2z$

Skill Practice Fill in the missing radicand to rationalize the radical expression.

4. $\sqrt{7} \cdot \sqrt{?}$ **5.** $\sqrt[5]{t^2} \cdot \sqrt[5]{?}$ **6.** $\sqrt[3]{5x^2} \cdot \sqrt[3]{?}$

To rationalize the denominator of an expression with a single radical term in the denominator, we have the following strategy. Multiply the numerator and denominator by an appropriate expression to create an nth root of an nth power in the denominator.

Answers

4. 7 **5.** t^3 **6.** 5^2x

Example 4 Rationalizing the Denominator—One Term

Simplify the expression. $\dfrac{5}{\sqrt[3]{a}}$ $\quad a \neq 0$

Solution:

To remove the radical from the denominator, a cube root of a perfect cube is needed in the denominator. Multiply numerator and denominator by $\sqrt[3]{a^2}$ because $\sqrt[3]{a} \cdot \sqrt[3]{a^2} = \sqrt[3]{a^3} = a$.

$$\frac{5}{\sqrt[3]{a}} = \frac{5}{\sqrt[3]{a}} \cdot \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}$$

$$= \frac{5\sqrt[3]{a^2}}{\sqrt[3]{a^3}}$$ Multiply the radicals.

$$= \frac{5\sqrt[3]{a^2}}{a}$$ Simplify.

Skill Practice Simplify the expression. Assume $y > 0$.

7. $\dfrac{2}{\sqrt[4]{y}}$

Note that for $a \neq 0$, the expression $\dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}} = 1$. In Example 4, multiplying the expression $\dfrac{5}{\sqrt[3]{a}}$ by this ratio does not change its value.

Example 5 Rationalizing the Denominator—One Term

Simplify the expression. $\sqrt{\dfrac{y^5}{7}}$

Solution:

$$\sqrt{\frac{y^5}{7}}$$ The radical contains an irreducible fraction.

$$= \frac{\sqrt{y^5}}{\sqrt{7}}$$ Apply the division property of radicals.

$$= \frac{y^2\sqrt{y}}{\sqrt{7}}$$ Simplify the radical in the numerator.

$$= \frac{y^2\sqrt{y}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$ Rationalize the denominator.

Note: $\sqrt{7} \cdot \sqrt{7} = \sqrt{7^2} = 7$

$$= \frac{y^2\sqrt{7y}}{\sqrt{7^2}}$$

$$= \frac{y^2\sqrt{7y}}{7}$$ Simplify.

Avoiding Mistakes

A factor within a radicand cannot be simplified with a factor outside the radicand. For example, $\dfrac{\sqrt{7y}}{7}$ cannot be simplified.

Answer

7. $\dfrac{2\sqrt[4]{y^3}}{y}$

Skill Practice Simplify the expression.

8. $\sqrt{\dfrac{8}{3}}$

Example 6 **Rationalizing the Denominator—One Term**

Simplify the expression. $\dfrac{15}{\sqrt[3]{25s}}$

Solution:

$$\frac{15}{\sqrt[3]{25s}}$$

$$= \frac{15}{\sqrt[3]{5^2s}} \cdot \frac{\sqrt[3]{5s^2}}{\sqrt[3]{5s^2}}$$

Because $25 = 5^2$, one additional factor of 5 is needed to form a perfect cube. Two additional factors of s are needed to make a perfect cube. Multiply numerator and denominator by $\sqrt[3]{5s^2}$.

$$= \frac{15\sqrt[3]{5s^2}}{\sqrt[3]{5^3s^3}}$$

$$= \frac{15\sqrt[3]{5s^2}}{5s}$$

Simplify the perfect cube.

$$= \frac{\overset{3}{\cancel{15}}\sqrt[3]{5s^2}}{\underset{1}{\cancel{5}}s}$$

Reduce to lowest terms.

$$= \frac{3\sqrt[3]{5s^2}}{s}$$

Skill Practice Simplify the expression.

9. $\dfrac{18}{\sqrt[3]{3y^2}}$

TIP: In the expression $\dfrac{15\sqrt[3]{5s^2}}{5s}$, the factor of 15 and the factor of 5 may be reduced because both are outside the radical.

$$\frac{15\sqrt[3]{5s^2}}{5s} = \frac{15}{5} \cdot \frac{\sqrt[3]{5s^2}}{s}$$
$$= \frac{3\sqrt[3]{5s^2}}{s}$$

4. Rationalizing the Denominator—Two Terms

Example 7 demonstrates how to rationalize a two-term denominator involving square roots.

First recall from the multiplication of polynomials that the product of two conjugates results in a difference of squares.

$$(a + b)(a - b) = a^2 - b^2$$

If either a or b has a square root factor, the expression will simplify without a radical. That is, the expression is *rationalized*. For example:

$$\begin{aligned}(2 + \sqrt{6})(2 - \sqrt{6}) &= (2)^2 - (\sqrt{6})^2\\ &= 4 - 6\\ &= -2\end{aligned}$$

Answers

8. $\dfrac{2\sqrt{6}}{3}$ **9.** $\dfrac{6\sqrt[3]{9y}}{y}$

Example 7 Rationalizing the Denominator—Two Terms

Simplify the expression by rationalizing the denominator. $\dfrac{-2}{2+\sqrt{6}}$

Solution:

$$\frac{-2}{2+\sqrt{6}}$$

$$= \frac{(-2)}{(2+\sqrt{6})} \cdot \frac{(2-\sqrt{6})}{(2-\sqrt{6})}$$ Multiply the numerator and denominator by the conjugate of the denominator.

conjugates

Avoiding Mistakes

When constructing the conjugate of an expression, change only the sign between the terms (not the sign of the leading term).

$$= \frac{-2(2-\sqrt{6})}{(2)^2-(\sqrt{6})^2}$$ In the denominator, apply the formula $(a+b)(a-b) = a^2 - b^2$.

$$= \frac{-2(2-\sqrt{6})}{4-6}$$ Simplify.

$$= \frac{-2(2-\sqrt{6})}{-2}$$

$$= \frac{-\cancel{2}(2-\sqrt{6})}{-\cancel{2}}$$

$$= 2-\sqrt{6}$$

Skill Practice Simplify by rationalizing the denominator.

10. $\dfrac{8}{3+\sqrt{5}}$

Example 8 Rationalizing the Denominator—Two Terms

Rationalize the denominator. $\dfrac{4+\sqrt{x}}{\sqrt{x}-7}$

Solution:

$$\frac{(4+\sqrt{x})}{(\sqrt{x}-7)} \cdot \frac{(\sqrt{x}+7)}{(\sqrt{x}+7)}$$ Multiply numerator and denominator by the conjugate of the denominator.

FOR REVIEW

Recall that the product of conjugates results in a difference of squares.

$$(a-b)(a+b) = a^2 - b^2$$

$$= \frac{(4)(\sqrt{x})+(4)(7)+(\sqrt{x})(\sqrt{x})+(\sqrt{x})(7)}{(\sqrt{x})^2-(7)^2}$$ Apply the distributive property.

$$= \frac{4\sqrt{x}+28+x+7\sqrt{x}}{x-49}$$ Multiply the radicals.

$$= \frac{x+11\sqrt{x}+28}{x-49}$$ Simplify.

Skill Practice Rationalize the denominator.

11. $\dfrac{\sqrt{y}-4}{8-\sqrt{y}}$

Answers

10. $6-2\sqrt{5}$

11. $\dfrac{y+4\sqrt{y}-32}{64-y}$

Example 9 Rationalizing the Denominator—Two Terms

Rationalize the denominator. $\dfrac{3\sqrt{2}+2\sqrt{5}}{\sqrt{2}-4\sqrt{5}}$

Solution:

$\dfrac{(3\sqrt{2}+2\sqrt{5})}{(\sqrt{2}-4\sqrt{5})}\cdot\dfrac{(\sqrt{2}+4\sqrt{5})}{(\sqrt{2}+4\sqrt{5})}$ Multiply numerator and denominator by the conjugate of the denominator.

$=\dfrac{(3\sqrt{2})\cdot(\sqrt{2})+(3\sqrt{2})(4\sqrt{5})+(2\sqrt{5})(\sqrt{2})+(2\sqrt{5})(4\sqrt{5})}{(\sqrt{2})^2-(4\sqrt{5})^2}$ Apply the distributive property.

$=\dfrac{6+12\sqrt{10}+2\sqrt{10}+40}{2-80}$ Multiply the radicals.

$=\dfrac{46+14\sqrt{10}}{-78}$ or $-\dfrac{46+14\sqrt{10}}{78}$

$=-\dfrac{\overset{1}{\cancel{2}}(23+7\sqrt{10})}{\underset{39}{\cancel{78}}}$ Factor and simplify.

$=-\dfrac{23+7\sqrt{10}}{39}$

Skill Practice Rationalize the denominator.

12. $\dfrac{5\sqrt{2}-2\sqrt{5}}{\sqrt{2}-\sqrt{5}}$

Answer
12. $-\sqrt{10}$

Section 6.6 Activity

A.1. The division property of radicals indicates that $\sqrt[n]{\dfrac{a}{b}}=$ ________, provided that the radicals represent real numbers and that $b\neq 0$.

For Exercises A.2–A.3, apply the division property of radicals and simplify the expression. Assume that all variables represent positive real numbers.

A.2. $\sqrt[3]{\dfrac{40x^5}{y^3}}$

A.3. $\dfrac{\sqrt{50x^3}}{\sqrt{2x}}$ (*Hint:* Write the quotient as a single radical and simplify the fraction.)

A.4. The expression $\dfrac{6}{\sqrt[4]{x}}$ is not simplified because there is a radical in the denominator.

a. Multiply the expression by a convenient ratio of 1 so that the product in the denominator is a perfect fourth power.

$$\frac{6}{\sqrt[4]{x}}\cdot\frac{\sqrt[4]{\square}}{\sqrt[4]{\square}}=\frac{\square}{\sqrt[4]{x^4}}$$

b. Simplify the expression.

A.5. The expression $\dfrac{14}{\sqrt[3]{49y}}$ is not simplified because there is a radical in the denominator.

a. Factor the denominator as a product of prime factors.

b. Multiply the expression by a convenient ratio of 1 so that the product in the denominator is a perfect square.

$$\frac{14}{\sqrt[3]{7^2y}}\cdot\frac{\sqrt[3]{\square}}{\sqrt[3]{\square}}=\frac{\square}{\sqrt[3]{7^3y^3}}$$

c. Simplify the expression.

In Exercises A.6–A.7, we investigate how to rationalize the denominator when the denominator has two terms with one or more square roots.

A.6. Recall that the product of conjugates results in a difference of squares.

a. Multiply. $(7x-2)(7x+2)$

b. Multiply. $(\sqrt{7}x-2)(\sqrt{7}x+2)$

c. Multiply. $(\sqrt{7}-2)(\sqrt{7}+2)$

d. After multiplying the expressions in parts (b) and (c), do radicals still appear in the expressions?

e. Now consider the expression $\dfrac{12}{\sqrt{7}-2}$. To eliminate the radical in the denominator, we want to multiply the expression by a convenient ratio of 1. Fill in the blank so that in the resulting product, the radical in the denominator would be eliminated.

$$\frac{12}{(\sqrt{7}-2)}\cdot\frac{(\boxed{})}{(\boxed{})}$$

f. Simplify the expression $\dfrac{12}{\sqrt{7}-2}$ by using the result from part (e). Be sure to write the final answer in lowest terms.

A.7. Consider an expression with two terms in the denominator where one or more terms is a square root, such as $\dfrac{\sqrt{t}-5}{\sqrt{t}+4}$.

a. To simplify the expression, we want to remove the radical(s) from the (choose one: numerator/ denominator).

b. Multiply numerator and denominator of the expression by the __________ of the denominator.

c. Simplify. $\dfrac{\sqrt{t}-5}{\sqrt{t}+4}$

Section 6.6 Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, simplify the expression. Assume that all variables represent positive real numbers.

R.1. $\sqrt[5]{x^5}$ **R.2.** $\sqrt[6]{y^6}$ **R.3.** $\sqrt{a^2b^4}$ **R.4.** $\sqrt{p^8q^6}$

For Exercises R.5–R.8, multiply. Assume that all variable expressions represent positive real numbers.

R.5. $(\sqrt{p}-4)(\sqrt{p}+4)$ **R.6.** $(\sqrt{w}+5)(\sqrt{w}-5)$

R.7. $(\sqrt{10}+\sqrt{7})(\sqrt{10}-\sqrt{7})$ **R.8.** $(\sqrt{11}-\sqrt{5})(\sqrt{11}+\sqrt{5})$

For the exercises in this set, assume that all variables represent positive real numbers unless otherwise stated.

Vocabulary and Key Concepts

1. **a.** In the simplified form of a radical expression, no __________ may appear in the denominator of a fraction.

b. The division property of radicals indicates that if $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\square}{\square}$ provided that $b \neq 0$.

c. The simplified form of the expression $\sqrt[3]{\dfrac{64}{x^6}}$ is __________.

d. The expression $\dfrac{\sqrt{3}}{3}$ (is/is not) in simplified form, whereas $\dfrac{3}{\sqrt{3}}$ (is/is not) in simplified form.

2. **a.** The process of removing a radical from the denominator of a fraction is called __________ the denominator.

b. To rationalize the denominator for the expression $\dfrac{\sqrt{x}+3}{\sqrt{x}-2}$, multiply numerator and denominator by the conjugate of the __________.

Concept 2: Division Property of Radicals

For Exercises 3–14, simplify using the division property of radicals. Assume all variables represent positive real numbers. **(See Examples 1–2.)**

3. $\sqrt{\dfrac{49x^4}{y^6}}$ **4.** $\sqrt{\dfrac{100p^2}{q^8}}$ **5.** $\sqrt{\dfrac{8a^2}{x^6}}$ **6.** $\sqrt{\dfrac{4w^3}{25y^4}}$

7. $\sqrt[3]{\dfrac{-16j^3}{k^3}}$ **8.** $\sqrt[5]{\dfrac{32x}{y^{10}}}$ **9.** $\dfrac{\sqrt{72ab^5}}{\sqrt{8ab}}$ **10.** $\dfrac{\sqrt{6x^3}}{\sqrt{24x}}$

11. $\dfrac{\sqrt[4]{3b^3}}{\sqrt[4]{48b^{11}}}$ **12.** $\dfrac{\sqrt[3]{128wz^8}}{\sqrt[3]{2wz^2}}$ **13.** $\dfrac{\sqrt{3yz^2}}{\sqrt{w^4}}$ **14.** $\dfrac{\sqrt{50x^3z}}{\sqrt{9y^4}}$

Concept 3: Rationalizing the Denominator—One Term

The radical expressions in Exercises 15–22 have radicals in the denominator. Fill in the missing radicands to rationalize the denominators. **(See Example 3.)**

15. $\dfrac{x}{\sqrt{5}} = \dfrac{x}{\sqrt{5}} \cdot \dfrac{\sqrt{?}}{\sqrt{?}}$ **16.** $\dfrac{2}{\sqrt{x}} = \dfrac{2}{\sqrt{x}} \cdot \dfrac{\sqrt{?}}{\sqrt{?}}$ **17.** $\dfrac{7}{\sqrt[3]{x}} = \dfrac{7}{\sqrt[3]{x}} \cdot \dfrac{\sqrt[3]{?}}{\sqrt[3]{?}}$ **18.** $\dfrac{5}{\sqrt[4]{y}} = \dfrac{5}{\sqrt[4]{y}} \cdot \dfrac{\sqrt[4]{?}}{\sqrt[4]{?}}$

19. $\dfrac{8}{\sqrt{3z}} = \dfrac{8}{\sqrt{3z}} \cdot \dfrac{\sqrt{??}}{\sqrt{??}}$ **20.** $\dfrac{10}{\sqrt{7w}} = \dfrac{10}{\sqrt{7w}} \cdot \dfrac{\sqrt{??}}{\sqrt{??}}$ **21.** $\dfrac{1}{\sqrt[4]{8a^2}} = \dfrac{1}{\sqrt[4]{8a^2}} \cdot \dfrac{\sqrt[4]{??}}{\sqrt[4]{??}}$ **22.** $\dfrac{1}{\sqrt[3]{9b^2}} = \dfrac{1}{\sqrt[3]{9b^2}} \cdot \dfrac{\sqrt[3]{??}}{\sqrt[3]{??}}$

For Exercises 23–50, rationalize the denominator. **(See Examples 4–6.)**

23. $\dfrac{1}{\sqrt{3}}$ **24.** $\dfrac{1}{\sqrt{7}}$ **25.** $\sqrt{\dfrac{1}{x}}$ **26.** $\sqrt{\dfrac{1}{z}}$

27. $\dfrac{6}{\sqrt{2y}}$ **28.** $\dfrac{9}{\sqrt{3t}}$ **29.** $\sqrt{\dfrac{a^3}{2}}$ **30.** $\sqrt{\dfrac{b^3}{3}}$

31. $\dfrac{6}{\sqrt{8}}$

32. $\dfrac{2}{\sqrt{48}}$

33. $\dfrac{3}{\sqrt[3]{2}}$

34. $\dfrac{2}{\sqrt[3]{7}}$

35. $\dfrac{-6}{\sqrt[4]{x}}$

36. $\dfrac{-2}{\sqrt[5]{y}}$

37. $\dfrac{7}{\sqrt[3]{4}}$

38. $\dfrac{1}{\sqrt[3]{9}}$

39. $\sqrt[3]{\dfrac{4}{w^2}}$

40. $\sqrt[3]{\dfrac{5}{z^2}}$

41. $\sqrt[4]{\dfrac{16}{3}}$

42. $\sqrt[4]{\dfrac{81}{8}}$

43. $\dfrac{2}{\sqrt[3]{4x^2}}$

44. $\dfrac{6}{\sqrt[3]{3y^2}}$

45. $\dfrac{8}{7\sqrt{24}}$

46. $\dfrac{5}{3\sqrt{50}}$

47. $\dfrac{1}{\sqrt{x^7}}$

48. $\dfrac{1}{\sqrt{y^5}}$

49. $\dfrac{2}{\sqrt{8x^5}}$

50. $\dfrac{6}{\sqrt{27t^7}}$

Concept 4: Rationalizing the Denominator—Two Terms

51. What is the conjugate of $\sqrt{2} - \sqrt{6}$?

52. What is the conjugate of $\sqrt{11} + \sqrt{5}$?

53. What is the conjugate of $\sqrt{x} + 23$?

54. What is the conjugate of $17 - \sqrt{y}$?

For Exercises 55–74, rationalize the denominator. **(See Examples 7–9.)**

55. $\dfrac{4}{\sqrt{2} + 3}$

56. $\dfrac{6}{4 - \sqrt{3}}$

57. $\dfrac{8}{\sqrt{6} - 2}$

58. $\dfrac{-12}{\sqrt{5} - 3}$

59. $\dfrac{\sqrt{7}}{\sqrt{3} + 2}$

60. $\dfrac{\sqrt{8}}{\sqrt{3} + 1}$

61. $\dfrac{-1}{\sqrt{p} + \sqrt{q}}$

62. $\dfrac{6}{\sqrt{a} - \sqrt{b}}$

63. $\dfrac{x - 5}{\sqrt{x} + \sqrt{5}}$

64. $\dfrac{y - 2}{\sqrt{y} - \sqrt{2}}$

65. $\dfrac{\sqrt{w} + 2}{9 - \sqrt{w}}$

66. $\dfrac{10 - \sqrt{t}}{\sqrt{t} - 6}$

67. $\dfrac{3\sqrt{x} - \sqrt{y}}{\sqrt{y} + \sqrt{x}}$

68. $\dfrac{2\sqrt{a} + \sqrt{b}}{\sqrt{b} - \sqrt{a}}$

69. $\dfrac{3\sqrt{10}}{2 + \sqrt{10}}$

70. $\dfrac{4\sqrt{7}}{3 + \sqrt{7}}$

71. $\dfrac{2\sqrt{3} + \sqrt{7}}{3\sqrt{3} - \sqrt{7}}$

72. $\dfrac{5\sqrt{2} - \sqrt{5}}{5\sqrt{2} + \sqrt{5}}$

73. $\dfrac{\sqrt{5} + 4}{2 - \sqrt{5}}$

74. $\dfrac{3 + \sqrt{2}}{\sqrt{2} - 5}$

Mixed Exercises

For Exercises 75–78, write the English phrase as an algebraic expression. Then simplify the expression.

75. 16 divided by the cube root of 4

76. 21 divided by the principal fourth root of 27

77. 4 divided by the difference of x and the principal square root of 2

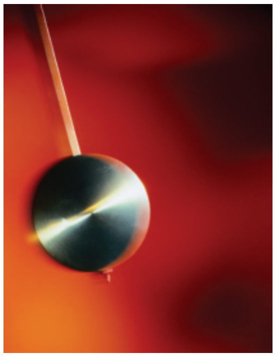
78. 8 divided by the sum of y and the principal square root of 3

79. The time $T(x)$ (in seconds) for a pendulum to make one complete swing back and forth is approximated by

$$T(x) = 2\pi\sqrt{\frac{x}{32}}$$

where x is the length of the pendulum in feet.

Determine the exact time required for one swing for a pendulum that is 1 ft long. Then approximate the time to the nearest hundredth of a second.



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80. An object is dropped off a building x meters tall. The time $T(x)$ (in seconds) required for the object to hit the ground is given by

$$T(x) = \sqrt{\frac{10x}{49}}$$

Find the exact time required for the object to hit the ground if it is dropped off Three First National Plaza in Chicago, a height of 230 m. Then round the time to the nearest hundredth of a second.

For Exercises 81–84, rationalize the denominator.

81. **a.** $\dfrac{1}{\sqrt{2}}$ **b.** $\dfrac{1}{\sqrt[3]{2}}$

82. **a.** $\dfrac{1}{\sqrt[3]{x}}$ **b.** $\dfrac{1}{\sqrt[3]{x^2}}$

83. **a.** $\dfrac{1}{\sqrt{5a}}$ **b.** $\dfrac{1}{\sqrt{5}+a}$

84. **a.** $\dfrac{4}{\sqrt{2x}}$ **b.** $\dfrac{4}{2-\sqrt{x}}$

Expanding Your Skills

For Exercises 85–90, simplify each term of the expression. Then add or subtract as indicated.

85. $\dfrac{\sqrt{6}}{2} + \dfrac{1}{\sqrt{6}}$

86. $\dfrac{1}{\sqrt{7}} + \sqrt{7}$

87. $\sqrt{15} - \sqrt{\dfrac{3}{5}} + \sqrt{\dfrac{5}{3}}$

88. $\sqrt{\dfrac{6}{2}} - \sqrt{12} + \sqrt{\dfrac{2}{6}}$

89. $\sqrt[3]{25} + \dfrac{3}{\sqrt[3]{5}}$

90. $\dfrac{1}{\sqrt[3]{4}} + \sqrt[3]{54}$

For Exercises 91–98, rationalize the numerator by multiplying both numerator and denominator by the conjugate of the numerator.

91. $\dfrac{\sqrt{3}+6}{2}$

92. $\dfrac{\sqrt{7}-2}{5}$

93. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

94. $\dfrac{\sqrt{p}+\sqrt{q}}{\sqrt{p}-\sqrt{q}}$

95. $\dfrac{\sqrt{5+3h}-\sqrt{5}}{h}$

96. $\dfrac{\sqrt{7+2h}-\sqrt{7}}{h}$

97. $\dfrac{\sqrt{4+5h}-2}{h}$

98. $\dfrac{\sqrt{9+4h}-3}{h}$

Problem Recognition Exercises

Operations on Radicals

As you work through the following exercises, you will perform a variety of operations on radicals. For each exercise, if you are unsure what to do, try thinking of an analogy of the same operation on polynomials. For example, for each row of the table, compare the exercise on the left to the exercise on the right. Assume that all variables represent positive real numbers.

$(4x)(5x)$ Product of monomials $= (4 \cdot 5)(x \cdot x)$ $= 20x^2$	$(4\sqrt{x})(5\sqrt{x})$ $= (4 \cdot 5)(\sqrt{x} \cdot \sqrt{x})$ $= 20\sqrt{x^2}$ $= 20x$
$(2a - 5)(2a + 5)$ Product of conjugates $= (2a)^2 - (5)^2$ $= 4a^2 - 25$	$(2\sqrt{a} - 5)(2\sqrt{a} + 5)$ $= (2\sqrt{a})^2 - (5)^2$ $= 4\sqrt{a^2} - 25$ $= 4a - 25$
$(2 - 3x)^2$ Square of a binomial $= (2)^2 - 2(2)(3x) + (3x)^2$ $= 4 - 12x + 9x^2$	$(2 - 3\sqrt{x})^2$ $= (2)^2 - 2(2)(3\sqrt{x}) + (3\sqrt{x})^2$ $= 4 - 12\sqrt{x} + 9\sqrt{x^2}$ $= 4 - 12\sqrt{x} + 9x$
$(4c + 2)(3c + 5)$ Product of polynomials $= (4c + 2)(3c + 5)$ $= (4c)(3c) + (4c)(5) + (2)(3c) + (2)(5)$ $= 12c^2 + 20c + 6c + 10$ $= 12c^2 + 26c + 10$	$(4\sqrt{c} + 2)(3\sqrt{c} + 5)$ $= (4\sqrt{c} + 2)(3\sqrt{c} + 5)$ $= (4\sqrt{c})(3\sqrt{c}) + (4\sqrt{c})(5) + (2)(3\sqrt{c}) + (2)(5)$ $= 12\sqrt{c^2} + 20\sqrt{c} + 6\sqrt{c} + 10$ $= 12c + 26\sqrt{c} + 10$

For Exercises 1–30, simplify each expression. Assume that all variable expressions represent positive real numbers.

1. **a.** $\sqrt{24}$ **b.** $\sqrt[3]{24}$

2. **a.** $\sqrt{54}$ **b.** $\sqrt[3]{54}$

3. **a.** $\sqrt{200y^6}$ **b.** $\sqrt[3]{200y^6}$

4. **a.** $\sqrt{32z^{15}}$ **b.** $\sqrt[3]{32z^{15}}$

5. **a.** $\sqrt{80}$ **b.** $\sqrt[3]{80}$ **c.** $\sqrt[4]{80}$

6. **a.** $\sqrt{48}$ **b.** $\sqrt[3]{48}$ **c.** $\sqrt[4]{48}$

7. **a.** $\sqrt{x^5y^6}$ **b.** $\sqrt[3]{x^5y^6}$ **c.** $\sqrt[4]{x^5y^6}$

8. **a.** $\sqrt{a^{10}b^9}$ **b.** $\sqrt[3]{a^{10}b^9}$ **c.** $\sqrt[4]{a^{10}b^9}$

9. **a.** $\sqrt[3]{32s^5t^6}$ **b.** $\sqrt[4]{32s^5t^6}$ **c.** $\sqrt[5]{32s^5t^6}$

10. **a.** $\sqrt[3]{96v^7w^{20}}$ **b.** $\sqrt[4]{96v^7w^{20}}$ **c.** $\sqrt[5]{96v^7w^{20}}$

11. **a.** $\sqrt{5} + \sqrt{5}$ **b.** $\sqrt{5} \cdot \sqrt{5}$

12. **a.** $\sqrt{10} + \sqrt{10}$ **b.** $\sqrt{10} \cdot \sqrt{10}$

13. **a.** $2\sqrt{6} - 5\sqrt{6}$ **b.** $2\sqrt{6} \cdot 5\sqrt{6}$

14. **a.** $3\sqrt{7} - 10\sqrt{7}$ **b.** $3\sqrt{7} \cdot 10\sqrt{7}$

15. **a.** $\sqrt{8} + \sqrt{2}$ **b.** $\sqrt{8} \cdot \sqrt{2}$

16. **a.** $\sqrt{12} + \sqrt{3}$ **b.** $\sqrt{12} \cdot \sqrt{3}$

17. **a.** $5\sqrt{18} - 4\sqrt{8}$ **b.** $5\sqrt{18} \cdot 4\sqrt{8}$

18. **a.** $\sqrt{50} - \sqrt{72}$ **b.** $\sqrt{50} \cdot \sqrt{72}$

19. **a.** $4\sqrt[3]{24} + 6\sqrt[3]{3}$ **b.** $4\sqrt[3]{24} \cdot 6\sqrt[3]{3}$

20. **a.** $2\sqrt[3]{54} - 5\sqrt[3]{2}$ **b.** $2\sqrt[3]{54} \cdot 5\sqrt[3]{2}$

21. **a.** $(\sqrt{3})(\sqrt{6})$ **b.** $\sqrt{3} + \sqrt{6}$ **c.** $\dfrac{\sqrt{6}}{\sqrt{3}}$

22. **a.** $\dfrac{\sqrt{14}}{\sqrt{2}}$ **b.** $(\sqrt{2})(\sqrt{14})$ **c.** $\sqrt{2} + \sqrt{14}$

23. **a.** $(3\sqrt{z})^2$ **b.** $(3 + \sqrt{z})^2$ **c.** $(3 + \sqrt{z})(3 - \sqrt{z})$

24. **a.** $(4 - \sqrt{x})^2$ **b.** $(4 - \sqrt{x})(4 + \sqrt{x})$ **c.** $(4\sqrt{x})^2$

25. **a.** $\dfrac{12}{\sqrt{2x}}$ **b.** $\sqrt{\dfrac{12}{2x}}$ **c.** $\dfrac{12}{\sqrt{2} + x}$

26. **a.** $\dfrac{15}{3 - \sqrt{y}}$ **b.** $\dfrac{15}{\sqrt{3y}}$ **c.** $\sqrt{\dfrac{15}{3y}}$

27. **a.** $(2\sqrt{5} + 1) + (\sqrt{5} - 2)$ **b.** $(2\sqrt{5} + 1)(\sqrt{5} - 2)$ **c.** $2\sqrt{5}(\sqrt{5} - 2)$

28. **a.** $(4\sqrt{3} - 5)(\sqrt{3} + 4)$ **b.** $4\sqrt{3}(\sqrt{3} + 4)$ **c.** $(4\sqrt{3} - 5) - (\sqrt{3} + 4)$

29. **a.** $\sqrt{16a^{15}}$ **b.** $\sqrt[3]{16a^{15}}$

30. **a.** $\sqrt[3]{27y^9}$ **b.** $\sqrt{27y^9}$

Solving Radical Equations

Section 6.7

Concepts

1. Solutions to Radical Equations
2. Solving Radical Equations Involving One Radical
3. Solving Radical Equations Involving More than One Radical
4. Applications of Radical Equations and Functions

1. Solutions to Radical Equations

An equation with one or more radicals containing a variable is called a **radical equation**. For example, $\sqrt[3]{x} = 5$ is a radical equation. Recall that $(\sqrt[n]{a})^n = a$, provided $\sqrt[n]{a}$ is a real number. The basis of solving a radical equation is to eliminate the radical by raising both sides of the equation to a power equal to the index of the radical.

To solve the equation $\sqrt[3]{x} = 5$, cube both sides of the equation.

$$\sqrt[3]{x} = 5$$

$$(\sqrt[3]{x})^3 = (5)^3$$

$$x = 125$$

By raising each side of a radical equation to a power equal to the index of the radical, a new equation is produced. Note, however, that some of (or all) the solutions to the new equation may *not* be solutions to the original radical equation. For this reason, it is necessary to *check all potential solutions* in the original equation. For example, consider the equation $\sqrt{x} = -7$. This equation has no solution because by definition, the principal square root of x must be a nonnegative number. However, if we square both sides of the equation, it appears as though a solution exists.

$$(\sqrt{x})^2 = (-7)^2$$

$x = 49$ The value 49 does not check in the original equation $\sqrt{x} = -7$. Therefore, 49 is an **extraneous solution**.

Solving a Radical Equation

Step 1 Isolate the radical. If an equation has more than one radical, choose one of the radicals to isolate.

Step 2 Raise each side of the equation to a power equal to the index of the radical.

Step 3 Solve the resulting equation. If the equation still has a radical, repeat steps 1 and 2.

***Step 4** Check the potential solutions in the original equation.

*In solving radical equations, extraneous solutions *potentially occur* only when each side of the equation is raised to an even power.

2. Solving Radical Equations Involving One Radical

Example 1 Solving an Equation Containing One Radical

Solve the equation. $\sqrt{p} + 5 = 9$

Solution:

$\sqrt{p} + 5 = 9$

$\sqrt{p} = 4$ Isolate the radical.

$(\sqrt{p})^2 = 4^2$ Because the index is 2, square both sides.

$p = 16$

Check: $p = 16$ Check $p = 16$ as a potential solution.

$\sqrt{p} + 5 = 9$

$\sqrt{16} + 5 \stackrel{?}{=} 9$

$4 + 5 \stackrel{?}{=} 9$ ✓ True, 16 is a solution to the original equation.

The solution set is $\{16\}$.

Skill Practice Solve.

1. $\sqrt{x} - 3 = 2$

Example 2 Solving an Equation Containing One Radical

Solve the equation. $\sqrt[3]{w-1} - 2 = 2$

Solution:

$\sqrt[3]{w-1} - 2 = 2$

$\sqrt[3]{w-1} = 4$ Isolate the radical.

$(\sqrt[3]{w-1})^3 = (4)^3$ Because the index is 3, cube both sides.

$w - 1 = 64$ Simplify.

$w = 65$

Answer

1. $\{25\}$

Check: $w = 65$

$\sqrt[3]{65 - 1} - 2 \stackrel{?}{=} 2$ Check $w = 65$ as a potential solution.

$\sqrt[3]{64} - 2 \stackrel{?}{=} 2$

$4 - 2 \stackrel{?}{=} 2$ ✓ True, 65 is a solution to the original equation.

The solution set is $\{65\}$.

Skill Practice Solve.

2. $\sqrt[3]{t + 2} + 5 = 3$

Example 3 Solving an Equation Containing One Radical

Solve the equation. $7 = (x + 3)^{1/4} + 9$

Solution:

$7 = (x + 3)^{1/4} + 9$

$7 = \sqrt[4]{x + 3} + 9$ Note that $(x + 3)^{1/4} = \sqrt[4]{x + 3}$.

$-2 = \sqrt[4]{x + 3}$ Isolate the radical.

$(-2)^4 = (\sqrt[4]{x + 3})^4$ Because the index is 4, raise both sides to the fourth power.

$16 = x + 3$

$x = 13$ Solve for x.

Check: $x = 13$

$7 = \sqrt[4]{x + 3} + 9$

$7 \stackrel{?}{=} \sqrt[4]{(13) + 3} + 9$

$7 \stackrel{?}{=} \sqrt[4]{16} + 9$

$7 \stackrel{?}{=} 2 + 9$ (false) 13 is *not* a solution to the original equation.

The equation $7 = \sqrt[4]{x + 3} + 9$ has no solution.

The solution set is the empty set, { }.

TIP: After isolating the radical in Example 3, the equation shows a fourth root equated to a negative number:

$-2 = \sqrt[4]{x + 3}$

By definition, a principal fourth root of any real number must be nonnegative. Therefore, there can be no real solution to this equation.

Skill Practice Solve.

3. $3 = 6 + (x - 1)^{1/4}$

Example 4 Solving an Equation Containing One Radical

Solve the equation. $y + 2\sqrt{4y - 3} = 3$

Solution:

$y + 2\sqrt{4y - 3} = 3$

$2\sqrt{4y - 3} = 3 - y$ Isolate the radical term.

$(2\sqrt{4y - 3})^2 = (3 - y)^2$ Because the index is 2, square both sides.

$4(4y - 3) = 9 - 6y + y^2$ Note that $(2\sqrt{4y - 3})^2 = 2^2(\sqrt{4y - 3})^2$ and $(3 - y)^2 = (3 - y)(3 - y) = 9 - 6y + y^2$.

FOR REVIEW

Recall that the square of a binomial is a perfect square trinomial.

$(a - b)^2 = a^2 - 2ab + b^2$
$(3 - y)^2 = (3)^2 - 2(3)(y) + (y)^2$
$= 9 - 6y + y^2$

Answers

2. $\{-10\}$
3. { } (The value 82 does not check.)

$$16y - 12 = 9 - 6y + y^2$$

$$0 = y^2 - 22y + 21$$ The equation is quadratic. Set one side equal to zero. Write the other side in descending order.

$$0 = (y - 21)(y - 1)$$ Factor.

$y - 21 = 0$ or $y - 1 = 0$ Set each factor equal to zero.

$y = 21$ or $y = 1$ Solve.

Check: $y = 21$

$$y + 2\sqrt{4y - 3} = 3$$

$$21 + 2\sqrt{4(21) - 3} \stackrel{?}{=} 3$$

$$21 + 2\sqrt{81} \stackrel{?}{=} 3$$

$$21 + 18 \stackrel{?}{=} 3$$

$$39 \stackrel{?}{=} 3 \text{ False}$$

Check: $y = 1$

$$y + 2\sqrt{4y - 3} = 3$$

$$1 + 2\sqrt{4(1) - 3} \stackrel{?}{=} 3$$

$$1 + 2\sqrt{1} \stackrel{?}{=} 3$$

$$1 + 2 \stackrel{?}{=} 3$$

$$3 \stackrel{?}{=} 3 \checkmark$$

The solution set is $\{1\}$. (The value 21 does not check.)

Skill Practice Solve.

4. $2\sqrt{m + 3} - m = 3$

3. Solving Radical Equations Involving More than One Radical

Example 5 Solving an Equation With Two Radicals

Solve the equation. $\sqrt[3]{2x - 4} = \sqrt[3]{1 - 8x}$

Solution:

$$\sqrt[3]{2x - 4} = \sqrt[3]{1 - 8x}$$

$$(\sqrt[3]{2x - 4})^3 = (\sqrt[3]{1 - 8x})^3$$ Because the index is 3, cube both sides.

$$2x - 4 = 1 - 8x$$ Simplify.

$$10x - 4 = 1$$ Solve the resulting equation.

$$10x = 5$$

$$x = \frac{1}{2}$$ Solve for x.

Check: $x = \frac{1}{2}$

$$\sqrt[3]{2x - 4} = \sqrt[3]{1 - 8x}$$

$$\sqrt[3]{2\left(\frac{1}{2}\right) - 4} \stackrel{?}{=} \sqrt[3]{1 - 8\left(\frac{1}{2}\right)}$$

$$\sqrt[3]{1 - 4} \stackrel{?}{=} \sqrt[3]{1 - 4}$$

$$\sqrt[3]{-3} \stackrel{?}{=} \sqrt[3]{-3} \checkmark \text{ (True)}$$ Therefore, $\frac{1}{2}$ is a solution to the original equation.

The solution set is $\left\{\frac{1}{2}\right\}$.

Answer

4. $\{-3, 1\}$

Skill Practice Solve.

5. $\sqrt[5]{2y-1} = \sqrt[5]{10y+3}$

Example 6 Solving an Equation With Two Radicals

Solve the equation. $\sqrt{3m+1} - \sqrt{m+4} = 1$

Solution:

$$\sqrt{3m+1} - \sqrt{m+4} = 1$$

$$\sqrt{3m+1} = \sqrt{m+4} + 1 \qquad \text{Isolate one of the radicals.}$$

$$(\sqrt{3m+1})^2 = (\sqrt{m+4} + 1)^2 \qquad \text{Square both sides.}$$

$$3m + 1 = m + 4 + 2\sqrt{m+4} + 1$$

Note: $(\sqrt{m+4} + 1)^2$

$$= (\sqrt{m+4})^2 + 2(1)\sqrt{m+4} + (1)^2$$

$$= m + 4 + 2\sqrt{m+4} + 1$$

$$3m + 1 = m + 5 + 2\sqrt{m+4} \qquad \text{Combine } like \text{ terms.}$$

$$2m - 4 = 2\sqrt{m+4} \qquad \text{Isolate the radical again.}$$

$$m - 2 = \sqrt{m+4} \qquad \text{Divide both sides by 2.}$$

$$(m-2)^2 = (\sqrt{m+4})^2 \qquad \text{Square both sides again.}$$

$$m^2 - 4m + 4 = m + 4 \qquad \text{The resulting equation is quadratic.}$$

$$m^2 - 5m = 0 \qquad \text{Set one side equal to zero.}$$

$$m(m-5) = 0 \qquad \text{Factor.}$$

$$m = 0 \quad \text{or} \quad m = 5$$

Check: $m = 0$

$$\sqrt{3(0)+1} - \sqrt{(0)+4} \stackrel{?}{=} 1$$

$$\sqrt{1} - \sqrt{4} \stackrel{?}{=} 1$$

$$1 - 2 \stackrel{?}{=} 1 \quad \text{(False)}$$

Check: $m = 5$

$$\sqrt{3(5)+1} - \sqrt{(5)+4} = 1$$

$$\sqrt{16} - \sqrt{9} \stackrel{?}{=} 1$$

$$4 - 3 \stackrel{?}{=} 1 \checkmark$$

The solution set is $\{5\}$. (The value 0 does not check.)

Skill Practice Solve.

6. $\sqrt{3c+1} - \sqrt{c-1} = 2$

TIP: In Example 6, we divided the equation by 2 because all coefficients were divisible by 2. This makes the coefficients smaller before we square both sides of the equation a second time.

Answers

5. $\left\{-\frac{1}{2}\right\}$ **6.** $\{1, 5\}$

Example 7 **Solving an Equation With Two Radicals**

Solve the equation. $\sqrt{5x+4} = 1 + \sqrt{5x-3}$

Solution:

$$\sqrt{5x+4} = 1 + \sqrt{5x-3} \qquad \text{One radical is already isolated.}$$

$$(\sqrt{5x+4})^2 = (1 + \sqrt{5x-3})^2 \qquad \text{Square both sides.}$$

$$5x + 4 = 1 + 2\sqrt{5x-3} + 5x - 3 \qquad \textit{Note: } (1+\sqrt{5x-3})^2 = (1)^2 + 2(1)\sqrt{5x-3} + (\sqrt{5x-3})^2$$

$$5x + 4 = 2\sqrt{5x-3} + 5x - 2 \qquad \text{Combine } \textit{like} \text{ terms.}$$

$$4 = 2\sqrt{5x-3} - 2 \qquad \text{Subtract } 5x \text{ from both sides.}$$

$$6 = 2\sqrt{5x-3} \qquad \text{Isolate the radical.}$$

$$3 = \sqrt{5x-3} \qquad \text{Divide both sides by 2.}$$

$$(3)^2 = (\sqrt{5x-3})^2 \qquad \text{Square both sides again.}$$

$$9 = 5x - 3 \qquad \text{The resulting equation is linear.}$$

$$12 = 5x \qquad \text{Solve for } x.$$

$$\frac{12}{5} = x \qquad \underline{\text{Check}}: x = \frac{12}{5}$$

$$\sqrt{5\left(\frac{12}{5}\right)+4} \stackrel{?}{=} 1 + \sqrt{5\left(\frac{12}{5}\right)-3}$$

$$\sqrt{16} \stackrel{?}{=} 1 + \sqrt{9}$$

$$4 \stackrel{?}{=} 1 + 3 \quad \checkmark \text{ True}$$

The solution set is $\left\{\frac{12}{5}\right\}$.

Skill Practice Solve.

7. $\sqrt{4x-3} = 2 - \sqrt{4x+1}$

4. Applications of Radical Equations and Functions

Example 8 **Applying a Radical Equation in Geometry**

For a pyramid with a square base, the length of a side of the base b is given by

$$b = \sqrt{\frac{3V}{h}}$$

where V is the volume and h is the height.

The Pyramid of the Pharaoh Khufu (known as the Great Pyramid) at Giza has a square base (Figure 6-3). If the distance around the bottom of the pyramid is 921.6 m and the height is 146.6 m, what is the volume of the pyramid?

Waj/Shutterstock

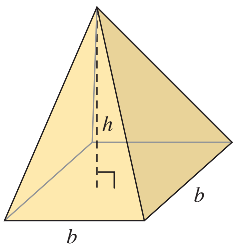


Figure 6-3

Answer

7. $\left\{\frac{3}{4}\right\}$

Solution:

$$b = \sqrt{\frac{3V}{h}}$$

$$b^2 = \left(\sqrt{\frac{3V}{h}}\right)^2 \qquad \text{Because the index is 2, square both sides.}$$

$$b^2 = \frac{3V}{h} \qquad \text{Simplify.}$$

$$b^2 \cdot h = \frac{3V}{\cancel{h}} \cdot \cancel{h} \qquad \text{Multiply both sides by } h.$$

$$b^2h = 3V$$

$$\frac{b^2h}{3} = \frac{\cancel{3}V}{\cancel{3}} \qquad \text{Divide both sides by 3.}$$

$$\frac{b^2h}{3} = V$$

The length of a side b (in meters) is given by $\frac{921.6}{4} = 230.4$ m.

$$\frac{(230.4)^2(146.6)}{3} = V \qquad \text{Substitute } b = 230.4 \text{ and } h = 146.6.$$

$$2{,}594{,}046 \approx V$$

The volume of the Great Pyramid at Giza is approximately 2,594,046 m^3.

Skill Practice

8. The length of the legs, s, of an isosceles right triangle is $s = \sqrt{2A}$, where A is the area. If the legs of the triangle are 9 in., find the area.

Example 9 Applying a Radical Function

On a certain surface, the speed $s(x)$ (in miles per hour) of a car before the brakes were applied can be approximated from the length of its skid marks x (in feet) by

$$s(x) = 3.8\sqrt{x} \qquad x \geq 0 \qquad \text{See Figure 6-4.}$$

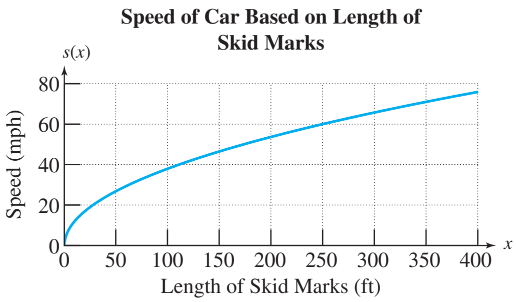


Figure 6-4

a. Find the speed of a car before the brakes were applied if its skid marks are 361 ft long.

b. How long would you expect the skid marks to be if the car had been traveling the speed limit of 50 mph? (Round to the nearest foot.)

Answer

8. 40.5 in.^{2}

Solution:

a. $s(x) = 3.8\sqrt{x}$

$s(361) = 3.8\sqrt{361}$ Substitute $x = 361$.

$= 3.8(19)$

$= 72.2$

If the skid marks are 361 ft, the car was traveling approximately 72.2 mph before the brakes were applied.

b. $s(x) = 3.8\sqrt{x}$

$50 = 3.8\sqrt{x}$ Substitute $s(x) = 50$ and solve for x.

$\dfrac{50}{3.8} = \sqrt{x}$ Isolate the radical.

$\left(\dfrac{50}{3.8}\right)^2 = x$

$x \approx 173$

If the car had been going the speed limit (50 mph), then the length of the skid marks would have been approximately 173 ft.

Skill Practice When an object is dropped from a height of 64 ft, the time $t(x)$ (in seconds) it takes to reach a height x (in feet) is given by

$$t(x) = \frac{1}{4}\sqrt{64 - x}$$

9. Find the time to reach a height of 28 ft from the ground.

10. What is the height after 1 sec?

Answers

9. $\frac{3}{2}$ sec **10.** 48 ft

Section 6.7 Activity

A.1. Consider the equations $x - 5 = 4$ and $3x = 6$.

a. To isolate the variable, we perform the *inverse* operation to the operation acting on x. For example, to solve $x - 5 = 4$, we would add 5 to both sides of the equation. How would you solve $3x = 6$?

b. By analogy, how would you solve the equation $\sqrt[3]{x} = 4$?

A.2. Consider the equation $\sqrt{x} = -5$.

a. By inspection, the solution set is ________ because no real number x has a negative square root.

b. However, if you were to square both sides of the equation $\sqrt{x} = -5$, the resulting equation would be ________. Does $x = 25$ check in the original equation $\sqrt{x} = -5$?

c. Part (b) illustrates that if we square both sides of an equation, we must check the potential solution(s). Likewise, if we raise both sides of an equation to an (choose one: even/odd) power, we must check the potential solutions.

A.3. **a.** Consider the equation $2x - 3 = 5$. What is the first step to solve the equation? What is the second step to solve the equation?

b. Consider the equation $2\sqrt{x} - 3 = 5$. What is the first step to solve the equation? What is the second step to solve the equation?

c. Solve the equation. $2\sqrt{x} - 3 = 5$

d. Solve the equation. $2\sqrt{3x + 1} - 3 = 5$

e. Solve the equation. $2\sqrt[3]{3x+1} - 3 = 5$

f. Solve the equation. $2(3x+1)^{1/4} - 3 = 5$

g. For which equations from parts (c)–(f) was it necessary to check the potential solution? Why?

A.4. **a.** Square the binomial. $(5-x)^2$

b. Solve the equation. $\sqrt{x-3} - 5 = -x$

A.5. The equation $\sqrt{5x-3} = \sqrt{x+1}$ contains two square roots, and each square root is isolated on one side of the equation. Solve the equation and check the potential solution.

A.6. The equation $\sqrt{x+2} + \sqrt{2x+5} = 1$ contains two square roots, but because the equation has a third term, it is not possible to isolate both square roots simultaneously. This significantly complicates the process to solve the equation. Let's walk through the process using the following steps.

a. First, we want to isolate one of the radicals. Write the equation with $\sqrt{x+2}$ isolated on the left side of the equation.

b. Square both sides of the equation and write the new equation.

c. The right side of the equation still has a radical. Isolate the term $-2\sqrt{2x+5}$ on the right side of the equation and write the new equation.

d. To make the equation a bit simpler, multiply both sides by –1 and write the new equation.

e. The term on the right side of the equation is $2\sqrt{2x+5}$. If we divided by 2 on both sides to isolate the radical, we would create fractional coefficients on the left. Instead, square both sides of the equation as is and write the new equation.

f. The new equation is quadratic. Solve this equation, but do not write the final solution set yet.

g. The solutions to the equation in part (f) are the *potential* solutions to the original equation $\sqrt{x+2} + \sqrt{2x+5} = 1$. Check the solutions from part (f) in the original equation and write the final solution set.

Practice Exercises — Section 6.7

Prerequisite Review

For Exercises R.1–R.6, square the expression as indicated. Assume all variable expressions represent positive real numbers.

R.1. $(t+8)^2$ **R.2.** $(y-10)^2$ **R.3.** $(\sqrt{3t+1})^2$

R.4. $(\sqrt{9-w})^2$ **R.5.** $(\sqrt{m}-7)^2$ **R.6.** $(3+\sqrt{x})^2$

For Exercises R.7–R.12, solve the equation.

R.7. $3y - 3 = 12$ **R.8.** $4w + 1 = 33$ **R.9.** $x^2 - 4x - 21 = 0$

R.10. $x^2 + 13x + 30 = 0$ **R.11.** $10x^2 - 7 = 33x$ **R.12.** $6x^2 - 4 = 5x$

Vocabulary and Key Concepts

1. **a.** The equation $\sqrt{x+5} + 7 = 11$ is an example of a ________ equation.

b. The first step to solve the equation $\sqrt{x+5} + 7 = 11$ is to ________ the radical by subtracting ________ from both sides of the equation.

c. When solving a radical equation, some potential solutions may not check in the original equation. These are called ________ solutions.

2. **a.** To solve the equation $\sqrt[3]{w-1} = 5$, raise both sides of the equation to the ________ power.

b. To solve the equation $\sqrt[4]{m-3} = 2$, raise both sides of the equation to the ________ power.

Concept 2: Solving Radical Equations Involving One Radical

For Exercises 3–10, match the equation with the best first step used to solve the equation.

3. $\sqrt{x+1} = 5$

4. $2\sqrt{x+1} = 5$

5. $x + 1 = 5$

6. $(x+1)^{1/4} = 5$

7. $\sqrt{x+1} - 2 = 5$

8. $\sqrt[3]{x+1} = 5$

9. $\dfrac{\sqrt{x+1}}{2} = 5$

10. $\sqrt{x+1} + 2 = 5$

a. Raise both sides to the fourth power.
b. Divide both sides by 2.
c. Subtract 2 from both sides.
d. Square both sides.
e. Subtract 1 from both sides.
f. Multiply both sides by 2.
g. Cube both sides.
h. Add 2 to both sides.

For Exercises 11–30, solve the equations. **(See Examples 1–3.)**

11. $\sqrt{x} = 10$ **12.** $\sqrt{y} = 7$ **13.** $\sqrt{x} + 4 = 6$ **14.** $\sqrt{x} + 2 = 8$

15. $\sqrt{5y+1} = 4$ **16.** $\sqrt{9z-5} - 2 = 9$ **17.** $6 = \sqrt{2z-3} - 3$ **18.** $4 = \sqrt{8+3a} - 1$

19. $\sqrt{x^2+5} = x + 1$ **20.** $\sqrt{y^2-8} = y - 2$ **21.** $\sqrt[3]{x-2} - 1 = 2$ **22.** $\sqrt[3]{2x-5} - 1 = 1$

23. $(15-w)^{1/3} + 7 = 2$ **24.** $(k+18)^{1/3} + 5 = 3$ **25.** $3 + \sqrt{x-16} = 0$ **26.** $12 + \sqrt{2x+1} = 0$

27. $2\sqrt{6a+7} - 2a = 0$ **28.** $2\sqrt{3-w} - w = 0$ **29.** $(2x-5)^{1/4} = -1$ **30.** $(x+16)^{1/4} = -4$

For Exercises 31–34, assume all variables represent positive real numbers.

31. Solve for V: $r = \sqrt[3]{\dfrac{3V}{4\pi}}$ **32.** Solve for V: $r = \sqrt{\dfrac{V}{h\pi}}$

33. Solve for h^2: $r = \pi\sqrt{r^2+h^2}$ **34.** Solve for d: $s = 1.3\sqrt{d}$

For Exercises 35–40, square the expression as indicated.

35. $(a+5)^2$ **36.** $(b+7)^2$ **37.** $(\sqrt{5a} - 3)^2$

38. $(2 + \sqrt{b})^2$ **39.** $(\sqrt{r-3} + 5)^2$ **40.** $(2 - \sqrt{2t-4})^2$

For Exercises 41–46, solve the radical equations, if possible. **(See Example 4.)**

41. $\sqrt{a^2+2a+1} = a + 5$ **42.** $\sqrt{b^2-5b-8} = b + 7$ **43.** $\sqrt{25w^2-2w-3} = 5w - 4$

44. $\sqrt{4p^2-2p+1} = 2p - 3$ **45.** $4\sqrt{p-2} - 2 = -p$ **46.** $x - 3\sqrt{x-5} = 5$

Concept 3: Solving Radical Equations Involving More than One Radical

For Exercises 47–70, solve the radical equations, if possible. **(See Examples 5–7.)**

47. $\sqrt[4]{h+4} = \sqrt[4]{2h-5}$ **48.** $\sqrt[4]{3b+6} = \sqrt[4]{7b-6}$ **49.** $\sqrt[3]{5a+3} - \sqrt[3]{a-13} = 0$

50. $\sqrt[3]{k-8} - \sqrt[3]{4k+1} = 0$ **51.** $\sqrt{5a-9} = \sqrt{5a} - 3$ **52.** $\sqrt{8+b} = 2 + \sqrt{b}$

53. $\sqrt{2h+5} - \sqrt{2h} = 1$ **54.** $\sqrt{3k-5} - \sqrt{3k} = -1$ **55.** $(t-9)^{1/2} - t^{1/2} = 3$

56. $(y-16)^{1/2} - y^{1/2} = 4$ **57.** $6 = \sqrt{x^2+3} - x$ **58.** $2 = \sqrt{y^2+5} - y$

59. $\sqrt{3t-7} = 2 - \sqrt{3t+1}$ **60.** $\sqrt{p-6} = \sqrt{p+2} - 4$ **61.** $\sqrt{z+1} + \sqrt{2z+3} = 1$

62. $\sqrt{2y+6} = \sqrt{7-2y} + 1$

63. $\sqrt{6m+7} - \sqrt{3m+3} = 1$

64. $\sqrt{5w+1} - \sqrt{3w} = 1$

65. $2 + 2\sqrt{2t+3} + 2\sqrt{3t-5} = 0$

66. $6 + 3\sqrt{3x+1} + 3\sqrt{x-1} = 0$

67. $3\sqrt{y-3} = \sqrt{4y+3}$

68. $\sqrt{5x-8} = 2\sqrt{x-1}$

69. $\sqrt{p+7} = \sqrt{2p} + 1$

70. $\sqrt{t} = \sqrt{t-12} + 2$

Concept 4: Applications of Radical Equations and Functions

71. If an object is dropped from an initial height h, its velocity at impact with the ground is given by

$$v = \sqrt{2gh}$$

where g is the acceleration due to gravity and h is the initial height. **(See Example 8.)**

a. Find the initial height (in feet) of an object if its velocity at impact is 44 ft/sec. (Assume that the acceleration due to gravity is $g = 32$ ft/sec^2.)

b. Find the initial height (in meters) of an object if its velocity at impact is 26 m/sec. (Assume that the acceleration due to gravity is $g = 9.8$ m/sec^2.) Round to the nearest tenth of a meter.

72. The time T (in seconds) required for a pendulum to make one complete swing back and forth is approximated by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where g is the acceleration due to gravity and L is the length of the pendulum (in feet).

a. Find the length of a pendulum that requires 1.36 sec to make one complete swing back and forth. (Assume that the acceleration due to gravity is $g = 32$ ft/sec^2.) Round to the nearest tenth of a foot.

b. Find the time required for a pendulum to complete one swing back and forth if the length of the pendulum is 4 ft. (Assume that the acceleration due to gravity is $g = 32$ ft/sec^2.) Round to the nearest tenth of a second.

73. The airline cost for x thousand passengers to travel round trip from New York to Atlanta is given by

$$C(x) = \sqrt{0.3x + 1}$$

where $C(x)$ is measured in millions of dollars and $x \geq 0$. **(See Example 9.)**

a. Find the airline's cost for 10,000 passengers ($x = 10$) to travel from New York to Atlanta.

b. If the airline charges $320 per passenger, find the profit made by the airline for flying 10,000 passengers from New York to Atlanta.

c. Approximate the number of passengers who traveled from New York to Atlanta if the total cost for the airline was $4 million.

74. The time $t(d)$ in seconds it takes an object to drop d meters is given by

$$t(d) = \sqrt{\frac{d}{4.9}}$$

a. Approximate the height of the JP Morgan Chase Tower in Houston if it takes an object 7.89 sec to drop from the top. Round to the nearest meter.

b. Approximate the height of the Willis Tower in Chicago if it takes an object 9.51 sec to drop from the top. Round to the nearest meter.

David Forman/Image Source/
Getty Images

Technology Connections

75. Graph Y_1 and Y_2 on a viewing window defined by $-10 \le x \le 40$ and $-5 \le y \le 10$.

$$Y_1 = \sqrt{2x} \qquad \text{and} \qquad Y_2 = 8$$

Use an *Intersect* feature to approximate the x-coordinate of the point of intersection of the two graphs. How does the point of intersection relate to the solution to the equation $\sqrt{2x} = 8$?

76. Graph Y_1 and Y_2 on a viewing window defined by $-10 \le x \le 20$ and $-5 \le y \le 10$.

$$Y_1 = \sqrt{4x} \qquad \text{and} \qquad Y_2 = 6$$

Use an *Intersect* feature to approximate the x-coordinate of the point of intersection of the two graphs. How does the point of intersection relate to the solution to the equation $\sqrt{4x} = 6$?

Expanding Your Skills

77. The number of hours needed to cook a turkey that weighs x pounds can be approximated by

$$t(x) = 0.90\sqrt[5]{x^3}$$

where $t(x)$ is the time in hours and x is the weight of the turkey in pounds.

a. Find the weight of a turkey that cooked for 4 hr. Round to the nearest pound.

b. Find $t(18)$ and interpret the result. Round to the nearest tenth of an hour.

For Exercises 78–81, use the Pythagorean theorem to find a, b, or c.

78. Find b when $a = 2$ and $c = y$.

79. Find b when $a = h$ and $c = 5$.

80. Find a when $b = x$ and $c = 8$.

81. Find a when $b = 14$ and $c = k$.

Section 6.8 Complex Numbers

Concepts

1. Definition of *i*
2. Powers of *i*
3. Definition of a Complex Number
4. Addition, Subtraction, and Multiplication of Complex Numbers
5. Division and Simplification of Complex Numbers

1. Definition of *i*

We have already learned that there are no real-valued square roots of a negative number. For example, $\sqrt{-9}$ is not a real number because no real number when squared equals -9. However, the square roots of a negative number are defined over another set of numbers called the **imaginary numbers**. The foundation of the set of imaginary numbers is the definition of the imaginary number i.

Definition of the Imaginary Number *i*

$$\boldsymbol{i} = \sqrt{-1}$$

Note: From the definition of i, it follows that $i^2 = -1$.

Using the imaginary number i, we can define the square root of any negative real number.

Definition of $\sqrt{-b}$ for $b > 0$

Let b be a positive real number. Then $\sqrt{-b} = i\sqrt{b}$.

Example 1 **Simplifying Expressions in Terms of *i***

Simplify the expressions in terms of i.

a. $\sqrt{-64}$ **b.** $\sqrt{-50}$ **c.** $-\sqrt{-4}$ **d.** $\sqrt{-29}$

Solution:

a. $\sqrt{-64} = i\sqrt{64}$
$= 8i$

b. $\sqrt{-50} = i\sqrt{50}$
$= i\sqrt{5^2 \cdot 2}$
$= 5i\sqrt{2}$

c. $-\sqrt{-4} = -1 \cdot \sqrt{-4}$
$= -1 \cdot i\sqrt{4}$
$= -1 \cdot 2i$
$= -2i$

d. $\sqrt{-29} = i\sqrt{29}$

Avoiding Mistakes

In an expression such as $i\sqrt{29}$, the i is often written in front of the square root. The expression $\sqrt{29}i$ is also correct, but may be misinterpreted as $\sqrt{29i}$ (with i incorrectly placed under the radical).

Skill Practice Simplify the expressions in terms of i.

1. $\sqrt{-81}$ **2.** $\sqrt{-20}$ **3.** $-\sqrt{-36}$ **4.** $\sqrt{-7}$

If a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad b \neq 0$$

The conditions that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ must both be real numbers prevent us from applying the multiplication and division properties of radicals for square roots with a negative radicand. Therefore, to multiply or divide radicals with a negative radicand, first write the radical in terms of the imaginary number i. This is demonstrated in Example 2.

Example 2 **Simplifying a Product or Quotient in Terms of *i***

Simplify the expressions.

a. $\dfrac{\sqrt{-100}}{\sqrt{-25}}$ **b.** $\sqrt{-25} \cdot \sqrt{-9}$ **c.** $\sqrt{-5} \cdot \sqrt{-5}$

Answers
1. $9i$ **2.** $2i\sqrt{5}$ **3.** $-6i$ **4.** $i\sqrt{7}$

Solution:

a. $\dfrac{\sqrt{-100}}{\sqrt{-25}}$

$= \dfrac{10i}{5i}$ Simplify each radical in terms of i *before* dividing.

$= 2$

b. $\sqrt{-25} \cdot \sqrt{-9}$

$= 5i \cdot 3i$ Simplify each radical in terms of i first *before* multiplying.

$= 15i^2$ Multiply.

$= 15(-1)$ Recall that $i^2 = -1$.

$= -15$ Simplify.

c. $\sqrt{-5} \cdot \sqrt{-5}$

$= i\sqrt{5} \cdot i\sqrt{5}$

$= i^2 \cdot (\sqrt{5})^2$

$= -1 \cdot 5$

$= -5$

Skill Practice Simplify the expressions.

5. $\dfrac{\sqrt{-36}}{\sqrt{-9}}$ **6.** $\sqrt{-16} \cdot \sqrt{-49}$ **7.** $\sqrt{-2} \cdot \sqrt{-2}$

Avoiding Mistakes

In Example 2, we wrote the radical expressions in terms of *i* first, before multiplying or dividing. If we had mistakenly applied the multiplication or division property first, we would have obtained an incorrect answer.

Correct: $\sqrt{-25} \cdot \sqrt{-9}$

$= (5i)(3i) = 15i^2$

$= 15(-1) = -15$

correct

Be careful: $\sqrt{-25} \cdot \sqrt{-9}$

$\neq \sqrt{225} = 15$

(incorrect answer)

$\sqrt{-25}$ and $\sqrt{-9}$ are not real numbers. Therefore, the multiplication property of radicals cannot be applied.

2. Powers of *i*

From the definition of $i = \sqrt{-1}$, it follows that

$i = i$

$i^2 = -1$

$i^3 = -i$ because $i^3 = i^2 \cdot i = (-1)i = -i$

$i^4 = 1$ because $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^5 = i$ because $i^5 = i^4 \cdot i = (1)i = i$

$i^6 = -1$ because $i^6 = i^4 \cdot i^2 = (1)(-1) = -1$

Answers

5. 2 **6.** −28 **7.** −2

This pattern of values $i, -1, -i, 1, i, -1, -i, 1, \ldots$ continues for all subsequent powers of i. Table 6-1 lists several powers of i.

Table 6-1 Powers of *i*

$i^1 = i$	$i^5 = i$	$i^9 = i$
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1$

To simplify higher powers of i, we can decompose the expression into multiples of i^4 ($i^4 = 1$) and write the remaining factors as i, i^2, or i^3.

Example 3 Simplifying Powers of *i*

Simplify the powers of i.

a. i^{13} **b.** i^{18} **c.** i^{107} **d.** i^{32}

Solution:

a. $i^{13} = (i^{12}) \cdot (i)$ Write the exponent as a multiple of 4 and a remainder.

$= (i^4)^3 \cdot (i)$

$= (1)^3(i)$ Recall that $i^4 = 1$.

$= i$ Simplify.

b. $i^{18} = (i^{16}) \cdot (i^2)$ Write the exponent as a multiple of 4 and a remainder.

$= (i^4)^4 \cdot (i^2)$

$= (1)^4 \cdot (-1)$ $i^4 = 1$ and $i^2 = -1$

$= -1$ Simplify.

c. $i^{107} = (i^{104}) \cdot (i^3)$ Write the exponent as a multiple of 4 and a remainder.

$= (i^4)^{26}(i^3)$

$= (1)^{26}(-i)$ $i^4 = 1$ and $i^3 = -i$

$= -i$ Simplify.

d. $i^{32} = (i^4)^8$

$= (1)^8$ $i^4 = 1$

$= 1$ Simplify.

Skill Practice Simplify the powers of i.

8. i^{45} **9.** i^{22} **10.** i^{31} **11.** i^{80}

3. Definition of a Complex Number

We have already learned the definitions of the integers, rational numbers, irrational numbers, and real numbers. In this section, we define the complex numbers.

Answers

8. i **9.** -1
10. $-i$ **11.** 1

Definition of a Complex Number

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

Notes:

- If $b = 0$, then the complex number $a + bi$ is a real number.
- If $b \neq 0$, then we say that $a + bi$ is an imaginary number.
- The complex number $a + bi$ is said to be written in standard form. The quantities a and b are called the real and imaginary parts (respectively) of the complex number.
- The complex numbers $a - bi$ and $a + bi$ are called **complex conjugates**.

From the definition of a complex number, it follows that all real numbers are complex numbers and all imaginary numbers are complex numbers. Figure 6-5 illustrates the relationship among the sets of numbers we have learned so far.

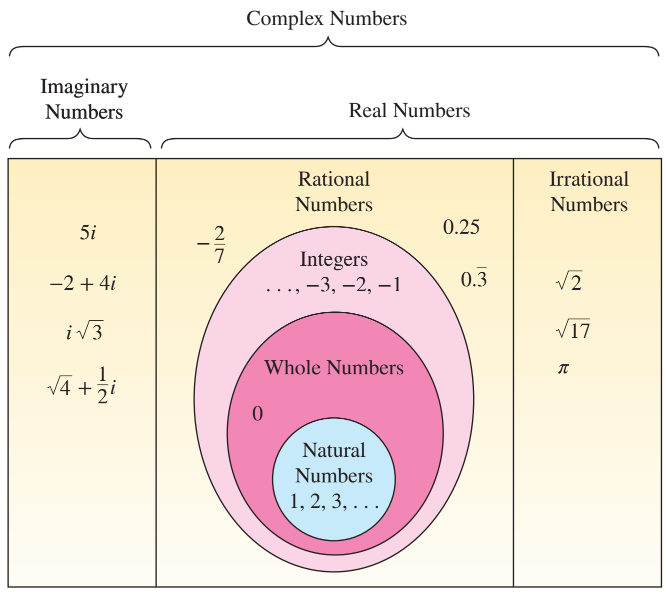


Figure 6-5

Example 4 Identifying the Real and Imaginary Parts of a Complex Number

Identify the real and imaginary parts of the complex numbers.

a. $-8 + 2i$ **b.** $\dfrac{3}{2}$ **c.** $-1.75i$

Solution:

a. $-8 + 2i$ — -8 is the real part, and 2 is the imaginary part.

b. $\dfrac{3}{2} = \dfrac{3}{2} + 0i$ — Rewrite $\frac{3}{2}$ in the form $a + bi$. $\frac{3}{2}$ is the real part, and 0 is the imaginary part.

c. $-1.75i$

$= 0 + -1.75i$ — Rewrite $-1.75i$ in the form $a + bi$. 0 is the real part, and -1.75 is the imaginary part.

Skill Practice Identify the real and imaginary parts of the complex numbers.

12. $22 - 14i$ **13.** -50 **14.** $15i$

TIP: Example 4(b) illustrates that a real number is also a complex number.

$$\frac{3}{2} = \frac{3}{2} + 0i$$

Example 4(c) illustrates that an imaginary number is also a complex number.

$$-1.75i = 0 + -1.75i$$

4. Addition, Subtraction, and Multiplication of Complex Numbers

The operations of addition, subtraction, and multiplication of real numbers also apply to imaginary numbers. To add or subtract complex numbers, combine the real parts and combine the imaginary parts. The commutative, associative, and distributive properties that apply to real numbers also apply to complex numbers.

Example 5 Adding and Subtracting Complex Numbers

Add or subtract as indicated. Write the answers in the form $a + bi$.

a. $(1 - 5i) + (-3 + 7i)$ **b.** $\left(-\frac{1}{4} + \frac{3}{5}i\right) - \left(\frac{1}{2} - \frac{1}{10}i\right)$

c. $\sqrt{-8} + \sqrt{-18}$

Solution:

real parts

a. $(1 - 5i) + (-3 + 7i) = (1 + -3) + (-5 + 7)i$ Add the real parts. Add the imaginary parts.

imaginary parts

$= -2 + 2i$ Simplify.

b. $\left(-\frac{1}{4} + \frac{3}{5}i\right) - \left(\frac{1}{2} - \frac{1}{10}i\right) = -\frac{1}{4} + \frac{3}{5}i - \frac{1}{2} + \frac{1}{10}i$ Apply the distributive property.

$= \left(-\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{3}{5} + \frac{1}{10}\right)i$ Add real parts. Add imaginary parts.

$= \left(-\frac{1}{4} - \frac{2}{4}\right) + \left(\frac{6}{10} + \frac{1}{10}\right)i$ Get common denominators.

$= -\frac{3}{4} + \frac{7}{10}i$ Simplify.

c. $\sqrt{-8} + \sqrt{-18} = 2i\sqrt{2} + 3i\sqrt{2}$ Simplify each radical in terms of i.

$= 5i\sqrt{2}$ Combine *like* radicals.

Skill Practice Perform the indicated operations.

15. $\left(\frac{1}{2} - \frac{1}{4}i\right) + \left(\frac{3}{5} + \frac{2}{3}i\right)$ **16.** $(-6 + 11i) - (-9 - 12i)$ **17.** $\sqrt{-45} - \sqrt{-20}$

Answers

12. real: 22; imaginary: −14
13. real: −50; imaginary: 0
14. real: 0; imaginary: 15
15. $\frac{11}{10} + \frac{5}{12}i$ **16.** $3 + 23i$
17. $i\sqrt{5}$

Example 6 Multiplying Complex Numbers

Multiply the complex numbers. Write the answers in the form $a + bi$.

a. $(10 - 5i)(2 + 3i)$ **b.** $(1.2 + 0.5i)(1.2 - 0.5i)$

Solution:

a. $(10 - 5i)(2 + 3i)$

$= (10)(2) + (10)(3i) + (-5i)(2) + (-5i)(3i)$ Apply the distributive property.

$= 20 + 30i - 10i - 15i^2$

$= 20 + 20i - (15)(-1)$ Recall $i^2 = -1$.

$= 20 + 20i + 15$

$= 35 + 20i$ Write in the form $a + bi$.

b. $(1.2 + 0.5i)(1.2 - 0.5i)$

The expressions $(1.2 + 0.5i)$ and $(1.2 - 0.5i)$ are complex conjugates. The product is a difference of squares.

$(a + b)(a - b) = a^2 - b^2$ Apply the formula, where $a = 1.2$ and $b = 0.5i$.

$(1.2 + 0.5i)(1.2 - 0.5i) = (1.2)^2 - (0.5i)^2$

$= 1.44 - 0.25i^2$

$= 1.44 - 0.25(-1)$ Recall $i^2 = -1$.

$= 1.44 + 0.25$

$= 1.69 + 0i$

Skill Practice Multiply.

18. $(4 - 6i)(2 - 3i)$ **19.** $(1.5 + 0.8i)(1.5 - 0.8i)$

5. Division and Simplification of Complex Numbers

The product of a complex number and its complex conjugate is a real number. For example:

$$
\begin{aligned}
(5 + 3i)(5 - 3i) &= 25 - 9i^2 \\
&= 25 - 9(-1) \\
&= 25 + 9 \\
&= 34
\end{aligned}
$$

To divide by a complex number, multiply the numerator and denominator by the complex conjugate of the denominator. This produces a real number in the denominator so that the resulting expression can be written in the form $a + bi$.

Answers

18. $-10 - 24i$ **19.** $2.89 + 0i$

Example 7 Dividing by a Complex Number

Divide the complex numbers and write the answer in the form $a + bi$.

$$\frac{4-3i}{5+2i}$$

Solution:

$$\frac{4-3i}{5+2i}$$ Multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{(4-3i)}{(5+2i)} \cdot \frac{(5-2i)}{(5-2i)} = \frac{(4)(5)+(4)(-2i)+(-3i)(5)+(-3i)(-2i)}{(5)^2-(2i)^2}$$

$$= \frac{20-8i-15i+6i^2}{25-4i^2}$$ Simplify numerator and denominator.

$$= \frac{20-23i+6(-1)}{25-4(-1)}$$ Recall $i^2 = -1$.

$$= \frac{20-23i-6}{25+4}$$

$$= \frac{14-23i}{29}$$ Simplify.

$$= \frac{14}{29} - \frac{23}{29}i$$ Write in the form $a + bi$.

Skill Practice Divide the complex numbers. Write the answer in the form $a + bi$.

20. $\dfrac{2+i}{3-2i}$

TIP: In Example 7, we asked for the answer in the form $a + bi$. However, if the second term is negative, we often leave an answer in terms of subtraction: $\frac{14}{29} - \frac{23}{29}i$. This is the same as $\frac{14}{29} + \left(-\frac{23}{29}i\right)$.

Example 8 Simplifying a Complex Number

Simplify the complex number. $\dfrac{6+\sqrt{-18}}{9}$

Solution:

$$\frac{6+\sqrt{-18}}{9} = \frac{6+i\sqrt{18}}{9}$$ Write the radical in terms of i.

$$= \frac{6+3i\sqrt{2}}{9}$$ Simplify $\sqrt{18} = 3\sqrt{2}$.

$$= \frac{3(2+i\sqrt{2})}{9}$$ Factor the numerator.

$$= \frac{\overset{1}{\cancel{3}}(2+i\sqrt{2})}{\underset{3}{\cancel{9}}}$$ Simplify.

$$= \frac{2+i\sqrt{2}}{3} \quad \text{or} \quad \frac{2}{3} + \frac{\sqrt{2}}{3}i$$ Write in the form $a + bi$.

Skill Practice Simplify the complex number.

21. $\dfrac{8-\sqrt{-24}}{6}$

TIP: As an alternative approach in Example 8, the expression $\dfrac{6+i\sqrt{18}}{9}$ can be written in the form $a + bi$ and then simplified.

$$\frac{6+i\sqrt{18}}{9} = \frac{6}{9} + \frac{3\sqrt{2}}{9}i$$
$$= \frac{2}{3} + \frac{\sqrt{2}}{3}i$$

Answers

20. $\frac{4}{13} + \frac{7}{13}i$

21. $\frac{4-i\sqrt{6}}{3}$ or $\frac{4}{3} - \frac{\sqrt{6}}{3}i$

Section 6.8 Activity

A.1. By definition, $\sqrt{-1} =$ ________ and $i^2 =$ ________.

A.2. **a.** By definition, if b is a positive real number, then $\sqrt{-b} =$ ________.

b. Simplify. $\sqrt{-13}$ **c.** Simplify. $\sqrt{-72}$

d. Simplify. $\dfrac{\sqrt{-50}}{\sqrt{25}}$ **e.** Simplify. $\dfrac{6+\sqrt{-28}}{4}$

A.3. Recall that the product property of radicals, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$, is true provided that the individual square roots $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are *real* numbers.

a. Therefore, what is the first step to simplify the expression $\sqrt{-4} \cdot \sqrt{-25}$?

b. Simplify the expression $\sqrt{-4} \cdot \sqrt{-25}$.

A.4. Recall that the quotient property of radicals, $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$, is true provided that the individual square roots $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are *real* numbers.

a. Therefore, what is the first step to simplify the expression $\dfrac{\sqrt{-50}}{\sqrt{-2}}$?

b. Simplify the expression $\dfrac{\sqrt{-50}}{\sqrt{-2}}$.

A.5. To simplify higher powers of i, we can use the fact that $i^2 = -1$. Decompose the higher power of i as a product of i^2 and another power.

a. $i^3 = i^2 \cdot \square = -1 \cdot \square =$ ________

b. $i^4 = i^2 \cdot \square = -1 \cdot \square =$ ________

A.6. From Exercise A.5, we see that $i^4 = 1$. Use the fact that $i^2 = -1$ and $i^4 = 1$ to decompose higher powers of i.

a. $i^5 = i^4 \cdot \square = 1 \cdot \square =$ ________

b. $i^6 = i^4 \cdot \square = 1 \cdot \square =$ ________

c. $i^7 = i^4 \cdot i^2 \cdot \square =$ ________

d. $i^8 = i^4 \cdot \square =$ ________

A.7. Simplify the powers of i.

a. i^{43} **b.** i^{72} **c.** i^{50} **d.** i^{37}

A.8. Write the number in the form $a + bi$ and identify the real and imaginary parts.

a. $3 - 7i$ **b.** $4 + \sqrt{-12}$ **c.** 9 **d.** $\dfrac{3}{4}i$

For Exercises A.9–A.14, simplify the expressions.

A.9. **a.** $(5 + 3x) - (4 - 7x)$ **b.** $(5 + 3i) - (4 - 7i)$ **A.10.** **a.** $(4 - 6x)(3 + 2x)$ **b.** $(4 - 6i)(3 + 2i)$

A.11. **a.** $(9 - 10x)(9 + 10x)$ **b.** $(9 - 10i)(9 + 10i)$ **A.12.** **a.** $(5 + 4x)^2$ **b.** $(5 + 4i)^2$

A.13. **a.** $\dfrac{5}{3 - 4\sqrt{x}}$ **b.** $\dfrac{5}{3 - 4i}$

A.14. $\dfrac{6 + \sqrt{-28}}{4}$

Practice Exercises

Section 6.8

Prerequisite Review

For Exercises R.1–R.10, perform the indicated operations.

R.1. $(3y + 4) + (-5y - 6)$

R.2. $(-7t + 8) + (9t - 11)$

R.3. $(-4x + 9) - (5x - 8)$

R.4. $(10n + 4) - (-5n - 3)$

R.5. $(2n - 7)(5n + 8)$

R.6. $(3z + 1)(z - 8)$

R.7. $(7d - 5)(7d + 5)$

R.8. $(11k + 4)(11k - 4)$

R.9. $(10b - 3)^2$

R.10. $(9x + 4)^2$

For Exercises R.11–R.14, simplify the expression.

R.11. $\sqrt{90}$

R.12. $\sqrt{112}$

R.13. $\dfrac{\sqrt{27}}{\sqrt{3}}$

R.14. $\dfrac{\sqrt{50}}{\sqrt{2}}$

For Exercises R.15–R.16, write the expression as a sum of two terms.

R.15. $\dfrac{10 + 5x}{15}$

R.16. $\dfrac{-2 - 4y}{6}$

Vocabulary and Key Concepts

1. **a.** A square root of a negative number is not a real number, but rather is an __________ number.

b. $i =$ __________, and $i^2 =$ __________.

c. For a positive number b, the value $\sqrt{-b} =$ __________.

d. A complex number is a number in the form __________, where a and b are real numbers and $i =$ __________.

e. Given a complex number $a + bi$, the value a is called the __________ part, and __________ is called the imaginary part.

f. The complex conjugate of $a - bi$ is __________.

2. **a.** Answer true or false. All real numbers are complex numbers.

b. Answer true or false. All imaginary numbers are complex numbers.

Concept 1: Definition of *i*

3. Simplify the expressions $\sqrt{-1}$ and $-\sqrt{1}$.

4. Simplify i^2.

For Exercises 5–30, simplify the expressions. **(See Examples 1–2.)**

5. $\sqrt{-49}$

6. $\sqrt{-121}$

7. $-\sqrt{49}$

8. $-\sqrt{121}$

9. $\sqrt{-\dfrac{1}{4}}$

10. $\sqrt{-\dfrac{9}{25}}$

11. $\sqrt{-144}$ **12.** $\sqrt{-81}$ **13.** $\sqrt{-3}$ **14.** $\sqrt{-17}$

15. $-\sqrt{-20}$ **16.** $-\sqrt{-75}$ **17.** $(2\sqrt{-25})(3\sqrt{-4})$ **18.** $(-4\sqrt{-9})(-3\sqrt{-1})$

19. $7\sqrt{-63} - 4\sqrt{-28}$ **20.** $7\sqrt{-3} - 4\sqrt{-27}$ **21.** $\sqrt{-7} \cdot \sqrt{-7}$ **22.** $\sqrt{-11} \cdot \sqrt{-11}$

23. $\sqrt{-9} \cdot \sqrt{-16}$ **24.** $\sqrt{-25} \cdot \sqrt{-36}$ **25.** $\sqrt{-15} \cdot \sqrt{-6}$ **26.** $\sqrt{-12} \cdot \sqrt{-50}$

27. $\dfrac{\sqrt{-50}}{\sqrt{25}}$ **28.** $\dfrac{\sqrt{-27}}{\sqrt{9}}$ **29.** $\dfrac{\sqrt{-90}}{\sqrt{-10}}$ **30.** $\dfrac{\sqrt{-125}}{\sqrt{-45}}$

Concept 2: Powers of *i*

For Exercises 31–42, simplify the powers of i. **(See Example 3.)**

31. i^7 **32.** i^{38} **33.** i^{64} **34.** i^{75}

35. i^{41} **36.** i^{25} **37.** i^{52} **38.** i^0

39. i^{23} **40.** i^{103} **41.** i^6 **42.** i^{82}

Concept 3: Definition of a Complex Number

43. What is the complex conjugate of a complex number $a + bi$?

44. True or false?

a. Every real number is a complex number. **b.** Every complex number is a real number.

For Exercises 45–52, identify the real and imaginary parts of the complex number. **(See Example 4.)**

45. $-5 + 12i$ **46.** $22 - 16i$ **47.** $-6i$ **48.** $10i$

49. 35 **50.** -1 **51.** $\dfrac{3}{5} + i$ **52.** $-\dfrac{1}{2} - \dfrac{1}{4}i$

Concept 4: Addition, Subtraction, and Multiplication of Complex Numbers

For Exercises 53–76, perform the indicated operations. Write the answer in the form $a + bi$. **(See Examples 5–6.)**

53. $(2 - i) + (5 + 7i)$ **54.** $(5 - 2i) + (3 + 4i)$ **55.** $\left(\dfrac{1}{2} + \dfrac{2}{3}i\right) - \left(\dfrac{1}{5} - \dfrac{5}{6}i\right)$

56. $\left(\dfrac{11}{10} - \dfrac{7}{5}i\right) - \left(-\dfrac{2}{5} + \dfrac{3}{5}i\right)$ **57.** $\sqrt{-98} - \sqrt{-8}$ **58.** $\sqrt{-75} + \sqrt{-12}$

59. $(2 + 3i) - (1 - 4i) + (-2 + 3i)$ **60.** $(2 + 5i) - (7 - 2i) + (-3 + 4i)$

61. $(8i)(3i)$ **62.** $(2i)(4i)$ **63.** $6i(1 - 3i)$ **64.** $-i(3 + 4i)$

65. $(2 - 10i)(3 + 2i)$ **66.** $(4 + 7i)(2 - 3i)$ **67.** $(-5 + 2i)(5 + 2i)$ **68.** $(4 - 11i)(4 + 11i)$

69. $(4 + 5i)^2$ **70.** $(3 - 2i)^2$ **71.** $(2 + i)(3 - 2i)(4 + 3i)$ **72.** $(3 - i)(3 + i)(4 - i)$

73. $(-4 - 6i)^2$ **74.** $(-3 - 5i)^2$ **75.** $\left(-\frac{1}{2} - \frac{3}{4}i\right)\left(-\frac{1}{2} + \frac{3}{4}i\right)$ **76.** $\left(-\frac{2}{3} + \frac{1}{6}i\right)\left(-\frac{2}{3} - \frac{1}{6}i\right)$

Concept 5: Division and Simplification of Complex Numbers

For Exercises 77–90, divide the complex numbers. Write the answer in the form $a + bi$. **(See Example 7.)**

77. $\dfrac{2}{1 + 3i}$ **78.** $\dfrac{-2}{3 + i}$ **79.** $\dfrac{-i}{4 - 3i}$ **80.** $\dfrac{3 - 3i}{1 - i}$

81. $\dfrac{5 + 2i}{5 - 2i}$ **82.** $\dfrac{7 + 3i}{4 - 2i}$ **83.** $\dfrac{3 + 7i}{-2 - 4i}$ **84.** $\dfrac{-2 + 9i}{-1 - 4i}$

85. $\dfrac{13i}{-5 - i}$ **86.** $\dfrac{15i}{-2 - i}$ **87.** $\dfrac{2 + 3i}{6i}$ (*Hint:* Consider multiplying numerator and denominator by i or by $-i$. This will make the denominator a real number.)

88. $\dfrac{4 - i}{2i}$ **89.** $\dfrac{-10 + i}{i}$ **90.** $\dfrac{-6 - i}{-i}$

For Exercises 91–98, simplify the complex numbers. Write the answer in the form $a + bi$. **(See Example 8.)**

91. $\dfrac{2 + \sqrt{-16}}{8}$ **92.** $\dfrac{6 - \sqrt{-4}}{4}$ **93.** $\dfrac{-6 + \sqrt{-72}}{6}$ **94.** $\dfrac{-20 + \sqrt{-500}}{10}$

95. $\dfrac{-8 - \sqrt{-48}}{4}$ **96.** $\dfrac{-18 - \sqrt{-72}}{3}$ **97.** $\dfrac{-5 + \sqrt{-50}}{10}$ **98.** $\dfrac{14 + \sqrt{-98}}{7}$

Expanding Your Skills

For Exercises 99–102, determine if the complex number is a solution to the equation.

99. $x^2 - 4x + 5 = 0;\quad 2 + i$ **100.** $x^2 - 6x + 25 = 0;\quad 3 - 4i$

101. $x^2 + 12 = 0;\quad -2i\sqrt{3}$ **102.** $x^2 + 18 = 0;\quad 3i\sqrt{2}$

Chapter 6 Summary

Section 6.1 Definition of an *n*th Root

Key Concepts

b is an **nth root** of a if $b^n = a$.

The expression $\sqrt{a}$ represents the **principal square root** of a.
The expression $\sqrt[n]{a}$ represents the principal nth root of a.

$\sqrt[n]{a^n} = |a|$ if n is even.

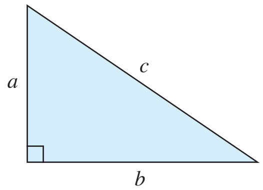
$\sqrt[n]{a^n} = a$ if n is odd.

$\sqrt[n]{a}$ is not a real number if $a < 0$ and n is even.

$f(x) = \sqrt[n]{x}$ defines a **radical function**.

The Pythagorean Theorem

$a^2 + b^2 = c^2$



Examples

Example 1

2 is a square root of 4.

-2 is a square root of 4.

-3 is a cube root of -27.

Example 2

$\sqrt{36} = 6 \qquad \sqrt[3]{-64} = -4$

Example 3

$\sqrt[4]{(x+3)^4} = |x+3| \qquad \sqrt[5]{(x+3)^5} = x+3$

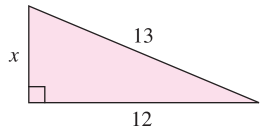
Example 4

$\sqrt[4]{-16}$ is not a real number.

Example 5

For $g(x) = \sqrt{x}$ the domain is $[0, \infty)$.
For $h(x) = \sqrt[3]{x}$ the domain is $(-\infty, \infty)$.

Example 6



$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 = 25$$
$$x = \sqrt{25}$$
$$x = 5$$

Section 6.2 Rational Exponents

Key Concepts

Let a be a real number and n be an integer such that $n > 1$. If $\sqrt[n]{a}$ exists, then

$$\boldsymbol{a^{1/n}} = \sqrt[n]{a}$$

$$\boldsymbol{a^{m/n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

All properties of integer exponents hold for rational exponents, provided the roots are real-valued.

Examples

Example 1

$$121^{1/2} = \sqrt{121} = 11$$

Example 2

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = (3)^2 = 9$$

Example 3

a. $p^{1/3}p^{1/4} = p^{1/3+1/4} = p^{4/12+3/12} = p^{7/12}$

b. $\dfrac{4^{4/3}}{4^{1/3}} = 4^{4/3-1/3} = 4^{3/3} = 4$

c. $(y^{-1/2})^6 = y^{(-1/2)(6)} = y^{-3} = \dfrac{1}{y^3}$

Section 6.3 Simplifying Radical Expressions

Key Concepts

Let a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real. Then,

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \qquad \text{Multiplication property}$$

A radical expression whose radicand is written as a product of prime factors is in simplified form if all the following conditions are met:

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand does not contain a fraction.
3. No radicals are in the denominator of a fraction.

Examples

Example 1

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \cdot 3}\\ &= \sqrt{4} \cdot \sqrt{3}\\ &= 2\sqrt{3}\end{aligned}$$

Example 2

$$\begin{aligned}&\sqrt[3]{16x^5y^7}\\ &= \sqrt[3]{2^4x^5y^7}\\ &= \sqrt[3]{2^3x^3y^6 \cdot 2x^2y}\\ &= \sqrt[3]{2^3x^3y^6} \cdot \sqrt[3]{2x^2y}\\ &= 2xy^2\sqrt[3]{2x^2y}\end{aligned}$$

Section 6.4 Addition and Subtraction of Radicals

Key Concepts

***Like* radicals** have radical factors with the same index and the same radicand.

Use the distributive property to add and subtract *like* radicals.

Examples

Example 1

$2x\sqrt{7} - 5x\sqrt{7} + x\sqrt{7}$

$= (2 - 5 + 1)x\sqrt{7}$

$= -2x\sqrt{7}$

Example 2

$x\sqrt[4]{16x} - 3\sqrt[4]{x^5}$

$= 2x\sqrt[4]{x} - 3x\sqrt[4]{x}$

$= (2 - 3)x\sqrt[4]{x}$

$= -x\sqrt[4]{x}$

Section 6.5 Multiplication of Radicals

Key Concepts

The Multiplication Property of Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

To multiply or divide radicals with different indices, convert to rational exponents and use the properties of exponents.

Examples

Example 1

$3\sqrt{2}(\sqrt{2} + 5\sqrt{7} - \sqrt{6})$

$= 3\sqrt{4} + 15\sqrt{14} - 3\sqrt{12}$

$= 3 \cdot 2 + 15\sqrt{14} - 3 \cdot 2\sqrt{3}$

$= 6 + 15\sqrt{14} - 6\sqrt{3}$

Example 2

$\sqrt{p} \cdot \sqrt[5]{p^2}$

$= p^{1/2} \cdot p^{2/5}$

$= p^{5/10} \cdot p^{4/10}$

$= p^{9/10}$

$= \sqrt[10]{p^9}$

Section 6.6 Division of Radicals and Rationalization

Key Concepts

The Division Property of Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad b \neq 0$$

The process of removing a radical from the denominator of an expression is called **rationalizing the denominator**.

- Rationalizing a denominator with one term
- Rationalizing a denominator with two terms involving square roots

Examples

Example 1

Simplify.

$$\sqrt{\frac{4x^5}{y^4}}$$

$$= \frac{\sqrt{4x^5}}{\sqrt{y^4}}$$

$$= \frac{2x^2\sqrt{x}}{y^2}$$

Example 2

Rationalize the denominator.

$$\frac{4}{\sqrt[3]{t}}$$

$$= \frac{4}{\sqrt[3]{t}} \cdot \frac{\sqrt[3]{t^2}}{\sqrt[3]{t^2}}$$

$$= \frac{4\sqrt[3]{t^2}}{\sqrt[3]{t^3}} = \frac{4\sqrt[3]{t^2}}{t}$$

Example 3

Rationalize the denominator.

$$\frac{\sqrt{2} - 3\sqrt{x}}{\sqrt{x} - \sqrt{2}}$$

$$= \frac{(\sqrt{2} - 3\sqrt{x})}{(\sqrt{x} - \sqrt{2})} \cdot \frac{(\sqrt{x} + \sqrt{2})}{(\sqrt{x} + \sqrt{2})}$$

$$= \frac{\sqrt{2} \cdot \sqrt{x} + \sqrt{2} \cdot \sqrt{2} - 3\sqrt{x} \cdot \sqrt{x} - 3\sqrt{x} \cdot \sqrt{2}}{(\sqrt{x})^2 - (\sqrt{2})^2}$$

$$= \frac{\sqrt{2x} + 2 - 3x - 3\sqrt{2x}}{x - 2}$$

$$= \frac{-3x - 2\sqrt{2x} + 2}{x - 2}$$

Section 6.7 Solving Radical Equations

Key Concepts

Steps to Solve a Radical Equation

1. Isolate the radical. If an equation has more than one radical, choose one of the radicals to isolate.
2. Raise each side of the equation to a power equal to the index of the radical.
3. Solve the resulting equation. If the equation still has a radical, repeat steps 1 and 2.
4. Check the potential solutions in the original equation.

Example 1

Solve.

$$\sqrt[3]{2x+5}+7=12$$
$$\sqrt[3]{2x+5}=5$$
$$(\sqrt[3]{2x+5})^3=(5)^3$$
$$2x+5=125$$
$$2x=120$$
$$x=60$$

Check:

$$\sqrt[3]{2(60)+5}+7\stackrel{?}{=}12$$
$$\sqrt[3]{125}+7\stackrel{?}{=}12$$
$$5+7\stackrel{?}{=}12 \quad \text{(true)}$$

The solution set is $\{60\}$.

Examples

Example 2

Solve.

$$\sqrt{b-5}-\sqrt{b+3}=2$$
$$\sqrt{b-5}=\sqrt{b+3}+2$$
$$\left(\sqrt{b-5}\right)^2=\left(\sqrt{b+3}+2\right)^2$$
$$b-5=b+3+4\sqrt{b+3}+4$$
$$b-5=b+7+4\sqrt{b+3}$$
$$-12=4\sqrt{b+3}$$
$$-3=\sqrt{b+3}$$
$$(-3)^2=\left(\sqrt{b+3}\right)^2$$
$$9=b+3$$
$$6=b$$

Check:

$$\sqrt{6-5}-\sqrt{6+3}\stackrel{?}{=}2$$
$$\sqrt{1}-\sqrt{9}\stackrel{?}{=}2$$
$$1-3\stackrel{?}{=}2 \quad \text{(false)}$$

No solution, $\{\ \}$

Section 6.8 Complex Numbers

Key Concepts

$\boldsymbol{i} = \sqrt{-1}$ and $i^2 = -1$

For a real number $b > 0$, $\sqrt{-b} = i\sqrt{b}$

A **complex number** is in the form $a + bi$, where a and b are real numbers. The value a is called the real part, and b is called the imaginary part.

To add or subtract complex numbers, combine the real parts and combine the imaginary parts.

Multiply complex numbers by using the distributive property.

Divide complex numbers by multiplying the numerator and denominator by the **complex conjugate** of the denominator.

Examples

Example 1

$\sqrt{-4} \cdot \sqrt{-9}$

$= (2i)(3i)$

$= 6i^2$

$= -6$

Example 2

$(3 - 5i) - (2 + i) + (3 - 2i)$

$= 3 - 5i - 2 - i + 3 - 2i$

$= 4 - 8i$

Example 3

$(1 + 6i)(2 + 4i)$

$= 2 + 4i + 12i + 24i^2$

$= 2 + 16i + 24(-1)$

$= -22 + 16i$

Example 4

$\dfrac{3}{2 - 5i}$

$= \dfrac{3}{(2 - 5i)} \cdot \dfrac{(2 + 5i)}{(2 + 5i)}$

$= \dfrac{6 + 15i}{4 - 25i^2}$

$= \dfrac{6 + 15i}{4 + 25}$

$= \dfrac{6 + 15i}{29}$

$= \dfrac{6}{29} + \dfrac{15}{29}i$

Chapter 6 Review Exercises

For the exercises in this set, assume that all variables represent positive real numbers unless otherwise stated.

Section 6.1

1. True or false?

   a. The principal nth root of an even-indexed root is always positive

   b. The principal nth root of an odd-indexed root is always positive.

2. Explain why $\sqrt{(-3)^2} \neq -3$.

3. For $a > 0$ and $b > 0$, are the following statements true or false?

   a. $\sqrt{a^2 + b^2} = a + b$

   b. $\sqrt{(a + b)^2} = a + b$

For Exercises 4–6, simplify the radicals.

4. $\sqrt{\dfrac{50}{32}}$ **5.** $\sqrt[4]{625}$ **6.** $\sqrt{(-6)^2}$

7. Evaluate the function values for $f(x) = \sqrt{x - 1}$.

   a. $f(10)$ **b.** $f(1)$ **c.** $f(8)$

   d. Write the domain of f in interval notation.

8. Evaluate the function values for $g(t) = \sqrt{5 + t}$.

   a. $g(-5)$ **b.** $g(-4)$ **c.** $g(4)$

   d. Write the domain of g in interval notation

9. Write the English expression as an algebraic expression: Four more than the quotient of the cube root of $2x$ and the principal fourth root of $2x$.

For Exercises 10–11, simplify the expression. Assume that x and y represent *any* real number.

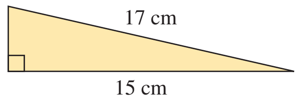
10. **a.** $\sqrt{x^2}$ **b.** $\sqrt[3]{x^3}$

c. $\sqrt[4]{x^4}$ **d.** $\sqrt[5]{(x + 1)^5}$

11. **a.** $\sqrt{4y^2}$ **b.** $\sqrt[3]{27y^3}$

c. $\sqrt[100]{y^{100}}$ **d.** $\sqrt[101]{y^{101}}$

12. Use the Pythagorean theorem to find the length of the third side of the triangle.



Section 6.2

13. Are the properties of exponents the same for rational exponents and integer exponents? Give an example. (Answers may vary.)

14. In the expression $x^{m/n}$ what does n represent?

15. Explain the process of eliminating a negative exponent from an algebraic expression.

For Exercises 16–21, simplify the expressions. Write the answers with positive exponents only.

16. $(-125)^{1/3}$ **17.** $16^{-1/4}$

18. $\left(\dfrac{1}{16}\right)^{-3/4} - \left(\dfrac{1}{8}\right)^{-2/3}$ **19.** $\left(b^{1/2} \cdot b^{1/3}\right)^{12}$

20. $\left(\dfrac{x^{-1/4}y^{-1/3}z^{3/4}}{2^{1/3}x^{-1/3}y^{2/3}}\right)^{-12}$ **21.** $\left(\dfrac{a^{12}b^{-4}c^7}{a^3b^2c^4}\right)^{1/3}$

For Exercises 22–23, rewrite the expressions by using rational exponents.

22. $\sqrt[4]{x^3}$ **23.** $\sqrt[3]{2y^2}$

For Exercises 24–26, use a calculator to approximate the expressions to four decimal places.

24. $10^{1/3}$ **25.** $17.8^{2/3}$ **26.** $\sqrt[5]{147^4}$

Section 6.3

27. List the criteria for a radical expression to be simplified.

For Exercises 28–31, simplify the radicals.

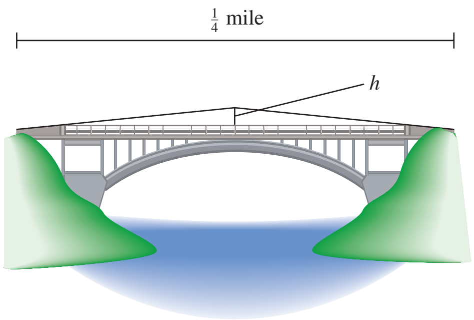
28. $\sqrt{108}$ **29.** $\sqrt[4]{x^5yz^4}$

30. $-2\sqrt[3]{250a^3b^{10}}$ **31.** $\sqrt[3]{\dfrac{-16a^4}{2ab^3}}$

32. Write an English phrase that describes the following mathematical expressions: (Answers may vary.)

a. $\sqrt{\dfrac{2}{x}}$ **b.** $(x+1)^3$

33. An engineering firm made a mistake when building a $\frac{1}{4}$-mi bridge in the Florida Keys. The bridge was made without adequate expansion joints to prevent buckling during the heat of summer. During mid-June, the bridge expanded 1.5 ft, causing a vertical bulge in the middle. Calculate the height of the bulge h in feet. (*Note:* 1 mi = 5280 ft.) Round to the nearest foot.



Section 6.4

34. Complete the following statement: Radicals may be added or subtracted if . . .

For Exercises 35–38, determine whether the radicals may be combined, and explain your answer.

35. $\sqrt[3]{2x} - 2\sqrt{2x}$ **36.** $2 + \sqrt{x}$

37. $\sqrt[4]{3xy} + 2\sqrt[4]{3xy}$ **38.** $-4\sqrt{32} + 7\sqrt{50}$

For Exercises 39–42, add or subtract as indicated.

39. $4\sqrt{7} - 2\sqrt{7} + 3\sqrt{7}$

40. $2\sqrt[3]{64} + 3\sqrt[3]{54} - 16$

41. $\sqrt{50} + 7\sqrt{2} - \sqrt{8}$

42. $x\sqrt[3]{16x^2} - 4\sqrt[3]{2x^5} + 5x\sqrt[3]{54x^2}$

For Exercises 43–44, answer true or false. If an answer is false, explain why. Assume all variables represent positive real numbers.

43. $5 + 3\sqrt{x} = 8\sqrt{x}$

44. $\sqrt{y} + \sqrt{y} = \sqrt{2y}$

Section 6.5

For Exercises 45–56, multiply the radicals and simplify the answer.

45. $\sqrt{3} \cdot \sqrt{12}$ **46.** $\sqrt[4]{4} \cdot \sqrt[4]{8}$

47. $-2\sqrt{3}\left(\sqrt{7} - 3\sqrt{11}\right)$ **48.** $-3\sqrt{5}\left(2\sqrt{3} - \sqrt{5}\right)$

49. $\left(2\sqrt{x} - 3\right)\left(2\sqrt{x} + 3\right)$ **50.** $(\sqrt{y} + 4)(\sqrt{y} - 4)$

51. $\left(\sqrt{7y} - \sqrt{3x}\right)^2$ **52.** $\left(2\sqrt{3w} + 5\right)^2$

53. $\left(-\sqrt{z} - \sqrt{6}\right)\left(2\sqrt{z} + 7\sqrt{6}\right)$

54. $\left(3\sqrt{a} - \sqrt{5}\right)\left(\sqrt{a} + 2\sqrt{5}\right)$

55. $\sqrt[3]{u} \cdot \sqrt{u^5}$ **56.** $\sqrt{2} \cdot \sqrt[4]{w^3}$

Section 6.6

For Exercises 57–60, simplify the radicals.

57. $\sqrt{\dfrac{3y^5}{25x^6}}$ **58.** $\sqrt[3]{\dfrac{-16x^7y^6}{z^9}}$

59. $\dfrac{\sqrt{324w^7}}{\sqrt{4w^3}}$ **60.** $\dfrac{\sqrt[3]{3t^{14}}}{\sqrt[3]{192t^2}}$

For Exercises 61–68, rationalize the denominator.

61. $\sqrt{\dfrac{7}{2y}}$ **62.** $\sqrt{\dfrac{5}{3w}}$

63. $\dfrac{4}{\sqrt[3]{9p^2}}$ **64.** $\dfrac{-2}{\sqrt[3]{2x}}$

65. $\dfrac{-5}{\sqrt{15} + \sqrt{10}}$ **66.** $\dfrac{-6}{\sqrt{7} + \sqrt{5}}$

67. $\dfrac{t-3}{\sqrt{t} - \sqrt{3}}$ **68.** $\dfrac{w-7}{\sqrt{w} - \sqrt{7}}$

69. Write the mathematical expression as an English phrase. (Answers may vary.)

$$\frac{\sqrt{2}}{x^2}$$

Section 6.7

Solve the radical equations in Exercises 70–77, if possible.

70. $\sqrt{2y} = 7$ **71.** $\sqrt{a-6} - 5 = 0$

72. $\sqrt[3]{2w-3} + 5 = 2$

73. $\sqrt[4]{p+12} - \sqrt[4]{5p-16} = 0$

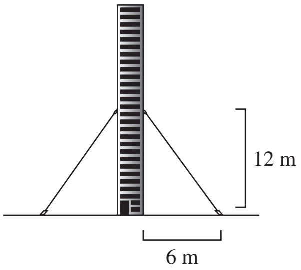
74. $\sqrt{t} + \sqrt{t-5} = 5$

75. $\sqrt{8x+1} = -\sqrt{x-13}$

76. $\sqrt{2m^2+4} - \sqrt{9m} = 0$

77. $\sqrt{x+2} = 1 - \sqrt{2x+5}$

78. A tower is supported by stabilizing wires. Find the exact length of each wire, and then round to the nearest tenth of a meter.



79. The velocity, $v(d)$, of an ocean wave depends on the water depth d as the wave approaches land.

$$v(d) = \sqrt{32d}$$

where $v(d)$ is in feet per second and d is in feet.

a. Find $v(20)$ and interpret its value. Round to one decimal place.

b. Find the depth of the water at a point where a wave is traveling at 16 ft/sec.

Section 6.8

80. Define a complex number.

81. Define an imaginary number.

82. Describe the first step in the process to simplify the expression.

$$\frac{3}{4+6i}$$

For Exercises 83–86, rewrite the expressions in terms of i.

83. $\sqrt{-16}$

84. $-\sqrt{-5}$

85. $\sqrt{-75} \cdot \sqrt{-3}$

86. $\dfrac{-\sqrt{-24}}{\sqrt{6}}$

For Exercises 87–90, simplify the powers of i.

87. i^{38}

88. i^{101}

89. i^{19}

90. $i^{1000} + i^{1002}$

For Exercises 91–94, perform the indicated operations. Write the final answer in the form $a + bi$.

91. $(-3 + i) - (2 - 4i)$

92. $(1 + 6i)(3 - i)$

93. $(4 - 3i)(4 + 3i)$

94. $(5 - i)^2$

For Exercises 95–96, write the expressions in the form $a + bi$, and determine the real and imaginary parts.

95. $\dfrac{17 - 4i}{-4}$

96. $\dfrac{-16 - 8i}{8}$

For Exercises 97–100, divide and simplify. Write the final answer in the form $a + bi$.

97. $\dfrac{2 - i}{3 + 2i}$

98. $\dfrac{10 + 5i}{2 - i}$

99. $\dfrac{5 + 3i}{-2i}$

100. $\dfrac{4i}{4 - i}$

For Exercises 101–102, simplify the expression.

101. $\dfrac{-8 + \sqrt{-40}}{12}$

102. $\dfrac{6 - \sqrt{-144}}{3}$

Chapter 6 Test

Study Skills Exercise

Give yourself enough uninterrupted time to complete the test review provided by your instructor and to complete the Chapter Test. Do this in a "test setting" if possible. Simulate the same environment by using only the resources allowed on the test and the time allotted for the test. After completing your practice test, check your answers. For each problem you answered incorrectly, go to the Review Exercises and do all of the problems that are similar. If you still have questions after completing the Review Exercises, seek help from one of your resources.

1. **a.** What is the principal square root of 36?

b. What is the negative square root of 36?

2. Which of the following are real numbers?

a. $-\sqrt{100}$ **b.** $\sqrt{-100}$

c. $-\sqrt[3]{1000}$ **d.** $\sqrt[3]{-1000}$

3. Simplify.

a. $\sqrt[3]{y^3}$ **b.** $\sqrt[4]{y^4}$

For Exercises 4–11, simplify the radicals. Assume that all variables represent positive numbers.

4. $\sqrt[4]{81}$ **5.** $\sqrt{\dfrac{16}{9}}$

6. $\sqrt[3]{32}$ **7.** $\sqrt{a^4b^3c^5}$

8. $\sqrt{18x^5y^3z^4}$ **9.** $\sqrt{\dfrac{32w^6}{2w}}$

10. $\sqrt[3]{\dfrac{x^6}{125y^3}}$ **11.** $\dfrac{2\sqrt{72}}{8}$

12. **a.** Evaluate the function values $f(-8)$, $f(-6)$, $f(-4)$, and $f(-2)$ for $f(x) = \sqrt{-2x-4}$.

b. Write the domain of f in interval notation.

13. Use a calculator to evaluate $\dfrac{-3-\sqrt{5}}{17}$ to four decimal places.

For Exercises 14–15, simplify the expressions. Assume that all variables represent positive numbers.

14. $-27^{1/3}$ **15.** $8^{2/3} \cdot \left(\dfrac{25x^4y^6}{z^2}\right)^{1/2}$

For Exercises 16–17, use rational exponents to multiply or divide. Write the final answer in radical form.

16. $\sqrt[6]{7} \cdot \sqrt{y}$ **17.** $\dfrac{\sqrt[3]{10}}{\sqrt[4]{10}}$

18. Add or subtract as indicated.

$$3\sqrt{5} + 4\sqrt{5} - 2\sqrt{20}$$

For Exercises 19–20, multiply the radicals.

19. $3\sqrt{x}(\sqrt{2} - \sqrt{5})$

20. $(2\sqrt{5} - 3\sqrt{x})(4\sqrt{5} + \sqrt{x})$

For Exercises 21–22, rationalize the denominator. Assume $x > 0$.

21. $\dfrac{-2}{\sqrt[3]{x}}$ **22.** $\dfrac{\sqrt{x}+2}{3-\sqrt{x}}$

23. Rewrite the expressions in terms of i.

a. $\sqrt{-8}$ **b.** $2\sqrt{-16}$ **c.** $\dfrac{2+\sqrt{-8}}{4}$

For Exercises 24–30, perform the indicated operations and simplify completely. Write the final answer in the form $a + bi$.

24. $(3 - 5i) - (2 + 6i)$

25. $(4 + i)(8 + 2i)$

26. $\sqrt{-16} \cdot \sqrt{-49}$

27. $(4 - 7i)^2$

28. $(2 - 10i)(2 + 10i)$

29. $\dfrac{3 - 2i}{3 - 4i}$

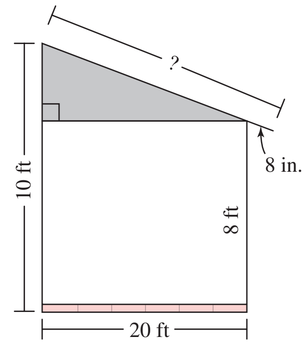
30. $\dfrac{6i}{3 - 5i}$

31. If the volume V of a sphere is known, the radius of the sphere can be computed by

$$r(V) = \sqrt[3]{\frac{3V}{4\pi}}$$

Find $r(10)$ to two decimal places. Interpret the meaning in the context of the problem.

32. A patio 20 ft wide has a slanted roof, as shown in the figure. Find the length of the roof if there is an 8-in. overhang. Round the answer to the nearest foot.



For Exercises 33–35, solve the radical equation.

33. $\sqrt[3]{2x + 5} = -3$

34. $\sqrt{5x + 8} = \sqrt{5x - 1} + 1$

35. $\sqrt{t + 7} - \sqrt{2t - 3} = 2$

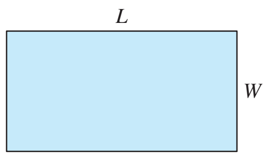
Quadratic Equations and Functions

7

CHAPTER OUTLINE

Mathematics in Art

A **golden rectangle** is a rectangle in which the ratio of its length L to its width W is equal to the ratio of the sum of its length and width to its length.



$$\frac{L}{W} = \frac{L + W}{L}$$

The values of L and W that meet this condition are said to be in the **golden ratio**. The golden ratio has been studied by artists and art historians for generations because the golden ratio represents an aesthetically pleasing ratio between the length and width of a figure. For example, the face of the Parthenon built in ancient Greece has the dimensions of a golden rectangle.

We can show that the length of a golden rectangle is approximately 1.62 times the width. Substituting 1 for the width, we have the proportion $\frac{L}{1} = \frac{L+1}{L}$. Then, clearing fractions and writing the quadratic equation in standard form gives $L^2 - L - 1 = 0$. The expression on the left is not factorable, but fortunately in this chapter, we will learn two techniques to solve a quadratic equation when factoring fails. The positive solution for L in this equation is the golden ratio, $\frac{1+\sqrt{5}}{2} \approx 1.62$.

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Section 7.1 Square Root Property and Completing the Square

Concepts

1. Solving Quadratic Equations by Using the Square Root Property
2. Solving Quadratic Equations by Completing the Square
3. Literal Equations

1. Solving Quadratic Equations by Using the Square Root Property

We have already learned how to solve a quadratic equation by factoring and applying the zero product rule. For example:

$$x^2 = 81$$

$$x^2 - 81 = 0 \qquad \text{Set one side equal to zero.}$$

$$(x-9)(x+9) = 0 \qquad \text{Factor.}$$

$$x - 9 = 0 \quad \text{or} \quad x + 9 = 0 \qquad \text{Set each factor equal to zero.}$$

$$x = 9 \quad \text{or} \quad x = -9$$

The solution set is $\{9, -9\}$.

It is important to note that the zero product rule can only be used if the equation is factorable. In this section, we will learn a method to solve quadratic equations containing expressions that are both factorable and nonfactorable.

Consider a quadratic equation of the form $x^2 = k$. The solutions are all numbers (real or imaginary) that when squared equal k, so for any nonzero constant k, there will be two solutions, $\sqrt{k}$ or $-\sqrt{k}$. For example:

$$x^2 = 25 \qquad \text{The solutions are 5 and } -5.$$

$$x^2 = -25 \qquad \text{The solutions are } 5i \text{ and } -5i.$$

This principle is stated formally as the **square root property**.

FOR REVIEW

The square root property states that if $x^2 = k$, then $x = \pm\sqrt{k}$. To understand why, recall that $\sqrt{x^2} = |x|$. Thus,

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$|x| = 5$$
$$x = 5 \text{ or } x = -5$$

The Square Root Property

For any real number k, if $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Note: The solution may also be written as $\pm\sqrt{k}$, read "plus or minus the square root of k."

Example 1 Solving a Quadratic Equation by Using the Square Root Property

Use the square root property to solve the equation. $4p^2 = 9$

Solution:

$$4p^2 = 9$$

$$p^2 = \frac{9}{4} \qquad \text{Isolate } p^2 \text{ by dividing both sides by 4.}$$

$$p = \pm\sqrt{\frac{9}{4}} \qquad \text{Apply the square root property.}$$

$$p = \pm\frac{3}{2} \qquad \text{Simplify the radical.}$$

The solution set is $\left\{\frac{3}{2}, -\frac{3}{2}\right\}$.

Skill Practice Solve using the square root property.

1. $25a^2 = 16$

For a quadratic equation, $ax^2 + bx + c = 0$, if $b = 0$, then the equation is easily solved by using the square root property. This is demonstrated in Example 2.

Example 2 Solving a Quadratic Equation by Using the Square Root Property

Use the square root property to solve the equation. $3x^2 + 75 = 0$

Solution:

$3x^2 + 75 = 0$ Rewrite the equation to fit the form $x^2 = k$.

$3x^2 = -75$

$x^2 = -25$ The equation is now in the form $x^2 = k$.

$x = \pm\sqrt{-25}$ Apply the square root property.

$= \pm 5i$

Check: $x = 5i$	Check: $x = -5i$
$3x^2 + 75 = 0$	$3x^2 + 75 = 0$
$3(5i)^2 + 75 \stackrel{?}{=} 0$	$3(-5i)^2 + 75 \stackrel{?}{=} 0$
$3(25i^2) + 75 \stackrel{?}{=} 0$	$3(25i^2) + 75 \stackrel{?}{=} 0$
$3(-25) + 75 \stackrel{?}{=} 0$	$3(-25) + 75 \stackrel{?}{=} 0$
$-75 + 75 \stackrel{?}{=} 0$ ✓	$-75 + 75 \stackrel{?}{=} 0$ ✓

The solution set is $\{\pm 5i\}$.

Avoiding Mistakes

A common mistake is to forget the ± symbol when solving the equation $x^2 = k$:

$x = \pm\sqrt{k}$

Skill Practice Solve using the square root property.

2. $8x^2 + 72 = 0$

Example 3 Solving a Quadratic Equation by Using the Square Root Property

Use the square root property to solve the equation. $(w + 3)^2 = 20$

Solution:

$(w + 3)^2 = 20$ The equation is in the form $x^2 = k$, where $x = (w + 3)$.

$w + 3 = \pm\sqrt{20}$ Apply the square root property.

$w + 3 = \pm\sqrt{4 \cdot 5}$ Simplify the radical.

$w + 3 = \pm 2\sqrt{5}$

$w = -3 \pm 2\sqrt{5}$ Solve for w.

The solution set is $\{-3 \pm 2\sqrt{5}\}$.

TIP: Recall that $-3 \pm 2\sqrt{5}$ represents two solutions:

$-3 + 2\sqrt{5}$ and $-3 - 2\sqrt{5}$

Skill Practice Solve using the square root property.

3. $(t - 5)^2 = 18$

Answers

1. $\left\{\frac{4}{5}, -\frac{4}{5}\right\}$ **2.** $\{\pm 3i\}$

3. $\{5 \pm 3\sqrt{2}\}$

2. Solving Quadratic Equations by Completing the Square

In Example 3 we used the square root property to solve an equation where the square of a binomial was equal to a constant.

$$\underbrace{(w+3)^2}_{\text{Square of a binomial}} = \underbrace{20}_{\text{Constant}}$$

The square of a binomial is the factored form of a perfect square trinomial. For example:

Perfect Square Trinomial		Factored Form
$x^2 + 10x + 25$	$\longrightarrow$	$(x+5)^2$
$t^2 - 6t + 9$	$\longrightarrow$	$(t-3)^2$
$p^2 - 14p + 49$	$\longrightarrow$	$(p-7)^2$

For a perfect square trinomial with a leading coefficient of 1, the constant term is the square of one-half the linear term coefficient. For example:

$$x^2 + 10x + 25$$

$$\left[\tfrac{1}{2}(10)\right]^2$$

In general, an expression of the form $x^2 + bx + n$ is a perfect square trinomial if $n = \left(\frac{1}{2}b\right)^2$. The process to create a perfect square trinomial is called **completing the square**.

Example 4 Completing the Square

Determine the value of n that makes the polynomial a perfect square trinomial. Then factor the expression as the square of a binomial.

a. $x^2 + 12x + n$ **b.** $x^2 - 26x + n$

c. $x^2 + 11x + n$ **d.** $x^2 - \dfrac{4}{7}x + n$

Solution:

The expressions are in the form $x^2 + bx + n$. The value of n equals the square of one-half the linear term coefficient $\left(\frac{1}{2}b\right)^2$.

a. $x^2 + 12x + n$

$x^2 + 12x + 36$	$n = \left[\frac{1}{2}(12)\right]^2 = (6)^2 = 36$
$(x+6)^2$	Factored form

b. $x^2 - 26x + n$

$x^2 - 26x + 169$	$n = \left[\frac{1}{2}(-26)\right]^2 = (-13)^2 = 169$
$(x-13)^2$	Factored form

c. $x^2 + 11x + n$

$x^2 + 11x + \dfrac{121}{4}$	$n = \left[\frac{1}{2}(11)\right]^2 = \left(\frac{11}{2}\right)^2 = \frac{121}{4}$
$\left(x + \dfrac{11}{2}\right)^2$	Factored form

d. $x^2 - \frac{4}{7}x + n$

$x^2 - \frac{4}{7}x + \frac{4}{49}$ $\quad n = [\frac{1}{2}(-\frac{4}{7})]^2 = (-\frac{2}{7})^2 = \frac{4}{49}$

$\left(x - \frac{2}{7}\right)^2$ $\quad$ Factored form

Skill Practice Determine the value of n that makes the polynomial a perfect square trinomial. Then factor.

4. $x^2 + 20x + n$ **5.** $y^2 - 16y + n$

6. $a^2 - 15a + n$ **7.** $w^2 + \frac{7}{3}w + n$

The process of completing the square can be used to write a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ in the form $(x - h)^2 = k$. Then the square root property can be used to solve the equation. The following steps outline the procedure.

Solving a Quadratic Equation $ax^2 + bx + c = 0$ by Completing the Square and Applying the Square Root Property

Step 1 Divide both sides by a to make the leading coefficient 1.

Step 2 Isolate the variable terms on one side of the equation.

Step 3 Complete the square.

- Add the square of one-half the linear term coefficient to both sides, $(\frac{1}{2}b)^2$.
- Factor the resulting perfect square trinomial.

Step 4 Apply the square root property and solve for x.

Example 5 Solving a Quadratic Equation by Completing the Square and Applying the Square Root Property

Solve by completing the square and applying the square root property.

$$x^2 - 6x + 13 = 0$$

Solution:

$x^2 - 6x + 13 = 0$ **Step 1:** Since the leading coefficient a is equal to 1, we do not have to divide by a. We can proceed to step 2.

$x^2 - 6x = -13$ **Step 2:** Isolate the variable terms on one side.

$x^2 - 6x + 9 = -13 + 9$ **Step 3:** To complete the square, add $[\frac{1}{2}(-6)]^2 = 9$ to both sides of the equation.

$(x - 3)^2 = -4$ Factor the perfect square trinomial.

Answers

4. $n = 100$; $(x + 10)^2$
5. $n = 64$; $(y - 8)^2$
6. $n = \frac{225}{4}$; $\left(a - \frac{15}{2}\right)^2$
7. $n = \frac{49}{36}$; $\left(w + \frac{7}{6}\right)^2$

FOR REVIEW

Recall that a solution to an equation can be checked by substitution. Also recall that $i^2 = -1$.

$$x^2 - 6x + 13 = 0$$
$$(3 + 2i)^2 - 6(3 + 2i) + 13 \stackrel{?}{=} 0$$
$$9 + 12i + 4i^2 - 18 - 12i + 13 \stackrel{?}{=} 0$$
$$9 + 12i - 4 - 18 - 12i + 13 \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$

$x - 3 = \pm\sqrt{-4}$	**Step 4:**	Apply the square root property.
$x - 3 = \pm 2i$		Simplify the radical.
$x = 3 \pm 2i$		Solve for x.

The solutions are imaginary numbers and can be written as $3 + 2i$ and $3 - 2i$.

The solution set is $\{3 \pm 2i\}$.

Skill Practice Solve by completing the square and applying the square root property.

8. $z^2 - 4z + 26 = 0$

Example 6 Solving a Quadratic Equation by Completing the Square and Applying the Square Root Property

Solve by completing the square and applying the square root property.

$$2m^2 + 10m = 3$$

Solution:

$2m^2 + 10m = 3$	The variable terms are already isolated on one side of the equation.
$\dfrac{2m^2}{2} + \dfrac{10m}{2} = \dfrac{3}{2}$	Divide by the leading coefficient, 2.
$m^2 + 5m = \dfrac{3}{2}$	
$m^2 + 5m + \dfrac{25}{4} = \dfrac{3}{2} + \dfrac{25}{4}$	Add $\left[\frac{1}{2}(5)\right]^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$ to both sides.
$\left(m + \dfrac{5}{2}\right)^2 = \dfrac{6}{4} + \dfrac{25}{4}$	Factor the left side and write the terms on the right with a common denominator.
$\left(m + \dfrac{5}{2}\right)^2 = \dfrac{31}{4}$	
$m + \dfrac{5}{2} = \pm\sqrt{\dfrac{31}{4}}$	Apply the square root property.
$m = -\dfrac{5}{2} \pm \dfrac{\sqrt{31}}{2}$	Subtract $\frac{5}{2}$ from both sides and simplify the radical.

The solution set is $\left\{-\dfrac{5}{2} \pm \dfrac{\sqrt{31}}{2}\right\}$. The solutions are irrational numbers.

TIP: The solutions to Example 6 can also be written as:

$$\frac{-5 \pm \sqrt{31}}{2}$$

Skill Practice Solve by completing the square and applying the square root property.

9. $4x^2 + 12x = 5$

Answers

8. $\{2 \pm i\sqrt{22}\}$

9. $\left\{-\dfrac{3}{2} \pm \dfrac{\sqrt{14}}{2}\right\}$

Example 7 Solving a Quadratic Equation by Completing the Square and Applying the Square Root Property

Solve by completing the square and applying the square root property.

$$2x(2x - 10) = -30 + 6x$$

Solution:

$2x(2x - 10) = -30 + 6x$

$4x^2 - 20x = -30 + 6x$ Clear parentheses.

$4x^2 - 26x + 30 = 0$ Write the equation in the form $ax^2 + bx + c = 0$.

$\dfrac{4x^2}{4} - \dfrac{26x}{4} + \dfrac{30}{4} = \dfrac{0}{4}$ **Step 1:** Divide both sides by the leading coefficient, 4.

$x^2 - \dfrac{13}{2}x + \dfrac{15}{2} = 0$

$x^2 - \dfrac{13}{2}x = -\dfrac{15}{2}$ **Step 2:** Isolate the variable terms on one side.

$x^2 - \dfrac{13}{2}x + \dfrac{169}{16} = -\dfrac{15}{2} + \dfrac{169}{16}$ **Step 3:** Add $\left[\frac{1}{2}\left(-\frac{13}{2}\right)\right]^2 = \left(-\frac{13}{4}\right)^2 = \frac{169}{16}$ to both sides.

$\left(x - \dfrac{13}{4}\right)^2 = -\dfrac{120}{16} + \dfrac{169}{16}$ Factor the perfect square trinomial. Rewrite the right-hand side with a common denominator.

$\left(x - \dfrac{13}{4}\right)^2 = \dfrac{49}{16}$

$x - \dfrac{13}{4} = \pm\sqrt{\dfrac{49}{16}}$ **Step 4:** Apply the square root property.

$x - \dfrac{13}{4} = \pm\dfrac{7}{4}$ Simplify the radical.

$x = \dfrac{13}{4} \pm \dfrac{7}{4}$

$x = \dfrac{13}{4} + \dfrac{7}{4} = \dfrac{20}{4} = 5$

$x = \dfrac{13}{4} - \dfrac{7}{4} = \dfrac{6}{4} = \dfrac{3}{2}$

The solution set is $\left\{5, \dfrac{3}{2}\right\}$. The solutions are rational numbers.

Skill Practice Solve by completing the square and applying the square root property.

10. $2y(y - 1) = 3 - y$

TIP: In general, if the solutions to a quadratic equation are rational numbers, the equation can be solved by factoring and using the zero product rule. Consider the equation from Example 7.

$$2x(2x - 10) = -30 + 6x$$
$$4x^2 - 20x = -30 + 6x$$
$$4x^2 - 26x + 30 = 0$$
$$2(2x^2 - 13x + 15) = 0$$
$$2(x - 5)(2x - 3) = 0$$
$$x = 5 \quad \text{or} \quad x = \frac{3}{2}$$

Avoiding Mistakes

When the solutions are rational, combine the *like* terms. That is, do not leave the solution with the ± sign.

Answer

10. $\left\{\frac{3}{2}, -1\right\}$

3. Literal Equations

Example 8 Solving a Literal Equation

Ignoring air resistance, the distance d (in meters) that an object falls in t sec is given by the equation

$$d = 4.9t^2 \qquad \text{where } t \geq 0$$

a. Solve the equation for t. Do not rationalize the denominator.

b. Using the equation from part (a), determine the amount of time required for an object to fall 500 m. Round to the nearest second.

Solution:

a. $d = 4.9t^2$

$\dfrac{d}{4.9} = t^2$ Isolate the quadratic term. The equation is in the form $t^2 = k$.

$t = \pm\sqrt{\dfrac{d}{4.9}}$ Apply the square root property.

$= \sqrt{\dfrac{d}{4.9}}$ Because t represents time, $t \geq 0$. We reject the negative solution.

b. $t = \sqrt{\dfrac{d}{4.9}}$

$= \sqrt{\dfrac{500}{4.9}}$ Substitute $d = 500$.

$t \approx 10.1$

The object will require approximately 10.1 sec to fall 500 m.

Skill Practice

11. Given $x = 2yz^2$, solve for z where $z > 0$. Do not rationalize the denominator.

12. Use the equation from the previous exercise to find z when $x = 54$ and $y = 3$.

Answers

11. $z = \sqrt{\dfrac{x}{2y}}$

12. $z = 3$

Section 7.1 Activity

A.1. **a.** Consider the equation $x^2 = 49$. One method to solve this equation algebraically is to set one side equal to zero, factor the other side, and then apply the zero product rule. Solve the equation using this process.

b. Now consider a similar equation $x^2 = 16$. By inspection, the solutions to this equation are the real numbers that when squared equal 16. Write the solution set.

c. Now consider the equation $x^2 = -25$. There is no real number x such that the square of x is negative. However, if we solve the equation over the set of complex numbers, the solutions are $\sqrt{-25}$ and $-\sqrt{-25}$ or equivalently, $5i$ and $-5i$. To show this, fill in the blanks.
$(5i)^2 =$ __________ and $(-5i)^2 =$ __________

d. We can now generalize the results from parts (a)–(c) as the **square root property**. That is, for a real number k, the solutions to the equation $x^2 = k$ are __________ and __________.

For Exercises A.2–A.3, solve the equations.

A.2. **a.** $u^2 = 64$ **b.** $(x + 3)^2 = 64$ **c.** $4(x + 3)^2 = 64$

A.3. **a.** $u^2 = 20$ **b.** $(x - 4)^2 = 20$ **c.** $(x - 4)^2 + 26 = 20$

A.4. Consider the trinomial $x^2 + 10x + \boxed{?}$. Suppose we want to fill in the blank so that the expression is a perfect square trinomial. Thus,

$$x^2 + 10x + \boxed{?} = (x + a)^2.$$

The right side expands to $x^2 + 2ax + a^2$, which means that the middle term $10x$ on the left must equal the middle term $2ax$ on the right.

$$\begin{aligned} x^2 + 10x + \boxed{?} &= (x + a)^2 \\ &= x^2 + 2ax + a^2 \end{aligned}$$

a. If $10x = 2ax$, what is the value of a?

b. What is the value of a^2?

c. Fill in the blank to form a perfect square trinomial and then factor the result.

$$x^2 + 10x + \square = (x + \square)^2$$

d. Given the expression $x^2 + bx + n$, what is the value of n that would make the expression a perfect square trinomial?

A.5. Determine the value of n that would make the expression a perfect square trinomial. Then factor the result.

a. $y^2 + 18y + n$ **b.** $t^2 - 11t + n$ **c.** $x^2 + \frac{3}{5}x + n$

A.6. Consider the quadratic equation $2x^2 + 8x + 20 = 0$. The left side is not factorable using the techniques we have learned thus far. Therefore, we cannot apply the zero product property. Instead, we will solve the equation by completing the square and applying the square root property.

a. Divide both sides by the leading coefficient, 2, and write the resulting equation.

b. Using the equation from part (a), isolate the terms containing x on the left side of the equation. Write the new equation.

c. Write the left side of the equation as a perfect square trinomial by adding an appropriate constant. To balance the equation, be sure to add the same constant to the right side.

d. Write the left side in factored form and simplify on the right.

e. Solve the equation by using the square root property.

Practice Exercises — Section 7.1

Prerequisite Review

For Exercises R.1–R.4, solve the equation.

R.1. $4x - 3 = 17$

R.2. $-6t + 4 = 64$

R.3. $n - \frac{2}{3} = \frac{\sqrt{5}}{3}$

R.4. $p + \frac{1}{5} = \frac{\sqrt{7}}{5}$

R.5. Identify the square roots of 121.

R.6. Identify the square roots of 64.

For Exercises R.7–R.10, simplify the radical.

R.7. $\sqrt{72}$ **R.8.** $\sqrt{500}$

R.9. $\sqrt{\dfrac{37}{64}}$ **R.10.** $\sqrt{\dfrac{10}{49}}$

For Exercises R.11–R.14, square the binomial.

R.11. $(y - 7)^2$ **R.12.** $(t + 9)^2$

R.13. $(3w + 4)^2$ **R.14.** $(2m - 5)^2$

For Exercises R.15–R.18, factor completely.

R.15. $p^2 + 22p + 121$ **R.16.** $w^2 - 26w + 169$

R.17. $64x^2 - 144x + 81$ **R.18.** $4x^2 + 40x + 100$

Vocabulary and Key Concepts

1. a. The zero product rule states that if $ab = 0$, then $a =$ __________ or $b =$ __________.

b. To apply the zero product rule, one side of the equation must be equal to __________ and the other side must be written in factored form.

c. The square root property states that for any real number k, if $x^2 = k$, then $x =$ __________ or $x =$ __________.

d. To apply the square root property to the equation $t^2 + 2 = 11$, first subtract __________ from both sides. The solution set is __________.

e. The process to create a perfect square trinomial is called __________ the square.

f. Fill in the blank to complete the square for the trinomial $x^2 + 20x +$ __________.

g. To use completing the square to solve the equation $4x^2 + 3x + 5 = 0$, the first step is to divide by __________ so that the coefficient of the x^2 term is __________.

h. Given the trinomial $y^2 + 8y + 16$, the coefficient of the linear term is __________.

Concept 1: Solving Quadratic Equations by Using the Square Root Property

For Exercises 2–21, solve the equations by using the square root property. Write imaginary solutions in the form $a + bi$. **(See Examples 1–3.)**

2. $x^2 = 100$ **3.** $y^2 = 4$ **4.** $a^2 = 5$ **5.** $k^2 - 7 = 0$

6. $4t^2 = 81$ **7.** $36u^2 = 121$ **8.** $3v^2 + 33 = 0$ **9.** $-2m^2 = 50$

10. $(p - 5)^2 = 9$ **11.** $(q + 3)^2 = 4$ **12.** $(3x - 2)^2 - 5 = 0$ **13.** $(2y + 3)^2 - 7 = 0$

14. $(h-4)^2 = -8$

15. $(t+5)^2 = -18$

16. $6p^2 - 3 = 2$

17. $15 = 4 + 3w^2$

18. $\left(x - \frac{3}{2}\right)^2 + \frac{7}{4} = 0$

19. $\left(m + \frac{4}{5}\right)^2 + \frac{3}{25} = 0$

20. $-x^2 + 4 = 13$

21. $-y^2 - 2 = 14$

22. Given the equation $x^2 = k$, match the following statements.

a. If $k > 0$, then _____ **i.** there will be one real solution.

b. If $k < 0$, then _____ **ii.** there will be two real solutions.

c. If $k = 0$, then _____ **iii.** there will be two imaginary solutions.

23. State two methods that can be used to solve the equation $x^2 - 36 = 0$. Then solve the equation by using both methods.

24. Explain the difference between solving the equations: $x = \sqrt{16}$ and $x^2 = 16$.

For Exercises 25–26, solve the equations.

25. **a.** $\sqrt{x} = 4$
b. $x^2 = 4$

26. **a.** $\sqrt{y} = 9$
b. $y^2 = 9$

Concept 2: Solving Quadratic Equations by Completing the Square

For Exercises 27–38, find the value of n so that the expression is a perfect square trinomial. Then factor the trinomial. **(See Example 4.)**

27. $x^2 - 6x + n$

28. $x^2 + 24x + n$

29. $t^2 + 8t + n$

30. $v^2 - 18v + n$

31. $c^2 - c + n$

32. $x^2 + 9x + n$

33. $y^2 + 5y + n$

34. $a^2 - 7a + n$

35. $b^2 + \frac{2}{5}b + n$

36. $m^2 - \frac{2}{7}m + n$

37. $p^2 - \frac{2}{3}p + n$

38. $w^2 + \frac{3}{4}w + n$

39. Summarize the steps used in solving a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square and applying the square root property.

40. What types of quadratic equations can be solved by completing the square and applying the square root property?

For Exercises 41–60, solve the quadratic equation by completing the square and applying the square root property. Write imaginary solutions in the form $a + bi$. **(See Examples 5–7.)**

41. $t^2 + 8t + 15 = 0$

42. $m^2 + 6m + 8 = 0$

43. $x^2 + 6x = -16$

44. $x^2 - 4x = -15$

45. $p^2 + 4p + 6 = 0$

46. $q^2 + 2q + 2 = 0$

47. $-3y - 10 = -y^2$

48. $-24 = -2y^2 + 2y$

49. $2a^2 + 4a + 5 = 0$

50. $3a^2 + 6a - 7 = 0$

51. $9x^2 - 36x + 40 = 0$

52. $9y^2 - 12y + 5 = 0$

53. $25p^2 - 10p = 2$

54. $9n^2 - 6n = 1$

55. $(2w + 5)(w - 1) = 2$

56. $(3p - 5)(p + 1) = -3$

57. $n(n - 4) = 7$

58. $m(m + 10) = 2$

59. $2x(x + 6) = 14$

60. $3x(x - 2) = 24$

Concept 3: Literal Equations

61. The distance d (in feet) that an object falls in t sec is given by the equation $d = 16t^2$, where $t \geq 0$.

- **a.** Solve the equation for t. **(See Example 8.)**
- **b.** Using the equation from part (a), determine the amount of time required for an object to fall 1024 ft.

62. The volume V (in cubic inches) of a can that is 4 in. tall is given by the equation $V = 4\pi r^2$, where r is the radius of the can, measured in inches.

- **a.** Solve the equation for r. Do not rationalize the denominator.
- **b.** Using the equation from part (a), determine the radius of a can with a volume of 12.56 in.^{3} Use 3.14 for π.

For Exercises 63–68, solve for the indicated variable.

63. $A = \pi r^2$ for r $(r > 0)$

64. $E = mc^2$ for c $(c > 0)$

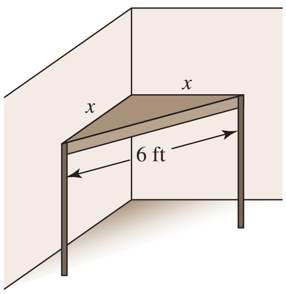
65. $a^2 + b^2 + c^2 = d^2$ for a $(a > 0)$

66. $a^2 + b^2 = c^2$ for b $(b > 0)$

67. $V = \frac{1}{3}\pi r^2 h$ for r $(r > 0)$

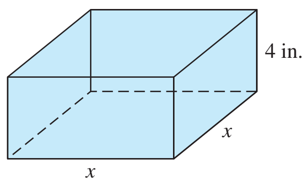
68. $V = \frac{1}{3}s^2 h$ for s $(s > 0)$

69. A corner shelf is to be made from a triangular piece of plywood, as shown in the diagram. Find the distance x that the shelf will extend along the walls. Assume that the walls are at right angles. Round the answer to a tenth of a foot.



70. The volume $V(x)$ (in cubic inches) of a box with a square bottom and a height of 4 in. is given by $V(x) = 4x^2$, where x is the length (in inches) of the sides of the bottom of the box.

- **a.** If the volume of the box is 289 in.^{3}, find the dimensions of the box.
- **b.** Are there two possible answers to part (a)? Why or why not?



71. A square has an area of 50 in.^{2} What are the lengths of the sides? (Round to one decimal place.)

72. The amount of money A in an account with an interest rate r compounded annually is given by

$$A = P(1 + r)^t$$

where P is the initial principal and t is the number of years the money is invested.

a. If a \$10,000 investment grows to \$11,664 after 2 years, find the interest rate.

b. If a \$6000 investment grows to \$7392.60 after 2 years, find the interest rate.

c. Jamal wants to invest \$5000. He wants the money to grow to at least \$6500 in 2 years to cover the cost of his son's first year at college. What interest rate does Jamal need for his investment to grow to \$6500 in 2 years? Round to the nearest hundredth of a percent.

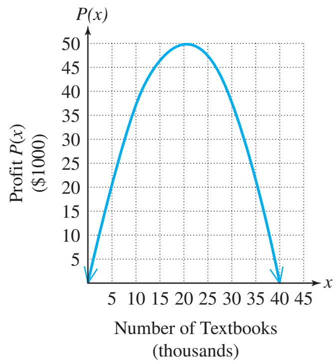
73. A textbook company has discovered that the profit for selling its books is given by

$$P(x) = -\frac{1}{8}x^2 + 5x$$

where x is the number of textbooks produced (in thousands) and $P(x)$ is the corresponding profit (in thousands of dollars).

a. Approximate the number of books required to make a profit of \$20,000. [*Hint:* Let $P(x) = 20$. Then complete the square to solve for x.] Round to one decimal place.

b. Why are there two answers to part (a)?



74. If we ignore air resistance, the distance $d(t)$ (in feet) that an object travels in free fall can be approximated by $d(t) = 16t^2$, where t is the time in seconds after the object was dropped.

a. If the CN Tower in Toronto is 1815 ft high, how long will it take an object to fall from the top of the building? Round to one decimal place.

b. If the Renaissance Tower in Dallas is 886 ft high, how long will it take an object to fall from the top of the building? Round to one decimal place.

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Section 7.2 Quadratic Formula

Concepts

1. Derivation of the Quadratic Formula
2. Solving Quadratic Equations by Using the Quadratic Formula
3. Using the Quadratic Formula in Applications
4. Discriminant
5. Mixed Review: Methods to Solve a Quadratic Equation

1. Derivation of the Quadratic Formula

If we solve a quadratic equation in standard form $ax^2 + bx + c = 0$, $a > 0$, by completing the square and using the square root property, the result is a formula that gives the solutions for x in terms of a, b, and c.

$ax^2 + bx + c = 0 \ (a \neq 0)$	Begin with a quadratic equation in standard form.
$\dfrac{ax^2}{a} + \dfrac{bx}{a} + \dfrac{c}{a} = \dfrac{0}{a}$	Divide by the leading coefficient.
$x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$	
$x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$	Isolate the terms containing x.
$x^2 + \dfrac{b}{a}x + \left(\dfrac{1}{2} \cdot \dfrac{b}{a}\right)^2 = \left(\dfrac{1}{2} \cdot \dfrac{b}{a}\right)^2 - \dfrac{c}{a}$	Add the square of $\frac{1}{2}$ the linear term coefficient to both sides of the equation.
$\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2} - \dfrac{c}{a}$	Factor the left side as a perfect square.
$\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}$	Combine fractions on the right side by getting a common denominator.
$x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}}$	Apply the square root property.
$x + \dfrac{b}{2a} = \dfrac{\pm\sqrt{b^2 - 4ac}}{2a}$	Simplify the denominator.
$x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$	Subtract $\dfrac{b}{2a}$ from both sides.
$= \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Combine fractions.

The solutions to the equation $ax^2 + bx + c = 0$ in terms of the coefficients a, b, and c are given by the **quadratic formula**.

TIP: When applying the quadratic formula, note that *a*, *b*, and *c* are constants. The variable is *x*.

The Quadratic Formula

For a quadratic equation of the form $ax^2 + bx + c = 0 \ (a \neq 0)$ the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula gives us another technique to solve a quadratic equation. This method will work regardless of whether the equation is factorable or not factorable.

2. Solving Quadratic Equations by Using the Quadratic Formula

Example 1 Solving a Quadratic Equation by Using the Quadratic Formula

Solve the quadratic equation by using the quadratic formula. $2x^2 - 3x = 5$

Solution:

$$2x^2 - 3x = 5$$

$$2x^2 - 3x - 5 = 0$$ Write the equation in the form $ax^2 + bx + c = 0$.

$a = 2, \quad b = -3, \quad c = -5$ Identify a, b, and c.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Apply the quadratic formula.

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$ Substitute $a = 2$, $b = -3$, and $c = -5$.

$$= \frac{3 \pm \sqrt{9 + 40}}{4}$$ Simplify.

$$= \frac{3 \pm \sqrt{49}}{4}$$

$$= \frac{3 \pm 7}{4} \quad \rightarrow \quad x = \frac{3 + 7}{4} = \frac{10}{4} = \frac{5}{2}, \qquad x = \frac{3 - 7}{4} = \frac{-4}{4} = -1$$

The solution set is $\left\{\frac{5}{2}, -1\right\}$. Both solutions check in the original equation.

Skill Practice Solve the equation by using the quadratic formula.

1. $6x^2 - 5x = 4$

Avoiding Mistakes

- The term $-b$ represents the *opposite* of b.
- Remember to write the *entire* numerator over $2a$.

Example 2 Solving a Quadratic Equation by Using the Quadratic Formula

Solve the quadratic equation by using the quadratic formula. $-x(x - 6) = 11$

Solution:

$$-x(x - 6) = 11$$

$$-x^2 + 6x - 11 = 0$$ Write the equation in the form $ax^2 + bx + c = 0$.

$$-1(-x^2 + 6x - 11) = -1(0)$$ If the leading coefficient of the quadratic polynomial is negative, we suggest multiplying both sides of the equation by -1. Although this is not mandatory, it is generally easier to simplify the quadratic formula when the value of a is positive.

$$x^2 - 6x + 11 = 0$$

Answer

1. $\left\{-\frac{1}{2}, \frac{4}{3}\right\}$

$a = 1, b = -6, \text{ and } c = 11$ Identify a, b, and c.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Apply the quadratic formula.

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(11)}}{2(1)}$$ Substitute $a = 1$, $b = -6$, and $c = 11$.

$$= \frac{6 \pm \sqrt{36 - 44}}{2}$$ Simplify.

$$= \frac{6 \pm \sqrt{-8}}{2}$$

$$= \frac{6 \pm 2i\sqrt{2}}{2}$$ Simplify the radical.

$$= \frac{2(3 \pm i\sqrt{2})}{2}$$ Factor the numerator.

$$= \frac{\cancel{2}(3 \pm i\sqrt{2})}{\cancel{2}}$$ Simplify the fraction to lowest terms.

$$= 3 \pm i\sqrt{2} \quad \begin{cases} x = 3 + i\sqrt{2} \\ x = 3 - i\sqrt{2} \end{cases}$$ The solutions are imaginary numbers.

The solution set is $\{3 \pm i\sqrt{2}\}$.

Avoiding Mistakes

When identifying a, b, and c, use the coefficients only, not the variable. For example, the value of b is -6, not $-6x$.

Avoiding Mistakes

Always simplify the radical completely before trying to reduce the fraction to lowest terms.

Skill Practice Solve the equation by using the quadratic formula.

2. $-y(y + 4) = 12$

3. Using the Quadratic Formula in Applications

Example 3 Using the Quadratic Formula in an Application

A delivery truck travels south from Hartselle, Alabama, to Birmingham, Alabama, along Interstate 65. The truck then heads east to Atlanta, Georgia, along Interstate 20. The distance from Birmingham to Atlanta is 8 mi less than twice the distance from Hartselle to Birmingham. If the straight-line distance from Hartselle to Atlanta is 165 mi, find the distance from Hartselle to Birmingham and from Birmingham to Atlanta. (Round the answers to the nearest mile.)

Solution:

The motorist travels due south and then due east. Therefore, the three cities form the vertices of a right triangle (Figure 7-1).

Let x represent the distance between Hartselle and Birmingham.

Then $2x - 8$ represents the distance between Birmingham and Atlanta.

Use the Pythagorean theorem to establish a relationship among the three sides of the triangle.

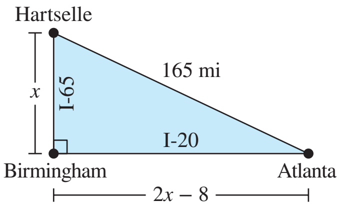


Figure 7-1

Answer

2. $\{-2 \pm 2i\sqrt{2}\}$

$$(x)^2 + (2x - 8)^2 = (165)^2$$

$$x^2 + 4x^2 - 32x + 64 = 27{,}225$$

$$5x^2 - 32x - 27{,}161 = 0$$ Write the equation in the form $ax^2 + bx + c = 0$.

$a = 5 \quad b = -32 \quad c = -27{,}161$ Identify a, b, and c.

$$x = \frac{-(-32) \pm \sqrt{(-32)^2 - 4(5)(-27{,}161)}}{2(5)}$$ Apply the quadratic formula.

$$= \frac{32 \pm \sqrt{1024 + 543{,}220}}{10}$$ Simplify.

$$= \frac{32 \pm \sqrt{544{,}244}}{10}$$

$$x = \frac{32 + \sqrt{544{,}244}}{10} \approx 76.97$$

$$x = \frac{32 - \sqrt{544{,}244}}{10} \approx -70.57$$

We reject the negative solution because distance cannot be negative. Rounding to the nearest whole unit, we have $x = 77$. Therefore, $2x - 8 = 2(77) - 8 = 146$.

The distance between Hartselle and Birmingham is 77 mi, and the distance between Birmingham and Atlanta is 146 mi.

Skill Practice

3. Steve and Tammy leave a campground, hiking on two different trails. Steve heads south and Tammy heads east. By lunchtime they are 9 mi apart. Steve walked 3 mi more than twice as many miles as Tammy. Find the distance each person hiked. (Round to the nearest tenth of a mile.)

Example 4 Analyzing a Quadratic Function

A model rocket is launched straight upward from the side of a 144-ft cliff (Figure 7-2). The initial velocity is 112 ft/sec. The height of the rocket $h(t)$ is given by

$$h(t) = -16t^2 + 112t + 144$$

where $h(t)$ is measured in feet and t is the time in seconds after launch. Find the time(s) at which the rocket is 208 ft above the ground.

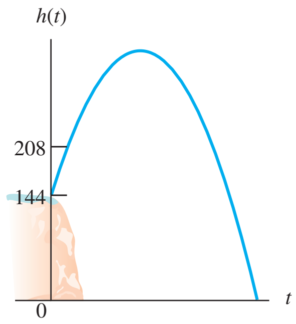


Figure 7-2

Solution:

$$h(t) = -16t^2 + 112t + 144$$

$$208 = -16t^2 + 112t + 144$$ Substitute 208 for $h(t)$.

$$16t^2 - 112t + 64 = 0$$ Write the equation in the form $at^2 + bt + c = 0$.

$$\frac{16t^2}{16} - \frac{112t}{16} + \frac{64}{16} = \frac{0}{16}$$ Divide by 16. This makes the coefficients smaller, and it is less cumbersome to solve.

$$t^2 - 7t + 4 = 0$$ The equation is not factorable. Apply the quadratic formula.

Answer

3. Tammy hiked 2.8 mi, and Steve hiked 8.6 mi.

$$t = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(4)}}{2(1)}$$ Let $a = 1$, $b = -7$, and $c = 4$.

$$= \frac{7 \pm \sqrt{33}}{2}$$

$$t = \frac{7 + \sqrt{33}}{2} \approx 6.37$$

$$t = \frac{7 - \sqrt{33}}{2} \approx 0.63$$

The rocket will reach a height of 208 ft after approximately 0.63 sec (on the way up) and after 6.37 sec (on the way down).

Skill Practice

4. A rocket is launched from the top of a 96-ft building with an initial velocity of 64 ft/sec. The height $h(t)$ of the rocket is given by $h(t) = -16t^2 + 64t + 96$. Find the time it takes for the rocket to hit the ground. [*Hint:* $h(t) = 0$ when the object hits the ground.]

4. Discriminant

The radicand within the quadratic formula is the expression $b^2 - 4ac$. This is called the **discriminant**. The discriminant can be used to determine the number of solutions to a quadratic equation as well as whether the solutions are rational, irrational, or imaginary numbers.

Using the Discriminant to Determine the Number and Type of Solutions to a Quadratic Equation

Consider the equation $ax^2 + bx + c = 0$, where a, b, and c are rational numbers and $a \neq 0$. The expression $b^2 - 4ac$ is called the *discriminant*. Furthermore,

- If $b^2 - 4ac > 0$, then there will be two real solutions.
  - **a.** If $b^2 - 4ac$ is a perfect square, the solutions will be rational numbers.
  - **b.** If $b^2 - 4ac$ is not a perfect square, the solutions will be irrational numbers.
- If $b^2 - 4ac < 0$, then there will be two imaginary solutions.
- If $b^2 - 4ac = 0$, then there will be one rational solution.

Example 5 Using the Discriminant

Use the discriminant to determine the type and number of solutions for each equation.

a. $2x^2 - 5x + 9 = 0$ **b.** $3x^2 = -x + 2$

c. $-2x(2x - 3) = -1$ **d.** $3.6x^2 = -1.2x - 0.1$

Solution:

For each equation, first write the equation in standard form $ax^2 + bx + c = 0$. Then determine the discriminant.

Equation	Discriminant	Solution Type and Number
a. $2x^2 - 5x + 9 = 0$	$b^2 - 4ac$ $= (-5)^2 - 4(2)(9)$ $= 25 - 72$ $= -47$	Because $-47 < 0$, there will be two imaginary solutions.

Answer

4. $2 + \sqrt{10} \approx 5.16$ sec

b. $3x^2 = -x + 2$

$3x^2 + x - 2 = 0$	$b^2 - 4ac$ $= (1)^2 - 4(3)(-2)$ $= 1 - (-24)$ $= 25$	$25 > 0$ and 25 is a perfect square. There will be two rational solutions.

c. $-2x(2x - 3) = -1$

$-4x^2 + 6x = -1$

$-4x^2 + 6x + 1 = 0$	$b^2 - 4ac$ $= (6)^2 - 4(-4)(1)$ $= 36 - (-16)$ $= 52$	$52 > 0$, but 52 is *not* a perfect square. There will be two irrational solutions.

d. $3.6x^2 = -1.2x - 0.1$

$3.6x^2 + 1.2x + 0.1 = 0$	$b^2 - 4ac$ $= (1.2)^2 - 4(3.6)(0.1)$ $= 1.44 - 1.44$ $= 0$	Because the discriminant equals 0, there will be only one rational solution.

Skill Practice Use the discriminant to determine the type and number of solutions for the equation.

5. $3y^2 + y + 3 = 0$

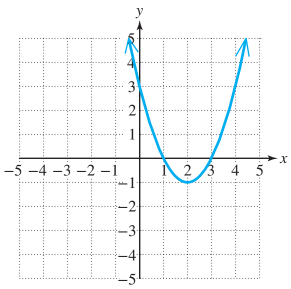
6. $4t^2 = 6t - 2$

7. $3t(t + 1) = 9$

8. $\frac{2}{3}x^2 - \frac{2}{3}x + \frac{1}{6} = 0$

With the discriminant we can determine the number of real-valued solutions to the equation $ax^2 + bx + c = 0$, and thus the number of x-intercepts to the graph of $f(x) = ax^2 + bx + c$. The following illustrations show the graphical interpretation of the three cases of the discriminant.

$f(x) = x^2 - 4x + 3$

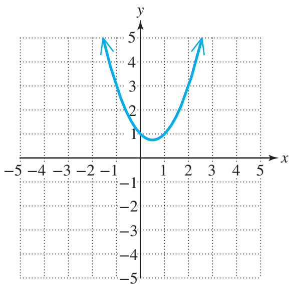


Use $x^2 - 4x + 3 = 0$ to find the value of the discriminant.

$$b^2 - 4ac = (-4)^2 - 4(1)(3)$$
$$= 4$$

Since the discriminant is positive, there are two real solutions to the quadratic equation. Therefore, there are two x-intercepts to the corresponding quadratic function, (1, 0) and (3, 0).

$f(x) = x^2 - x + 1$



Use $x^2 - x + 1 = 0$ to find the value of the discriminant.

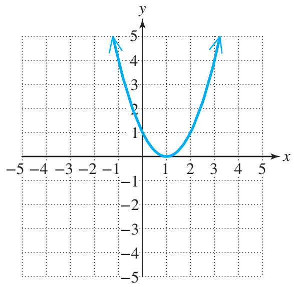
$$b^2 - 4ac = (-1)^2 - 4(1)(1)$$
$$= -3$$

Since the discriminant is negative, there are no real solutions to the quadratic equation. Therefore, there are no x-intercepts to the corresponding quadratic function.

Answers

5. -35; two imaginary solutions
6. 4; two rational solutions
7. 117; two irrational solutions
8. 0; one rational solution

$f(x) = x^2 - 2x + 1$



Use $x^2 - 2x + 1 = 0$ to find the value of the discriminant.

$$\begin{aligned} b^2 - 4ac &= (-2)^2 - 4(1)(1) \\ &= 0 \end{aligned}$$

Since the discriminant is zero, there is one real solution to the quadratic equation. Therefore, there is one x-intercept to the corresponding quadratic function, (1, 0).

Example 6 Finding *x*- and *y*-Intercepts of a Quadratic Function

Given $f(x) = x^2 - 3x + 1$

a. Find the discriminant and use it to determine if there are any x-intercepts.

b. Find the x-intercept(s), if they exist.

c. Find the y-intercept.

Solution:

a. $a = 1$, $b = -3$, and $c = 1$.

The discriminant is $b^2 - 4ac = (-3)^2 - 4(1)(1)$

$$= 9 - 4$$

$$= 5$$

Since $5 > 0$, there are two x-intercepts.

TIP: Recall that an x-intercept is a point $(a, 0)$ where the graph of a function intersects the x-axis. A y-intercept is a point $(0, b)$ where the graph intersects the y-axis.

b. The x-intercepts are given by the real solutions to the equation $f(x) = 0$. In this case, we have

$f(x) = x^2 - 3x + 1 = 0$

$x^2 - 3x + 1 = 0$ The equation is in the form $ax^2 + bx + c = 0$.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - (4)(1)(1)}}{2(1)}$$ Apply the quadratic formula.

$$= \frac{3 \pm \sqrt{5}}{2}$$

$$x = \frac{3 + \sqrt{5}}{2} \approx 2.6$$

$$x = \frac{3 - \sqrt{5}}{2} \approx 0.4$$

The solutions are $\dfrac{3 + \sqrt{5}}{2}$ and $\dfrac{3 - \sqrt{5}}{2}$. Therefore, the x-intercepts are

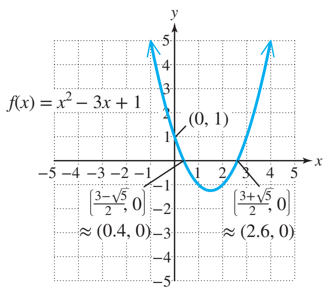
$\left(\dfrac{3 + \sqrt{5}}{2}, 0\right)$ and $\left(\dfrac{3 - \sqrt{5}}{2}, 0\right)$.

c. To find the y-intercept, evaluate $f(0)$.

$f(0) = (0)^2 - 3(0) + 1 = 1$

The y-intercept is located at (0, 1).

The parabola is shown in the graph with the x- and y-intercepts labeled.



Skill Practice Given $f(x) = x^2 + 5x + 2$,

9. Find the discriminant and use it to determine if there are any x-intercepts.

10. Find the x- and y-intercepts.

Answers

9. Discriminant: 17; there are two x-intercepts.

10. x-intercepts: $\left(\dfrac{-5 + \sqrt{17}}{2}, 0\right)$, $\left(\dfrac{-5 - \sqrt{17}}{2}, 0\right)$

y-intercept: (0, 2)

Example 7 Finding *x*- and *y*-Intercepts of a Quadratic Function

Given $f(x) = x^2 + x + 2$

a. Find the discriminant and use it to determine if there are any *x*-intercepts.

b. Find the *x*-intercept(s), if they exist.

c. Find the *y*-intercept.

Solution:

a. $a = 1$, $b = 1$, and $c = 2$.

The discriminant is $b^2 - 4ac = (1)^2 - 4(1)(2)$

$$= 1 - 8$$

$$= -7$$

Since $-7 < 0$, there are no *x*-intercepts.

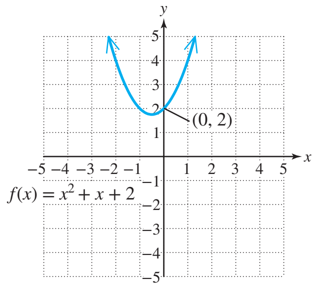
b. There are no *x*-intercepts.

c. $f(0) = (0)^2 + (0) + 2$

$$= 2$$

The *y*-intercept is located at (0, 2).

The parabola is shown. *Note:* The graph does not intersect the *x*-axis.



Skill Practice Given $f(x) = 2x^2 - 3x + 5$,

11. Find the discriminant and use it to determine if there are any *x*-intercepts.

12. Find the *y*-intercept.

5. Mixed Review: Methods to Solve a Quadratic Equation

Three methods have been presented to solve quadratic equations.

Methods to Solve a Quadratic Equation

Factor and use the zero product rule. • This method works well if you can factor the equation easily.	Example: $x^2 + 8x + 15 = 0$ factors as $(x + 3)(x + 5) = 0$
Use the square root property. Complete the square if necessary. This method is particularly good if • the equation is of the form $ax^2 + c = 0$ or • the equation is of the form $x^2 + bx + c = 0$, where b is even.	Examples: $4x^2 + 9 = 0$ $x^2 + 10x - 3 = 0$
Quadratic formula • This method works in all cases—just be sure to write the equation in the form $ax^2 + bx + c = 0$ before applying the formula.	Example: $7x^2 - 3x + 11 = 0$

Answers

11. Discriminant: -31; there are no *x*-intercepts.

12. (0, 5)

Before solving a quadratic equation, take a minute to analyze it first. Each problem must be evaluated individually before choosing the most efficient method to find its solutions.

FOR REVIEW

Recall that a second-degree polynomial equation such as $2x^2 - 3x + 1 = 0$ is quadratic. A first-degree equation such as $2x - 1 = 0$ is linear.

Example 8 Solving a Quadratic Equation by Using Any Method

Solve the equation by using any method. $(x + 3)^2 + x^2 - 9x = 8$

Solution:

$(x + 3)^2 + x^2 - 9x = 8$

$x^2 + 6x + 9 + x^2 - 9x - 8 = 0$ Clear parentheses and write the equation in the form $ax^2 + bx + c = 0$.

$2x^2 - 3x + 1 = 0$ This equation is factorable.

$(2x - 1)(x - 1) = 0$ Factor.

$2x - 1 = 0$ or $x - 1 = 0$ Apply the zero product rule.

$x = \frac{1}{2}$ or $x = 1$ Solve for x.

The solution set is $\{\frac{1}{2}, 1\}$.

This equation could have been solved by using any of the three methods, but factoring was the most efficient method.

Skill Practice Solve using any method.

13. $2t(t - 1) + t^2 = 5$

Example 9 Solving a Quadratic Equation by Using Any Method

Solve the equation by using any method. $x^2 + 5 = -2x$

Solution:

$x^2 + 2x + 5 = 0$ The equation does not factor.

$x^2 + 2x = -5$ Because $a = 1$ and b is even, we can easily complete the square.

$x^2 + 2x + 1 = -5 + 1$ Add $\left[\frac{1}{2}(2)\right]^2 = 1^2 = 1$ to both sides.

$(x + 1)^2 = -4$

$x + 1 = \pm\sqrt{-4}$ Apply the square root property.

$x = -1 \pm 2i$ Solve for x.

The solution set is $\{-1 \pm 2i\}$.

This equation could also have been solved by using the quadratic formula.

Skill Practice Solve using any method.

14. $x^2 - 4x = -7$

Answers

13. $\left\{-1, \frac{5}{3}\right\}$

14. $\{2 \pm i\sqrt{3}\}$

Example 10 **Solving a Quadratic Equation by Using Any Method**

Solve the equation by using any method. $\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{3} = 0$

Solution:

$12 \cdot \left(\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{3}\right) = 12 \cdot (0)$ Clear fractions.

$3x^2 - 6x + 4 = 0$ The equation is in the form $ax^2 + bx + c = 0$.
The left-hand side does not factor.

$a = 3$, $b = -6$, and $c = 4$

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(4)}}{2(3)}$ Apply the quadratic formula.

$= \frac{6 \pm \sqrt{-12}}{6}$ Simplify.

$= \frac{6 \pm 2i\sqrt{3}}{6}$ Simplify the radical.

$= \frac{\overset{1}{\cancel{2}}(3 \pm i\sqrt{3})}{\underset{3}{\cancel{6}}}$ Factor and simplify.

$= \frac{3 \pm i\sqrt{3}}{3}$

$= \frac{3}{3} \pm \frac{\sqrt{3}}{3}i$ Write in the form $a \pm bi$.

$= 1 \pm \frac{\sqrt{3}}{3}i$ Simplify.

The solution set is $\left\{1 \pm \frac{\sqrt{3}}{3}i\right\}$.

Skill Practice Solve using any method.

15. $\frac{1}{5}x^2 - \frac{4}{5}x + \frac{1}{2} = 0$

Answer

15. $\left\{\frac{4 \pm \sqrt{6}}{2}\right\}$

Example 11 **Solving a Quadratic Equation by Using Any Method**

Solve the equation by using any method. $9p^2 - 11 = 0$

Solution:

$9p^2 - 11 = 0$	Because $b = 0$, use the square root property.
$9p^2 = 11$	Isolate the variable term.
$p^2 = \frac{11}{9}$	The equation is in the form $x^2 = k$.
$p = \pm\sqrt{\frac{11}{9}}$	Apply the square root property.
$p = \pm\frac{\sqrt{11}}{3}$	Simplify the radical.

The solution set is $\left\{\pm\frac{\sqrt{11}}{3}\right\}$.

Skill Practice Solve using any method.

16. $4y^2 - 13 = 0$

Answer

16. $\left\{\pm\frac{\sqrt{13}}{2}\right\}$

Section 7.2 Activity

A.1. Given a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, the solutions are given by the **quadratic formula**:

$$x = \boxed{}$$

For Exercises A.2–A.4, an equation is given in the form $ax^2 + bx + c = 0$.

a. Identify the values of a, b, and c.

b. Solve the equation by using the quadratic formula.

A.2. $2x^2 - 12x + 16 = 0$ **A.3.** $x^2 - 4x + 5 = 0$ **A.4.** $-x^2 - 6x - 9 = 0$

A.5. Based on the results from Exercises A.2–A.4, the solutions to a quadratic equation $ax^2 + bx + c = 0$ may either be real numbers or imaginary numbers. One way to determine the type of solution is to analyze the radicand from the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

a. If the radicand is negative, what type of solution does the equation have? (real or imaginary)

b. If the radicand is nonnegative, what type of solution does the equation have? (real or imaginary)

c. If the radicand is zero, *how many* solutions will the equation have? Otherwise, if the radicand is nonzero, how many solutions will there be?

d. The radicand, $b^2 - 4ac$, is given a special name called the ____________. It helps us determine the number and types of solutions to a quadratic equation.

For Exercises A.6–A.8, refer to the equations from Exercises A.2–A.4.

a. Find the discriminant.

b. Based on the discriminant, determine the number and types of solutions to the equation.

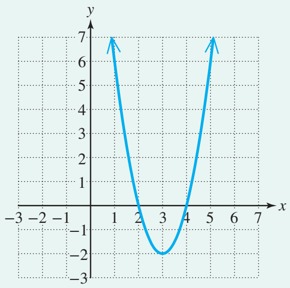
c. Do your results from part (b) agree with the solutions you found in Exercises A.2–A.4?

A.6. $2x^2 - 12x + 16 = 0$ **A.7.** $x^2 - 4x + 5 = 0$ **A.8.** $-x^2 - 6x - 9 = 0$

A.9. Consider the graph of the quadratic function defined by $f(x) = 2x^2 - 12x + 16$ and the related quadratic equation $2x^2 - 12x + 16 = 0$ from Exercises A.2 and A.6.

a. How are the solutions to the equation related to the graph?

b. What does the discriminant to the equation tell you about the number of x-intercepts of the graph?

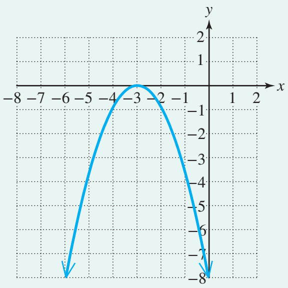


A.10. **a.** Based on the value of the discriminant from Exercise A.7, how many x-intercepts would the graph of $g(x) = x^2 - 4x + 5$ have?

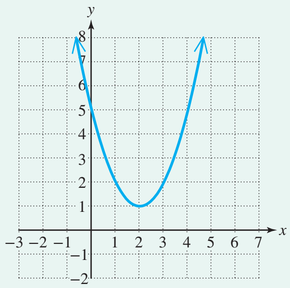
b. Based on the value of the discriminant from Exercise A.8, how many x-intercepts would the graph of $h(x) = -x^2 - 6x - 9$ have?

c. Based on the results of parts (a) and (b), match $g(x) = x^2 - 4x + 5$ and $h(x) = -x^2 - 6x - 9$ with their respective graphs.

i.



ii.



Practice Exercises

Section 7.2

Prerequisite Review

For Exercises R.1–R.6, simplify the expression.

R.1. $\sqrt{24}$

R.2. $\sqrt{128}$

R.3. $\dfrac{10 + \sqrt{125}}{5}$

R.4. $\dfrac{12 - \sqrt{48}}{4}$

R.5. $\sqrt{(-8)^2 - 4(1)(2)}$

R.6. $\sqrt{(-10)^2 - 4(4)(-3)}$

For Exercises R.7–R.10, solve the equation by first clearing fractions or decimals.

R.7. $\frac{1}{4}x - \frac{5}{12} = \frac{1}{2}x + 2$

R.8. $\frac{1}{5} - \frac{2}{15}y = 1 + \frac{2}{3}y$

R.9. $0.07x - 0.38 = 0.09x + 0.8$

R.10. $2.8w - 7.2 = 1.8w + 3.6$

For Exercises R.11–R.12, solve the equation by applying the square root property.

R.11. $(x + 7)^2 = 44$

R.12. $(y - 6)^2 = 50$

Vocabulary and Key Concepts

1. **a.** For the equation $ax^2 + bx + c = 0$ $(a \neq 0)$, the ____________ formula gives the solutions as $x =$ ____________.

b. To apply the quadratic formula, a quadratic equation must be written in the form ____________ where $a \neq 0$.

c. To apply the quadratic formula to solve the equation $8x^2 - 42x - 27 = 0$, the value of a is ____________, the value of b is ____________, and the value of c is ____________.

d. To apply the quadratic formula to solve the equation $3x^2 - 7x - 4 = 0$, the value of $-b$ is ____________ and the value of the radicand is ____________.

2. **a.** The radicand within the quadratic formula is ____________ and is called the ____________.

b. If the discriminant is negative, then the solutions to a quadratic equation will be (real/imaginary) numbers.

c. If the discriminant is positive, then the solutions to a quadratic equation will be (real/imaginary) numbers.

d. Given a quadratic function $f(x) = ax^2 + bx + c = 0$, the function will have no x-intercepts if the discriminant is (less than, greater than, equal to) zero.

Concept 2: Solving Quadratic Equations by Using the Quadratic Formula

For Exercises 3–8, determine whether the equation is linear, quadratic, or neither.

3. $5x^2 - 3x = 8 + x$

4. $-4x + 5 = 2x^2 + 7$

5. $\frac{4}{x^2} + \frac{2}{x} + \frac{1}{3} = 0$

6. $x + 5\sqrt{x} + 4 = 0$

7. $2(x - 5) + x^2 = 3x(x - 4) - 2x^2$

8. $5x(x + 3) - 9 = 4x^2 - 3(x + 1)$

For Exercises 9–34, solve the equation by using the quadratic formula. Write imaginary solutions in the form $a + bi$. **(See Examples 1–2.)**

9. $x^2 + 11x - 12 = 0$

10. $5x^2 - 14x - 3 = 0$

11. $9y^2 - 2y + 5 = 0$

12. $2t^2 + 3t - 7 = 0$

13. $12p^2 - 4p + 5 = 0$

14. $-5n^2 + 4n - 6 = 0$

15. $-z^2 = -2z - 35$

16. $12x^2 - 5x = 2$

17. $y^2 + 3y = 8$

18. $k^2 + 4 = 6k$

19. $25x^2 - 20x + 4 = 0$

20. $9y^2 = -12y - 4$

21. $w(w - 6) = -14$

22. $m(m + 6) = -11$

23. $(x + 2)(x - 3) = 1$

24. $3y(y + 1) - 7y(y + 2) = 6$

25. $-4a^2 - 2a + 3 = 0$

26. $-2m^2 - 5m + 3 = 0$

27. $\frac{1}{2}y^2 + \frac{2}{3} = -\frac{2}{3}y$
(*Hint:* Clear fractions first.)

28. $\frac{2}{3}p^2 - \frac{1}{6}p + \frac{1}{2} = 0$

29. $\frac{1}{5}h^2 + h + \frac{3}{5} = 0$

30. $\frac{1}{4}w^2 + \frac{7}{4}w + 1 = 0$

31. $0.01x^2 + 0.06x + 0.08 = 0$
(*Hint:* Clear decimals first.)

32. $0.5y^2 - 0.7y + 0.2 = 0$

33. $0.3t^2 + 0.7t - 0.5 = 0$

34. $0.01x^2 + 0.04x - 0.07 = 0$

For Exercises 35–38, factor the expression. Then use the zero product rule and the quadratic formula to solve the equation. There should be three solutions to each equation. Write imaginary solutions in the form $a + bi$.

35. **a.** Factor. $x^3 - 27$
b. Solve. $x^3 - 27 = 0$

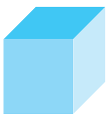
36. **a.** Factor. $64x^3 + 1$
b. Solve. $64x^3 + 1 = 0$

37. **a.** Factor. $3x^3 - 6x^2 + 6x$
b. Solve. $3x^3 - 6x^2 + 6x = 0$

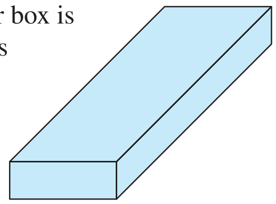
38. **a.** Factor. $5x^3 + 5x^2 + 10x$
b. Solve. $5x^3 + 5x^2 + 10x = 0$

Concept 3: Using the Quadratic Formula in Applications

39. The volume of a cube is 27 ft^3. Find the lengths of the sides.



40. The volume of a rectangular box is 64 ft^3. If the width is 3 times longer than the height, and the length is 9 times longer than the height, find the dimensions of the box.



41. The hypotenuse of a right triangle measures 4 in. One leg of the triangle is 2 in. longer than the other leg. Find the lengths of the legs of the triangle. Round to one decimal place. **(See Example 3.)**

42. The length of one leg of a right triangle is 1 cm more than twice the length of the other leg. The hypotenuse measures 6 cm. Find the lengths of the legs. Round to one decimal place.

43. The hypotenuse of a right triangle is 10.2 m long. One leg is 2.1 m shorter than the other leg. Find the lengths of the legs. Round to one decimal place.

44. The hypotenuse of a right triangle is 17 ft long. One leg is 3.4 ft longer than the other leg. Find the lengths of the legs.

45. The fatality rate (in fatalities per 100 million vehicle miles driven) can be approximated for drivers x years old according to the function $F(x) = 0.0036x^2 - 0.35x + 9.2$. (*Source:* U.S. Department of Transportation)

a. Approximate the fatality rate for drivers 16 years old.

b. Approximate the fatality rate for drivers 40 years old.

c. Approximate the fatality rate for drivers 80 years old.

d. For what age(s) is the fatality rate approximately 2.5?

46. The braking distance (in feet) of a car going v mph is given by

$$d(v) = \frac{v^2}{20} + v \qquad v \geq 0$$

a. Find the speed for a braking distance of 150 ft. Round to the nearest mile per hour.

b. Find the speed for a braking distance of 100 ft. Round to the nearest mile per hour.

47. Mitch throws a baseball straight up in the air from a cliff that is 48 ft high. The initial velocity is 48 ft/sec. The height (in feet) of the object after t sec is given by $h(t) = -16t^2 + 48t + 48$. Find the time at which the height of the object is 64 ft. **(See Example 4.)**

48. An astronaut on the moon throws a rock into the air from the deck of a spacecraft that is 8 m high. The initial velocity of the rock is 2.4 m/sec. The height (in meters) of the rock after t sec is given by $h(t) = -0.8t^2 + 2.4t + 8$. Find the time at which the height of the rock is 6 m.

Concept 4: Discriminant

For Exercises 49–56,

a. Write the equation in the form $ax^2 + bx + c = 0, a > 0$.

b. Find the value of the discriminant.

c. Use the discriminant to determine the number and type of the solutions. Choose from imaginary, rational, and irrational. **(See Example 5.)**

49. $x^2 + 2x = -1$

50. $12y - 9 = 4y^2$

51. $19m^2 = 8m$

52. $2n - 5n^2 = 0$

53. $5p^2 - 21 = 0$

54. $3k^2 = 7$

55. $4n(n - 2) - 5n(n - 1) = 4$

56. $(2x + 1)(x - 3) = -9$

For Exercises 57–62, determine the discriminant. Then use the discriminant to determine the number of x-intercepts for the function.

57. $f(x) = x^2 - 6x + 5$

58. $g(x) = -x^2 - 4x - 3$

59. $h(x) = 4x^2 + 12x + 9$

60. $k(x) = 25x^2 - 10x + 1$

61. $p(x) = 2x^2 + 3x + 6$

62. $m(x) = 3x^2 + 4x + 7$

For Exercises 63–68, find the x- and y-intercepts of the quadratic function. **(See Examples 6–7.)**

63. $f(x) = x^2 - 5x + 3$

64. $g(x) = 2x^2 + 7x + 2$

65. $g(x) = -x^2 + x - 1$

66. $f(x) = 2x^2 + x + 5$

67. $p(x) = 2x^2 + 5x - 2$

68. $h(x) = 3x^2 + 2x - 2$

Concept 5: Mixed Review: Methods to Solve a Quadratic Equation

For Exercises 69–86, solve the quadratic equation by using any method. Write imaginary solutions in the form $a + bi$. **(See Examples 8–11.)**

69. $a^2 + 2a + 10 = 0$

70. $4z^2 + 7z = 0$

71. $(x - 2)^2 + 2x^2 - 13x = 10$

72. $(x - 3)^2 + 3x^2 - 5x = 12$

73. $4y^2 - 20y + 43 = 0$

74. $k^2 + 18 = 4k$

75. $\left(x + \frac{1}{2}\right)^2 + 4 = 0$

76. $(2y + 3)^2 = 9$

77. $2y(y - 3) = -1$

78. $w(w - 5) = 4$

79. $(2t + 5)(t - 1) = (t - 3)(t + 8)$

80. $(b - 1)(b + 4) = (3b + 2)(b + 1)$

81. $\frac{1}{8}x^2 - \frac{1}{2}x + \frac{1}{4} = 0$

82. $\frac{1}{6}x^2 - \frac{1}{2}x + \frac{1}{4} = 0$

83. $32z^2 - 20z - 3 = 0$

84. $8k^2 - 14k + 3 = 0$

85. $4p^2 - 21 = 0$

86. $5h^2 - 120 = 0$

Sometimes students shy away from completing the square and using the square root property to solve a quadratic equation. However, sometimes this process leads to a simple solution. For Exercises 87–88, solve the equations two ways.

a. Solve the equation by completing the square and applying the square root property.

b. Solve the equation by applying the quadratic formula.

c. Which technique was easier for you?

87. $x^2 + 6x = 5$

88. $x^2 - 10x = -27$

Technology Connections

89. Graph $Y_1 = x^3 - 27$. Compare the x-intercepts with the solutions to the equation $x^3 - 27 = 0$ found in Exercise 35.

90. Graph $Y_1 = 64x^3 + 1$. Compare the x-intercepts with the solutions to the equation $64x^3 + 1 = 0$ found in Exercise 36.

91. Graph $Y_1 = 3x^3 - 6x^2 + 6x$. Compare the x-intercepts with the solutions to the equation $3x^3 - 6x^2 + 6x = 0$ found in Exercise 37.

92. Graph $Y_1 = 5x^3 + 5x^2 + 10x$. Compare the x-intercepts with the solutions to the equation $5x^3 + 5x^2 + 10x = 0$ found in Exercise 38.

Section 7.3 Equations in Quadratic Form

Concepts

1. Solving Equations by Using Substitution
2. Solving Equations Reducible to a Quadratic

1. Solving Equations by Using Substitution

We have learned to solve a variety of different types of equations, including linear, radical, rational, and polynomial equations. Sometimes, however, it is necessary to use a quadratic equation as a tool to solve other types of equations.

In Example 1, we will solve the equation $(2x^2 - 5)^2 - 16(2x^2 - 5) + 39 = 0$. Notice that the terms in the equation are written in descending order by degree. Furthermore, the first two terms have the same base, $2x^2 - 5$, and the exponent on the first term is exactly double the exponent on the second term. The third term is a constant. An equation in this pattern is called **quadratic in form**.

exponent is double   third term is constant

$$(2x^2 - 5)^{2} - 16(2x^2 - 5)^{1} + 39 = 0.$$

To solve this equation we will use substitution as demonstrated in Example 1.

Example 1 Solving an Equation in Quadratic Form

Solve the equation. $(2x^2 - 5)^2 - 16(2x^2 - 5) + 39 = 0$

Solution:

$(2x^2 - 5)^2 - 16(2x^2 - 5) + 39 = 0$ Once the substitution is made, the equation becomes quadratic in the variable u.

Substitute $u = (2x^2 - 5)$.

$u^2 \quad - \quad 16u \quad + 39 = 0$ The equation is in the form $au^2 + bu + c = 0$.

$(u - 13)(u - 3) = 0$ Factor.

$u = 13 \quad$ or $\quad u = 3$ Apply the zero product rule.

Reverse substitute.

$2x^2 - 5 = 13 \quad$ or $\quad 2x^2 - 5 = 3$ Reverse substitute.

$2x^2 = 18 \quad$ or $\quad 2x^2 = 8$

$x^2 = 9 \quad$ or $\quad x^2 = 4$ Write the equations in the form $x^2 = k$.

$x = \pm\sqrt{9} \quad$ or $\quad x = \pm\sqrt{4}$ Apply the square root property.

$= \pm 3 \quad$ or $\quad = \pm 2$

The solution set is $\{3, -3, 2, -2\}$. All solutions check in the original equation.

Avoiding Mistakes

When using substitution, it is critical to reverse substitute to solve the equation in terms of the original variable.

Skill Practice Solve the equation.

1. $(3t^2 - 10)^2 + 5(3t^2 - 10) - 14 = 0$

Answer

1. $\{1, -1, 2, -2\}$

For an equation written in descending order, notice that u was set equal to the variable factor on the middle term. This is generally the case.

Example 2 Solving an Equation in Quadratic Form

Solve the equation. $p^{2/3} - 2p^{1/3} = 8$

Solution:

$$p^{2/3} - 2p^{1/3} = 8$$

$$p^{2/3} - 2p^{1/3} - 8 = 0$$ Set the equation equal to zero.

$$(p^{1/3})^2 - 2(p^{1/3})^1 - 8 = 0$$ Make the substitution $u = p^{1/3}$.

Substitute $u = p^{1/3}$.

$$u^2 - 2u - 8 = 0$$ Then the equation is in the form $au^2 + bu + c = 0$.

$$(u - 4)(u + 2) = 0$$ Factor.

$$u = 4 \quad \text{or} \quad u = -2$$ Apply the zero product rule.

Reverse substitute.

$$p^{1/3} = 4 \quad \text{or} \quad p^{1/3} = -2$$

$$\sqrt[3]{p} = 4 \quad \text{or} \quad \sqrt[3]{p} = -2$$ The equations are radical equations.

$$(\sqrt[3]{p})^3 = (4)^3 \quad \text{or} \quad (\sqrt[3]{p})^3 = (-2)^3$$ Cube both sides.

$$p = 64 \quad \text{or} \quad p = -8$$

The solution set is $\{64, -8\}$. All solutions check in the original equation.

Skill Practice Solve the equation.

2. $y^{2/3} - y^{1/3} = 12$

Example 3 Solving an Equation in Quadratic Form

Solve the equation. $x - \sqrt{x} - 12 = 0$

Solution:

The equation can be solved by first isolating the radical and then squaring both sides (this is saved for Exercise 25). However, this equation is also quadratic in form. By writing $\sqrt{x}$ as $x^{1/2}$, we see that the exponent on the first term is exactly double the exponent on the middle term.

$$x^1 - x^{1/2} - 12 = 0$$

$$(x^{1/2})^2 - (x^{1/2})^1 - 12 = 0$$ Let $u = x^{1/2}$.

$$u^2 - u - 12 = 0$$

$$(u - 4)(u + 3) = 0$$ Factor.

$$u = 4 \quad \text{or} \quad u = -3$$ Solve for u.

$$x^{1/2} = 4 \quad \text{or} \quad x^{1/2} = -3$$ Reverse substitute.

Answer

2. $\{64, -27\}$

Avoiding Mistakes

Recall that when each side of an equation is raised to an even power, we must check the potential solutions.

$\sqrt{x} = 4$ or $\cancel{\sqrt{x} = -3}$ — Solve each equation for x. Recall that the principal square root of a number cannot be negative.

$x = 16$

The solution set is $\{16\}$. — The value 16 checks in the original equation.

Skill Practice Solve the equation.

3. $z - \sqrt{z} - 2 = 0$

2. Solving Equations Reducible to a Quadratic

Some equations are reducible to a quadratic equation. In Example 4, we solve a polynomial equation by factoring. The resulting factors are quadratic.

FOR REVIEW

If a quadratic equation cannot be solved by factoring and applying the zero product rule, then

- apply the square root property (sometimes complete the square first), or
- apply the quadratic formula.

Example 4 Solving a Polynomial Equation

Solve the equation. $4x^4 + 7x^2 - 2 = 0$

Solution:

$4x^4 + 7x^2 - 2 = 0$ — This is a polynomial equation.

$(4x^2 - 1)(x^2 + 2) = 0$ — Factor.

$4x^2 - 1 = 0$ or $x^2 + 2 = 0$ — Set each factor equal to zero. Notice that the two equations are quadratic. Each can be solved by the square root property.

$x^2 = \frac{1}{4}$ or $x^2 = -2$

$x = \pm\sqrt{\frac{1}{4}}$ or $x = \pm\sqrt{-2}$ — Apply the square root property.

$x = \pm\frac{1}{2}$ or $x = \pm i\sqrt{2}$ — Simplify the radicals.

The solution set is $\left\{\frac{1}{2}, -\frac{1}{2}, i\sqrt{2}, -i\sqrt{2}\right\}$.

Skill Practice Solve the equation.

4. $9x^4 + 35x^2 - 4 = 0$

Example 5 Solving a Rational Equation

Solve the equation. $\frac{3y}{y+2} - \frac{2}{y-1} = 1$

Solution:

$\frac{3y}{y+2} - \frac{2}{y-1} = 1$ — This is a rational equation. The LCD is $(y + 2)(y - 1)$. Also note that the restrictions on y are $y \neq -2$ and $y \neq 1$.

$\left(\frac{3y}{y+2} - \frac{2}{y-1}\right) \cdot (y+2)(y-1) = 1 \cdot (y+2)(y-1)$ — Multiply both sides by the LCD.

$\frac{3y}{\cancel{y+2}} \cdot (\cancel{y+2})(y-1) - \frac{2}{\cancel{y-1}} \cdot (y+2)(\cancel{y-1}) = 1 \cdot (y+2)(y-1)$

Answers

3. {4} (The value 1 does not check.)

4. $\left\{\pm\frac{1}{3}, \pm 2i\right\}$

$3y(y-1)-2(y+2)=(y+2)(y-1)$	Clear fractions.
$3y^2-3y-2y-4=y^2-y+2y-2$	Apply the distributive property.
$3y^2-5y-4=y^2+y-2$	The equation is quadratic.
$2y^2-6y-2=0$	Write the equation in descending order.
$\dfrac{2y^2}{2}-\dfrac{6y}{2}-\dfrac{2}{2}=\dfrac{0}{2}$ $y^2-3y-1=0$	Each coefficient in the equation is divisible by 2. Therefore, if we divide both sides by 2, the coefficients in the equation are smaller. This will make it easier to apply the quadratic formula.
$y=\dfrac{-(-3)\pm\sqrt{(-3)^2-4(1)(-1)}}{2(1)}$	Apply the quadratic formula.

$$=\frac{3\pm\sqrt{9+4}}{2}$$

$$=\frac{3\pm\sqrt{13}}{2} \quad y=\frac{3+\sqrt{13}}{2} \qquad y=\frac{3-\sqrt{13}}{2}$$

The solution set is $\left\{\dfrac{3+\sqrt{13}}{2}, \dfrac{3-\sqrt{13}}{2}\right\}$.

Skill Practice Solve the equation.

5. $\dfrac{t}{2t-1}-\dfrac{1}{t+4}=1$

Answer

5. $\left\{\dfrac{-5\pm3\sqrt{5}}{2}\right\}$

Section 7.3 Activity

A.1. Consider the equation $(2x-1)^2-13(2x-1)+36=0$.

a. Notice that the factor $(2x-1)$ appears in the first two terms. In the first term, $2x-1$ is raised to the second power, and in the middle term, $2x-1$ is raised to the first power. In such a case, the equation is said to be ________ in form.

b. By making an appropriate substitution, this equation can be expressed as a quadratic equation in a new variable. Rewrite the equation by making the substitution $u=2x-1$.

c. Find the values of u that are solutions to the equation from part (b).

d. From part (c) you found that $u=4$ and $u=9$. In each equation, reverse the substitution. That is, replace u with $2x-1$ and then solve each equation for x. Write the solution set to the original equation.

For Exercises A.2–A.5, determine an appropriate substitution to make the given equation quadratic in the variable u.

	Equation in Quadratic Form	Substitution	New Equation
A.2.	$(x^2-3)^2-9(x^2-3)-52=0$	Let $u=$ ________	
A.3.	$3x^{2/3}-x^{1/3}-4=0$	Let $u=$ ________	
A.4.	$x^4-29x^2+100=0$	Let $u=$ ________	
A.5.	$\left(2+\dfrac{3}{x}\right)^2-\left(2+\dfrac{3}{x}\right)-12=0$	Let $u=$ ________	

For Exercises A.6–A.9, solve the equation.

A.6. $(x^2 - 3)^2 - 9(x^2 - 3) - 52 = 0$

A.7. $3x^{2/3} - x^{1/3} - 4 = 0$

A.8. $x^4 - 29x^2 + 100 = 0$

A.9. $\left(2 + \frac{3}{x}\right)^2 - \left(2 + \frac{3}{x}\right) - 12 = 0$

Section 7.3 Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, write the expression in radical form. Assume that all variables represent positive real numbers.

R.1. $x^{1/2}$

R.2. $y^{1/3}$

R.3. $z^{2/3}$

R.4. $p^{3/4}$

For Exercises R.5–R.16, solve the equation.

R.5. $u^2 - 7u - 18 = 0$

R.6. $u^2 + 14u + 40 = 0$

R.7. $6u^2 + 2u = 6$

R.8. $2x^2 = 8x + 44$

R.9. $3x^2 - 7 = 20$

R.10. $4x^2 + 1 = 65$

R.11. $w^2 + 5 = 0$

R.12. $p^2 + 7 = 0$

R.13. $x^{1/3} = 5$

R.14. $y^{1/5} = 2$

R.15. $\frac{2}{x+1} - \frac{3}{x-2} = \frac{1}{x^2 - x - 2}$

R.16. $\frac{5}{t+3} - \frac{4}{t+2} = \frac{6}{t^2 + 5t + 6}$

Vocabulary and Key Concepts

1. **a.** An equation that can be written in the form $au^2 + bu + c = 0$, where u represents an algebraic expression, is said to be in ______________ form.

b. To use the method of substitution to solve the equation $(3x - 1)^2 + 2(3x - 1) - 8 = 0$, let $u =$ ______________.

c. To use the method of substitution to solve the equation $p^{2/3} - 2p^{1/3} - 15 = 0$, let $u =$ ______________.

Concept 1: Solving Equations by Using Substitution

For Exercises 2–7, identify a substitution $u =$ _____ that would make the equation quadratic in the variable u. Then write the equation in terms of u.

2. $(2t - 3)^2 - 8(2t - 3) - 20 = 0$

3. $(3m + 4)^2 - 4(3m + 4) - 5 = 0$

4. $(x^2 - 3)^2 - 4(x^2 - 3) + 3 = 0$

5. $(y^2 + 1)^2 - 7(y^2 + 1) + 10 = 0$

6. $2\left(\frac{1}{x}\right)^2 - 5\left(\frac{1}{x}\right) - 3 = 0$

7. $\frac{3}{x^2} + \frac{10}{x} - 8 = 0$

8. **a.** Solve the quadratic equation by factoring. $u^2 - 2u - 24 = 0$

b. Solve the equation by using substitution. $(x - 5)^2 - 2(x - 5) - 24 = 0$

9. **a.** Solve the quadratic equation by factoring. $u^2 + 10u + 24 = 0$

b. Solve the equation by using substitution. $(y^2 + 5y)^2 + 10(y^2 + 5y) + 24 = 0$

10. **a.** Solve the quadratic equation by factoring. $u^2 - 2u - 35 = 0$

b. Solve the equation by using substitution. $(w^2 - 6w)^2 - 2(w^2 - 6w) - 35 = 0$

For Exercises 11–24, solve the equation by using substitution. **(See Examples 1–3.)**

11. $(x^2 - 2x)^2 + 2(x^2 - 2x) = 3$

12. $(x^2 + x)^2 - 8(x^2 + x) = -12$

13. $(y^2 - 4y)^2 - (y^2 - 4y) = 20$

14. $(w^2 - 2w)^2 - 11(w^2 - 2w) = -24$

15. $m^{2/3} - m^{1/3} - 6 = 0$

16. $2n^{2/3} + 7n^{1/3} - 15 = 0$

17. $2t^{2/5} + 7t^{1/5} + 3 = 0$

18. $p^{2/5} + p^{1/5} - 2 = 0$

19. $y + 6\sqrt{y} = 16$

20. $p - 8\sqrt{p} = -15$

21. $2x + 3\sqrt{x} - 2 = 0$

22. $3t + 5\sqrt{t} - 2 = 0$

23. $16\left(\frac{x+6}{4}\right)^2 + 8\left(\frac{x+6}{4}\right) + 1 = 0$

24. $9\left(\frac{x+3}{2}\right)^2 - 6\left(\frac{x+3}{2}\right) + 1 = 0$

25. In Example 3, we solved the equation $x - \sqrt{x} - 12 = 0$ by using substitution. Now solve this equation by first isolating the radical and then squaring both sides. Don't forget to check the potential solutions in the original equation. Do you obtain the same solution as in Example 3?

Concept 2: Solving Equations Reducible to a Quadratic

For Exercises 26–36, solve the equations. **(See Examples 4–5.)**

26. $t^4 + t^2 - 12 = 0$

27. $w^4 + 4w^2 - 45 = 0$

28. $x^2(9x^2 + 7) = 2$

29. $y^2(4y^2 + 17) = 15$

30. $\frac{y}{10} - 1 = -\frac{12}{5y}$

31. $1 + \frac{5}{x} = -\frac{3}{x^2}$

32. $\frac{x+5}{x} + \frac{x}{2} = \frac{x+19}{4x}$

33. $\frac{3x}{x+1} - \frac{2}{x-3} = 1$

34. $\frac{2t}{t-3} - \frac{1}{t+4} = 1$

35. $\frac{x}{2x-1} = \frac{1}{x-2}$

36. $\frac{z}{3z+2} = \frac{2}{z+1}$

Mixed Exercises

For Exercises 37–60, solve the equations.

37. $x^4 - 16 = 0$

38. $t^4 - 625 = 0$

39. $(4x + 5)^2 + 3(4x + 5) + 2 = 0$

40. $2(5x+3)^2 - (5x+3) - 28 = 0$

41. $4m^4 - 9m^2 + 2 = 0$

42. $x^4 - 7x^2 + 12 = 0$

43. $x^6 - 9x^3 + 8 = 0$

44. $x^6 - 26x^3 - 27 = 0$

45. $\sqrt{x^2+20} = 3\sqrt{x}$

46. $\sqrt{x^2+60} = 4\sqrt{x}$

47. $\sqrt{4t+1} = t+1$

48. $\sqrt{t+10} = t+4$

49. $2\left(\frac{t-4}{3}\right)^2 - \left(\frac{t-4}{3}\right) - 3 = 0$

50. $\left(\frac{x+1}{5}\right)^2 - 3\left(\frac{x+1}{5}\right) - 10 = 0$

51. $x^{2/3} + x^{1/3} = 20$

52. $x^{2/5} - 3x^{1/5} = -2$

53. $m^4 + 2m^2 - 8 = 0$

54. $2c^4 + c^2 - 1 = 0$

55. $a^3 + 16a - a^2 - 16 = 0$
(*Hint:* Factor by grouping first.)

56. $b^3 + 9b - b^2 - 9 = 0$

57. $x^3 + 5x - 4x^2 - 20 = 0$

58. $y^3 + 8y - 3y^2 - 24 = 0$

59. $\left(\frac{2}{x-3}\right)^2 + 8\left(\frac{2}{x-3}\right) + 12 = 0$

60. $\left(\frac{5}{x+1}\right)^2 - 6\left(\frac{5}{x+1}\right) - 16 = 0$

Technology Connections

61. **a.** Solve the equation $x^4 + 4x^2 + 4 = 0$.

b. How many solutions are real and how many solutions are imaginary?

c. How many x-intercepts do you anticipate for the function defined by $y = x^4 + 4x^2 + 4$?

d. Graph $Y_1 = x^4 + 4x^2 + 4$ on a standard viewing window.

62. **a.** Solve the equation $x^4 - 2x^2 + 1 = 0$.

b. How many solutions are real and how many solutions are imaginary?

c. How many x-intercepts do you anticipate for the function defined by $y = x^4 - 2x^2 + 1$?

d. Graph $Y_1 = x^4 - 2x^2 + 1$ on a standard viewing window.

63. **a.** Solve the equation $x^4 - x^3 - 6x^2 = 0$.

b. How many solutions are real and how many solutions are imaginary?

c. How many x-intercepts do you anticipate for the function defined by $y = x^4 - x^3 - 6x^2$?

d. Graph $Y_1 = x^4 - x^3 - 6x^2$ on a standard viewing window.

64. **a.** Solve the equation $x^4 - 10x^2 + 9 = 0$.

b. How many solutions are real and how many solutions are imaginary?

c. How many x-intercepts do you anticipate for the function defined by $y = x^4 - 10x^2 + 9$?

d. Graph $Y_1 = x^4 - 10x^2 + 9$ on a standard viewing window.

Problem Recognition Exercises

Quadratic and Quadratic Type Equations

For Exercises 1–4, solve each equation by

a. Completing the square and applying the square root property.

b. Using the quadratic formula.

1. $x^2 + 10x + 3 = 0$ **2.** $v^2 - 16v + 5 = 0$ **3.** $3t^2 + t + 4 = 0$ **4.** $4y^2 + 3y + 5 = 0$

In Exercises 5–24, we have presented all types of equations that you have learned up to this point. For each equation,

a. First determine the type of equation that is presented. Choose from: linear equation, quadratic equation, quadratic in form, rational equation, or radical equation.

b. Solve the equation by using a suitable method.

5. $t^2 + 5t - 14 = 0$ **6.** $a^2 - 9a + 20 = 0$ **7.** $a^4 - 10a^2 + 9 = 0$

8. $x^4 - 3x^2 - 4 = 0$ **9.** $x - 3x^{1/2} - 4 = 0$ **10.** $x^2 - 9x - 2 = 0$

11. $8b(b + 1) + 2(3b - 4) = 4b(2b + 3)$ **12.** $6x(x + 1) - 3(x + 4) = 3x(2x + 5)$

13. $5a(a + 6) = 10(3a - 1)$ **14.** $4x(x + 3) = 6(2x - 4)$ **15.** $\dfrac{t}{t + 5} + \dfrac{3}{t - 4} = \dfrac{17}{t^2 + t - 20}$

16. $\dfrac{v}{v + 4} + \dfrac{12}{v^2 + 7v + 12} = \dfrac{5}{v + 3}$ **17.** $c^2 - 20c - 1 = 0$ **18.** $d^2 + 18d + 4 = 0$

19. $2u(u - 3) = 4(2 - u)$ **20.** $3y(y + 2) = 9(y + 1)$ **21.** $\sqrt{2b + 3} = b$

22. $\sqrt{5t + 6} = t$ **23.** $x^{2/3} + 2x^{1/3} - 15 = 0$ **24.** $y^{2/3} + 5y^{1/3} + 4 = 0$

Graphs of Quadratic Functions

Section 7.4

Concepts

1. **Quadratic Functions of the Form $f(x) = x^2 + k$**
2. **Quadratic Functions of the Form $f(x) = (x - h)^2$**
3. **Quadratic Functions of the Form $f(x) = ax^2$**
4. **Quadratic Functions of the Form $f(x) = a(x - h)^2 + k$**

A quadratic function is defined as a function of the form $f(x) = ax^2 + bx + c$ $(a \neq 0)$. The graph of a quadratic function is a **parabola**. In this section, we will learn how to graph parabolas.

A parabola opens upward if $a > 0$ (Figure 7-3) and opens downward if $a < 0$ (Figure 7-4). If a parabola opens upward, the **vertex** is the lowest point on the graph. If a parabola opens downward, the **vertex** is the highest point on the graph. The **axis of symmetry** is the vertical line that passes through the vertex.

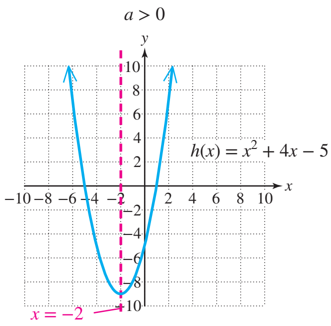


Figure 7-3

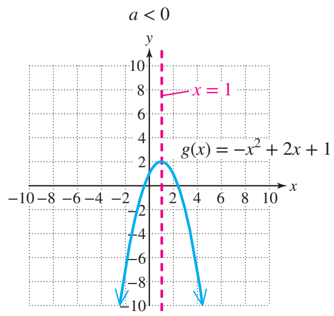


Figure 7-4

1. Quadratic Functions of the Form $f(x) = x^2 + k$

One technique for graphing a function is to plot a sufficient number of points on the graph until the general shape and defining characteristics can be determined. Then sketch a curve through the points.

Example 1 Graphing Quadratic Functions of the Form $f(x) = x^2 + k$

Graph the functions f, g, and h on the same coordinate system.

$$f(x) = x^2 \qquad g(x) = x^2 + 1 \qquad h(x) = x^2 - 2$$

Solution:

Several function values for f, g, and h are shown in Table 7-1 for selected values of x. The corresponding graphs are pictured in Figure 7-5.

Table 7-1

x	$f(x) = x^2$	$g(x) = x^2 + 1$	$h(x) = x^2 - 2$
−3	9	10	7
−2	4	5	2
−1	1	2	−1
0	0	1	−2
1	1	2	−1
2	4	5	2
3	9	10	7

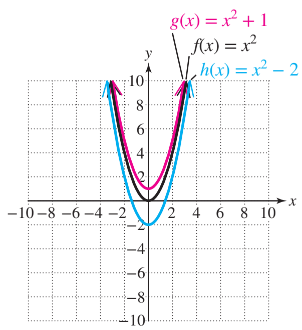
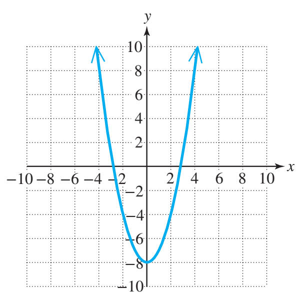


Figure 7-5

Skill Practice Refer to the graph of $f(x) = x^2 + k$ to determine the value of k.

1.



Answer

1. $k = -8$

Notice that the graphs of $g(x) = x^2 + 1$ and $h(x) = x^2 - 2$ take on the same shape as $f(x) = x^2$. However, the y values of g are 1 greater than the y values of f. Hence, the graph of $g(x) = x^2 + 1$ is the same as the graph of $f(x) = x^2$ shifted *up* 1 unit. Likewise the y values of h are 2 less than those of f. The graph of $h(x) = x^2 - 2$ is the same as the graph of $f(x) = x^2$ shifted *down* 2 units.

A function of the type $f(x) = x^2 + k$ represents a vertical shift of the graph of $f(x) = x^2$. The functions in Example 1 illustrate the following properties of quadratic functions of the form $f(x) = x^2 + k$.

Graphs of $f(x) = x^2 + k$

A function of the type $f(x) = x^2 + k$ represents a vertical shift of the graph of $f(x) = x^2$.

- If $k > 0$, then the graph of $f(x) = x^2 + k$ is the same as the graph of $f(x) = x^2$ shifted *up* k units.
- If $k < 0$, then the graph of $f(x) = x^2 + k$ is the same as the graph of $f(x) = x^2$ shifted *down* $|k|$ units.

Example 2 Graphing Quadratic Functions of the Form $f(x) = x^2 + k$

Sketch the functions defined by

a. $m(x) = x^2 - 4$ **b.** $n(x) = x^2 + \dfrac{7}{2}$

Solution:

a. $m(x) = x^2 - 4$

$m(x) = x^2 + (-4)$

Because $k = -4$ (negative), the graph is obtained by shifting the graph of $f(x) = x^2$ down $|-4|$ units (Figure 7-6).

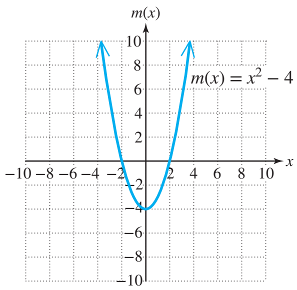


Figure 7-6

b. $n(x) = x^2 + \dfrac{7}{2}$

Because $k = \frac{7}{2}$ (positive), the graph is obtained by shifting the graph of $f(x) = x^2$ up $\frac{7}{2}$ units (Figure 7-7).

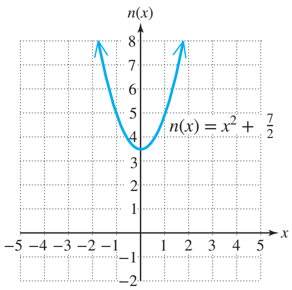


Figure 7-7

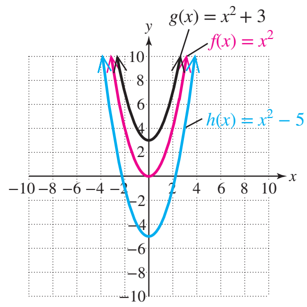
TIP: For more accuracy in the graph, plot one or two points near the vertex and use the symmetry of the curve to find additional points on the graph.

Skill Practice

2. Graph the functions f, g, and h on the same coordinate system.
$f(x) = x^2$ $g(x) = x^2 + 3$ $h(x) = x^2 - 5$

Answer

2.



2. Quadratic Functions of the Form $f(x) = (x - h)^2$

The graph of $f(x) = x^2 + k$ represents a vertical shift (up or down) of the graph of $f(x) = x^2$. Example 3 shows that functions of the form $f(x) = (x - h)^2$ represent a horizontal shift (left or right) of the graph of $f(x) = x^2$.

Example 3 Graphing Quadratic Functions of the Form $f(x) = (x - h)^2$

Graph the functions f, g, and h on the same coordinate system.

$$f(x) = x^2 \qquad g(x) = (x + 1)^2 \qquad h(x) = (x - 2)^2$$

Solution:

Several function values for f, g, and h are shown in Table 7-2 for selected values of x. The corresponding graphs are pictured in Figure 7-8.

Table 7-2

x	$f(x) = x^2$	$g(x) = (x + 1)^2$	$h(x) = (x - 2)^2$
-4	16	9	36
-3	9	4	25
-2	4	1	16
-1	1	0	9
0	0	1	4
1	1	4	1
2	4	9	0
3	9	16	1
4	16	25	4
5	25	36	9

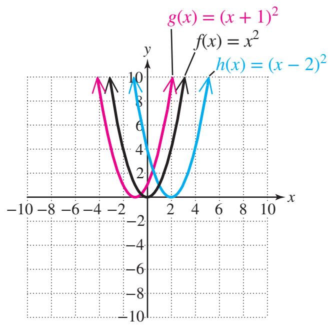
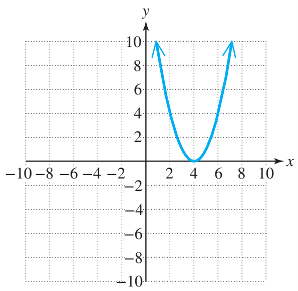


Figure 7-8

Skill Practice Refer to the graph of $f(x) = (x - h)^2$ to determine the value of h.

3.



Example 3 illustrates the following properties of quadratic functions of the form $f(x) = (x - h)^2$.

Answer

3. $h = 4$

Graphs of $f(x) = (x - h)^2$

A function of the type $f(x) = (x - h)^2$ represents a horizontal shift of the graph of $f(x) = x^2$.

If $h > 0$, then the graph of $f(x) = (x - h)^2$ is the same as the graph of $f(x) = x^2$ shifted h units to the *right*.

If $h < 0$, then the graph of $f(x) = (x - h)^2$ is the same as the graph of $f(x) = x^2$ shifted $|h|$ units to the *left*.

From Example 3 we have

$h(x) = (x - 2)^2$ and $g(x) = (x + 1)^2$

$g(x) = [x - (-1)]^2$

$y = x^2$ shifted 2 units to the right

$y = x^2$ shifted $|-1|$ unit to the left

Example 4 Graphing Functions of the Form $f(x) = (x - h)^2$

Sketch the functions p and q.

a. $p(x) = (x - 7)^2$ **b.** $q(x) = (x + 1.6)^2$

Solution:

a. $p(x) = (x - 7)^2$

Because $h = 7 > 0$, shift the graph of $f(x) = x^2$ to the *right* 7 units (Figure 7-9).

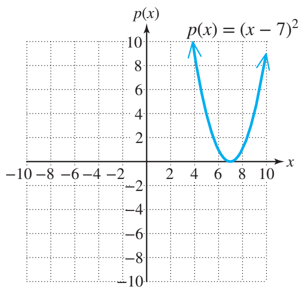


Figure 7-9

b. $q(x) = (x + 1.6)^2$

$q(x) = [x - (-1.6)]^2$

Because $h = -1.6 < 0$, shift the graph of $f(x) = x^2$ to the *left* 1.6 units (Figure 7-10).

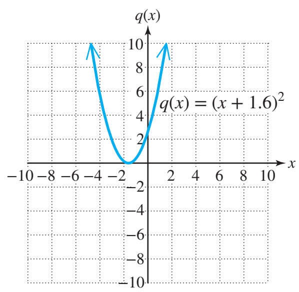


Figure 7-10

Skill Practice

4. Graph the functions f, g, and h on the same coordinate system.

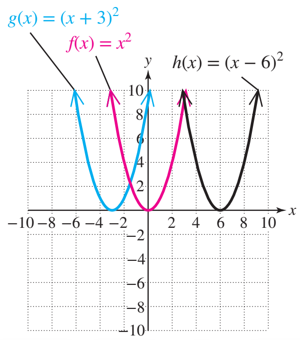
$f(x) = x^2$

$g(x) = (x + 3)^2$

$h(x) = (x - 6)^2$

Answer

4.



3. Quadratic Functions of the Form $f(x) = ax^2$

Examples 5 and 6 investigate functions of the form $f(x) = ax^2 \quad (a \neq 0)$.

Example 5 Graphing Functions of the Form $f(x) = ax^2$

Graph the functions f, g, and h on the same coordinate system.

$$f(x) = x^2 \qquad g(x) = 2x^2 \qquad h(x) = \frac{1}{2}x^2$$

Solution:

Several function values for f, g, and h are shown in Table 7-3 for selected values of x. The corresponding graphs are pictured in Figure 7-11.

Table 7-3

x	$f(x) = x^2$	$g(x) = 2x^2$	$h(x) = \frac{1}{2}x^2$
-3	9	18	$\frac{9}{2}$
-2	4	8	2
-1	1	2	$\frac{1}{2}$
0	0	0	0
1	1	2	$\frac{1}{2}$
2	4	8	2
3	9	18	$\frac{9}{2}$

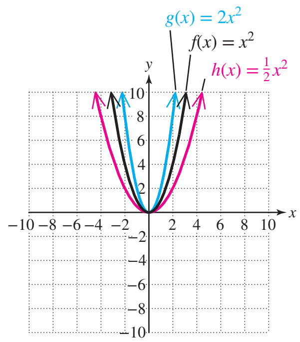


Figure 7-11

Skill Practice

5. Graph the functions f, g, and h on the same coordinate system.

$f(x) = x^2$

$g(x) = 3x^2$

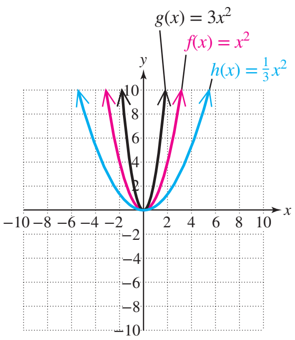
$h(x) = \frac{1}{3}x^2$

In Example 5, the function values defined by $g(x) = 2x^2$ are twice those of $f(x) = x^2$. The graph of $g(x) = 2x^2$ is the same as the graph of $f(x) = x^2$ *stretched vertically* by a factor of 2 (the graph appears narrower than $f(x) = x^2$).

In Example 5, the function values defined by $h(x) = \frac{1}{2}x^2$ are one-half those of $f(x) = x^2$. The graph of $h(x) = \frac{1}{2}x^2$ is the same as the graph of $f(x) = x^2$ *shrunk vertically* by a factor of $\frac{1}{2}$ (the graph appears wider than $f(x) = x^2$).

Answer

5.



Example 6 Graphing Functions of the Form $f(x) = ax^2$

Graph the functions f, g, and h on the same coordinate system.

$$f(x) = -x^2 \qquad g(x) = -3x^2 \qquad h(x) = -\frac{1}{3}x^2$$

Solution:

Several function values for f, g, and h are shown in Table 7-4 for selected values of x. The corresponding graphs are pictured in Figure 7-12.

Table 7-4

x	$f(x) = -x^2$	$g(x) = -3x^2$	$h(x) = -\frac{1}{3}x^2$
-3	-9	-27	-3
-2	-4	-12	$-\frac{4}{3}$
-1	-1	-3	$-\frac{1}{3}$
0	0	0	0
1	-1	-3	$-\frac{1}{3}$
2	-4	-12	$-\frac{4}{3}$
3	-9	-27	-3

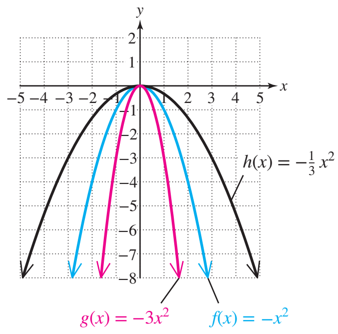


Figure 7-12

Skill Practice

6. Graph the functions f, g, and h on the same coordinate system.

$f(x) = -x^2 \qquad g(x) = -2x^2 \qquad h(x) = -\dfrac{1}{2}x^2$

Example 6 illustrates that if the coefficient of the squared term is negative, the parabola opens downward. The graph of $g(x) = -3x^2$ is the same as the graph of $f(x) = -x^2$ with a *vertical stretch* by a factor of $|-3|$. The graph of $h(x) = -\frac{1}{3}x^2$ is the same as the graph of $f(x) = -x^2$ with a *vertical shrink* by a factor of $|-\frac{1}{3}|$.

Graphs of $f(x) = ax^2$

1. If $a > 0$, the parabola opens upward. Furthermore,
   - If $0 < a < 1$, then the graph of $f(x) = ax^2$ is the same as the graph of $f(x) = x^2$ with a *vertical shrink* by a factor of a.
   - If $a > 1$, then the graph of $f(x) = ax^2$ is the same as the graph of $f(x) = x^2$ with a *vertical stretch* by a factor of a.
2. If $a < 0$, the parabola opens downward. Furthermore,
   - If $0 < |a| < 1$, then the graph of $f(x) = ax^2$ is the same as the graph of $f(x) = -x^2$ with a *vertical shrink* by a factor of $|a|$.
   - If $|a| > 1$, then the graph of $f(x) = ax^2$ is the same as the graph of $f(x) = -x^2$ with a *vertical stretch* by a factor of $|a|$.

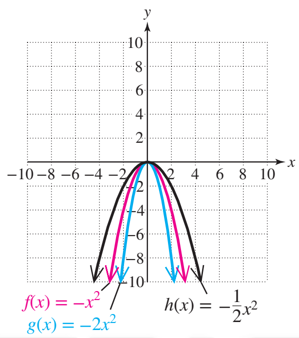
4. Quadratic Functions of the Form $f(x) = a(x - h)^2 + k$

We can summarize our findings from Examples 1–6 by graphing functions of the form $f(x) = a(x - h)^2 + k \quad (a \neq 0)$.

The graph of $f(x) = x^2$ has its vertex at the origin (0, 0). The graph of $f(x) = a(x - h)^2 + k$ is the same as the graph of $f(x) = x^2$ shifted to the right or left h units and shifted up or down k units. Therefore, the vertex is shifted from (0, 0) to (h, k). The axis of symmetry is the vertical line through the vertex. Therefore, the axis of symmetry must be the line $x = h$.

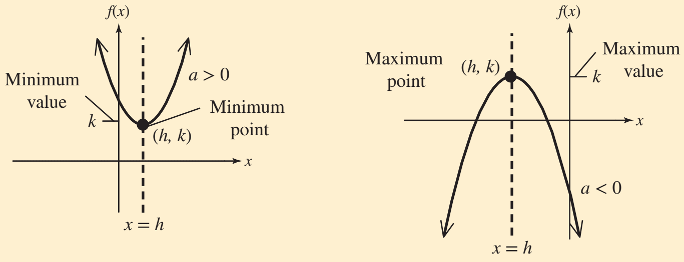
Answer

6.



Graphs of $f(x) = a(x - h)^2 + k$

1. The vertex is (h, k).
2. The axis of symmetry is the line $x = h$.
3. If $a > 0$, the parabola opens upward, and k is the **minimum value** of the function.
4. If $a < 0$, the parabola opens downward, and k is the **maximum value** of the function.



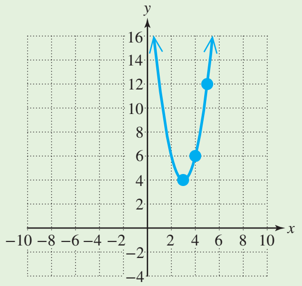
Example 7 Graphing a Function of the Form $f(x) = a(x - h)^2 + k$

Given the function defined by $f(x) = 2(x - 3)^2 + 4$

a. Identify the vertex.
b. Sketch the function.
c. Identify the axis of symmetry.
d. Identify the maximum or minimum value of the function.
e. Use the graph to determine the domain and range of f in interval notation.

TIP: We can find additional points to the right or left of the vertex and use the symmetry of the parabola to refine the sketch of the curve.

x	$f(x)$
4	6
5	12



Solution:

a. The function is in the form $f(x) = a(x - h)^2 + k$, where $a = 2$, $h = 3$, and $k = 4$. Therefore, the vertex is $(3, 4)$.

b. The graph of f is the same as the graph of $f(x) = x^2$ shifted to the right 3 units, shifted up 4 units, and stretched vertically by a factor of 2 (Figure 7-13).

c. The axis of symmetry is the line $x = 3$.

d. Because $a > 0$, the function opens upward. Therefore, the minimum function value is 4. Notice that the minimum value is the minimum y value on the graph.

e. The domain is all real numbers: $(-\infty, \infty)$. The vertex is the lowest point on the parabola, and its y-coordinate is 4. Therefore, the range is $[4, \infty)$.

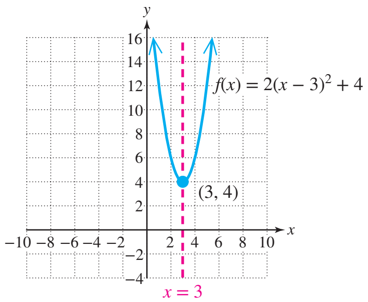


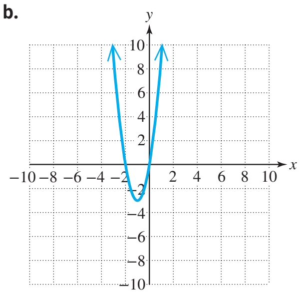
Figure 7-13

Skill Practice

7. Given the function defined by $g(x) = 3(x + 1)^2 - 3$
   a. Identify the vertex.
   b. Sketch the graph.
   c. Identify the axis of symmetry.
   d. Identify the maximum or minimum value of the function.
   e. Determine the domain and range in interval notation.

Answers

7. a. Vertex: $(-1, -3)$
b.



c. Axis of symmetry: $x = -1$
d. Minimum value: -3
e. Domain: $(-\infty, \infty)$; range: $[-3, \infty)$

Example 8 Graphing a Function of the Form $f(x) = a(x - h)^2 + k$

Given the function defined by $\quad g(x) = -(x + 2)^2 - \frac{7}{4}$

a. Identify the vertex.
b. Sketch the function.
c. Identify the axis of symmetry.
d. Identify the maximum or minimum value of the function.
e. Use the graph to determine the domain and range of f in interval notation.

Solution:

a. $g(x) = -(x + 2)^2 - \frac{7}{4}$

$$= -1[x - (-2)]^2 + \left(-\frac{7}{4}\right)$$

The function is in the form $g(x) = a(x - h)^2 + k$, where $a = -1$, $h = -2$, and $k = -\frac{7}{4}$. Therefore, the vertex is $(-2, -\frac{7}{4})$.

b. The graph of g is the same as the graph of $f(x) = x^2$ shifted to the left 2 units, shifted down $\frac{7}{4}$ units, and opening downward (Figure 7-14).

c. The axis of symmetry is the line $x = -2$.

d. The parabola opens downward, so the maximum function value is $-\frac{7}{4}$.

e. The domain is all real numbers: $(-\infty, \infty)$.

The vertex is the highest point on the parabola, and its y-coordinate is $-\frac{7}{4}$. Therefore, the range is $(-\infty, -\frac{7}{4}]$.

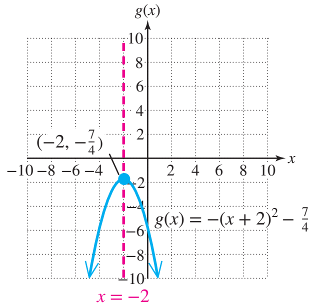


Figure 7-14

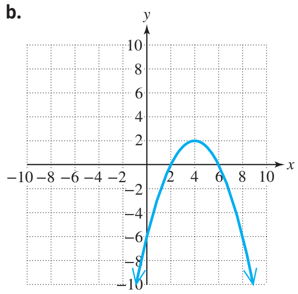
Skill Practice

8. Given the function defined by $h(x) = -\frac{1}{2}(x - 4)^2 + 2$

a. Identify the vertex.
b. Sketch the graph.
c. Identify the axis of symmetry.
d. Identify the maximum or minimum value of the function.
e. Determine the domain and range in interval notation.

Answers

8. a. Vertex: (4, 2)
b.

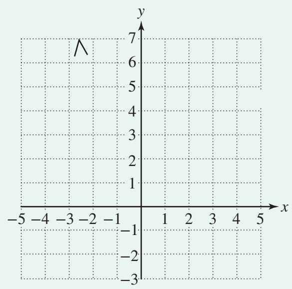


c. Axis of symmetry: $x = 4$
d. Maximum value: 2
e. Domain: $(-\infty, \infty)$; range: $(-\infty, 2]$

Section 7.4 Activity

A.1. Graph the functions defined by $f(x) = x^2$, $g(x) = x^2 + 1$, and $h(x) = x^2 - 2$ by completing the table of points. Graph the functions on the same coordinate system for comparison.

x	$f(x) = x^2$	$g(x) = x^2 + 1$	$h(x) = x^2 - 2$
0			
1			
2			
3			
−1			
−2			
−3			

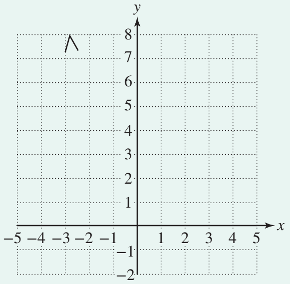


A.2. **a.** Based on the result of Exercise A.1, for $k > 0$, describe the graph of $y = x^2 + k$.

b. Based on the result of Exercise A.1, for $k > 0$, describe the graph of $y = x^2 - k$.

A.3. Graph the functions defined by $f(x) = x^2$, $g(x) = (x + 1)^2$, and $h(x) = (x - 2)^2$ by completing the table of points. Graph the functions on the same coordinate system for comparison.

x	$f(x) = x^2$	$g(x) = (x + 1)^2$	$h(x) = (x - 2)^2$
0			
1			
2			
3			
−1			
−2			
−3			

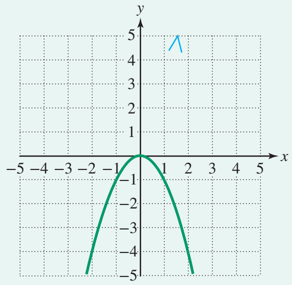


A.4. **a.** Based on the result of Exercise A.3, for $h > 0$, describe the graph of $y = (x + h)^2$.

b. Based on the result of Exercise A.3, for $h > 0$, describe the graph of $y = (x - h)^2$.

A.5. Graph the functions defined by $f(x) = x^2$, $g(x) = 2x^2$, $h(x) = \frac{1}{2}x^2$, and $k(x) = -x^2$ by completing the table of points. Graph the functions on the same coordinate system for comparison.

x	$f(x) = x^2$	$g(x) = 2x^2$	$h(x) = \frac{1}{2}x^2$	$k(x) = -x^2$
0				
1				
2				
3				
−1				
−2				
−3				



A.6. **a.** Based on the result of Exercise A.5, for $a > 1$, describe the graph of $y = ax^2$.

b. Based on the result of Exercise A.5, for $0 < a < 1$, describe the graph of $y = ax^2$.

c. Based on the result of Exercise A.5, for $a < 0$, describe the graph of $y = ax^2$.

A.7. Exercises A.1–A.6 illustrate *transformations* of graphs. These include shifting a graph right, left, upward, or downward; stretching or shrinking a graph vertically; and reflecting a graph over the x-axis. Given $y = a(x - h)^2 + k$, match the statement on the left with the conclusion on the right.

a. The value of a ________ i. shrinks or stretches the graph of $y = x^2$ vertically and/or reflects the graph over the x-axis.

b. The value of h ________ ii. shifts the graph vertically.

c. The value of k ________ iii. shifts the graph horizontally.

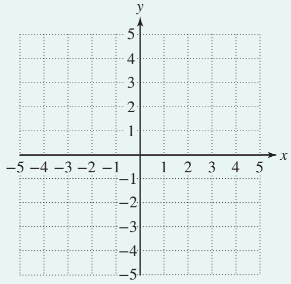
A.8. **a.** Use transformations to graph $f(x) = -(x + 2)^2 + 4$.

b. Identify the vertex of the parabola.

c. Identify the axis of symmetry.

d. Identify the maximum or minimum value of f.

e. Use the graph to determine the domain and range of f.



Practice Exercises

Section 7.4

Prerequisite Review

For Exercises R.1–R.4,

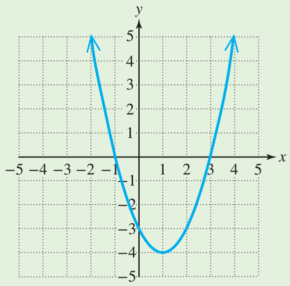
a. Write the domain in interval notation.

b. Write the range in interval notation.

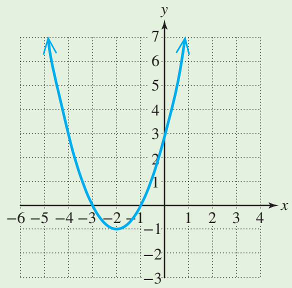
c. Identify the x-intercepts if any exist.

d. Identify the y-intercpet.

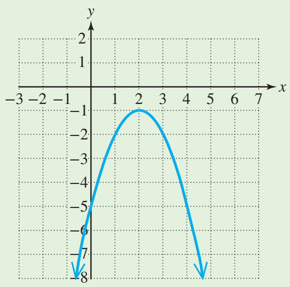
R.1.



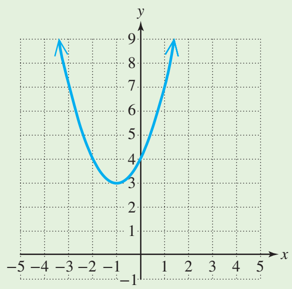
R.2.



R.3.



R.4.



Vocabulary and Key Concepts

1. **a.** The graph of a quadratic function, $f(x) = ax^2 + bx + c$, is a ____________.

b. The parabola defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ will open upward if a ____________ 0 and will open downward if a ____________ 0.

c. If a parabola opens upward, the vertex is the (highest/lowest) point on the graph. If a parabola opens downward, the vertex is the (highest/lowest) point on the graph.

2. **a.** Given $f(x) = a(x - h)^2 + k$ $(a \neq 0)$, the vertex of the parabola is given by the ordered pair ____________. If the parabola opens ____________, the value of k is the minimum value of the function. If the parabola opens ____________, the value of k is the maximum value of the function.

b. Given $f(x) = a(x - h)^2 + k$ $(a \neq 0)$, the axis of symmetry of the parabola is the vertical line that passes through the ____________. An equation of the axis of symmetry is ____________.

For Exercises 3–4, determine if the parabola opens upward or downward.

3. $f(x) = -2x^2 + 4$

4. $g(x) = x^2 - 9$

For Exercises 5–6, determine if the vertex of the parabola is a maximum point or a minimum point.

5. $f(x) = -2x^2 + 4$

6. $g(x) = x^2 - 9$

For Exercises 7–8, identify the vertex of the parabola and the axis of symmetry.

7. $h(x) = (x + 3)^2 + 1$

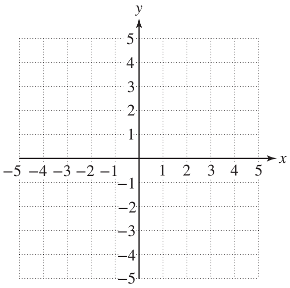
8. $k(x) = (x - 4)^2 - 2$

Concept 1: Quadratic Functions of the Form $f(x) = x^2 + k$

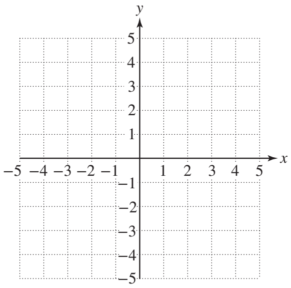
9. Describe how the value of k affects the graph of a function defined by $f(x) = x^2 + k$.

For Exercises 10–17, graph the functions. **(See Examples 1–2.)**

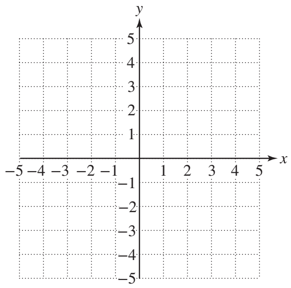
10. $g(x) = x^2 + 1$



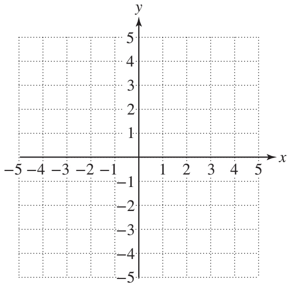
11. $f(x) = x^2 + 2$



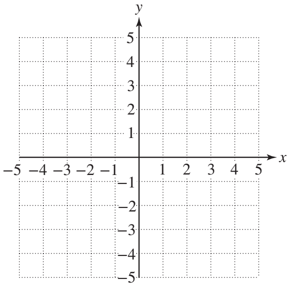
12. $p(x) = x^2 - 3$



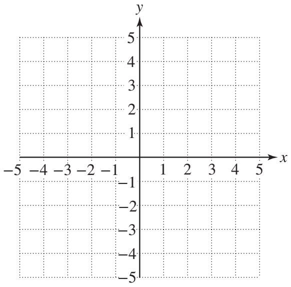
13. $q(x) = x^2 - 4$



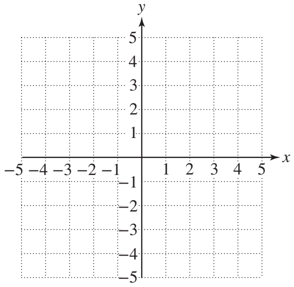
14. $T(x) = x^2 + \frac{3}{4}$



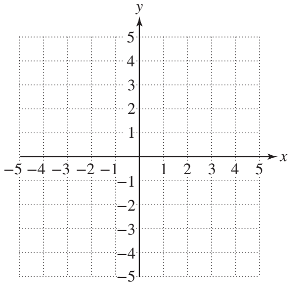
15. $S(x) = x^2 + \frac{3}{2}$



16. $M(x) = x^2 - \frac{5}{4}$



17. $n(x) = x^2 - \frac{1}{3}$

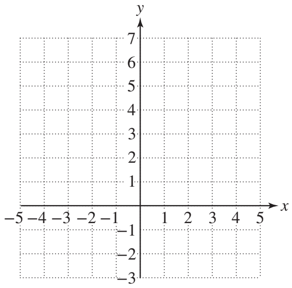


Concept 2: Quadratic Functions of the Form $f(x) = (x - h)^2$

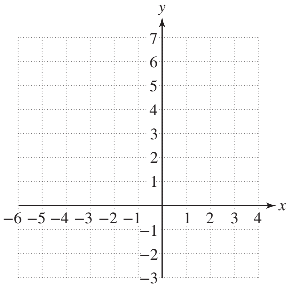
18. Describe how the value of h affects the graph of a function defined by $f(x) = (x - h)^2$.

For Exercises 19–26, graph the functions. **(See Examples 3–4.)**

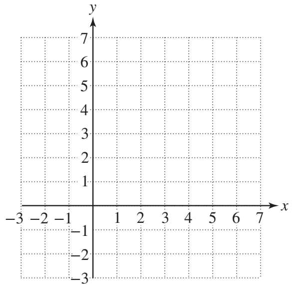
19. $r(x) = (x + 1)^2$



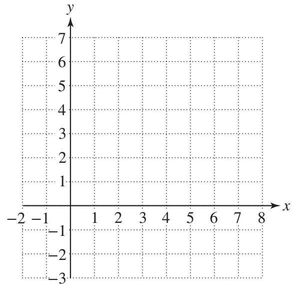
20. $h(x) = (x + 2)^2$



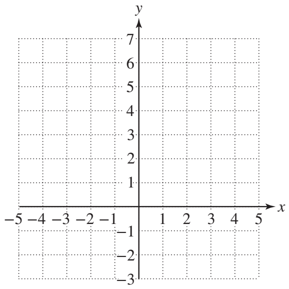
21. $k(x) = (x - 3)^2$



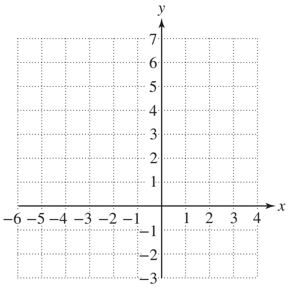
22. $L(x) = (x - 4)^2$



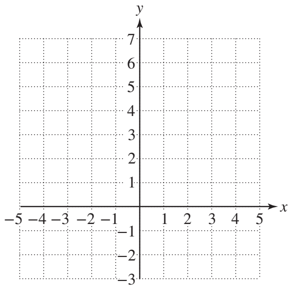
23. $A(x) = \left(x + \frac{3}{4}\right)^2$



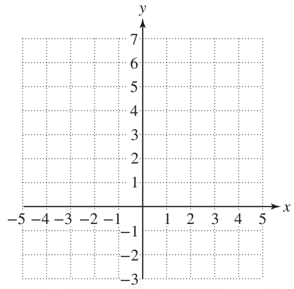
24. $r(x) = \left(x + \frac{3}{2}\right)^2$



25. $W(x) = (x - 1.25)^2$



26. $V(x) = (x - 2.5)^2$



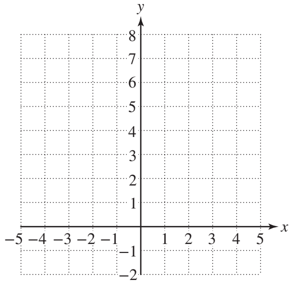
Concept 3: Quadratic Functions of the Form $f(x) = ax^2$

27. Describe how the value of a affects the graph of a function defined by $f(x) = ax^2$, where $a \neq 0$.

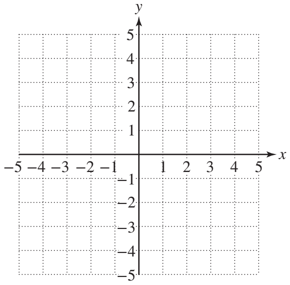
28. How do you determine whether the graph of a function defined by $h(x) = ax^2 + bx + c$ $(a \neq 0)$ opens upward or downward?

For Exercises 29–36, graph the functions. **(See Examples 5–6.)**

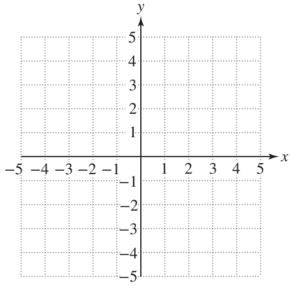
29. $f(x) = 4x^2$



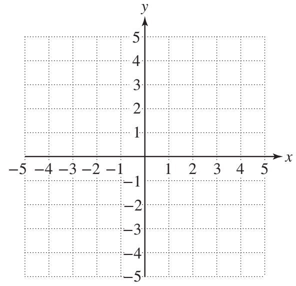
30. $g(x) = 3x^2$



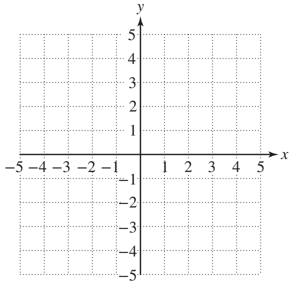
31. $h(x) = \frac{1}{4}x^2$



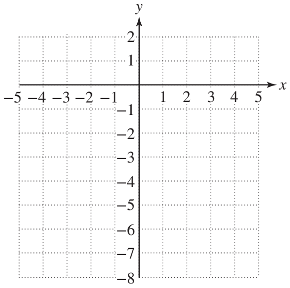
32. $f(x) = \frac{1}{5}x^2$



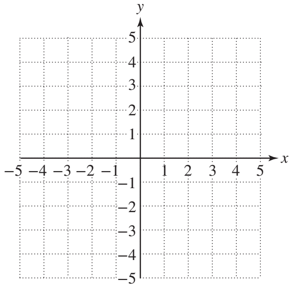
33. $c(x) = -x^2$



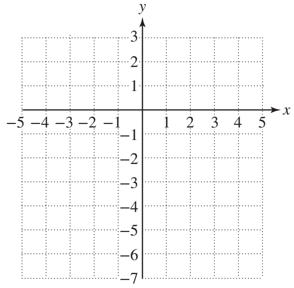
34. $g(x) = -4x^2$



35. $v(x) = -\frac{1}{5}x^2$



36. $f(x) = -\frac{1}{4}x^2$



Concept 4: Quadratic Functions of the Form $f(x) = a(x - h)^2 + k$

For Exercises 37–44, match the function with its graph.

37. $f(x) = -\frac{1}{4}x^2$

38. $g(x) = (x + 3)^2$

39. $k(x) = (x - 3)^2$

40. $h(x) = \frac{1}{4}x^2$

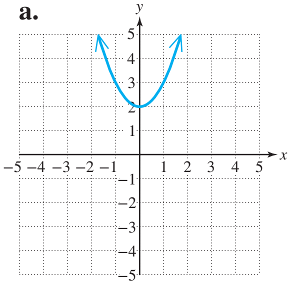
41. $t(x) = x^2 + 2$

42. $m(x) = x^2 - 4$

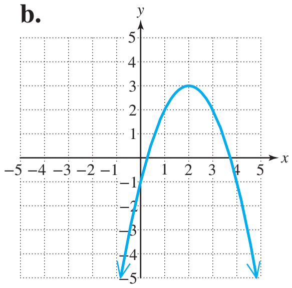
43. $p(x) = (x + 1)^2 - 3$

44. $n(x) = -(x - 2)^2 + 3$

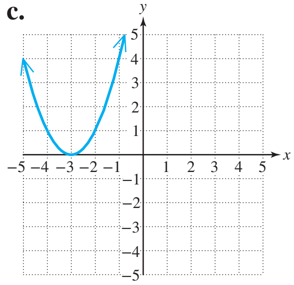
a.



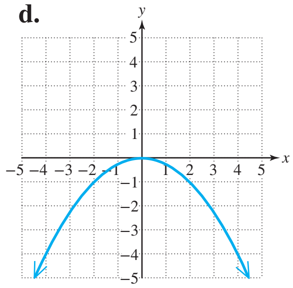
b.



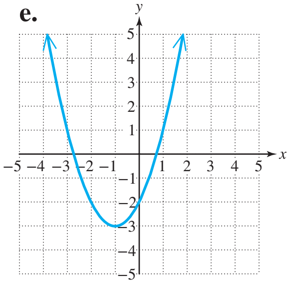
c.



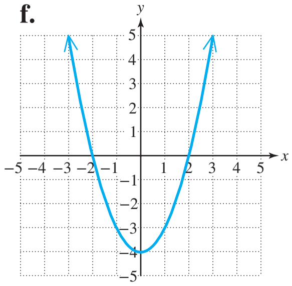
d.



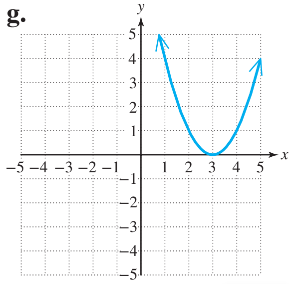
e.



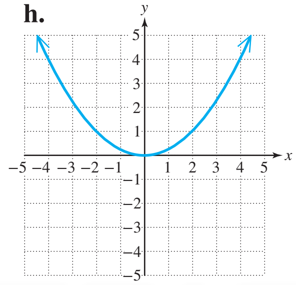
f.



g.

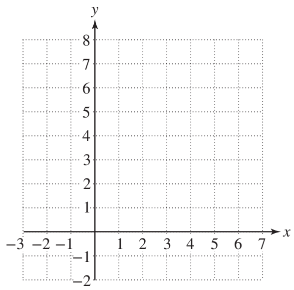


h.

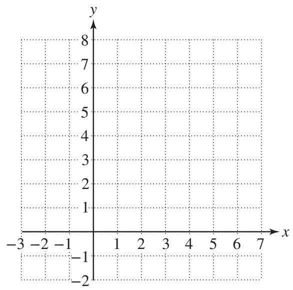


For Exercises 45–64, graph the parabola and the axis of symmetry. Label the coordinates of the vertex, and write the equation of the axis of symmetry. Use the graph to write the domain and range in interval notation. **(See Examples 7–8.)**

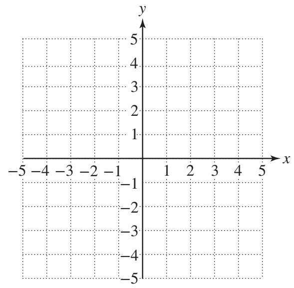
45. $f(x) = (x - 3)^2 + 2$



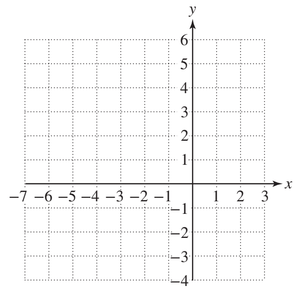
46. $f(x) = (x - 2)^2 + 3$



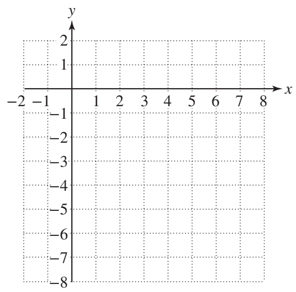
47. $f(x) = (x + 1)^2 - 3$



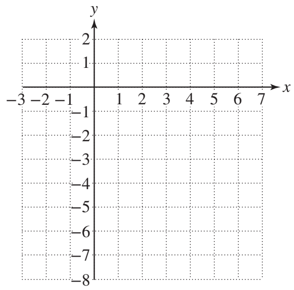
48. $f(x) = (x + 3)^2 - 1$



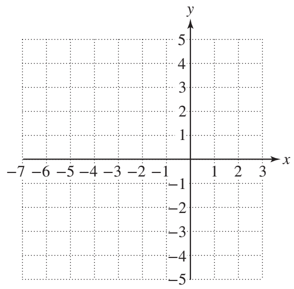
49. $f(x) = -(x - 4)^2 - 2$



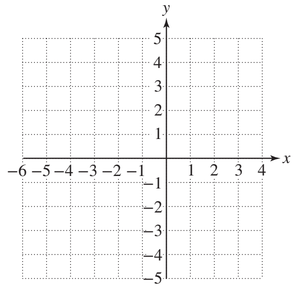
50. $f(x) = -(x - 2)^2 - 4$



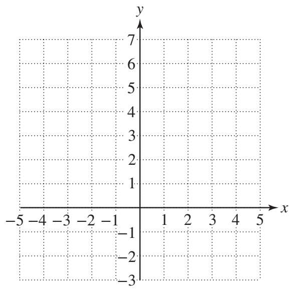
51. $f(x) = -(x + 3)^2 + 3$



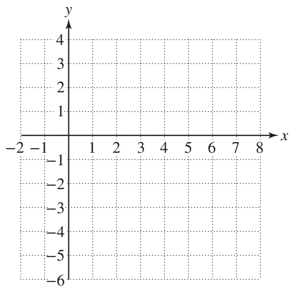
52. $f(x) = -(x + 2)^2 + 2$



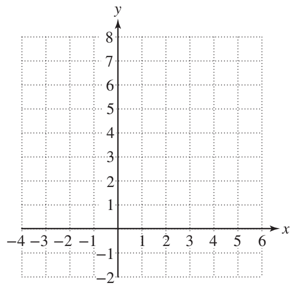
53. $f(x) = (x + 1)^2 + 1$



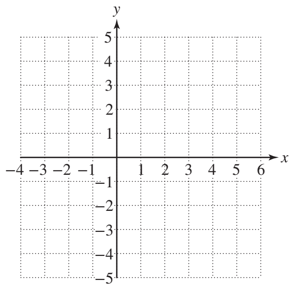
54. $f(x) = (x - 4)^2 - 4$



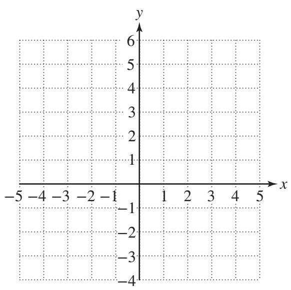
55. $f(x) = 3(x - 1)^2$



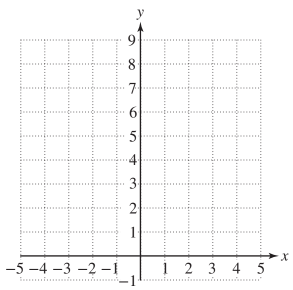
56. $f(x) = -3(x - 1)^2$



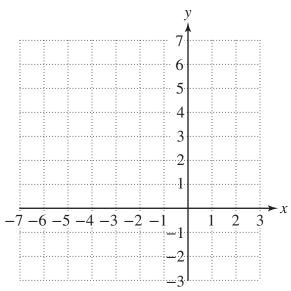
57. $f(x) = -4x^2 + 3$



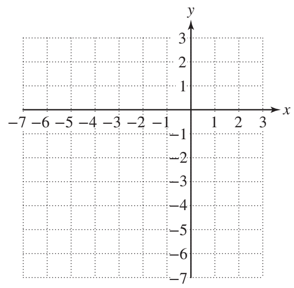
58. $f(x) = 4x^2 + 3$



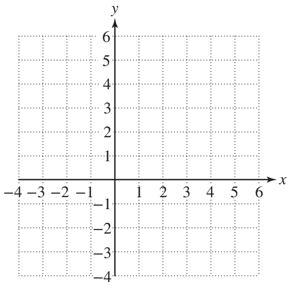
59. $f(x) = 2(x + 3)^2 - 1$



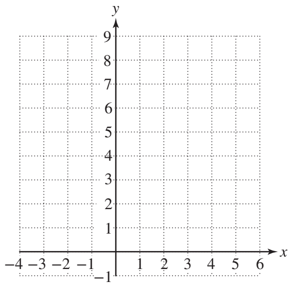
60. $f(x) = -2(x + 3)^2 - 1$



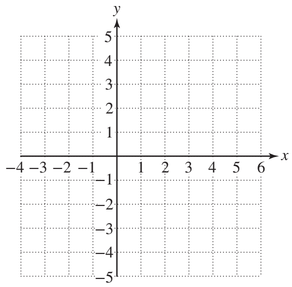
61. $f(x) = -\frac{1}{4}(x - 1)^2 + 2$



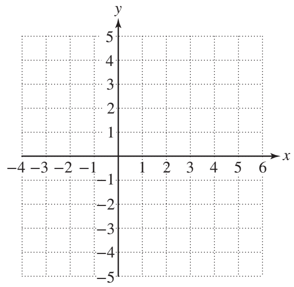
62. $f(x) = \frac{1}{4}(x - 1)^2 + 2$



63. $f(x) = \frac{1}{3}(x - 2)^2 + 1$



64. $f(x) = -\frac{1}{3}(x - 2)^2 + 1$



65. Compare the graphs of the following equations to the graph of $y = x^2$.

a. $y = x^2 + 3$

b. $y = (x + 3)^2$

c. $y = 3x^2$

66. Compare the graphs of the following equations to the graph of $y = x^2$.

a. $y = (x - 2)^2$

b. $y = 2x^2$

c. $y = x^2 - 2$

For Exercises 67–78, write the coordinates of the vertex and determine if the vertex is a maximum point or a minimum point. Then write the maximum or minimum value.

67. $f(x) = 4(x - 6)^2 - 9$

68. $g(x) = 3(x - 4)^2 - 7$

69. $p(x) = -\frac{2}{5}(x - 2)^2 + 5$

70. $h(x) = -\frac{3}{7}(x - 5)^2 + 10$

71. $k(x) = \frac{1}{2}(x + 8)^2$

72. $m(x) = \frac{2}{9}(x + 11)^2$

73. $n(x) = -6x^2 + \frac{21}{4}$

74. $q(x) = -4x^2 + \frac{1}{6}$

75. $A(x) = 2(x - 7)^2 - \frac{3}{2}$

76. $B(x) = 5(x - 3)^2 - \frac{1}{4}$

77. $F(x) = 7x^2$

78. $G(x) = -7x^2$

79. True or false: The function defined by $g(x) = -5x^2$ has a maximum value but no minimum value.

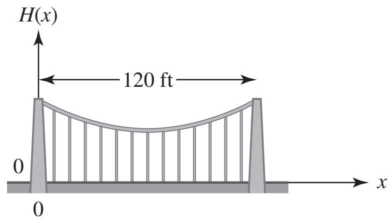
80. True or false: The function defined by $f(x) = 2(x - 5)^2$ has a maximum value but no minimum value.

81. True or false: If the vertex $(-2, 8)$ represents a minimum point, then the minimum value is -2.

82. True or false: If the vertex $(-2, 8)$ represents a maximum point, then the maximum value is 8.

Expanding Your Skills

83. A suspension bridge is 120 ft long. Its supporting cable hangs in a shape that resembles a parabola. The function defined by $H(x) = \frac{1}{90}(x - 60)^2 + 30$ (where $0 \le x \le 120$) approximates the height $H(x)$ (in feet) of the supporting cable a distance of x ft from the end of the bridge (see figure).

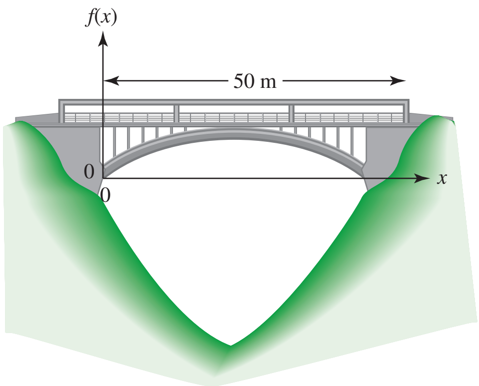


a. What is the location of the vertex of the parabolic cable?

b. What is the minimum height of the cable?

c. How high are the towers at either end of the supporting cable?

84. A 50-m bridge over a crevasse is supported by a parabolic arch. The function defined by $f(x) = -0.16(x - 25)^2 + 100$ (where $0 \le x \le 50$) approximates the height $f(x)$ (in meters) of the supporting arch x meters from the end of the bridge (see figure).



a. What is the location of the vertex of the arch?

b. What is the maximum height of the arch (relative to the origin)?

85. The staging platform for a fireworks display is 6 ft above ground, and the mortars leave the platform at 96 ft/sec. The height of the mortars $h(t)$ (in feet) can be modeled by $h(t) = -16(t - 3)^2 + 150$, where t is the time in seconds after launch.

a. If the fuses are set for 3 sec after launch, at what height will the fireworks explode?

b. Will the fireworks explode at their maximum height? Explain.

Section 7.5 Vertex of a Parabola: Applications and Modeling

Concepts

1. **Writing a Quadratic Function in the Form $f(x) = a(x - h)^2 + k$**
2. **Vertex Formula**
3. **Determining the Vertex and Intercepts of a Quadratic Function**
4. **Applications and Modeling of Quadratic Functions**

1. Writing a Quadratic Function in the Form $f(x) = a(x - h)^2 + k$

The graph of a quadratic function is a parabola, and if the function is written in the form $f(x) = a(x - h)^2 + k$ $(a \ne 0)$, then the vertex is (h, k). A quadratic function can be written in the form $f(x) = a(x - h)^2 + k$ $(a \ne 0)$ by completing the square. The process is similar to the steps to solve a quadratic equation by completing the square. However, in this context working with a function, we will perform all algebraic manipulations on the right side of the equation.

Example 1 Writing a Quadratic Function in the Form $f(x) = a(x - h)^2 + k \quad (a \neq 0)$

Given $f(x) = x^2 + 8x + 13$

a. Write the function in the form $f(x) = a(x - h)^2 + k$.

b. Identify the vertex, axis of symmetry, and minimum function value.

Solution:

a. $f(x) = x^2 + 8x + 13$

$= 1(x^2 + 8x) + 13$

Rather than dividing by the leading coefficient on both sides, we will factor out the leading coefficient from the variable terms on the right-hand side.

$= 1(x^2 + 8x \qquad) + 13$

Next, complete the square on the expression within the parentheses: $\left[\frac{1}{2}(8)\right]^2 = 16$.

$= 1(x^2 + 8x + 16 - 16) + 13$

Rather than add 16 to both sides of the function, we *add and subtract* 16 within the parentheses on the right-hand side. This has the effect of adding 0 to the right-hand side.

$= 1(x^2 + 8x + 16) - 16 + 13$

Use the associative property of addition to regroup terms and isolate the perfect square trinomial within the parentheses.

$= (x + 4)^2 - 3$

Factor and simplify.

b. $f(x) = (x + 4)^2 - 3$

The vertex is $(-4, -3)$.

The axis of symmetry is $x = -4$.

Because $a > 0$, the parabola opens upward.

The minimum value is -3 (Figure 7-15).

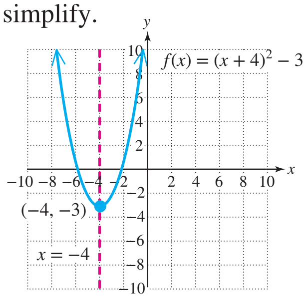


Figure 7-15

Avoiding Mistakes

Do not factor out the leading coefficient from the constant term.

Skill Practice

1. Given: $f(x) = x^2 + 8x - 1$

a. Write the function in the form $f(x) = a(x - h)^2 + k$.

b. Identify the vertex, axis of symmetry, and minimum value of the function.

Answers

1. a. $f(x) = (x + 4)^2 - 17$

b. Vertex: $(-4, -17)$; axis of symmetry: $x = -4$; minimum value: -17

Example 2 Analyzing a Quadratic Function

Given $f(x) = -2x^2 + 12x - 16$

a. Write the function in the form $f(x) = a(x - h)^2 + k$.

b. Find the vertex, axis of symmetry, and maximum function value.

c. Find the x- and y-intercepts.

d. Sketch the graph of the function.

Solution:

a. $f(x) = -2x^2 + 12x - 16$ — To find the vertex, write the function in the form $f(x) = a(x - h)^2 + k$.

$= -2(x^2 - 6x \quad) - 16$ — If the leading coefficient is not 1, factor the coefficient from the variable terms.

$= -2(x^2 - 6x + 9 - 9) - 16$ — Add and subtract the quantity $\left[\frac{1}{2}(-6)\right]^2 = 9$ within the parentheses.

$= -2(x^2 - 6x + 9) + (-2)(-9) - 16$ — To remove the term -9 from the parentheses, we must first apply the distributive property. When -9 is removed from the parentheses, it carries with it a factor of -2.

$= -2(x - 3)^2 + 18 - 16$ — Factor and simplify.

$= -2(x - 3)^2 + 2$

b. $f(x) = -2(x - 3)^2 + 2$

The vertex is $(3, 2)$. The axis of symmetry is $x = 3$. Because $a < 0$, the parabola opens downward and the maximum value is 2.

TIP: In Example 2(c), we could have used the form of the equation found in part (a) to find the intercepts.

c. The y-intercept is given by $f(0) = -2(0)^2 + 12(0) - 16 = -16$.

The y-intercept is $(0, -16)$.

To find the x-intercept(s), find the real solutions to the equation $f(x) = 0$.

$f(x) = -2x^2 + 12x - 16$

$0 = -2x^2 + 12x - 16$ — Substitute $f(x) = 0$.

$0 = -2(x^2 - 6x + 8)$ — Factor.

$0 = -2(x - 4)(x - 2)$

$x = 4 \quad \text{or} \quad x = 2$

The x-intercepts are $(4, 0)$ and $(2, 0)$.

d. Using the information from parts (a)–(c), sketch the graph (Figure 7-16).

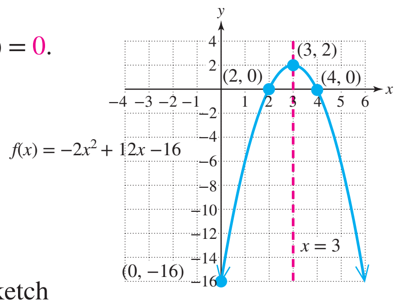


Figure 7-16

Skill Practice

2. Given $g(x) = -x^2 + 6x - 5$

a. Write the function in the form $g(x) = a(x - h)^2 + k$.

b. Identify the vertex, axis of symmetry, and maximum value of the function.

c. Determine the x- and y-intercepts.

d. Graph the function.

2. Vertex Formula

Completing the square and writing a quadratic function in the form $f(x) = a(x - h)^2 + k$ $(a \neq 0)$ is one method to find the vertex of a parabola. Another method is to use the vertex formula. The **vertex formula** can be derived by completing the square on $f(x) = ax^2 + bx + c$ $(a \neq 0)$.

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

$$= a\left(x^2 + \frac{b}{a}x \qquad\qquad \right) + c$$
Factor a from the variable terms.

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$
Add and subtract $\left[\frac{1}{2}(b/a)\right]^2 = b^2/(4a^2)$ within the parentheses.

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + (a)\left(-\frac{b^2}{4a^2}\right) + c$$
Apply the distributive property and remove the term $-b^2/(4a^2)$ from the parentheses.

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$
Factor the trinomial and simplify.

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$
Apply the commutative property of addition to reverse the last two terms.

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac}{4a} - \frac{b^2}{4a}$$
Obtain a common denominator.

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$= a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \frac{4ac - b^2}{4a}$$

$$f(x) = a(x \quad - \quad h)^2 \quad + \quad k$$

The function is in the form $f(x) = a(x - h)^2 + k$, where

$$h = \frac{-b}{2a} \qquad \text{and} \qquad k = \frac{4ac - b^2}{4a}$$

Therefore, the vertex is $\left(\dfrac{-b}{2a}, \dfrac{4ac - b^2}{4a}\right)$.

Although the y-coordinate of the vertex is given by $\dfrac{4ac - b^2}{4a}$, it is usually easier to determine the x-coordinate of the vertex first and then find y by evaluating the function at $x = -b/(2a)$.

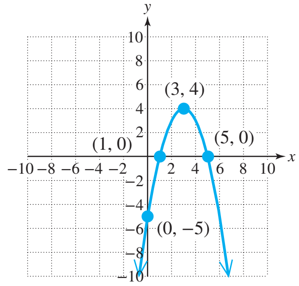
Answers

2. a. $g(x) = -(x - 3)^2 + 4$

b. Vertex: (3, 4); axis of symmetry: $x = 3$; maximum value: 4

c. x-intercepts: (5, 0) and (1, 0); y-intercept: (0, −5)

d.



The Vertex Formula

For $f(x) = ax^2 + bx + c$ $(a \neq 0)$, the vertex is given by

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) \quad \text{or} \quad \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

3. Determining the Vertex and Intercepts of a Quadratic Function

Example 3 **Determining the Vertex and Intercepts of a Quadratic Function**

Given: $h(x) = x^2 - 2x + 5$

a. Use the vertex formula to find the vertex.

b. Find the x- and y-intercepts.

c. Sketch the function.

Solution:

a. $h(x) = x^2 - 2x + 5$

$a = 1 \qquad b = -2 \qquad c = 5$ Identify a, b, and c.

The x-coordinate of the vertex is $\dfrac{-b}{2a} = \dfrac{-(-2)}{2(1)} = 1$.

The y-coordinate of the vertex is $h(1) = (1)^2 - 2(1) + 5 = 4$.

The vertex is $(1, 4)$.

b. The y-intercept is given by $h(0) = (0)^2 - 2(0) + 5 = 5$.

The y-intercept is $(0, 5)$.

To find the x-intercept(s), find the real solutions to the equation $h(x) = 0$.

$h(x) = x^2 - 2x + 5$

$0 = x^2 - 2x + 5$ This quadratic equation is not factorable. Apply the quadratic formula: $a = 1$, $b = -2$, $c = 5$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

The solutions to the equation $h(x) = 0$ are not real numbers. Therefore, there are no x-intercepts.

FOR REVIEW

Given a quadratic equation $ax^2 + bx + c = 0$, the quadratic formula gives the solutions for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

c.

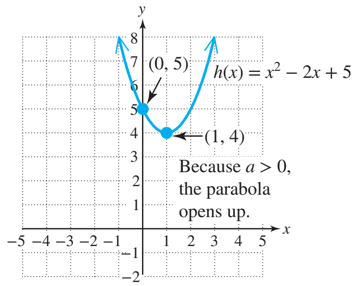


Figure 7-17

TIP: The location of the vertex and the direction that the parabola opens can be used to determine whether the function has any x-intercepts.

Given $h(x) = x^2 - 2x + 5$, the vertex $(1, 4)$ is above the x-axis. Furthermore, because $a > 0$, the parabola opens upward. Therefore, it is not possible for the graph to cross the x-axis (Figure 7-17).

Skill Practice

3. Given: $f(x) = x^2 + 4x + 6$

a. Use the vertex formula to find the vertex of the parabola.

b. Determine the x- and y-intercepts.

c. Sketch the graph.

4. Applications and Modeling of Quadratic Functions

Example 4 Applying a Quadratic Function

The crew from Extravaganza Entertainment launches fireworks at an angle of 60° from the horizontal. The height of one particular type of display can be approximated by

$$h(t) = -16t^2 + 128\sqrt{3}t$$

where $h(t)$ is measured in feet and t is measured in seconds.

a. How long will it take the fireworks to reach their maximum height? Round to the nearest second.

b. Find the maximum height. Round to the nearest foot.

Lena Kofoed

Solution:

$h(t) = -16t^2 + 128\sqrt{3}t$ — This parabola opens downward; therefore, the maximum height of the fireworks will occur at the vertex of the parabola.

$a = -16 \qquad b = 128\sqrt{3} \qquad c = 0$ — Identify a, b, and c, and apply the vertex formula.

The x-coordinate of the vertex is

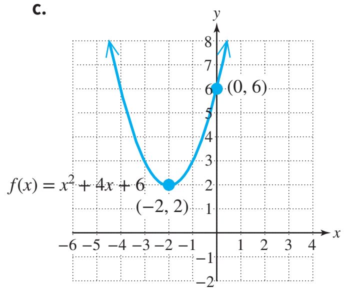
$$\frac{-b}{2a} = \frac{-128\sqrt{3}}{2(-16)} = \frac{-128\sqrt{3}}{-32} \approx 6.9$$

Answers

3. a. Vertex: $(-2, 2)$

b. x-intercepts: none; y-intercept: $(0, 6)$

c.



The y-coordinate of the vertex is approximately

$$h(6.9) = -16(6.9)^2 + 128\sqrt{3}(6.9) \approx 768$$

The vertex is (6.9, 768).

a. The fireworks will reach their maximum height in 6.9 sec.

b. The maximum height is 768 ft.

Skill Practice

4. An object is launched into the air with an initial velocity of 48 ft/sec from the top of a building 288 ft high. The height $h(t)$ of the object after t seconds is given by

$$h(t) = -16t^2 + 48t + 288$$

a. Find the time it takes for the object to reach its maximum height.
b. Find the maximum height.

In Example 5 we will learn how to write a quadratic model of a parabola given three points by using a system of equations and the standard form of a parabola: $y = ax^2 + bx + c$.

This process involves substituting the x- and y-coordinates of the three given points into the quadratic model. Then we solve the resulting system of three equations.

Example 5 Writing a Quadratic Model

Write an equation of a parabola that passes through the points $(1, -1)$, $(-1, -5)$, and $(2, 4)$.

Solution:

Substitute $(1, -1)$ into the equation

$$y = ax^2 + bx + c \longrightarrow \begin{aligned} (-1) &= a(1)^2 + b(1) + c \\ -1 &= a + b + c \\ a + b + c &= -1 \end{aligned}$$

Substitute $(-1, -5)$ into the equation

$$y = ax^2 + bx + c \longrightarrow \begin{aligned} (-5) &= a(-1)^2 + b(-1) + c \\ -5 &= a - b + c \\ a - b + c &= -5 \end{aligned}$$

Substitute $(2, 4)$ into the equation

$$y = ax^2 + bx + c \longrightarrow \begin{aligned} (4) &= a(2)^2 + b(2) + c \\ 4 &= 4a + 2b + c \\ 4a + 2b + c &= 4 \end{aligned}$$

Solve the system:

$$\begin{aligned} \boxed{A} \quad a + b + c &= -1 \\ \boxed{B} \quad a - b + c &= -5 \\ \boxed{C} \quad 4a + 2b + c &= 4 \end{aligned}$$

Answers
4. a. 1.5 sec **b.** 324 ft

Notice that the c variables all have a coefficient of 1. Therefore, we choose to eliminate the c variable.

$$\begin{array}{lrcl}
\boxed{\text{A}} & a+b+c=-1 & \longrightarrow & a+b+c=-1 \\
\boxed{\text{B}} & a-b+c=-5 & \xrightarrow[\text{Multiply by } -1.]{} & \underline{-a+b-c=5} \\
 & & & 2b \quad = \quad 4 \\
 & & & b = \quad 2 \quad \boxed{\text{D}}
\end{array}$$

$$\begin{array}{lrcl}
\boxed{\text{B}} & a-b+c=-5 & \longrightarrow & a-b+c=-5 \\
\boxed{\text{C}} & 4a+2b+c=4 & \xrightarrow[\text{Multiply by } -1.]{} & \underline{-4a-2b-c=-4} \\
 & & & -3a-3b \quad =-9 \quad \boxed{\text{E}}
\end{array}$$

Substitute b with 2 in equation $\boxed{\text{E}}$:

$$\begin{aligned}
-3a-3(2)&=-9\\
-3a-6&=-9\\
-3a&=-3\\
a&=1
\end{aligned}$$

Substitute a and b in equation $\boxed{\text{A}}$ to solve for c:

$$\begin{aligned}
(1)+(2)+c&=-1\\
3+c&=-1\\
c&=-4
\end{aligned}$$

Substitute $a=1$, $b=2$, and $c=-4$ in the standard form of the parabola for the final answer.

$$y=ax^2+bx+c$$

$$y=(1)x^2+(2)x+(-4) \longrightarrow y=x^2+2x-4$$

A graph of $y=x^2+2x-4$ is shown in Figure 7-18. Notice that the graph of the function passes through the points $(1, -1)$, $(-1, -5)$, and $(2, 4)$.

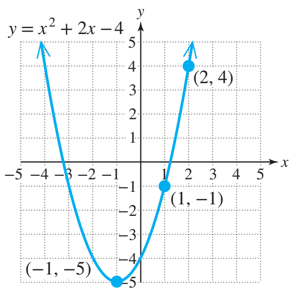


Figure 7-18

Skill Practice

5. Write an equation of the parabola that passes through the points $(1, 1)$, $(-2, 1)$, and $(3, -9)$.

Answer

5. $a=-1$, $b=-1$, $c=3$; $y=-x^2-x+3$

Section 7.5 Activity

A.1. Given a quadratic function defined by $f(x)=ax^2+bx+c$, state two methods to find the vertex of the parabola.

A.2. Given $f(x)=ax^2+bx+c$, explain how to use the vertex formula.

A.3. Consider the function defined by $f(x)=2x^2-4x-6$. One method to graph the function is to first write the function in vertex form $f(x)=a(x-h)^2+k$. Follow these steps.

a. Factor out the leading coefficient from the variable terms.

$$f(x)=2(\qquad\qquad)-6$$

b. What value would be added to the expression x^2-2x to complete the square?

c. Complete the square within the parentheses. However, rather than adding the constant from part (b) to both sides of the equation, *add* and *subtract* this value within the parentheses. This has the effect of adding 0 to the expression within the parentheses, which does not change the value of the expression.

d. Next, remove the term –1 from inside the parentheses to combine it with the constant outside the parentheses. However, to remove –1 from inside the parentheses, first apply the distributive property. When –1 is removed from the parentheses, it carries with it a factor of 2.

e. Rewrite the function with the expression in parentheses factored.

f. The function should now be written in vertex form. Identify the vertex.

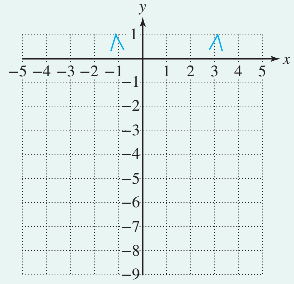
g. Does the graph of the function open upward or downward?

h. Identify the minimum or maximum value of the function.

i. Identify the axis of symmetry.

j. Find the x- and y-intercepts.

k. Graph the function.



A.4. Refer to $f(x) = 2x^2 - 4x - 6$ from Exercise A.3. Use the vertex formula to find the vertex. Then compare the result to Exercise A.3(f).

A.5. Consider the function defined by $f(x) = -x^2 - 6x - 10$.

a. Use the vertex formula to find the vertex of the parabola.

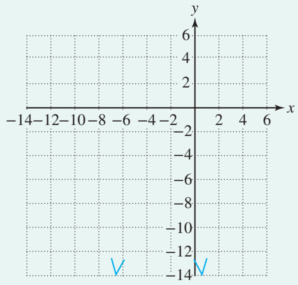
b. Does the graph of the function open upward or downward?

c. Identify the minimum or maximum value of the function.

d. Identify the axis of symmetry.

e. Find the x- and y-intercepts.

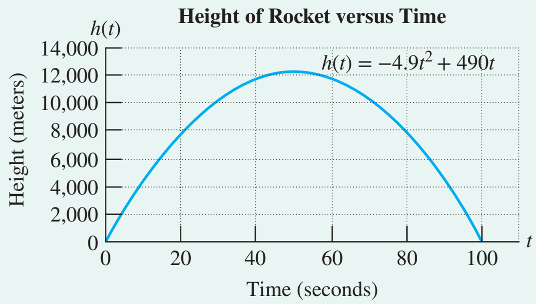
f. Graph the function.



A.6. A rocket is fired upward from ground level with an initial velocity of 490 m/sec. The height of the rocket is given by $h(t) = -4.9t^2 + 490t$, where $h(t)$ is measured in meters and t is the time in seconds after launch.

a. Function h is a quadratic function. Find the vertex.

b. Interpret the meaning of the vertex in the context of this problem.



Section 7.5 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, simplify the expression.

R.1. $\dfrac{-(-6)}{2(4)}$

R.2. $\dfrac{-(-7)}{2(-14)}$

For Exercises R.3–R.4, find the x- and y-intercepts.

R.3. $f(x) = -4x - 8$

R.4. $f(x) = \dfrac{1}{2}x + 3$

For Exercises R.5–R.10, solve the equation.

R.5. $x^2 - 4x - 32 = 0$

R.6. $x^2 + 15x + 56 = 0$

R.7. $x^2 - 4 = 0$

R.8. $x^2 - 49 = 0$

R.9. $x^2 + 24x + 144 = 0$

R.10. $x^2 - 16x + 64 = 0$

For Exercises R.11–R.12, find the value of n so that the expression is a perfect square trinomial. Then factor the trinomial.

R.11. $x^2 - 14x + n$

R.12. $x^2 + 20x + n$

Vocabulary and Key Concepts

1. **a.** Given $f(x) = ax^2 + bx + c$ $(a \neq 0)$, the vertex formula gives the x-coordinate of the vertex as _____. The y-coordinate can be found by evaluating the function at $x =$ ______.

b. Answer true or false. A parabola may have no x-intercepts.

c. Answer true or false. A parabola may have one x-intercept.

d. Answer true or false. A parabola may have two x-intercepts.

e. Answer true or false. A parabola may have three x-intercepts.

2. Consider the quadratic function defined by $f(x) = x^2 - 6x + 10$.

a. Identify the values of a and b that would be used to apply the vertex formula.

b. Determine the x-coordinate of the vertex.

c. Write the vertex as an ordered pair.

Concept 1: Writing a Quadratic Function in the Form $f(x) = a(x - h)^2 + k$

For Exercises 3–14, write the function in the form $f(x) = a(x - h)^2 + k$ by completing the square. Then identify the vertex. **(See Examples 1–2.)**

3. $g(x) = x^2 - 8x + 5$ **4.** $h(x) = x^2 + 4x + 5$ **5.** $n(x) = 2x^2 + 12x + 13$

6. $f(x) = 4x^2 + 16x + 19$ **7.** $p(x) = -3x^2 + 6x - 5$ **8.** $q(x) = -2x^2 + 12x - 11$

9. $k(x) = x^2 + 7x - 10$ **10.** $m(x) = x^2 - x - 8$ **11.** $F(x) = 5x^2 + 10x + 1$

12. $G(x) = 4x^2 + 4x - 7$ **13.** $P(x) = -2x^2 + x$ **14.** $Q(x) = -3x^2 + 12x$

Concept 2: Vertex Formula

For Exercises 15–26, find the vertex by using the vertex formula. **(See Example 3.)**

15. $Q(x) = x^2 - 4x + 7$ **16.** $T(x) = x^2 - 8x + 17$ **17.** $r(x) = -3x^2 - 6x - 5$

18. $s(x) = -2x^2 - 12x - 19$ **19.** $N(x) = x^2 + 8x + 1$ **20.** $M(x) = x^2 + 6x - 5$

21. $m(x) = \frac{1}{2}x^2 + x + \frac{5}{2}$ **22.** $n(x) = \frac{1}{2}x^2 + 2x + 3$ **23.** $k(x) = -x^2 + 2x + 2$

24. $h(x) = -x^2 + 4x - 3$ **25.** $A(x) = -\frac{1}{3}x^2 + x$ **26.** $B(x) = -\frac{2}{3}x^2 - 2x$

For Exercises 27–30, find the vertex two ways:

a. by completing the square and writing in the form $f(x) = a(x - h)^2 + k$, and

b. by using the vertex formula.

27. $p(x) = x^2 + 8x + 1$ **28.** $F(x) = x^2 + 4x - 2$ **29.** $f(x) = 2x^2 + 4x + 6$ **30.** $g(x) = 3x^2 + 12x + 9$

Concept 3: Determining the Vertex and Intercepts of a Quadratic Function

For Exercises 31–38

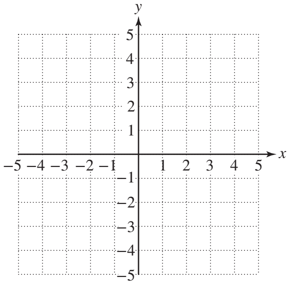
a. Find the vertex.

b. Find the y-intercept.

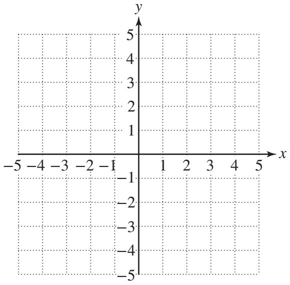
c. Find the x-intercept(s), if they exist.

d. Use this information to graph the function. **(See Examples 2–3.)**

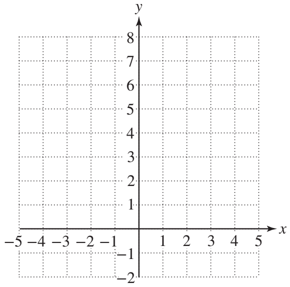
31. $f(x) = x^2 + 2x - 3$



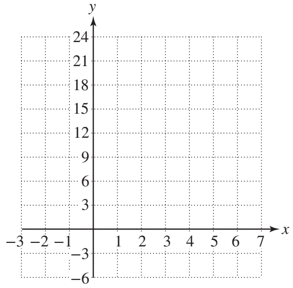
32. $f(x) = x^2 + 4x + 3$



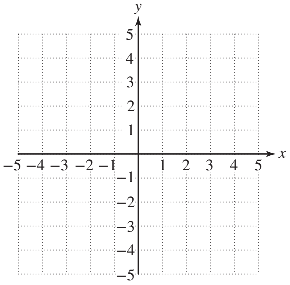
33. $f(x) = 2x^2 - 2x + 4$



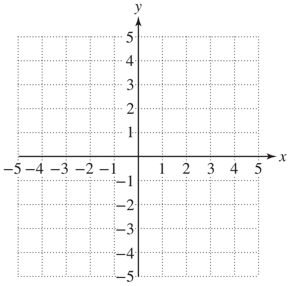
34. $f(x) = 2x^2 - 12x + 19$



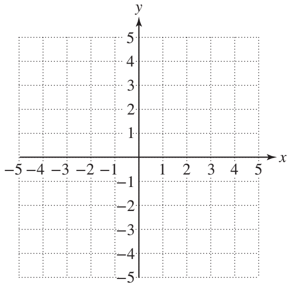
35. $f(x) = -x^2 + 3x - \dfrac{9}{4}$



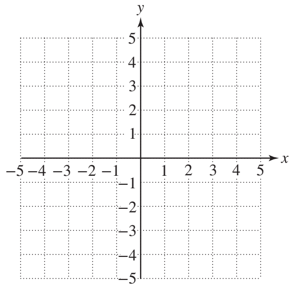
36. $f(x) = -x^2 - \dfrac{3}{2}x - \dfrac{9}{16}$



37. $f(x) = -x^2 - 2x + 3$



38. $f(x) = -x^2 - 4x$



Concept 4: Applications and Modeling of Quadratic Functions

39. A set of fireworks mortar shells is launched from the staging platform at 100 ft/sec from an initial height of 8 ft above the ground. The height of the fireworks, $h(t)$, can be modeled by

$h(t) = -16t^2 + 100t + 8$, where t is the time in seconds after launch. **(See Example 4.)**

a. What is the maximum height that the shells can reach before exploding?

b. For how many seconds after launch would the fuses need to be set so that the mortar shells would in fact explode at the maximum height?

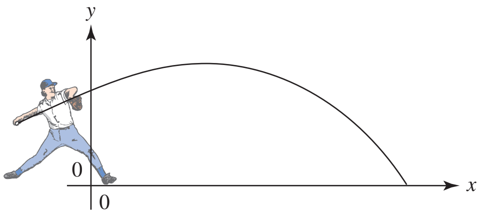
40. A baseball player throws a ball, and the height of the ball (in feet) can be approximated by

$$y(x) = -0.011x^2 + 0.577x + 5$$

where x is the horizontal position of the ball measured in feet from the origin.

a. For what value of x will the ball reach its highest point? Round to the nearest foot.

b. What is the maximum height of the ball? Round to the nearest tenth of a foot.



41. Gas mileage depends in part on the speed of the car. The gas mileage of a subcompact car is given by $m(x) = -0.04x^2 + 3.6x - 49$, where $20 \le x \le 70$ represents the speed in miles per hour and $m(x)$ is given in miles per gallon.

a. At what speed will the car get the maximum gas mileage?

b. What is the maximum gas mileage?

42. Gas mileage depends in part on the speed of the car. The gas mileage of a luxury car is given by $L(x) = -0.015x^2 + 1.44x - 21$, where $25 \le x \le 70$ represents the speed in miles per hour and $L(x)$ is given in miles per gallon.

a. At what speed will the car get the maximum gas mileage?

b. What is the maximum gas mileage? Round to the nearest whole unit.

43. The *Clostridium tetani* bacterium is cultured to produce tetanus toxin used in an inactive form for the tetanus vaccine. The amount of toxin produced per batch increases with time and then becomes unstable. The amount of toxin $b(t)$ (in grams) as a function of time t (in hours) can be approximated by:

$$b(t) = -\frac{1}{1152}t^2 + \frac{1}{12}t$$

a. How many hours will it take to produce the maximum yield?

b. What is the maximum yield?

44. The bacterium *Pseudomonas aeruginosa* is cultured with an initial population of 10^4 active organisms. The population of active bacteria increases up to a point, and then due to a limited food supply and an increase of waste products, the population of living organisms decreases. Over the first 48 hr, the population $P(t)$ can be approximated by:

$$P(t) = -1718.75t^2 + 82{,}500t + 10{,}000 \quad \text{where } 0 \le t \le 48$$

a. Find the time required for the population to reach its maximum value.

b. What is the maximum population?

For Exercises 45–50, use the standard form of a parabola given by $y = ax^2 + bx + c$ to write an equation of a parabola that passes through the given points. **(See Example 5.)**

45. (0, 4), (1, 0) and (−1, −10)

46. (0, 3), (3, 0), (−1, 8)

47. (2, 1), (−2, 5), and (1, −4)

48. (1, 2), (−1, −6), and (2, −3)

49. (2, −4), (1, 1), and (−1, −7)

50. (1, 4), (−1, 6), and (2, 18)

Technology Connections

For Exercises 51–56, graph the functions in Exercises 31–36 on a graphing calculator. Use the *Max* or *Min* feature to approximate the vertex.

51. $Y_1 = x^2 + 2x - 3$ (Exercise 31)

52. $Y_1 = x^2 + 4x + 3$ (Exercise 32)

53. $Y_1 = 2x^2 - 2x + 4$ (Exercise 33)

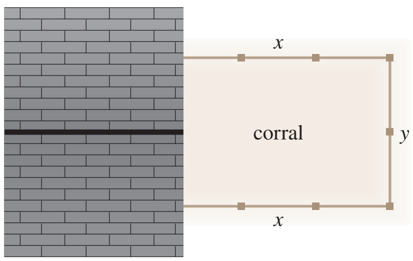
54. $Y_1 = 2x^2 - 12x + 19$ (Exercise 34)

55. $Y_1 = -x^2 + 3x - \frac{9}{4}$ (Exercise 35)

56. $Y_1 = -x^2 - \frac{3}{2}x - \frac{9}{16}$ (Exercise 36)

Expanding Your Skills

57. A farmer wants to fence a rectangular corral adjacent to the side of a barn; however, she has only 200 ft of fencing and wants to enclose the largest possible area. See the figure.

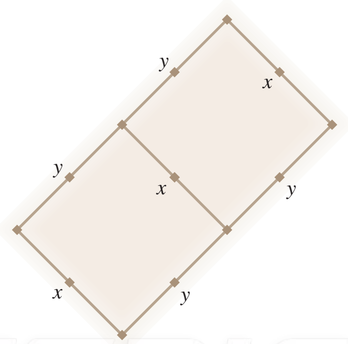


a. If x represents the length of the corral and y represents the width, explain why the dimensions of the corral are subject to the constraint $2x + y = 200$.

b. The area of the corral is given by $A = xy$. Use the constraint equation from part (a) to express A as a function of x, where $0 < x < 100$.

c. Use the function from part (b) to find the dimensions of the corral that will yield the maximum area. [*Hint:* Find the vertex of the function from part (b).]

58. A veterinarian wants to construct two equal-sized pens of maximum area out of 240 ft of fencing. See the figure.



a. If x represents the length of each pen and y represents the width of each pen, explain why the dimensions of the pens are subject to the constraint $3x + 4y = 240$.

b. The area of each individual pen is given by $A = xy$. Use the constraint equation from part (a) to express A as a function of x, where $0 < x < 80$.

c. Use the function from part (b) to find the dimensions of an individual pen that will yield the maximum area. [*Hint:* Find the vertex of the function from part (b).]

Section 7.6 Polynomial and Rational Inequalities

Concepts

1. Solving Quadratic and Polynomial Inequalities
2. Solving Rational Inequalities
3. Inequalities with "Special Case" Solution Sets

1. Solving Quadratic and Polynomial Inequalities

In this section, we will expand our study of solving inequalities.

Quadratic inequalities are inequalities that can be written in one of the following forms:

$$ax^2 + bx + c \geq 0 \qquad ax^2 + bx + c \leq 0$$
$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c < 0 \qquad \text{where} \quad a \neq 0$$

Recall that the graph of a quadratic function defined by $f(x) = ax^2 + bx + c$ is a parabola that opens upward or downward.

- The inequality $ax^2 + bx + c > 0$ is asking "For what values of x is the function positive (above the x-axis)?"
- The inequality $ax^2 + bx + c < 0$ is asking "For what values of x is the function negative (below the x-axis)?"

The graph of a quadratic function can be used to answer these questions.

Example 1 Using a Graph to Solve a Quadratic Inequality

Use the graph of $f(x) = x^2 - 6x + 5$ in Figure 7-19 to solve the inequalities.

a. $x^2 - 6x + 5 < 0$ **b.** $x^2 - 6x + 5 > 0$

Solution:

From Figure 7-19, we see that the graph of $f(x) = x^2 - 6x + 5$ is a parabola opening upward. The function factors as $f(x) = (x - 1)(x - 5)$. The x-intercepts are (1, 0) and (5, 0), and the y-intercept is (0, 5).

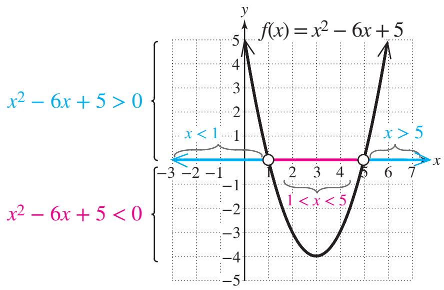


Figure 7-19

a. The solution to $x^2 - 6x + 5 < 0$ is the set of real numbers, x, for which $f(x) < 0$. Graphically, this is the set of all x values corresponding to the points where the parabola is below the x-axis (shown in red).

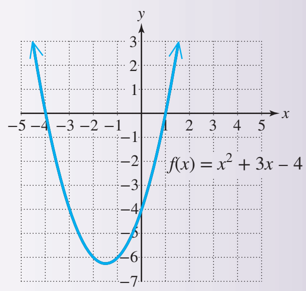
$$x^2 - 6x + 5 < 0 \qquad \text{for } \{x \mid 1 < x < 5\} \text{ or in interval notation, } (1, 5)$$

b. The solution to $x^2 - 6x + 5 > 0$ is the set of real numbers, x, for which $f(x) > 0$. This is the set of x values where the parabola is above the x-axis (shown in blue).

$$x^2 - 6x + 5 > 0 \qquad \text{for } \{x \mid x < 1 \text{ or } x > 5\} \quad \text{or} \quad (-\infty, 1) \cup (5, \infty)$$

Skill Practice Refer to the graph of $f(x) = x^2 + 3x - 4$ to solve the inequalities.

1. $x^2 + 3x - 4 > 0$
2. $x^2 + 3x - 4 < 0$



Answers

1. $\{x \mid x < -4 \text{ or } x > 1\}$; $(-\infty, -4) \cup (1, \infty)$
2. $\{x \mid -4 < x < 1\}$; $(-4, 1)$

TIP: The inequalities in Example 1 are strict inequalities. Therefore, $x = 1$ and $x = 5$ (where $f(x) = 0$) are not included in the solution set. However, the corresponding inequalities using the symbols $\le$ and $\ge$ *do* include the values where $f(x) = 0$.

The solution to $x^2 - 6x + 5 \le 0$ is $\{x \mid 1 \le x \le 5\}$ or equivalently, $[1, 5]$.

The solution to $x^2 - 6x + 5 \ge 0$ is $\{x \mid x \le 1 \text{ or } x \ge 5\}$ or $(-\infty, 1] \cup [5, \infty)$.

Notice that $x = 1$ and $x = 5$ are the boundaries of the solution set to the inequalities in Example 1. These values are the solutions to the related equation $x^2 - 6x + 5 = 0$.

Definition of Boundary Points

The **boundary points** of an inequality consist of the real solutions to the related equation and the points where the inequality is undefined.

Recall that testing points on intervals bounded by these points is the basis of the **test point method** to solve inequalities.

Solving Inequalities by Using the Test Point Method

Step 1 Find the boundary points of the inequality.

Step 2 Plot the boundary points on the number line. This divides the number line into intervals.

Step 3 Select a test point from each interval and substitute it into the original inequality.

- If a test point makes the original inequality true, then that interval is part of the solution set.

Step 4 Test the boundary points in the original inequality.

- If the original inequality is strict ($<$ or $>$), do not include the boundary points in the solution set.
- If the original inequality is defined using $\le$ or $\ge$, then include the boundary points that are defined within the inequality.

Note: Any boundary point that makes an expression within the inequality undefined must *always* be excluded from the solution set.

TIP: Recall that we also used the test point method to solve absolute value inequalities.

Example 2 Solving a Quadratic Inequality by Using the Test Point Method

Solve the inequality by using the test point method. $2x^2 + 5x < 12$

Solution:

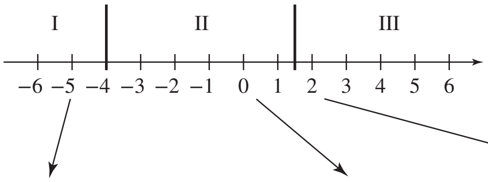
$2x^2 + 5x < 12$ **Step 1:** Find the boundary points. Because polynomials are defined for all values of x, the only boundary points are the real solutions to the related equation.

$2x^2 + 5x = 12$ Solve the related equation.

$2x^2 + 5x - 12 = 0$

$(2x - 3)(x + 4) = 0$

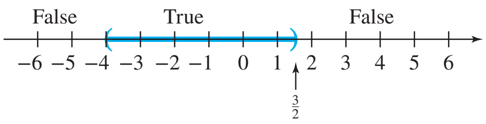
$x = \dfrac{3}{2} \qquad x = -4$ The boundary points are $\frac{3}{2}$ and -4.



Step 2: Plot the boundary points.

Step 3: Select a test point from each interval.

Test $x = -5$	**Test $x = 0$**	**Test $x = 2$**
$2x^2 + 5x < 12$	$2x^2 + 5x < 12$	$2x^2 + 5x < 12$
$2(-5)^2 + 5(-5) \overset{?}{<} 12$	$2(0)^2 + 5(0) \overset{?}{<} 12$	$2(2)^2 + 5(2) \overset{?}{<} 12$
$50 - 25 \overset{?}{<} 12$	$0 + 0 \overset{?}{<} 12$	$8 + 10 \overset{?}{<} 12$
$25 \overset{?}{<} 12$ False	$0 \overset{?}{<} 12$ True	$18 \overset{?}{<} 12$ False



Step 4: Test the boundary points. The strict inequality excludes values of x for which $2x^2 + 5x = 12$. Therefore, the boundary points are *not* included in the solution set.

The solution set is $\{x \mid -4 < x < \frac{3}{2}\}$ or equivalently in interval notation $(-4, \frac{3}{2})$.

Skill Practice Solve the inequality.

3. $x^2 + x > 6$

Example 3 Solving a Polynomial Inequality by Using the Test Point Method

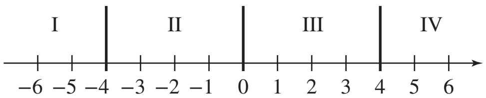
Solve the inequality by using the test point method. $x(x + 4)^2(x - 4) \ge 0$

Solution:

$x(x + 4)^2(x - 4) \ge 0$

$x(x + 4)^2(x - 4) = 0$ **Step 1:** Find the boundary points.

$x = 0 \qquad x = -4 \qquad x = 4$



Step 2: Plot the boundary points.

Step 3: Select a test point from each interval.

Test $x = -5$: $-5(-5 + 4)^2(-5 - 4) \overset{?}{\ge} 0$ $\qquad 45 \overset{?}{\ge} 0$ True

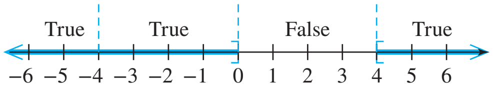
Test $x = -1$: $-1(-1 + 4)^2(-1 - 4) \overset{?}{\ge} 0$ $\qquad 45 \overset{?}{\ge} 0$ True

Test $x = 1$: $1(1 + 4)^2(1 - 4) \overset{?}{\ge} 0$ $\qquad -75 \overset{?}{\ge} 0$ False

Test $x = 5$: $5(5 + 4)^2(5 - 4) \overset{?}{\ge} 0$ $\qquad 405 \overset{?}{\ge} 0$ True

Answer

3. $(-\infty, -3) \cup (2, \infty)$



Step 4: The inequality symbol, $\geq$, includes equality. Therefore, include the boundary points in the solution set.

The solution set is $\{x \mid x \leq 0 \text{ and } x \geq 4\}$, or equivalently in interval notation, $(-\infty, 0] \cup [4, \infty)$.

Skill Practice Solve the inequality.

4. $t(t-5)(t+2)^2 \geq 0$

TIP: In Example 3, one side of the inequality is factored, and the other side is zero. For inequalities written in this form, we can use a sign chart to determine the sign of each factor. Then the sign of the product (bottom row) is easily determined.

Sign of x	−	−	+	+
Sign of $(x+4)^2$	+	+	+	+
Sign of $(x-4)$	−	−	−	+
Sign of $x(x+4)^2(x-4)$	+	+	−	+
	−4	0	4	

The solution to the inequality $x(x+4)^2(x-4) \geq 0$ includes the intervals for which the product is positive (shown in blue).

The solution is $(-\infty, 0] \cup [4, \infty)$.

Example 4 Solving a Polynomial Inequality by Using the Test Point Method

Solve the inequality by using the test point method. $x^2 + x - 4 \geq 0$

Solution:

$$x^2 + x - 4 \geq 0$$

$$x^2 + x - 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(-4)}}{2(1)}$$

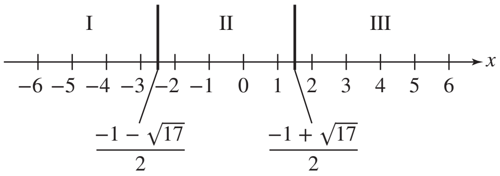
$$x = \frac{-1 \pm \sqrt{17}}{2}$$

$$x = \frac{-1 - \sqrt{17}}{2} \approx -2.56 \quad \text{and} \quad x = \frac{-1 + \sqrt{17}}{2} \approx 1.56$$

Step 1: Find the boundary points in the related equation. Since this equation is not factorable, use the quadratic formula to find the solutions.

Answer

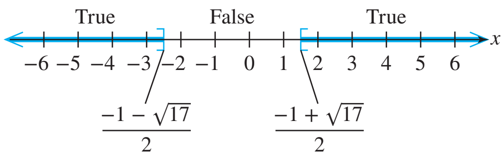
4. $(-\infty, 0] \cup [5, \infty)$



Step 2: Plot the boundary points.

Step 3: Select a test point from each interval.

Test $x = -3$	**Test $x = 0$**	**Test $x = 2$**
$x^2 + x - 4 \ge 0$	$x^2 + x - 4 \ge 0$	$x^2 + x - 4 \ge 0$
$(-3)^2 + (-3) - 4 \stackrel{?}{\ge} 0$	$(0)^2 + (0) - 4 \stackrel{?}{\ge} 0$	$(2)^2 + (2) - 4 \stackrel{?}{\ge} 0$
$2 \stackrel{?}{\ge} 0$ True	$-4 \stackrel{?}{\ge} 0$ False	$2 \stackrel{?}{\ge} 0$ True



Step 4: Test the boundary points. Both boundary points make the inequality true. Therefore, both boundary points are included in the solution set.

The solution set is $\left\{x \mid x \le \frac{-1-\sqrt{17}}{2} \text{ or } x \ge \frac{-1+\sqrt{17}}{2}\right\}$ or equivalently in interval notation: $\left(-\infty, \frac{-1-\sqrt{17}}{2}\right] \cup \left[\frac{-1+\sqrt{17}}{2}, \infty\right)$.

Skill Practice Solve the inequality.

5. $x^2 - 3x - 1 \le 0$

2. Solving Rational Inequalities

The test point method can be used to solve rational inequalities. A **rational inequality** is an inequality in which one or more terms is a rational expression. The solution set to a rational inequality must exclude all values of the variable that make the inequality undefined. That is, exclude all values that make the denominator equal to zero for any rational expression in the inequality.

Example 5 Solving a Rational Inequality by Using the Test Point Method

Solve the inequality. $\frac{3}{x-1} > 0$

Solution:

$$\frac{3}{x-1} > 0$$

Step 1: Find the boundary points. Note that the inequality is undefined for $x = 1$, so $x = 1$ is a boundary point. To find any other boundary points, solve the related equation.

$$\frac{3}{x-1} = 0$$

$$\cancel{(x-1)} \cdot \left(\frac{3}{\cancel{x-1}}\right) = (x-1) \cdot 0$$

Clear fractions.

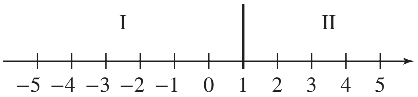
$$3 = 0$$

There is no solution to the related equation.

The only boundary point is $x = 1$.

Answer

5. $\left[\frac{3-\sqrt{13}}{2}, \frac{3+\sqrt{13}}{2}\right]$



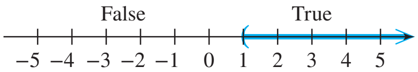
Step 2: Plot boundary points.

Test $x = 0$ **Test $x = 2$**

Step 3: Select test points.

$\dfrac{3}{(0) - 1} \overset{?}{>} 0$ $\dfrac{3}{(2) - 1} \overset{?}{>} 0$

$\dfrac{3}{-1} \overset{?}{>} 0$ False $\dfrac{3}{1} > 0$ True



Step 4: The boundary point $x = 1$ cannot be included in the solution set because it is undefined in the original inequality.

The solution set is $\{x \mid x > 1\}$ or equivalently in interval notation, $(1, \infty)$.

Skill Practice Solve the inequality.

6. $\dfrac{-5}{y + 2} < 0$

TIP: Using a sign chart we see that the quotient of the factors 3 and $(x - 1)$ is positive on the interval $(1, \infty)$.

Therefore, the solution to the inequality $\dfrac{3}{x - 1} > 0$ is $(1, \infty)$.

Sign of 3	+	+
Sign of $(x - 1)$	−	+
Sign of $\dfrac{3}{x - 1}$	−	+
	1 (undefined)	

The solution to the inequality $\dfrac{3}{x - 1} > 0$ can be confirmed from the graph of the related rational function, $f(x) = \dfrac{3}{x - 1}$ (see Figure 7-20).

- The graph is above the x-axis where $f(x) = \dfrac{3}{x - 1} > 0$ for $x > 1$(shaded red).
- Also note that $x = 1$ cannot be included in the solution set because 1 is not in the domain of f.

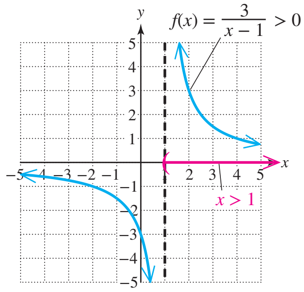


Figure 7-20

Example 6 Solving a Rational Inequality by Using the Test Point Method

Solve the inequality by using the test point method. $\dfrac{x + 2}{x - 4} \leq 3$

Answer

6. $(-2, \infty)$

Solution:

$$\frac{x+2}{x-4} \le 3$$

Step 1: Find the boundary points. Note that the inequality is undefined for $x = 4$. Therefore, $x = 4$ is automatically a boundary point. To find any other boundary points, solve the related equation.

$$\frac{x+2}{x-4} = 3$$

$$\cancel{(x-4)}\left(\frac{x+2}{\cancel{x-4}}\right) = (x-4)(3)$$ Clear fractions.

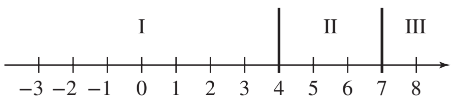
$$x + 2 = 3(x - 4)$$ Solve for x.

$$x + 2 = 3x - 12$$

$$-2x = -14$$

$$x = 7$$

The solution to the related equation is $x = 7$, and the inequality is undefined for $x = 4$. Therefore, the boundary points are $x = 4$ and $x = 7$.



Step 2: Plot boundary points.

Step 3: Select test points.

Test $x = 0$

$$\frac{x+2}{x-4} \le 3$$

$$\frac{0+2}{0-4} \stackrel{?}{\le} 3$$

$$-\frac{1}{2} \stackrel{?}{\le} 3 \quad \text{True}$$

Test $x = 5$

$$\frac{x+2}{x-4} \le 3$$

$$\frac{5+2}{5-4} \stackrel{?}{\le} 3$$

$$\frac{7}{1} \stackrel{?}{\le} 3 \quad \text{False}$$

Test $x = 8$

$$\frac{x+2}{x-4} \le 3$$

$$\frac{8+2}{8-4} \stackrel{?}{\le} 3$$

$$\frac{10}{4} \stackrel{?}{\le} 3$$

$$\frac{5}{2} \stackrel{?}{\le} 3 \quad \text{True}$$

Test $x = 4$

$$\frac{x+2}{x-4} \le 3$$

$$\frac{4+2}{4-4} \stackrel{?}{\le} 3$$

$$\frac{6}{0} \stackrel{?}{\le} 3 \quad \text{Undefined}$$

Test $x = 7$

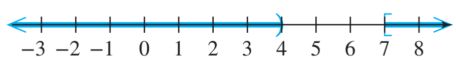
$$\frac{x+2}{x-4} \le 3$$

$$\frac{7+2}{7-4} \stackrel{?}{\le} 3$$

$$\frac{9}{3} \stackrel{?}{\le} 3 \quad \text{True}$$

Step 4: Test the boundary points.

The boundary point $x = 4$ cannot be included in the solution set, because it is undefined in the inequality. The boundary point $x = 7$ makes the original inequality true and must be included in the solution set.



The solution is $\{x \mid x < 4 \text{ or } x \ge 7\}$, or equivalently in interval notation, $(-\infty, 4) \cup [7, \infty)$.

Skill Practice Solve the inequality.

7. $\dfrac{x-5}{x+4} \le -1$

Answer

7. $\left(-4, \frac{1}{2}\right]$

3. Inequalities with "Special Case" Solution Sets

The solution to an inequality is often one or more intervals on the real number line. Sometimes, however, the solution to an inequality may be a single point on the number line, the empty set, or the set of all real numbers.

Example 7 Solving Inequalities

Solve the inequalities.

a. $x^2 + 6x + 9 \geq 0$ **b.** $x^2 + 6x + 9 > 0$

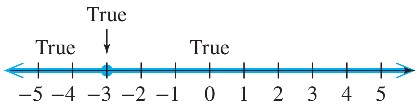
c. $x^2 + 6x + 9 \leq 0$ **d.** $x^2 + 6x + 9 < 0$

Solution:

a. $x^2 + 6x + 9 \geq 0$ Notice that $x^2 + 6x + 9$ is a perfect square trinomial.

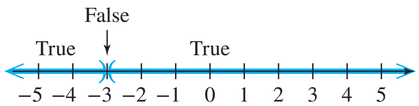
$(x + 3)^2 \geq 0$ Factor $x^2 + 6x + 9 = (x + 3)^2$.

The quantity $(x + 3)^2$ is a perfect square and is greater than or equal to zero for all real numbers x. The solution set is all real numbers, $(-\infty, \infty)$.



b. $x^2 + 6x + 9 > 0$

$(x + 3)^2 > 0$ This is the same inequality as in part (a) with the exception that the inequality is strict. The solution set does not include the point where $x^2 + 6x + 9 = 0$. Therefore, the boundary point $x = -3$ is *not* included in the solution set.

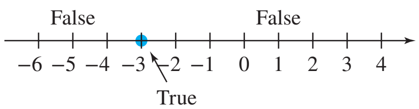


The solution set is $\{x \mid x \neq -3\}$ or equivalently $(-\infty, -3) \cup (-3, \infty)$.

c. $x^2 + 6x + 9 \leq 0$

$(x + 3)^2 \leq 0$

A perfect square cannot be less than zero. However, $(x + 3)^2$ is equal to zero at $x = -3$. Therefore, the solution set is $\{-3\}$.



d. $x^2 + 6x + 9 < 0$

$(x + 3)^2 < 0$

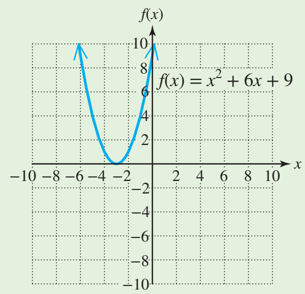
A perfect square cannot be negative; therefore, there are no real numbers x such that $(x + 3)^2 < 0$. There is no solution, $\{\ \}$.

Skill Practice Solve the inequalities.

8. $x^2 - 4x + 4 \geq 0$ **9.** $x^2 - 4x + 4 > 0$

10. $x^2 - 4x + 4 \leq 0$ **11.** $x^2 - 4x + 4 < 0$

TIP: The graph of $f(x) = x^2 + 6x + 9$, or equivalently $f(x) = (x + 3)^2$, is equal to zero at $x = -3$ and positive (above the x-axis) for all other values of x in its domain.



Answers

8. All real numbers; $(-\infty, \infty)$

9. $(-\infty, 2) \cup (2, \infty)$

10. $\{2\}$

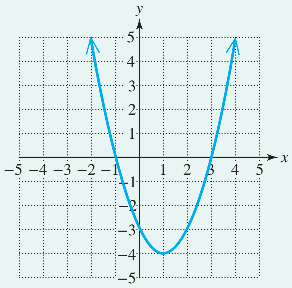
11. $\{\ \}$

Section 7.6 Activity

A.1. **a.** Consider the expression $x^2 - 2x - 3$. For different values of x, the expression may be zero, positive, or negative. To show this, evaluate the expression for $x = -1$, $x = 4$, and $x = 1$.

b. Consider the function defined by $f(x) = x^2 - 2x - 3$. For different values of x in the domain, the function may be zero, positive, or negative. To show this, find $f(-1)$, $f(4)$, and $f(1)$.

c. The graph of $f(x) = x^2 - 2x - 3$ is shown. For what value(s) of x is the function equal to zero? (These are the values of x that are solutions to $x^2 - 2x - 3 = 0$).



d. Refer to the graph of $f(x) = x^2 - 2x - 3$. For what values of x is the function negative (below the x-axis)? These are the values of x that are solutions to the inequality $x^2 - 2x - 3 < 0$. (*Hint:* There are infinitely many values of x that make the function negative. Express your answer in interval notation.)

e. For what values of x is the function positive (above the x-axis)? These are the values of x that are solutions to the inequality $x^2 - 2x - 3 > 0$. (*Hint:* The function is positive over two intervals of x. Write the answer as the union of two intervals.)

A.2. We will now walk through an algebraic approach to solve the quadratic inequality $x^2 - 2x - 3 > 0$.

a. First solve the related *equation* $x^2 - 2x - 3 = 0$.

b. Plot the solutions to the equation $x^2 - 2x - 3 = 0$ on the number line to divide the number line into intervals.

c. Test an arbitrary point from each interval on the number line to determine if it makes the inequality $x^2 - 2x - 3 > 0$ true or false. Which intervals make the inequality true?

d. Test the boundary points in the inequality $x^2 - 2x - 3 > 0$. In this case, the inequality is strict. Therefore, the boundary points (values for which $x^2 - 2x - 3$ *equals* zero) should be (choose one: included/excluded) from the solution set.

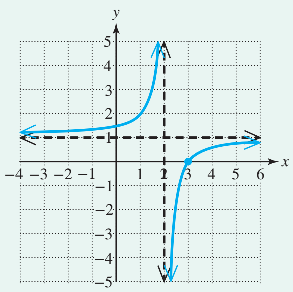
e. Write the solution set in interval notation.

f. Based on the outcomes of the test points in part (c), write the solution set to the inequality $x^2 - 2x - 3 < 0$.

g. Compare the algebraic process to solve the inequality $x^2 - 2x - 3 > 0$ with the graphical approach taken in Exercise A.1.

A.3. Solve the rational inequality $\dfrac{x-3}{x-2} \geq 0$ algebraically by following these steps. As you work through the process, you can check your answers by referring to the graph of $f(x) = \dfrac{x-3}{x-2}$.

a. First solve the related *equation* $\dfrac{x-3}{x-2} = 0$.



b. Identify value(s) of x for which $\dfrac{x-3}{x-2}$ is undefined. Can this value be a solution to the original inequality?

c. Plot the solutions to the equation $\dfrac{x-3}{x-2} = 0$ on the number line along with any values of x for which the expression is undefined. These are the boundary points that divide the number line into intervals.

d. Test an arbitrary point from each interval on the number line to determine if it makes the inequality $\dfrac{x-3}{x-2} \geq 0$ true or false. Which interval(s) make the inequality true?

e. The expression $\dfrac{x-3}{x-2}$ equals zero for $x = 3$. Therefore, is 3 a solution to the inequality $\dfrac{x-3}{x-2} \geq 0$?

f. Write the solution set to the inequality $\dfrac{x-3}{x-2} \geq 0$ in interval notation.

g. How would the solution set change for the inequality $\dfrac{x-3}{x-2} > 0$?

h. Write the solution set to the inequality $\dfrac{x-3}{x-2} \leq 0$.

A.4. Consider the expression $x^2 - 6x + 9$.

a. Write the expression in factored form.

b. Write the inequality $x^2 - 6x + 9 < 0$ with the left side in factored form. Is it possible for a perfect square to be negative?

c. Write the solution set to the inequality $x^2 - 6x + 9 < 0$ and explain your answer.

d. How does the solution set to the inequality $x^2 - 6x + 9 \leq 0$ differ from the result of part (c)?

e. Write the solution set to $x^2 - 6x + 9 \geq 0$.

f. Write the solution set to $x^2 - 6x + 9 > 0$.

Practice Exercises Section 7.6

Prerequisite Review

For Exercises R.1–R.8, solve the equation.

R.1. $x^2 + 9x + 14 = 0$

R.2. $y^2 - 19y + 18 = 0$

R.3. $4t^2 + 4t + 1 = 0$

R.4. $9p^2 - 30p + 25 = 0$

R.5. $2x(x - 6) = x - 6$

R.6. $3x(x + 3) = 2x + 6$

R.7. $\dfrac{x}{2x-5} = 1$

R.8. $\dfrac{4y}{3y+14} = 2$

R.9. $2t^2 - 3t + 4 = 0$

R.10. $3w^2 - w + 5 = 0$

Vocabulary and Key Concepts

1. An inequality of the form $ax^2 + bx + c > 0$ $(a \neq 0)$ is an example of a __________ inequality.

2. The boundary points of an inequality consist of the real __________ to the related equation and the points where the inequality is __________.

3. In solving an inequality by using the __________ __________ method, a point is selected from each interval formed by the boundary points and substituted into the original inequality.

4. If a test point makes the original inequality (true/false) then that interval is part of the solution set.

5. The inequality $\dfrac{4}{x+7} > 0$ is an example of a __________ inequality.

6. The solution set to a rational inequality must exclude all values that make the __________ equal to zero for any rational expression in the original inequality.

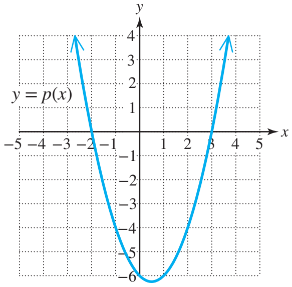
7. The solution set to the inequality $(x + 3)^2 \leq -4$ is __________, whereas the solution set to the inequality $(x + 3)^2 \geq -4$ is __________.

8. The solution set to the inequality $(x + 3)^2 > 0$ (includes/excludes) -3, whereas the solution set to the inequality $(x + 3)^2 \leq 0$ (includes/excludes) -3.

Concept 1: Solving Quadratic and Polynomial Inequalities

For Exercises 9–12, estimate from the graph the intervals for which the inequality is true. **(See Example 1.)**

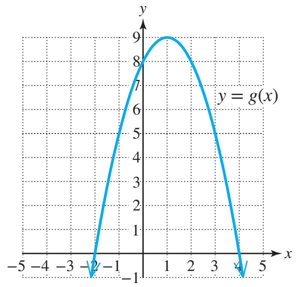
9.



a. $p(x) > 0$ **b.** $p(x) < 0$

c. $p(x) \leq 0$ **d.** $p(x) \geq 0$

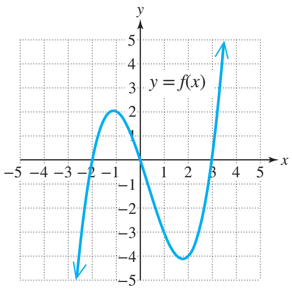
10.



a. $g(x) > 0$ **b.** $g(x) < 0$

c. $g(x) \leq 0$ **d.** $g(x) \geq 0$

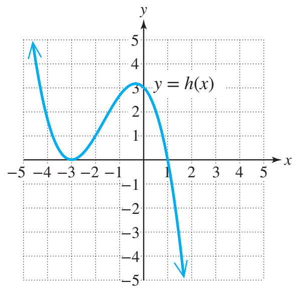
11.



a. $f(x) > 0$ **b.** $f(x) < 0$

c. $f(x) \leq 0$ **d.** $f(x) \geq 0$

12.



a. $h(x) > 0$ **b.** $h(x) < 0$

c. $h(x) \leq 0$ **d.** $h(x) \geq 0$

For Exercises 13–18, solve the equation and related inequalities. **(See Examples 2–4.)**

13. **a.** $3(4 - x)(2x + 1) = 0$
b. $3(4 - x)(2x + 1) < 0$
c. $3(4 - x)(2x + 1) > 0$

14. **a.** $5(y + 6)(3 - 5y) = 0$
b. $5(y + 6)(3 - 5y) < 0$
c. $5(y + 6)(3 - 5y) > 0$

15. **a.** $x^2 + 7x = 30$
b. $x^2 + 7x < 30$
c. $x^2 + 7x > 30$

16. **a.** $q^2 - 4q = 5$
b. $q^2 - 4q \leq 5$
c. $q^2 - 4q \geq 5$

17. **a.** $2p(p - 2) = p + 3$
b. $2p(p - 2) \leq p + 3$
c. $2p(p - 2) \geq p + 3$

18. **a.** $3w(w + 4) = 10 - w$
b. $3w(w + 4) < 10 - w$
c. $3w(w + 4) > 10 - w$

For Exercises 19–38, solve the polynomial inequality. Write the answer in interval notation. **(See Examples 2–4.)**

19. $(t - 7)(t - 1) < 0$

20. $(p - 4)(p - 2) > 0$

21. $-6(4 + 2x)(5 - x) > 0$

22. $-8(2t + 5)(6 - t) < 0$

23. $m(m + 1)^2(m + 5) \leq 0$

24. $w^2(3 - w)(w + 2) \geq 0$

25. $a^2 - 12a \leq -32$

26. $w^2 + 20w \geq -64$

27. $5x^2 - 2x - 1 > 0$

28. $3x^2 + 2x - 4 < 0$

29. $x^2 + 3x \leq 6$

30. $x^2 - 8 \leq 3x$

31. $b^2 - 121 < 0$

32. $c^2 - 25 < 0$

33. $3p(p - 2) - 3 \geq 2p$

34. $2t(t + 3) - t \leq 12$

35. $x^3 - x^2 \leq 12x$

36. $x^3 + 36 > 4x^2 + 9x$

37. $w^3 + w^2 > 4w + 4$

38. $2p^3 - 5p^2 \leq 3p$

Concept 2: Solving Rational Inequalities

For Exercises 39–42, solve the equation and related inequalities. **(See Examples 5–6.)**

39. **a.** $\dfrac{10}{x - 5} = 5$ **b.** $\dfrac{10}{x - 5} < 5$ **c.** $\dfrac{10}{x - 5} > 5$

40. **a.** $\dfrac{8}{a + 1} = 4$ **b.** $\dfrac{8}{a + 1} > 4$ **c.** $\dfrac{8}{a + 1} < 4$

41. **a.** $\dfrac{z + 2}{z - 6} = -3$ **b.** $\dfrac{z + 2}{z - 6} \leq -3$ **c.** $\dfrac{z + 2}{z - 6} \geq -3$

42. **a.** $\dfrac{w - 8}{w + 6} = 2$ **b.** $\dfrac{w - 8}{w + 6} \leq 2$ **c.** $\dfrac{w - 8}{w + 6} \geq 2$

For Exercises 43–54, solve the rational inequality. Write the answer in interval notation. **(See Examples 5–6.)**

43. $\dfrac{2}{x - 1} \geq 0$

44. $\dfrac{-3}{x + 2} \leq 0$

45. $\dfrac{b + 4}{b - 4} > 0$

46. $\dfrac{a + 1}{a - 3} < 0$

47. $\dfrac{3}{2x - 7} < -1$

48. $\dfrac{8}{4x + 9} > 1$

49. $\dfrac{x + 1}{x - 5} \geq 4$

50. $\dfrac{x - 2}{x + 6} \leq 5$

51. $\dfrac{1}{x} \leq 2$

52. $\dfrac{1}{x} \geq 3$

53. $\dfrac{(x + 2)^2}{x} > 0$

54. $\dfrac{(x - 3)^2}{x} < 0$

Concept 3: Inequalities with "Special Case" Solution Sets

For Exercises 55–70, solve the inequalities. **(See Example 7.)**

55. $x^2 + 10x + 25 \geq 0$

56. $x^2 + 6x + 9 < 0$

57. $x^2 + 2x + 1 < 0$

58. $x^2 + 8x + 16 \geq 0$

59. $x^4 + 3x^2 \leq 0$

60. $x^4 + 2x^2 \leq 0$

61. $x^2 + 12x + 36 < 0$

62. $x^2 + 12x + 36 \geq 0$

63. $x^2 + 3x + 5 < 0$

64. $2x^2 + 3x + 3 > 0$

65. $-5x^2 + x < 1$

66. $-3x^2 - x > 6$

67. $x^2 + 22x + 121 > 0$

68. $y^2 - 24y + 144 > 0$

69. $4t^2 - 12t \leq -9$

70. $9y^2 - 30y \leq -25$

Mixed Exercises

For Exercises 71–94, identify the inequality as one of the following types: linear, quadratic, rational, or polynomial (degree > 2). Then solve the inequality and write the answer in interval notation.

71. $2y^2 - 8 \leq 24$

72. $8p^2 - 18 > 0$

73. $(5x + 2)^2 > -4$

74. $(3 - 7x)^2 < -1$

75. $4(x - 2) < 6x - 3$

76. $-7(3 - y) > 4 + 2y$

77. $\dfrac{2x + 3}{x + 1} \leq 2$

78. $\dfrac{5x - 1}{x + 3} \geq 5$

79. $4x^3 - 40x^2 + 100x > 0$

80. $2y^3 - 12y^2 + 18y < 0$

81. $2p^3 > 4p^2$

82. $w^3 \leq 5w^2$

83. $-3(x+4)^2(x-5) \geq 0$

84. $5x(x-2)(x-6)^2 \geq 0$

85. $x^2 - 2 < 0$

86. $y^2 - 3 > 0$

87. $x^2 + 5x - 2 \geq 0$

88. $t^2 + 7t + 3 \leq 0$

89. $\dfrac{a+2}{a-5} \geq 0$

90. $\dfrac{t+1}{t-2} \leq 0$

91. $2 \geq t - 3$

92. $-5p + 8 < p$

93. $4x^2 + 2 \geq -x$

94. $5x - 4 \geq 2x^2$

Expanding Your Skills

95. A fireworks display is planned for the 4th of July. The fire department wants to make sure that all fireworks explode at least 120 ft above the ground. The staging platform is 8 ft high, and all the mortar shells will be launched at 108 ft/sec. The height of the mortars $h(t)$ (in feet), can be modeled by

$$h(t) = -16t^2 + 108t + 8 \qquad \text{where } t \text{ is the time in seconds after launch.}$$

For what interval of time should the fuses be set? Round to the nearest tenth of a second.

Problem Recognition Exercises

Recognizing Equations and Inequalities

At this point, you have learned how to solve a variety of equations and inequalities. Being able to distinguish the type of problem being posed is the first step in successfully solving it.

For Exercises 1–20,

a. Identify the problem type. Choose from

- linear equation
- quadratic equation
- rational equation
- absolute value equation
- radical equation
- equation quadratic in form
- polynomial equation
- linear inequality
- polynomial inequality
- rational inequality
- absolute value inequality
- compound inequality

b. Solve the equation or inequality. Write the solution to each inequality in interval notation if possible.

1. $(z^2 - 4)^2 - (z^2 - 4) - 12 = 0$

2. $3 + |4t - 1| < 6$

3. $2y(y - 4) \leq 5 + y$

4. $\sqrt[3]{11x - 3} + 4 = 6$

5. $-5 = -|w - 4|$

6. $\dfrac{5}{x-2} + \dfrac{3}{x+2} = 1$

7. $m^3 + 5m^2 - 4m - 20 \geq 0$

8. $-x - 4 > -5$ and $2x - 3 \leq 23$

9. $5 - 2[3 - (x - 4)] \leq 3x + 14$

10. $|2x - 6| = |x + 3|$

11. $\dfrac{3}{x-2} \leq 1$

12. $9 < |x + 4|$

13. $\sqrt{t+8} - 6 = t$

14. $(4x - 3)^2 = -10$

15. $-4 - x > 2$ or $8 < 2x$

16. $\dfrac{1}{3}x - 2 = \dfrac{3}{4} + \dfrac{5}{6}x$

17. $x^2 - 10x \leq -25$

18. $\dfrac{10}{x^2 + 1} < 0$

19. $x - 13\sqrt{x} + 36 = 0$

20. $x^4 - 13x^2 + 36 = 0$

Chapter 7 Summary

Section 7.1 Square Root Property and Completing the Square

Key Concepts

The **square root property** states that

If $x^2 = k$ then $x = \pm\sqrt{k}$

Follow these steps to solve a quadratic equation in the form $ax^2 + bx + c = 0$ $(a \neq 0)$ by completing the square and applying the square root property:

1. Divide both sides by a to make the leading coefficient 1.
2. Isolate the variable terms on one side of the equation.
3. Complete the square: Add the square of one-half the linear term coefficient to both sides of the equation. Then factor the resulting perfect square trinomial.
4. Apply the square root property and solve for x.

Examples

Example 1

$$(x - 5)^2 = -13$$

$$x - 5 = \pm\sqrt{-13} \qquad \text{(square root property)}$$

$$x = 5 \pm i\sqrt{13}$$

The solution set is $\{5 \pm i\sqrt{13}\}$.

Example 2

$$2x^2 - 12x - 16 = 0$$

$$\frac{2x^2}{2} - \frac{12x}{2} - \frac{16}{2} = \frac{0}{2}$$

$$x^2 - 6x - 8 = 0$$

$$x^2 - 6x = 8$$

$$x^2 - 6x + 9 = 8 + 9 \qquad \textit{Note: } [\tfrac{1}{2}(-6)]^2 = 9$$

$$(x - 3)^2 = 17$$

$$x - 3 = \pm\sqrt{17}$$

$$x = 3 \pm \sqrt{17}$$

The solution set is $\{3 \pm \sqrt{17}\}$.

Section 7.2 Quadratic Formula

Key Concepts

The solutions to a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **discriminant** of a quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$. If a, b, and c are rational numbers, then

1. If $b^2 - 4ac > 0$, then there will be two real solutions. Moreover,
   a. If $b^2 - 4ac$ is a perfect square, the solutions will be rational numbers.
   b. If $b^2 - 4ac$ is not a perfect square, the solutions will be irrational numbers.
2. If $b^2 - 4ac < 0$, then there will be two imaginary solutions.
3. If $b^2 - 4ac = 0$, then there will be one rational solution.

Three methods to solve a quadratic equation are

1. Factoring and applying the zero product rule.
2. Completing the square and applying the square root property.
3. Using the quadratic formula.

Example

Example 1

$$3x^2 - 2x + 4 = 0$$

$a = 3 \qquad b = -2 \qquad c = 4$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(4)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 - 48}}{6}$$

$$= \frac{2 \pm \sqrt{-44}}{6}$$

The discriminant is -44. Therefore, there will be two imaginary solutions.

$$= \frac{2 \pm 2i\sqrt{11}}{6}$$

$$= \frac{\overset{1}{\cancel{2}}(1 \pm i\sqrt{11})}{\underset{3}{\cancel{6}}}$$

$$= \frac{1 \pm i\sqrt{11}}{3}$$

$$= \frac{1}{3} \pm \frac{\sqrt{11}}{3}i$$

The solution set is $\left\{\frac{1}{3} \pm \frac{\sqrt{11}}{3}i\right\}$.

Section 7.3 Equations in Quadratic Form

Key Concepts

Substitution can be used to solve equations that are in quadratic form.

Example

Example 1

$x^{2/3} - x^{1/3} - 12 = 0 \qquad$ Let $u = x^{1/3}$.

$$u^2 - u - 12 = 0$$

$$(u - 4)(u + 3) = 0$$

$u = 4 \qquad$ or $\qquad u = -3$

$x^{1/3} = 4 \qquad$ or $\qquad x^{1/3} = -3$

$x = 64 \qquad$ or $\qquad x = -27 \qquad$ Cube both sides.

The solution set is $\{64, -27\}$.

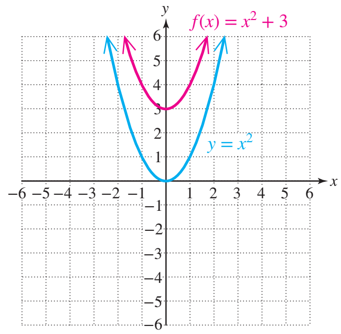
Section 7.4 Graphs of Quadratic Functions

Key Concepts

A quadratic function of the form $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ up k units if $k > 0$ and down $|k|$ units if $k < 0$.

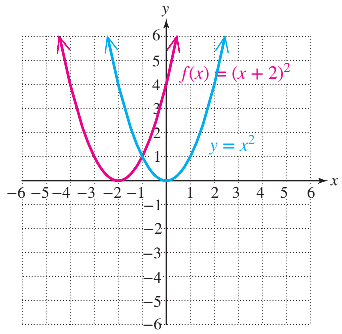
Examples

Example 1



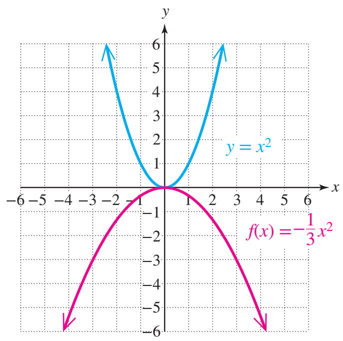
A quadratic function of the form $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ to the right h units if $h > 0$ and to the left $|h|$ units if $h < 0$.

Example 2



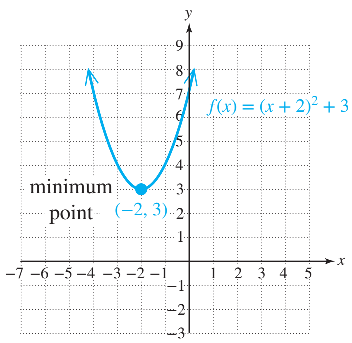
The graph of a quadratic function of the form $f(x) = ax^2$ is a parabola that opens upward when $a > 0$ and opens downward when $a < 0$. If $|a| > 1$, the graph of $f(x) = x^2$ is stretched vertically by a factor of $|a|$. If $0 < |a| < 1$, the graph of $f(x) = x^2$ is shrunk vertically by a factor of $|a|$.

Example 3



A quadratic function of the form $f(x) = a(x - h)^2 + k$ has vertex (h, k). If $a > 0$, the vertex represents the minimum point. If $a < 0$, the vertex represents the maximum point.

Example 4



Section 7.5 Vertex of a Parabola: Applications and Modeling

Key Concepts

Completing the square is a technique used to write a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in the form $f(x) = a(x - h)^2 + k$ for the purpose of identifying the vertex (h, k).

Examples

Example 1

$$\begin{aligned} f(x) &= 3x^2 + 6x + 11 \\ &= 3(x^2 + 2x \qquad\quad) + 11 \\ &= 3(x^2 + 2x + 1 - 1) + 11 \\ &= 3(x^2 + 2x + 1) - 3 + 11 \\ &= 3(x + 1)^2 + 8 \\ &= 3[x - (-1)]^2 + 8 \end{aligned}$$

The vertex is $(-1, 8)$. Because $a = 3 > 0$, the parabola opens upward and the vertex $(-1, 8)$ is a minimum point.

The **vertex formula** finds the vertex of a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$.

The vertex is

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) \quad \text{or} \quad \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

Example 2

$f(x) = 3x^2 + 6x + 11$

$a = 3 \qquad b = 6 \qquad c = 11$

$$x = \frac{-6}{2(3)} = -1$$

$f(-1) = 3(-1)^2 + 6(-1) + 11 = 8$

The vertex is $(-1, 8)$.

Section 7.6 Polynomial and Rational Inequalities

Key Concepts

The Test Point Method to Solve Polynomial and Rational Inequalities

1. Find the boundary points of the inequality. (Boundary points are the real solutions to the related equation and points where the inequality is undefined.)
2. Plot the boundary points on the number line. This divides the number line into intervals.
3. Select a test point from each interval and substitute it into the original inequality.
   - If a test point makes the original inequality true, then that interval is part of the solution set.
4. Test the boundary points in the original inequality.
   - If the original inequality is strict ($<$ or $>$), do not include the boundary points in the solution set.
   - If the original inequality is defined using $\le$ or $\ge$, then include the boundary points that are defined within the inequality.

Note: Any boundary point that makes an expression within the inequality undefined must *always* be excluded from the solution set.

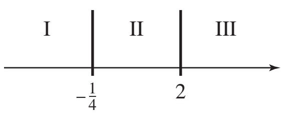
Example 1

$4x^2 - 7x - 2 < 0$

$(4x + 1)(x - 2) = 0$ Related equation

$4x + 1 = 0 \quad x - 2 = 0$

$x = -\dfrac{1}{4} \quad x = 2$ The boundaries are $-\dfrac{1}{4}$ and 2.

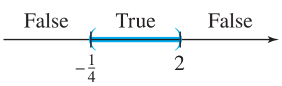


Interval I: Test $x = -1$: $4(-1)^2 - 7(-1) - 2 \overset{?}{<} 0$ False

Interval II: Test $x = 0$: $4(0)^2 - 7(0) - 2 \overset{?}{<} 0$ True

Interval III: Test $x = 3$: $4(3)^2 - 7(3) - 2 \overset{?}{<} 0$ False

The strict inequality excludes the boundary points as solutions.



The solution is $\left(-\dfrac{1}{4}, 2\right)$.

Examples

Example 2

$\dfrac{28}{2x - 3} \le 4$ The inequality is undefined for $x = \frac{3}{2}$. Find other possible boundary points by solving the related equation.

$\dfrac{28}{2x - 3} = 4$ Related equation

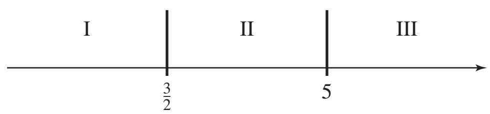
$$(2x - 3) \cdot \frac{28}{2x - 3} = (2x - 3) \cdot 4$$

$$28 = 8x - 12$$

$$40 = 8x$$

$$x = 5$$

The boundaries are $x = \dfrac{3}{2}$ and $x = 5$.



Interval I: Test $x = 1$: $\dfrac{28}{2(1) - 3} \overset{?}{\le} 4$ True

Interval II: Test $x = 2$: $\dfrac{28}{2(2) - 3} \overset{?}{\le} 4$ False

Interval III: Test $x = 6$: $\dfrac{28}{2(6) - 3} \overset{?}{\le} 4$ True

The boundary point $x = \frac{3}{2}$ is not included because $\dfrac{28}{2x - 3}$ is undefined there. The boundary $x = 5$ does check in the original inequality.

The solution is $(-\infty, \frac{3}{2}) \cup [5, \infty)$.

Chapter 7 Review Exercises

Section 7.1

For Exercises 1–8, solve the equations by using the square root property.

1. $x^2 = 5$

2. $2y^2 = -8$

3. $a^2 = 81$

4. $3b^2 = -19$

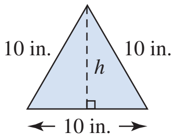
5. $(x - 2)^2 = 72$

6. $(2x - 5)^2 = -9$

7. $(3y - 1)^2 = 3$

8. $3(m - 4)^2 = 15$

9. The length of each side of an equilateral triangle is 10 in. Find the exact height of the triangle. Then round to the nearest tenth of an inch.



10. Use the square root property to find the length of the sides of a square whose area is 81 in^2.

11. Use the square root property to find the exact length of the sides of a square whose area is 150 in^2. Then round to the nearest tenth of an inch.

For Exercises 12–15, find the value of n so that the expression is a perfect square trinomial. Then factor the trinomial.

12. $x^2 + 16x + n$

13. $x^2 - 9x + n$

14. $y^2 + \frac{1}{2}y + n$

15. $z^2 - \frac{2}{5}z + n$

For Exercises 16–21, solve the equation by completing the square and applying the square root property.

16. $w^2 + 4w + 13 = 0$

17. $4y^2 - 8y - 20 = 0$

18. $2x^2 = 12x + 6$

19. $-t^2 + 8t - 25 = 0$

20. $3x^2 + 2x = 1$

21. $b^2 + \frac{7}{2}b = 2$

22. Solve for r. $V = \pi r^2 h \quad (r > 0)$

23. Solve for s. $A = 6s^2 \quad (s > 0)$

Section 7.2

24. Describe the type and number of solutions to a quadratic equation whose discriminant is less than zero.

For Exercises 25–30, determine the type (rational, irrational, or imaginary) and number of solutions for the equations by using the discriminant.

25. $x^2 - 5x = -6$

26. $2y^2 = -3y$

27. $z^2 + 23 = 17z$

28. $a^2 + a + 1 = 0$

29. $10b + 1 = -25b^2$

30. $3x^2 + 15 = 0$

For Exercises 31–38, solve the equations by using the quadratic formula.

31. $y^2 - 4y + 1 = 0$

32. $m^2 - 5m + 25 = 0$

33. $6a(a - 1) = 10 + a$

34. $3x(x - 3) = x - 8$

35. $b^2 - \frac{4}{25} = \frac{3}{5}b$

36. $k^2 + 0.4k = 0.05$

37. $-32 + 4x - x^2 = 0$

38. $8y - y^2 = 10$

For Exercises 39–42, solve using any method.

39. $3x^2 - 4x = 6$

40. $\frac{w}{8} - \frac{2}{w} = \frac{3}{4}$

41. $y^2 + 14y = -46$

42. $(a + 1)^2 = 11$

43. The landing distance that a certain plane will travel on a runway is determined by the initial landing speed at the instant the plane touches down. The function D relates landing distance in feet to initial landing speed s:

$$D(s) = \frac{1}{10}s^2 - 3s + 22 \text{ for } s \geq 50$$

where s is in feet per second.

a. Find the landing distance for a plane traveling 150 ft/sec at touchdown.

b. If the landing speed is too fast, the pilot may run out of runway. If the speed is too slow, the plane may stall. Find the maximum initial landing speed of a plane for a runway that is 1000 ft long. Round to one decimal place.

44. The recent population of Kenya (in thousands) can be approximated by $P(t) = 4.62t^2 + 564.6t + 13{,}128$ where t is the number of years since 1974.

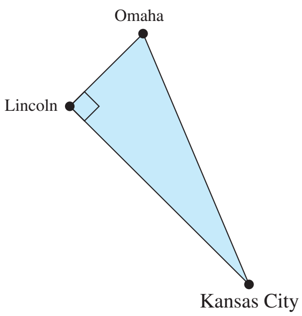
a. If this trend continues, predict the number of people in Kenya for the year 2025.

b. In what year after 1974 will the population of Kenya reach 50 million? (*Hint:* 50 million equals 50,000 thousand.)

45. A custom-built kitchen island is in the shape of a rectangle. The length is 1 ft more than twice the width. If the area is 22.32 ft^2, determine the dimensions of the island. Round the length and width to the nearest tenth of a foot.

Monkey Business Images/Shutterstock

46. Lincoln, Nebraska, Kansas City, Missouri, and Omaha, Nebraska, form the vertices of a right triangle. The distance between Lincoln and Kansas City is 10 mi more than 3 times the distance between Lincoln and Omaha. If the distance from Omaha and Kansas City is 167 mi, find the distance between Lincoln and Omaha. Round to the nearest mile.



Section 7.3

For Exercises 47–56, solve the equations.

47. $x - 4\sqrt{x} - 21 = 0$

48. $n - 6\sqrt{n} + 8 = 0$

49. $y^4 - 11y^2 + 18 = 0$

50. $2m^4 - m^2 - 3 = 0$

51. $t^{2/5} + t^{1/5} - 6 = 0$

52. $p^{2/5} - 3p^{1/5} + 2 = 0$

53. $\dfrac{2t}{t+1} + \dfrac{-3}{t-2} = 1$

54. $\dfrac{1}{m-2} - \dfrac{m}{m+3} = 2$

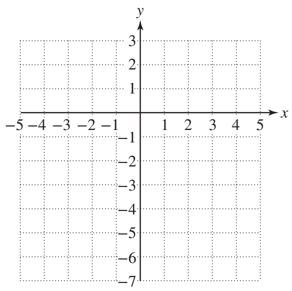
55. $(x^2 + 5)^2 + 2(x^2 + 5) - 8 = 0$

56. $(x^2 - 3)^2 - 5(x^2 - 3) + 4 = 0$

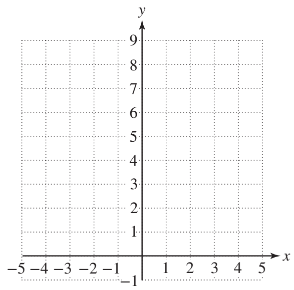
Section 7.4

For Exercises 57–64, graph the function and write the domain and range in interval notation.

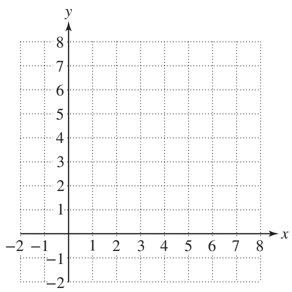
57. $g(x) = x^2 - 5$



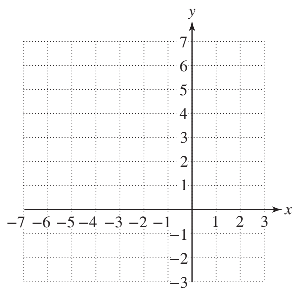
58. $f(x) = x^2 + 3$



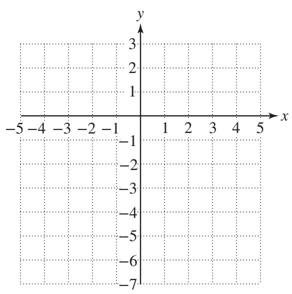
59. $h(x) = (x - 5)^2$



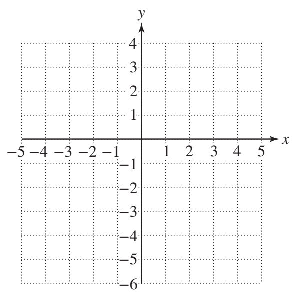
60. $k(x) = (x + 3)^2$



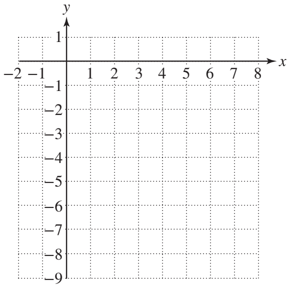
61. $m(x) = -2x^2$



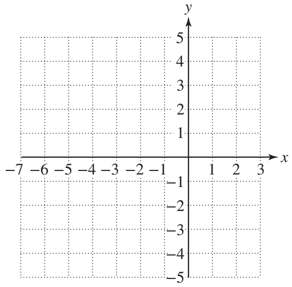
62. $n(x) = -4x^2$



63. $p(x) = -2(x - 5)^2 - 5$



64. $q(x) = -4(x + 3)^2 + 3$



For Exercises 65–66, write the coordinates of the vertex of the parabola and determine if the vertex is a maximum point or a minimum point. Then write the maximum or the minimum value.

65. $t(x) = \frac{1}{3}(x - 4)^2 + \frac{5}{3}$

66. $s(x) = -\frac{5}{7}(x - 1)^2 - \frac{1}{7}$

For Exercises 67–68, write the equation of the axis of symmetry of the parabola.

67. $a(x) = -\frac{3}{2}\left(x + \frac{2}{11}\right)^2 - \frac{4}{13}$

68. $w(x) = -\frac{4}{3}\left(x - \frac{3}{16}\right)^2 + \frac{2}{9}$

Section 7.5

For Exercises 69–72, write the function in the form $f(x) = a(x - h)^2 + k$ by completing the square. Then write the coordinates of the vertex.

69. $z(x) = x^2 - 6x + 7$

70. $b(x) = x^2 - 4x - 44$

71. $p(x) = -5x^2 - 10x - 13$

72. $q(x) = -3x^2 - 24x - 54$

For Exercises 73–76, find the coordinates of the vertex of each parabola by using the vertex formula.

73. $f(x) = -2x^2 + 4x - 17$

74. $g(x) = -4x^2 - 8x + 3$

75. $m(x) = 3x^2 - 3x + 11$

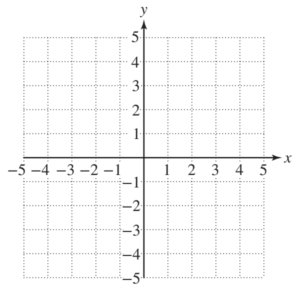
76. $n(x) = 3x^2 + 2x - 7$

77. For the quadratic equation $y = \frac{3}{4}x^2 - 3x$,

a. Write the coordinates of the vertex.

b. Find the x- and y-intercepts.

c. Use this information to sketch a graph of the parabola.

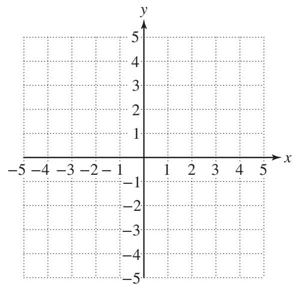


78. For the quadratic equation $y = -(x + 2)^2 + 4$,

a. Write the coordinates of the vertex.

b. Find the x- and y-intercepts.

c. Use this information to sketch a graph of the parabola.



79. The height $h(t)$ (in feet) of a projectile fired vertically into the air from the ground is given by the equation $h(t) = -16t^2 + 96t$, where t represents the number of seconds after launch.

a. How long will it take the projectile to reach its maximum height?

b. What is the maximum height?

80. The weekly profit, $P(x)$ (in dollars), for a catering service is given by $P(x) = -0.053x^2 + 15.9x + 7.5$. In this context, x is the number of meals prepared.

a. Find the number of meals that should be prepared to obtain the maximum profit.

b. What is the maximum profit?

81. Write an equation of a parabola that passes through the points $(-3, -4)$, $(-2, -5)$, and $(1, 4)$.

82. Write an equation of a parabola that passes through the points $(4, 18)$, $(-2, 12)$, and $(-1, 8)$.

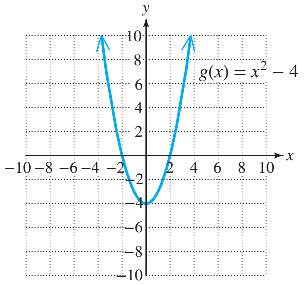
Section 7.6

83. Solve the equation and inequalities. How do your answers to parts (a), (b), and (c) relate to the graph of $g(x) = x^2 - 4$?

a. $x^2 - 4 = 0$

b. $x^2 - 4 < 0$

c. $x^2 - 4 > 0$



84. **a.** For what values of x is $\dfrac{4x}{x-2}$ undefined?

b. Solve the equation. $\dfrac{4x}{x-2} = 0$

c. Solve the inequality. $\dfrac{4x}{x-2} \geq 0$

d. Solve the inequality. $\dfrac{4x}{x-2} \leq 0$

For Exercises 85–96, solve the inequalities. Write the answers in interval notation.

85. $w^2 - 4w - 12 < 0$

86. $t^2 + 6t + 9 \geq 0$

87. $\dfrac{12}{x+2} \leq 6$

88. $\dfrac{8}{p-1} \geq -4$

89. $3y(y-5)(y+2) > 0$

90. $-3c(c+2)(2c-5) < 0$

91. $-x^2 - 4x \leq 1$

92. $y^2 + 4y > 5$

93. $\dfrac{w+1}{w-3} > 1$

94. $\dfrac{2a}{a+3} \leq 2$

95. $t^2 + 10t + 25 \leq 0$

96. $-x^2 - 4x < 4$

Chapter 7 Test

For Exercises 1–3, solve the equation by using the square root property.

1. $(x+3)^2 = 25$

2. $(p-2)^2 = 12$

3. $(m+1)^2 = -1$

4. Find the value of n so that the expression is a perfect square trinomial. Then factor the trinomial $d^2 + 11d + n$.

For Exercises 5–6, solve the equation by completing the square and applying the square root property.

5. $2x^2 + 12x - 36 = 0$

6. $2x^2 = 3x - 7$

For Exercises 7–8,

a. Write the equation in standard form $ax^2 + bx + c = 0$.

b. Identify a, b, and c.

c. Find the discriminant.

d. Determine the number and type (rational, irrational, or imaginary) of solutions.

7. $x^2 - 3x = -12$

8. $y(y-2) = -1$

For Exercises 9–10, solve the equation by using the quadratic formula.

9. $3x^2 - 4x + 1 = 0$

10. $x(x+6) = -11 - x$

11. The base of a triangle is 3 ft less than twice the height. The area of the triangle is 14 ft^2. Find the base and the height. Round the answers to the nearest tenth of a foot.

12. A circular garden has an area of approximately 450 ft^2. Find the radius. Round the answer to the nearest tenth of a foot.

For Exercises 13–21, solve the equation.

13. $x - \sqrt{x} - 6 = 0$

14. $y^{2/3} + 2y^{1/3} = 8$

15. $(3y-8)^2 - 13(3y-8) + 30 = 0$

16. $p^4 - 15p^2 = -54$

17. $3 = \frac{y}{2} - \frac{1}{y+1}$

18. $2x^2 - 9x = 5$

19. $x^2 - 8x + 1 = 0$

20. $(x + 7)^2 = -24$

21. $x(x - 12) = -13$

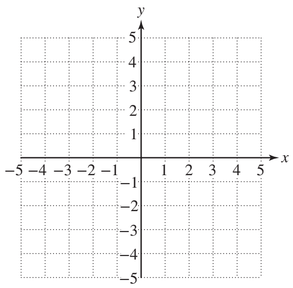
22. Find the vertex of $y = x^2 - 6x - 8$ two ways,

a. By completing the square.

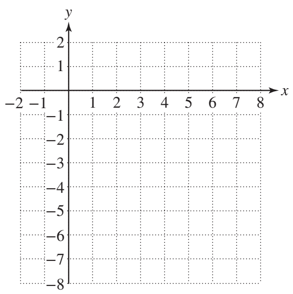
b. By using the vertex formula.

For Exercises 23–25, graph the function. Use the graph to write the domain and range in interval notation.

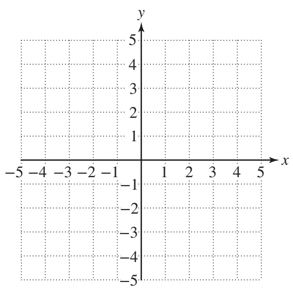
23. $h(x) = x^2 - 4$



24. $f(x) = -(x - 4)^2$

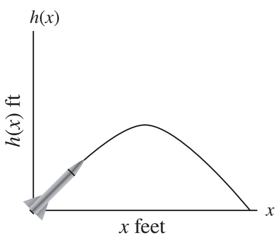


25. $g(x) = \frac{1}{2}(x + 2)^2 - 3$



26. A child launches a toy rocket from the ground. The height of the rocket can be determined by its horizontal distance from the launch pad x by

$$h(x) = -\frac{x^2}{256} + x$$



where x and $h(x)$ are in feet. How many feet from the launch pad will the rocket hit the ground?

27. The recent population of India (in millions) can be approximated by $P(t) = 0.135t^2 + 12.6t + 600$, where $t = 0$ corresponds to the year 1974.

a. Use the function to estimate the number of people in India in the year 2014.

b. Approximate the year in which the population of India reached 1 billion (1000 million). (Round to the nearest year.)

28. Explain the relationship between the graphs of $y = x^2$ and $y = x^2 - 2$.

29. Explain the relationship between the graphs of $y = x^2$ and $y = (x + 3)^2$.

30. Explain the relationship between the graphs of $y = 4x^2$ and $y = -4x^2$.

31. Given the function defined by

$$f(x) = -(x - 4)^2 + 2$$

a. Identify the vertex of the parabola.

b. Does this parabola open upward or downward?

c. Does the vertex represent the maximum or minimum point of the function?

d. What is the maximum or minimum value of the function f?

e. Write an equation for the axis of symmetry.

32. For the function defined by $g(x) = 2x^2 - 20x + 51$, find the vertex by using two methods.

a. Complete the square to write $g(x)$ in the form $g(x) = a(x - h)^2 + k$. Identify the vertex.

b. Use the vertex formula to find the vertex.

33. Given $f(x) = x^2 + 4x - 12$

a. Write the equation for the function in the form $f(x) = a(x - h)^2 + k$.

b. Determine the vertex.

c. Find the x- and y-intercepts.

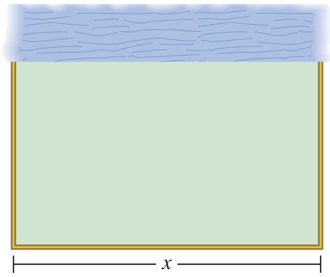
d. Determine the maximum or minimum value.

e. Write an equation for the axis of symmetry.

34. A farmer has 400 ft of fencing with which to enclose a rectangular field. The field is situated such that one of its sides is adjacent to a river and requires no fencing. The area of the field (in square feet) can be modeled by

$$A(x) = -\frac{x^2}{2} + 200x$$

where x is the length of the side parallel to the river (measured in feet).



a. Determine the value of x that maximizes the area of the field.

b. Determine the maximum area that can be enclosed.

For Exercises 35–40, solve the inequalities.

35. $\dfrac{2x - 1}{x - 6} \le 0$

36. $50 - 2a^2 > 0$

37. $y^3 + 3y^2 - 4y - 12 < 0$

38. $\dfrac{3}{w + 3} > 2$

39. $5x^2 - 2x + 2 < 0$

40. $t^2 + 22t + 121 \le 0$

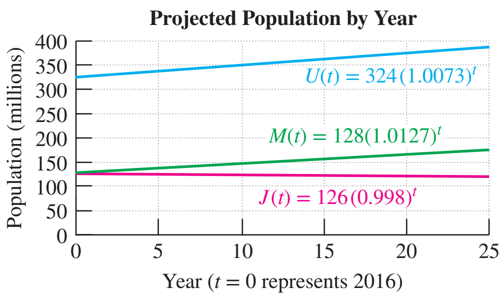
Exponential and Logarithmic Functions and Applications

8

CHAPTER OUTLINE

Mathematics and Population Growth

Over short periods of time, human population growth is proportional to the number of humans in the population at a given time. That is, the more people in a population, the faster the population will grow. For example, a country with 10 million people will typically have a greater yearly increase in population than a country with only 1 million people. However, other variables such as food supply, mortality rates, birth control, and immigration also affect population growth.



The graph shows the projected population growth for the United States, Mexico, and Japan based on growth rates of 0.73%, 1.27%, and −0.2%, respectively. Since Japan has a negative growth rate, its population is *decreasing*.

The functions U, M, and J that describe the population values versus time are called **exponential functions**. Exponential functions are characterized by having a constant base and variable exponent. In this chapter, we study exponential functions and their inverses, **logarithmic functions**. These functions have far-reaching applications, including the study of population growth, the growth of investments, and radioactive decay.

Section 8.1 Algebra of Functions and Composition

Concepts

1. Algebra of Functions
2. Composition of Functions
3. Operations on Functions

1. Algebra of Functions

Addition, subtraction, multiplication, and division can be used to create a new function from two or more functions. The domain of the new function will be the intersection of the domains of the original functions.

Sum, Difference, Product, and Quotient of Functions

Given two functions f and g, the functions $f+g$, $f-g$, $f \cdot g$, and $\frac{f}{g}$ are defined as

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{provided } g(x) \neq 0$$

For example, suppose $f(x) = |x|$ and $g(x) = 3$. Taking the sum of the functions produces a new function denoted by $f + g$. In this case, $(f+g)(x) = |x| + 3$. Graphically, the y values of the function $f + g$ are given by the sum of the corresponding y values of f and g. This is depicted in Figure 8-1. The function $f + g$ appears in red. In particular, notice that $(f+g)(2) = f(2) + g(2) = 2 + 3 = 5$.

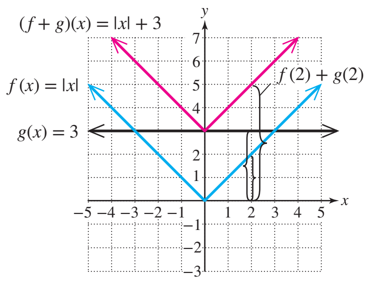


Figure 8-1

Example 1 Adding, Subtracting, Multiplying and Dividing Functions

Given: $g(x) = 4x$ $\quad h(x) = x^2 - 3x$ $\quad k(x) = x - 2$

a. Find $(g + h)(x)$. **b.** Find $(h - g)(x)$.

c. Find $(g \cdot k)(x)$. **d.** Find $\left(\frac{k}{h}\right)(x)$.

Solution:

a. $(g + h)(x) = g(x) + h(x)$

$= (4x) + (x^2 - 3x)$

$= 4x + x^2 - 3x$

$= x^2 + x$ Simplify.

b. $(h - g)(x) = h(x) - g(x)$

$= (x^2 - 3x) - (4x)$

$= x^2 - 3x - 4x$

$= x^2 - 7x$ Simplify.

c. $(g \cdot k)(x) = g(x) \cdot k(x)$

$= (4x)(x - 2)$

$= 4x^2 - 8x$ Simplify.

d. $\left(\frac{k}{h}\right)(x) = \frac{k(x)}{h(x)}$

$= \frac{x - 2}{x^2 - 3x}$ From the denominator we have $x^2 - 3x \neq 0$ or, equivalently, $x(x - 3) \neq 0$. Hence, $x \neq 3$ and $x \neq 0$.

Skill Practice Given $f(x) = x - 1$, $g(x) = 5x^2 + x$, and $h(x) = x^2$, find

1. $(f + g)(x)$ **2.** $(g - f)(x)$ **3.** $(g \cdot h)(x)$ **4.** $\left(\frac{f}{h}\right)(x)$

2. Composition of Functions

Composition of Functions

The **composition** of f and g, denoted $f \circ g$, is defined by the rule

$(f \circ g)(x) = f(g(x))$ provided that $g(x)$ is in the domain of f

Note: $f \circ g$ is read as "f of g" or "f compose g."

For example, given $f(x) = 2x - 3$ and $g(x) = x + 5$, we have

$(f \circ g)(x) = f(g(x))$

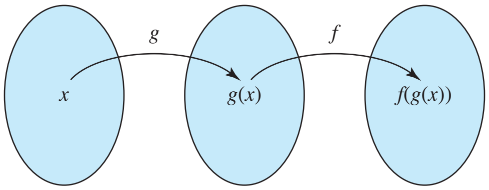
$= f(x + 5)$ Substitute $g(x) = x + 5$ into the function f.

$= 2(x + 5) - 3$

$= 2x + 10 - 3$

$= 2x + 7$

In this composition, the function g is the innermost operation and acts on x first. Then the output value of function g becomes the domain element of the function f, as shown in the figure.



Example 2 Composing Functions

Given: $f(x) = x - 5$, $g(x) = x^2$, and $n(x) = \sqrt{x + 2}$, find

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

c. $(n \circ f)(x)$

Answers

1. $5x^2 + 2x - 1$ **2.** $5x^2 + 1$

3. $5x^4 + x^3$ **4.** $\frac{x - 1}{x^2}, x \neq 0$

Solution:

a. $(f \circ g)(x) = f(g(x))$

$= f(x^2)$ Evaluate the function f at x^2.

$= (x^2) - 5$ Replace x with x^2 in function f.

$= x^2 - 5$

TIP: Examples 2(a) and 2(b) illustrate that the order in which two functions are composed may result in different functions. That is, $f \circ g$ does not necessarily equal $g \circ f$.

b. $(g \circ f)(x) = g(f(x))$

$= g(x - 5)$ Evaluate the function g at $(x - 5)$.

$= (x - 5)^2$ Replace x with $(x - 5)$ in function g.

$= x^2 - 10x + 25$ Simplify.

FOR REVIEW

Recall that to square a binomial $(a - b)^2$ we have:

$$(a - b)^2 = a^2 - 2ab + b^2.$$

Thus,

$$(x - 5)^2 = x^2 - 2(x)(5) + (5)^2 = x^2 - 10x + 25$$

c. $(n \circ f)(x) = n(f(x))$

$= n(x - 5)$ Evaluate the function n at $x - 5$.

$= \sqrt{(x - 5) + 2}$ Replace x with the quantity $(x - 5)$ in function n.

$= \sqrt{x - 3}$

Skill Practice Given $f(x) = 2x^2$, $g(x) = x + 3$, and $h(x) = \sqrt{x - 1}$, find

5. $(f \circ g)(x)$ **6.** $(g \circ f)(x)$ **7.** $(h \circ g)(x)$

3. Operations on Functions

Example 3 Combining Functions

Given the functions defined by $f(x) = x - 7$ and $h(x) = 2x^3$, find the function values, if possible.

a. $(f \cdot h)(3)$ **b.** $\left(\frac{h}{f}\right)(7)$ **c.** $(h \circ f)(2)$

Solution:

a. $(f \cdot h)(3) = f(3) \cdot h(3)$ $(f \cdot h)(3)$ is a product (not a composition).

$= (3 - 7) \cdot 2(3)^3$

$= (-4) \cdot 2(27)$

$= -216$

b. $\left(\frac{h}{f}\right)(7) = \frac{h(7)}{f(7)} = \frac{2(7)^3}{7 - 7}$

The value $x = 7$ will make the denominator equal to 0. Therefore, $\left(\frac{h}{f}\right)(7)$ is undefined.

c. $(h \circ f)(2) = h(f(2))$ Evaluate $f(2)$ first. $f(2) = 2 - 7 = -5$

$= h(-5)$ Substitute the result into function h.

$= 2(-5)^3$

$= 2(-125)$

$= -250$

Answers

5. $2x^2 + 12x + 18$ **6.** $2x^2 + 3$

7. $\sqrt{x + 2}$

Skill Practice Given $h(x) = x + 4$ and $k(x) = x^2 - 3$, find

8. $(h \cdot k)(-2)$ **9.** $\left(\frac{k}{h}\right)(-4)$ **10.** $(k \circ h)(1)$

Example 4 Finding Function Values From a Graph

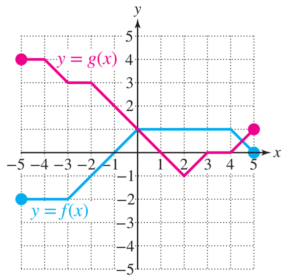
For the functions f and g pictured, find the function values if possible.

a. $g(2)$

b. $(f - g)(-3)$

c. $\left(\frac{g}{f}\right)(5)$

d. $(f \circ g)(4)$



Solution:

a. $g(2) = -1$ — The value $g(2)$ represents the y value of $y = g(x)$ (the red graph) when $x = 2$. Because the point $(2, -1)$ lies on the graph, $g(2) = -1$.

b. $(f - g)(-3) = f(-3) - g(-3)$ — Evaluate the difference of $f(-3)$ and $g(-3)$.

$= -2 - (3)$ — Estimate function values from the graph.

$= -5$

c. $\left(\frac{g}{f}\right)(5) = \frac{g(5)}{f(5)}$ — Evaluate the quotient of $g(5)$ and $f(5)$.

$= \frac{1}{0}$ (undefined)

The function $\frac{g}{f}$ is undefined at 5 because the denominator is zero.

d. $(f \circ g)(4) = f(g(4))$ — From the red graph, find the value of $g(4)$ first. We see that $g(4) = 0$.

$= f(0)$ — From the blue graph, find the value of f at $x = 0$.

$= 1$

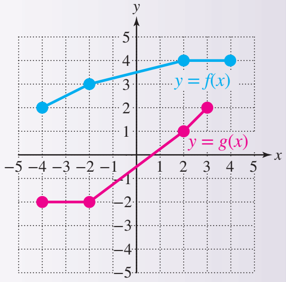
Skill Practice Find the values from the graph.

11. $g(3)$

12. $(f + g)(-4)$

13. $\left(\frac{f}{g}\right)(2)$

14. $(g \circ f)(-2)$

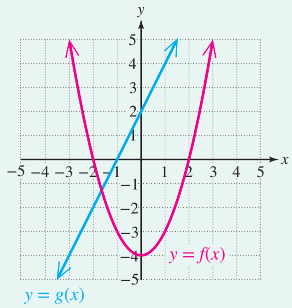


Answers

8. 2 **9.** undefined **10.** 22
11. 2 **12.** 0 **13.** 4 **14.** 2

Section 8.1 Activity

A.1. Consider the graphs of f and g given in the figure.



a. Find $f(0)$.

b. Find $g(0)$.

c. Find $f(0) + g(0)$.

d. Now suppose that a new function, $f + g$, is defined as the sum of functions f and g for all real numbers for which f and g are defined. That is, $(f + g)(x) = f(x) + g(x)$. Find $(f + g)(1)$.

e. Find $(f + g)(-2)$.

A.2. Given two functions f and g, write the definitions of the functions $f - g$, $f \cdot g$, and $\frac{f}{g}$.

a. $(f - g)(x) =$ __________

b. $(f \cdot g)(x) =$ __________

c. $\left(\frac{f}{g}\right)(x) =$ __________

A.3. Use the functions defined by $f(x) = x^2 - x - 12$ and $g(x) = x - 4$.

a. Explain the process to find $f(x - 4)$.

b. Find $f(x - 4)$.

c. Find $g(x^2 - x - 12)$.

A.4. **a.** The composition of functions f and g is denoted by $f \circ g$ and is defined by $(f \circ g)(x) =$ __________.
The composition of functions g and f is denoted by $g \circ f$ and is defined by $(g \circ f)(x) =$ __________.

b. Refer to the functions defined in Exercise A.3: $f(x) = x^2 - x - 12$ and $g(x) = x - 4$. Find $(f \circ g)(x)$.

c. Find $(g \circ f)(x)$.

A.5. Given $h(x) = 3x - 4$ and $k(x) = \frac{1}{x - 5}$, fill in the blanks to find the function value.

a. $(h \circ k)(4) = h[k(\square)] = h(\square) =$ __________

b. $(k \circ h)(0) = k[h(\square)] = k(\square) =$ __________

A.6. Given $h(x) = 3x - 4$ and $k(x) = \frac{1}{x - 5}$, find the function values if possible. If a function value does not exist, explain why.

a. $(h \circ k)(6)$

b. $(h \circ k)(5)$

c. $(k \circ h)(1)$

d. $(k \circ h)(3)$

A.7. Refer to the graphs in Exercise A.1.

a. Find $(f \cdot g)(-1)$.

b. Find $(g - f)(-3)$.

c. Find $\left(\frac{f}{g}\right)(0)$.

d. Explain why $\left(\frac{g}{f}\right)(-2)$ is undefined.

e. Find $(f \circ g)(-1)$.

f. Find $(g \circ f)(-2)$.

Section 8.1 Practice Exercises

Prerequisite Review

For Exercises R.1–R.8, perform the indicated operations.

R.1. $(3x^2 - 4x - 5) + (6x^2 + x - 3)$

R.2. $(-x^3 + 6x - 7) + (x^3 - 7x + 3)$

R.3. $(-x^4 + 2x^2 - 1) - (x^4 - 3x^2 + 5)$

R.4. $(3x^2 + 5x - 7) - (-x^2 - 4x - 1)$

R.5. $\dfrac{1}{x-3} \cdot \dfrac{4x-12}{x-1}$

R.6. $\dfrac{2x-3}{x} \cdot \dfrac{x^2}{16x-24}$

R.7. $\dfrac{x^2-25}{5-x} \div \dfrac{x^2+5x}{x^2}$

R.8. $\dfrac{3-x}{x^2} \div \dfrac{x^2-9}{2x^2+6x}$

For Exercises R.9–R.16, write the domain in interval notation.

R.9. $f(x) = 3x + 2$

R.10. $g(x) = -5x - 1$

R.11. $h(x) = \dfrac{1}{3x+2}$

R.12. $k(x) = \dfrac{x}{-5x-1}$

R.13. $m(x) = \dfrac{3-x}{x^2+9}$

R.14. $n(x) = \dfrac{2+x}{x^2+4}$

R.15. $p(x) = \sqrt{4-2x}$

R.16. $q(x) = \sqrt{8-x}$

For Exercises R.17–R.20, given $f(x) = -3x - 4$ and $g(x) = -x^2 + 4x + 6$, evaluate the functions.

R.17. $f(-4)$ **R.18.** $f(6)$ **R.19.** $g(-2)$ **R.20.** $g(-3)$

Vocabulary and Key Concepts

1. **a.** Given the functions f and g, the function $(f+g)(x) =$ ____________ + ____________.

   b. Given the functions f and g, the function $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{\square}$, provided ____________ $\neq 0$.

   c. The composition of functions f and g is defined by the rule $(f \circ g)(x) =$ ____________.

Concept 1: Algebra of Functions

2. Given $f(x) = x^2$ and $g(x) = 2x - 3$, find

   a. $f(-2)$ **b.** $g(-2)$ **c.** $(f+g)(-2)$

For Exercises 3–14, refer to the functions defined below.

$$f(x) = x + 4 \qquad g(x) = 2x^2 + 4x \qquad h(x) = x^2 + 1 \qquad k(x) = \frac{1}{x}$$

Find the indicated functions. **(See Example 1.)**

3. $(f+g)(x)$ **4.** $(f-g)(x)$ **5.** $(g-f)(x)$

6. $(f+h)(x)$ **7.** $(f \cdot h)(x)$ **8.** $(h \cdot k)(x)$

9. $(g \cdot f)(x)$ **10.** $(f \cdot k)(x)$ **11.** $\left(\dfrac{h}{f}\right)(x)$

12. $\left(\dfrac{g}{f}\right)(x)$ **13.** $\left(\dfrac{f}{g}\right)(x)$ **14.** $\left(\dfrac{f}{h}\right)(x)$

Concept 2: Composition of Functions

For Exercises 15–22, find the indicated functions. Use f, g, h, and k as defined in Exercises 3–14. **(See Example 2.)**

15. $(f \circ g)(x)$ **16.** $(f \circ k)(x)$ **17.** $(g \circ f)(x)$

18. $(k \circ f)(x)$ **19.** $(k \circ h)(x)$ **20.** $(h \circ k)(x)$

21. $(k \circ g)(x)$ **22.** $(g \circ k)(x)$

23. Based on your answers to Exercises 15 and 17, is it true in general that $(f \circ g)(x) = (g \circ f)(x)$?

24. Based on your answers to Exercises 16 and 18, is it true in general that $(f \circ k)(x) = (k \circ f)(x)$?

For Exercises 25–28, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

25. $f(x) = x^2 - 3x + 1,\ g(x) = 5x$

26. $f(x) = 3x^2 + 8,\ g(x) = 2x - 4$

27. $f(x) = |x|,\ g(x) = x^3 - 1$

28. $f(x) = \dfrac{1}{x+2},\ g(x) = |x + 2|$

29. For $h(x) = 5x - 4$, find $(h \circ h)(x)$.

30. For $k(x) = -x^2 + 1$, find $(k \circ k)(x)$.

Concept 3: Operations on Functions

For Exercises 31–46, refer to the functions defined below.

$$m(x) = x^3 \qquad n(x) = x - 3 \qquad r(x) = \sqrt{x + 4} \qquad p(x) = \frac{1}{x+2}$$

Find each function value if possible. **(See Example 3.)**

31. $(m \cdot r)(0)$

32. $(n \cdot p)(0)$

33. $(m + r)(-4)$

34. $(n - m)(4)$

35. $(r \circ n)(3)$

36. $(n \circ r)(5)$

37. $(p \circ m)(-1)$

38. $(m \circ n)(5)$

39. $(m \circ p)(2)$

40. $(r \circ m)(2)$

41. $(r + p)(-3)$

42. $(n + p)(-2)$

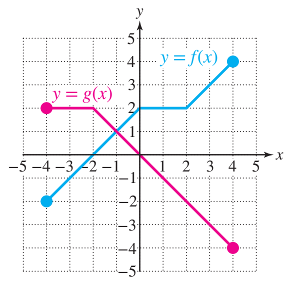
43. $(m \circ p)(-2)$

44. $(r \circ m)(-2)$

45. $\left(\dfrac{r}{n}\right)(12)$

46. $\left(\dfrac{n}{m}\right)(2)$

For Exercises 47–64, approximate each function value from the graph, if possible. **(See Example 4.)**



47. $f(-4)$

48. $f(1)$

49. $g(-2)$

50. $g(3)$

51. $(f + g)(2)$

52. $(g - f)(3)$

53. $(f \cdot g)(-1)$

54. $(g \cdot f)(-4)$

55. $\left(\dfrac{g}{f}\right)(0)$

56. $\left(\dfrac{f}{g}\right)(-2)$

57. $\left(\dfrac{f}{g}\right)(0)$

58. $\left(\dfrac{g}{f}\right)(-2)$

59. $(g \circ f)(-1)$

60. $(f \circ g)(0)$

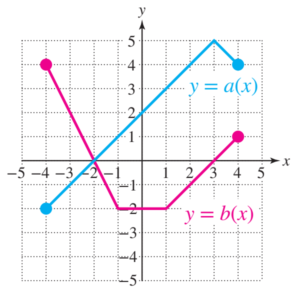
61. $(f \circ g)(-4)$

62. $(g \circ f)(-4)$

63. $(g \circ g)(2)$

64. $(f \circ f)(-2)$

For Exercises 65–80, approximate each function value from the graph, if possible. **(See Example 4.)**



65. $a(-3)$

66. $a(1)$

67. $b(-1)$

68. $b(3)$

69. $(a - b)(-1)$

70. $(a + b)(0)$

71. $(b \cdot a)(1)$

72. $(a \cdot b)(2)$

73. $(b \circ a)(0)$

74. $(a \circ b)(-2)$

75. $(a \circ b)(-4)$

76. $(b \circ a)(-3)$

77. $\left(\dfrac{b}{a}\right)(3)$

78. $\left(\dfrac{a}{b}\right)(4)$

79. $(a \circ a)(-2)$

80. $(b \circ b)(1)$

81. The cost in dollars of producing x toy cars is $C(x) = 2.2x + 1$. The revenue for selling x toy cars is $R(x) = 5.98x$. To calculate profit, subtract the cost from the revenue.

a. Write and simplify a function P that represents profit in terms of x.

b. Find the profit of producing 50 toy cars.

82. The cost in dollars of producing x lawn chairs is $C(x) = 2.5x + 10.1$. The revenue for selling x lawn chairs is $R(x) = 6.99x$. To calculate profit, subtract the cost from the revenue.

a. Write and simplify a function P that represents profit in terms of x.

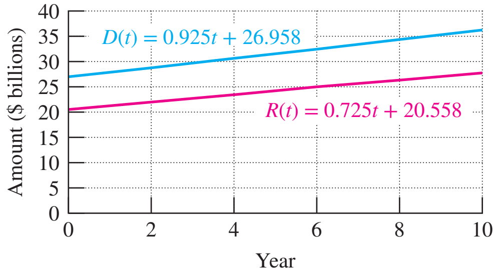
b. Find the profit in producing 100 lawn chairs.

83. The functions defined by $D(t) = 0.925t + 26.958$ and $R(t) = 0.725t + 20.558$ approximate the amount of child support (in billions of dollars) that was due $D(t)$ and the amount of child support actually received $R(t)$ in the United States for a selected number of years. The value $t = 0$ represents the first year of the study.

a. Find the function defined by $F(t) = D(t) - R(t)$. What does $F(t)$ represent in the context of this problem?

b. Find $F(4)$. What does this function value represent in the context of this problem?

Difference Between Child Support Due and Child Support Paid, United States



Source: U.S. Bureau of the Census

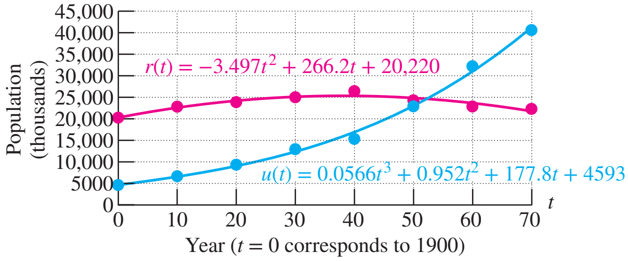
84. The rural and urban populations in the South (in the United States) between the years 1900 and 1970 can be modeled by the following:

$$r(t) = -3.497t^2 + 266.2t + 20{,}220$$

$$u(t) = 0.0566t^3 + 0.952t^2 + 177.8t + 4593$$

The variable t represents the number of years since 1900, and $0 \le t \le 70$. The variable $r(t)$ represents the rural population in thousands, and $u(t)$ represents the urban population in thousands.

Rural and Urban Populations in the South, United States, 1900–1970



Source: Historical Abstract of the United States

a. Find the function defined by $T(t) = r(t) + u(t)$. What does $T(t)$ represent in the context of this problem?

b. Use the function T to approximate the total population in the South for the year 1940.

85. Joe rides a bicycle and his wheels revolve at 80 revolutions per minute (rpm). Therefore, the total number of revolutions, r, is given by $r(t) = 80t$, where t is the time in minutes. For each revolution of the wheels of the bike, he travels approximately 7 ft. Therefore, the total distance he travels $D(r)$ (in feet) depends on the total number of revolutions, r, according to the function $D(r) = 7r$.

a. Find $(D \circ r)(t)$ and interpret its meaning in the context of this problem.

b. Find Joe's total distance in feet after 10 min.

86. The area of a square is given by the function $a(x) = x^2$, where x is the length of the sides of the square. If carpeting costs \$9.95 per square yard, then the cost $C(a)$ (in dollars) to carpet a square room is given by $C(a) = 9.95a$, where a is the area of the room in square yards.

a. Find $(C \circ a)(x)$ and interpret its meaning in the context of this problem.

b. Find the cost to carpet a square room if its floor dimensions are 15 yd by 15 yd.

Section 8.2 Inverse Functions

Concepts

1. Introduction to Inverse Functions
2. Definition of a One-to-One Function
3. Definition of the Inverse of a Function
4. Finding an Equation of the Inverse of a Function

1. Introduction to Inverse Functions

A function is a set of ordered pairs (x, y) such that for every element x in the domain, there corresponds exactly one element y in the range. For example, the function f relates the weight of a package of deli meat x to its cost y.

$$f = \{(1, 4), (2.5, 10), (4, 16)\}$$

That is, 1 lb of meat sells for \$4, 2.5 lb sells for \$10, and 4 lb sells for \$16. Now suppose we create a new function in which the values of x and y are interchanged. The new function, called the **inverse of *f***, denoted $\boldsymbol{f^{-1}}$, relates the price of meat x to its weight y.

Avoiding Mistakes

f^{-1} denotes the inverse of a function. The -1 does not represent an exponent.

$$f^{-1} = \{(4, 1), (10, 2.5), (16, 4)\}$$

Notice that interchanging the x and y values has the following outcome. The domain of f is the same as the range of f^{-1}, and the range of f is the domain of f^{-1}.

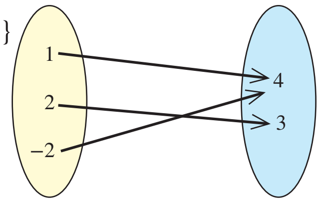
2. Definition of a One-to-One Function

A necessary condition for a function f to have an inverse function is that no two ordered pairs in f have different x-coordinates and the same y-coordinate. A function that satisfies this condition is called a **one-to-one function**. The function relating the weight of a package of meat to its price is a one-to-one function. However, consider the function g defined by

$$g = \{(1, 4), (2, 3), (-2, 4)\}$$

same y

different x



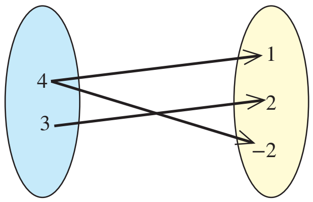
Not a one-to-one function because two different x values map to the same y value.

This function is not one-to-one because the range element 4 has two different x-coordinates, 1 and -2. Interchanging the x and y values produces a relation that violates the definition of a function.

$$\{(4, 1), (3, 2), (4, -2)\}$$

same x

different y



This relation is not a function because for $x = 4$, there are two different y values.

We have already learned that the vertical line test can be used to visually determine if a graph represents a function. Similarly, we use a **horizontal line test** to determine whether a function is one-to-one.

Using the Horizontal Line Test

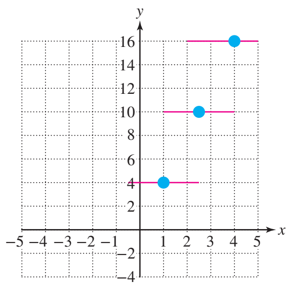
Consider a function defined by a set of points (x, y) in a rectangular coordinate system. Then y is a *one-to-one* function of x if no horizontal line intersects the graph in more than one point.

FOR REVIEW

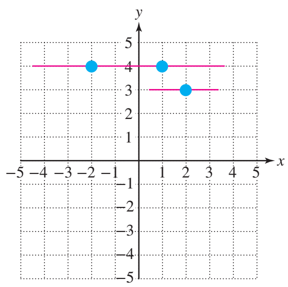
Recall that the vertical line test is used to determine if y is a function of x. If a vertical line crosses a graph in more than one point, the graph does not define y as a function of x.

To understand the horizontal line test, consider the functions f and g.

$$f = \{(1, 4), (2.5, 10), (4, 16)\} \qquad g = \{(1, 4), (2, 3), (-2, 4)\}$$



This function is one-to-one.
No horizontal line intersects more than once.

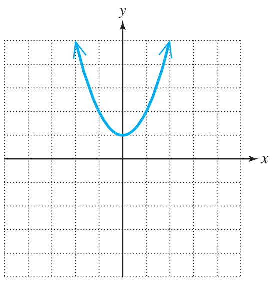


This function is *not* one-to-one.
A horizontal line intersects more than once.

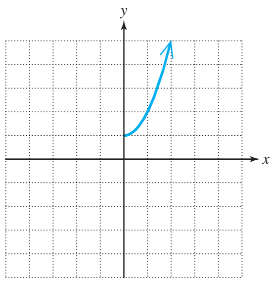
Example 1 Identifying a One-to-One Function

For each function, determine if the function is one-to-one.

a.

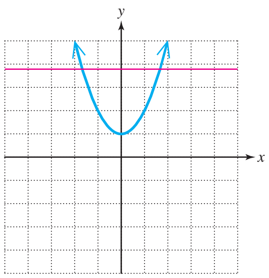


b.

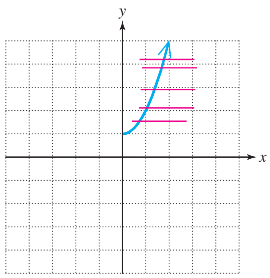


Solution:

a. Function is not one-to-one. A horizontal line intersects in more than one point.

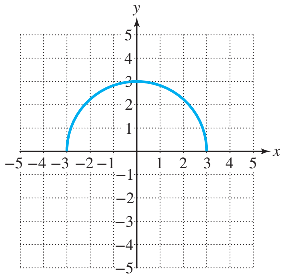


b. Function is one-to-one. No horizontal line intersects more than once.

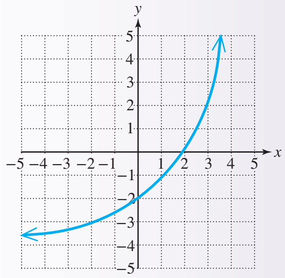


Skill Practice For each function determine if the function is one-to-one.

1.



2.



3. Definition of the Inverse of a Function

All one-to-one functions have an inverse function.

Definition of an Inverse Function

If f is a one-to-one function represented by ordered pairs of the form (x, y), then the inverse function, denoted by f^{-1}, is the set of ordered pairs given by (y, x).

Because the values of x and y are interchanged between a function f and its inverse f^{-1}, we have the following important relationship (Figure 8-2).

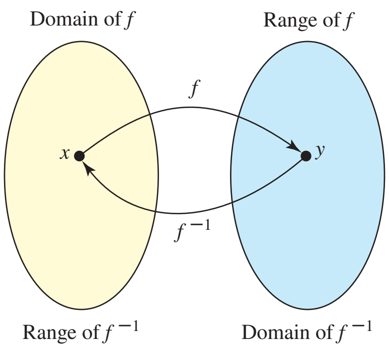


Figure 8-2

From Figure 8-2, we see that the operations performed by f are reversed by f^{-1}. This observation leads to the following important property.

Inverse Function Property

If f is a one-to-one function, then g is the inverse of f if and only if

$$(f \circ g)(x) = x \quad \text{for all } x \text{ in the domain of } g$$

and

$$(g \circ f)(x) = x \quad \text{for all } x \text{ in the domain of } f.$$

Answers

1. Not one-to-one
2. One-to-one

Example 2 **Composing a Function With Its Inverse**

Show that the functions are inverses.

$$f(x) = 5x + 4 \qquad g(x) = \frac{x-4}{5}$$

Solution:

To show that the functions f and g are inverses, confirm that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x-4}{5}\right) \\ &= \cancel{5}\left(\frac{x-4}{\cancel{5}}\right) + 4 \\ &= x - 4 + 4 \\ &= x \checkmark\end{aligned} \qquad \begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(5x+4) \\ &= \frac{(5x + \cancel{4}) - \cancel{4}}{5} \\ &= \frac{5x}{5} \\ &= x \checkmark\end{aligned}$$

Skill Practice Show that the functions are inverses.

3. $f(x) = 3x - 2$ and $g(x) = \dfrac{x+2}{3}$

FOR REVIEW

Recall the composition of functions

$$(f \circ g)(x) = f(g(x))$$

means that function f is applied to the result of function g. That is, x is input into function g. Then the result, $g(x)$, is input into function f. Similarly,

$$(g \circ f)(x) = g(f(x))$$

means that function g is applied to the result of function f.

4. Finding an Equation of the Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the inverse is a function $y = f^{-1}(x)$ that performs the inverse operations in the reverse order. For example, the function defined by $f(x) = 2x + 1$ multiplies x by 2 and then adds 1. Therefore, the inverse function must *subtract* 1 from x and *divide* by 2. We have

$$f^{-1}(x) = \frac{x-1}{2}$$ The expression $f^{-1}(x)$ is read as "f inverse of x."

To facilitate the process of finding an equation of the inverse of a one-to-one function, we offer the following steps.

Finding an Equation of an Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows:

Step 1 Replace $f(x)$ with y.

Step 2 Interchange x and y.

Step 3 Solve for y.

Step 4 Replace y with $f^{-1}(x)$.

Answer

3. $(f \circ g)(x) = f(g(x))$

$$= 3\left(\frac{x+2}{3}\right) - 2 = x$$

$(g \circ f)(x) = g(f(x))$

$$= \frac{3x - 2 + 2}{3} = x$$

Example 3 Finding an Equation of the Inverse of a Function

Find the inverse. $f(x) = 2x + 1$

Solution:

Avoiding Mistakes

It is important to check that a function is one-to-one before finding the inverse function.

Foremost, we know the graph of f is a nonvertical line. Therefore, $f(x) = 2x + 1$ defines a one-to-one function. To find the inverse we have

$y = 2x + 1$	**Step 1:**	Replace $f(x)$ with y.
$x = 2y + 1$	**Step 2:**	Interchange x and y.
$x - 1 = 2y$	**Step 3:**	Solve for y. Subtract 1 from both sides.
$\dfrac{x-1}{2} = y$		Divide both sides by 2.
$f^{-1}(x) = \dfrac{x-1}{2}$	**Step 4:**	Replace y with $f^{-1}(x)$.

Skill Practice Find the inverse.

4. $f(x) = 4x + 6$

In Example 3, we can verify that $f(x) = 2x + 1$ and $f^{-1}(x) = \dfrac{x-1}{2}$ are inverses by using function composition.

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = 2\left(\frac{x-1}{2}\right) + 1 = x \quad \checkmark \quad \text{and}$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = \frac{(2x+1)-1}{2} = x \quad \checkmark$$

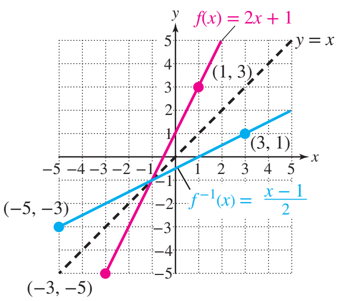


Figure 8-3

The key step in determining the equation of the inverse of a function is to interchange x and y. By so doing, a point (a, b) on f corresponds to a point (b, a) on f^{-1}. For this reason, the graphs of f and f^{-1} are symmetric with respect to the line $y = x$ (Figure 8-3). Notice that the point $(-3, -5)$ of the function f corresponds to the point $(-5, -3)$ of f^{-1}. Likewise, $(1, 3)$ of f corresponds to $(3, 1)$ of f^{-1}.

Example 4 Finding an Equation of the Inverse of a Function

Find the inverse of the one-to-one function. $g(x) = \sqrt[3]{5x} - 4$

Solution:

$y = \sqrt[3]{5x} - 4$	**Step 1:**	Replace $g(x)$ with y.
$x = \sqrt[3]{5y} - 4$	**Step 2:**	Interchange x and y.
$x + 4 = \sqrt[3]{5y}$	**Step 3:**	Solve for y. Add 4 to both sides.
$(x+4)^3 = (\sqrt[3]{5y})^3$		To eliminate the cube root, cube both sides.
$(x+4)^3 = 5y$		Simplify the right side.
$\dfrac{(x+4)^3}{5} = y$		Divide both sides with 5.
$g^{-1}(x) = \dfrac{(x+4)^3}{5}$	**Step 4:**	Replace y with $g^{-1}(x)$.

Answer

4. $f^{-1}(x) = \dfrac{x-6}{4}$

Skill Practice Find the inverse.

5. $h(x) = \sqrt[3]{2x - 1}$

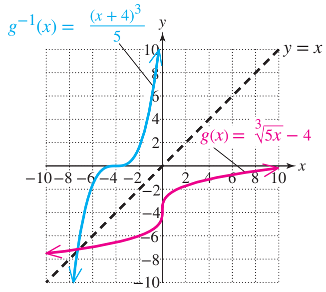


Figure 8-4

The graphs of g and g^{-1} from Example 4 are shown in Figure 8-4. Once again we see that the graphs of a function and its inverse are symmetric with respect to the line $y = x$.

For a function that is not one-to-one, sometimes we can restrict its domain to create a new function that is one-to-one. This is demonstrated in Example 5.

Example 5 Finding the Equation of an Inverse of a Function With a Restricted Domain

Given the function defined by $m(x) = x^2 + 4$ for $x \geq 0$, find an equation defining m^{-1}.

Solution:

We know that $y = x^2 + 4$ is a parabola with vertex at (0, 4) (Figure 8-5). The graph represents a function that is not one-to-one. However, with the restriction on the domain $x \geq 0$, the graph of $m(x) = x^2 + 4$, $x \geq 0$, consists of only the "right" branch of the parabola (Figure 8-6). This *is* a one-to-one function.

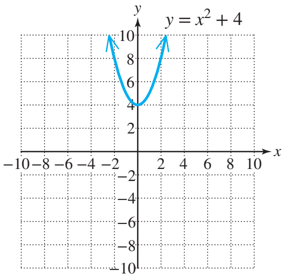


Figure 8-5

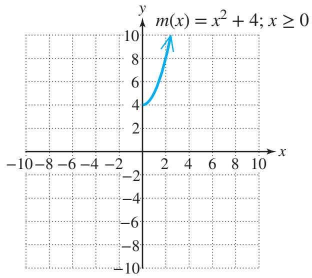


Figure 8-6

To find the inverse, we have

$y = x^2 + 4 \quad x \geq 0$	**Step 1:**	Replace $m(x)$ with y.
$x = y^2 + 4 \quad y \geq 0$	**Step 2:**	Interchange x and y. Notice that the restriction $x \geq 0$ becomes $y \geq 0$.
$x - 4 = y^2 \quad y \geq 0$	**Step 3:**	Solve for y. Subtract 4 from both sides.
$\sqrt{x - 4} = y \quad y \geq 0$		Apply the square root property. Notice that we obtain the *positive* square root of $x - 4$ because of the restriction $y \geq 0$.
$m^{-1}(x) = \sqrt{x - 4}$	**Step 4:**	Replace y with $m^{-1}(x)$. Notice that the domain of m^{-1} has the same values as the range of m.

Figure 8-7 shows the graphs of m and m^{-1}.

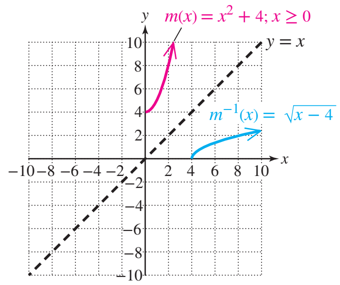


Figure 8-7

Skill Practice Find the inverse.

6. $g(x) = x^2 - 2 \quad x \geq 0$

Answers

5. $h^{-1}(x) = \frac{x^3 + 1}{2}$

6. $g^{-1}(x) = \sqrt{x + 2}$

Section 8.2 Activity

A.1. If a function defined by $y = f(x)$ has the property that no two elements in the domain of f correspond to the same element in the range of f, then f is said to be __________.

A.2. To determine whether a function is one-to-one, use the __________ line test.

For Exercises A.3–A.4, determine whether the function is one-to-one. If the function is not one-to-one, explain why.

A.3. $\{(2, -4), (-3, 1), (0, -2), (-4, 3)\}$

A.4. $\{(4, 1), (5, -3), (-1, -3), (-2, 5)\}$

For Exercises A.5–A.6, consider the given function.

a. Write each ordered pair with the x- and y-coordinates interchanged.

b. Is the new relation a function? If not, explain why.

A.5. $\{(2, -4), (-3, 1), (0, -2), (-4, 3)\}$ (Refer to Exercise A.3.)

A.6. $\{(4, 1), (5, -3), (-1, -3), (-2, 5)\}$ (Refer to Exercise A.4.)

A.7. Given $f(x) = \dfrac{x-4}{2}$,

a. Describe the graph of the function.

b. Is the function one-to-one? Does the function have an inverse function?

A.8. Find the inverse of $f(x) = \dfrac{x-4}{2}$ by following these steps.

a. Replace $f(x)$ with y.

b. Interchange x and y in the equation. This is the key step to finding the inverse.

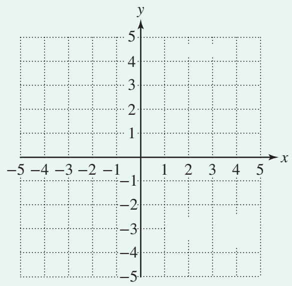
c. Solve for y in the new equation.

d. Replace y with $f^{-1}(x)$.

e. To check your answer to part (d), find $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$.

A.9. **a.** Refer to Exercise A.8. Graph $f(x) = \dfrac{x-4}{2}$ and $f^{-1}(x) = 2x + 4$ along with the line $y = x$ on the same coordinate system.

b. Describe the symmetry between a function and its inverse.

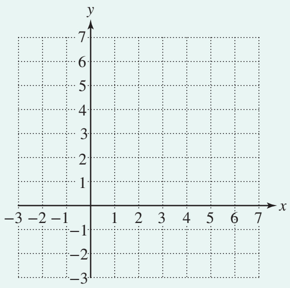


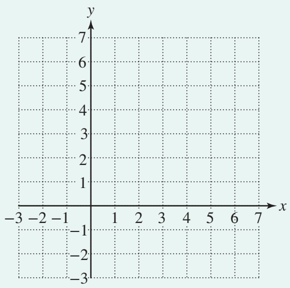
A.10. **a.** Graph $f(x) = x^2 + 2$ for $x \geq 0$ (right branch of the parabola only).

b. Is the function graphed in part (a) one-to-one?

c. Find the inverse of f.

d. Graph f^{-1} on the same coordinate system as the graph in part (a). Also graph the line $y = x$. Are the graphs of the function and its inverse symmetric with respect to $y = x$?





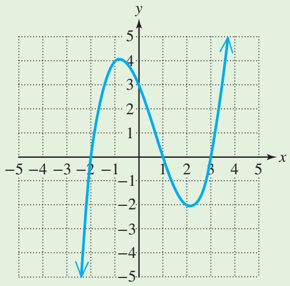
Practice Exercises

Section 8.2

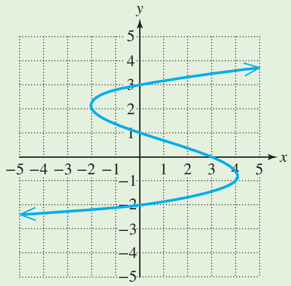
Prerequisite Review

For Exercises R.1–R.2, use the vertical line test to determine if the graph defines y as a function of x.

R.1.



R.2.



For Exercises R.3–R.6, solve for x.

R.3. $y = -3x - 4$

R.4. $y = 5x - 15$

R.5. $y = \sqrt[3]{x - 1}$

R.6. $y = (x - 4)^3$

R.7. Given $f(x) = 2x + 4$, find the function values.

a. $f(1)$ **b.** $f(-2)$ **c.** $f(t)$ **d.** $f\left(\frac{x-4}{2}\right)$

R.8. Given $g(x) = \sqrt[3]{x + 5}$, find the function values.

a. $g(3)$ **b.** $g(-6)$ **c.** $g(a)$ **d.** $g(x^3 - 5)$

For Exercises R.9–R.10, refer to the functions defined by $f(x) = 4x - 3$ and $g(x) = x^2 + 5$.

R.9. Find $(f \circ g)(x)$.

R.10. Find $(g \circ f)(x)$.

Vocabulary and Key Concepts

1. **a.** Given the function $f = \{(1, 2), (2, 3), (3, 4)\}$, write the set of ordered pairs representing f^{-1}.

b. A necessary condition for a function f to be a _____-_____-_____ function is that no two ordered pairs in f have different x-coordinates and the same _____-coordinate.

c. The function $f = \{(1, 5), (-2, 3), (-4, 2), (2, 5)\}$ (is/is not) a one-to-one function.

d. A function defined by $y = f(x)$ (is/is not) a one-to-one function if no horizontal line intersects the graph of f in more than one point.

e. The graph of a function and its inverse are symmetric with respect to the line __________.

2. **a.** Let f be a one-to-one function and let g be the inverse of f. Then $(f \circ g)(x) =$ __________ and $(g \circ f)(x) =$ __________.

b. The notation __________ is often used to represent the inverse of a function f and not the reciprocal of f.

c. If (a, b) is a point on the graph of a one-to-one function f, then the corresponding ordered pair __________ is a point on the graph of f^{-1}.

Concept 1: Introduction to Inverse Functions

For Exercises 3–6, write the inverse function for each function.

3. $g = \{(3, 5), (8, 1), (-3, 9), (0, 2)\}$

4. $f = \{(-6, 2), (-9, 0), (-2, -1), (3, 4)\}$

5. $r = \{(a, 3), (b, 6), (c, 9)\}$

6. $s = \{(-1, x), (-2, y), (-3, z)\}$

Concept 2: Definition of a One-to-One Function

7. The table relates a state x to the number of representatives in the House of Representatives y for a recent year. Does this relation define a one-to-one function? If so, write a function defining the inverse as a set of ordered pairs.

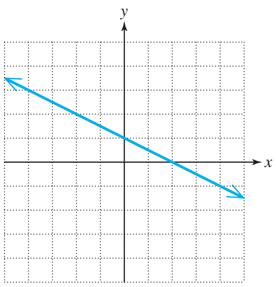
State x	Number of Representatives y
Colorado	7
California	53
Texas	32
Louisiana	7
Pennsylvania	19

8. The table relates a city x to its average January temperature y. Does this relation define a one-to-one function? If so, write a function defining the inverse as a set of ordered pairs.

City x	Temperature y (°C)
Gainesville, Florida	13.6
Keene, New Hampshire	−6.0
Wooster, Ohio	−4.0
Rock Springs,Wyoming	−6.0
Lafayette, Louisiana	10.9

For Exercises 9–14, determine if the function is one-to-one by using the horizontal line test. **(See Example 1.)**

9.



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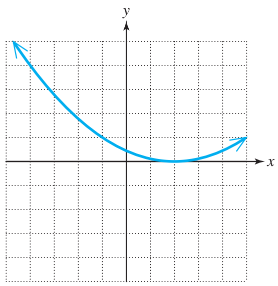

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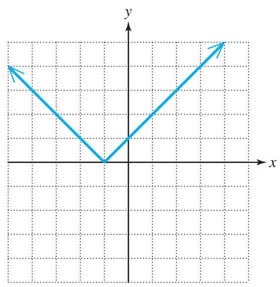

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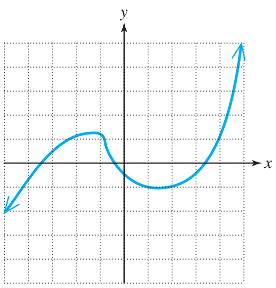

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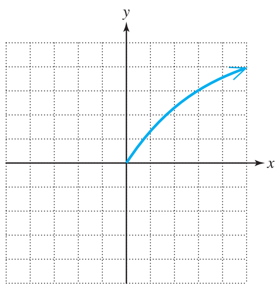

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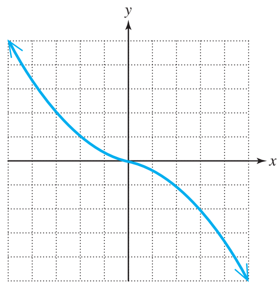


Concept 3: Definition of the Inverse of a Function

For Exercises 15–20, verify that f and g are inverse functions by showing that

a. $(f \circ g)(x) = x$ **b.** $(g \circ f)(x) = x$ **(See Example 2.)**

15. $f(x) = 6x + 1$ and $g(x) = \dfrac{x - 1}{6}$

16. $f(x) = 5x - 2$ and $g(x) = \dfrac{x + 2}{5}$

17. $f(x) = \dfrac{\sqrt[3]{x}}{2}$ and $g(x) = 8x^3$

18. $f(x) = \dfrac{\sqrt[3]{x}}{3}$ and $g(x) = 27x^3$

19. $f(x) = x^2 + 1, x \geq 0$, and $g(x) = \sqrt{x - 1}, x \geq 1$

20. $f(x) = x^2 - 3, x \geq 0$, and $g(x) = \sqrt{x + 3}, x \geq -3$

Concept 4: Finding an Equation of the Inverse of a Function

21. If function f adds 4 to x, then f^{-1} __________ 4 from x. Function f is defined by $f(x) = x + 4$, and $f^{-1}(x) =$ __________.

22. If function g subtracts 10 from x, then g^{-1} __________ 10 to x. Function g is defined by $g(x) = x - 10$, and $g^{-1}(x) =$ __________.

23. If function h multiplies x by 5, then h^{-1} __________ x by 5. Function h is defined by $h(x) = 5x$, and $h^{-1}(x) =$ __________.

24. If function k multiplies x by $\frac{1}{3}$, then k^{-1} multiplies x by __________. Function k is defined by $k(x) = \frac{1}{3}x$, and $k^{-1}(x) =$ __________.

25. Suppose that function f multiplies x by 9 and subtracts 2 from the result.

a. Write an equation for f.

b. Write an equation for f^{-1}.

26. Suppose that function g cubes x and adds 1 to the result.

a. Write an equation for g.

b. Write an equation for g^{-1}.

For Exercises 27–42, write an equation of the inverse for each one-to-one function as defined. **(See Examples 3–5.)**

27. $h(x) = x + 4$

28. $k(x) = x - 3$

29. $m(x) = \frac{1}{3}x - 2$

30. $n(x) = 4x + 2$

31. $p(x) = -x + 10$

32. $q(x) = -x - \frac{2}{3}$

33. $n(x) = \frac{3x + 2}{5}$

34. $p(x) = \frac{2x - 7}{4}$

35. $h(x) = \frac{4x - 1}{3}$

36. $f(x) = \frac{6x + 3}{2}$

37. $f(x) = x^3 + 1$

38. $g(x) = \sqrt[3]{x} + 2$

39. $g(x) = \sqrt[3]{2x - 1}$

40. $f(x) = x^3 - 4$

41. $g(x) = x^2 + 9 \quad x \geq 0$

42. $m(x) = x^2 - 1 \quad x \geq 0$

43. The function defined by $f(x) = 0.3048x$ converts a length of x feet into $f(x)$ meters.

a. Find the equivalent length in meters for a 4-ft board and a 50-ft wire.

b. Find an equation defining $y = f^{-1}(x)$.

c. Use the inverse function from part (b) to find the equivalent length in feet for a 1500-m race in track and field. Round to the nearest tenth of a foot.

44. The function defined by $s(x) = 1.47x$ converts a speed of x mph to $s(x)$ ft/sec.

a. Find the equivalent speed in feet per second for a car traveling 60 mph.

b. Find an equation defining $y = s^{-1}(x)$.

c. Use the inverse function from part (b) to find the equivalent speed in miles per hour for a train traveling 132 ft/sec. Round to the nearest tenth.

For Exercises 45–51, answer true or false.

45. The function defined by $y = 2$ has an inverse function defined by $x = 2$.

46. The domain of any one-to-one function is the same as the domain of its inverse.

47. All linear functions with a nonzero slope have an inverse function.

48. The function defined by $g(x) = |x|$ is one-to-one.

49. The function defined by $k(x) = x^2$ is one-to-one.

50. The function defined by $h(x) = |x|$ for $x \geq 0$ is one-to-one.

51. The function defined by $L(x) = x^2$ for $x \geq 0$ is one-to-one.

52. Explain how the domain and range of a one-to-one function and its inverse are related.

53. If $(0, b)$ is the y-intercept of a one-to-one function, what is the x-intercept of its inverse?

54. If $(a, 0)$ is the x-intercept of a one-to-one function, what is the y-intercept of its inverse?

55. **a.** Find the domain and range of the function defined by $f(x) = \sqrt{x - 1}$.

b. Find the domain and range of the function defined by $f^{-1}(x) = x^2 + 1, x \geq 0$.

56. **a.** Find the domain and range of the function defined by $g(x) = x^2 - 4, x \leq 0$.

b. Find the domain and range of the function defined by $g^{-1}(x) = -\sqrt{x + 4}$.

For Exercises 57–60, the graph of $y = f(x)$ is given.

a. State the domain of f.

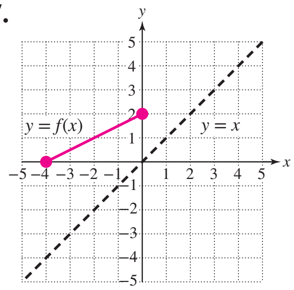
b. State the range of f.

c. State the domain of f^{-1}.

d. State the range of f^{-1}.

e. Graph the function defined by $y = f^{-1}(x)$. The line $y = x$ is provided for your reference.

57.



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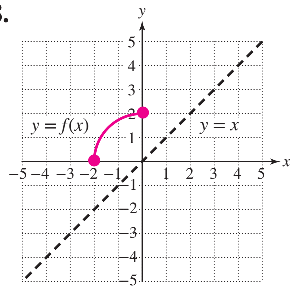

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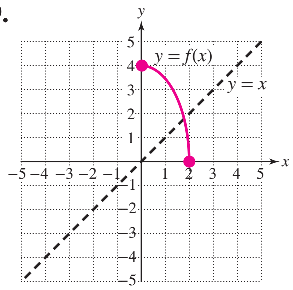

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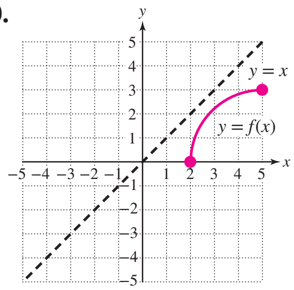


Technology Connections

For Exercises 61–64, use a graphing calculator to graph each function on the standard viewing window. Use the graph of the function to determine if the function is one-to-one on the interval $-10 \le x \le 10$. If the function is one-to-one, find its inverse and graph both functions on the standard viewing window.

61. $f(x) = \sqrt[3]{x+5}$

62. $k(x) = x^3 - 4$

63. $g(x) = 0.5x^3 - 2$

64. $m(x) = 3x - 4$

Expanding Your Skills

For Exercises 65–74, write an equation of the inverse of the one-to-one function.

65. $q(x) = \sqrt{x+4}$

66. $v(x) = \sqrt{x+16}$

67. $z(x) = -\sqrt{x+4}$

68. $u(x) = -\sqrt{x+16}$

69. $f(x) = \dfrac{x-1}{x+1}$

70. $p(x) = \dfrac{3-x}{x+3}$

71. $t(x) = \dfrac{2}{x-1}$

72. $w(x) = \dfrac{4}{x+2}$

73. $n(x) = x^2 + 9 \quad x \le 0$

74. $g(x) = x^2 - 1 \quad x \le 0$

Exponential Functions

Section 8.3

Concepts

1. Definition of an Exponential Function
2. Approximating Exponential Expressions With a Calculator
3. Graphs of Exponential Functions
4. Applications of Exponential Functions

1. Definition of an Exponential Function

We have already learned to recognize several categories of functions, including constant, linear, rational, and quadratic functions. In this section and the next, we will define two new types of functions called exponential and logarithmic functions.

To introduce the concept of an exponential function, consider the following salary plans for a new job. Plan A pays \$1 million for a month's work. Plan B starts with 2¢ on the first day, and every day thereafter the salary is doubled.

At first glance, the million-dollar plan appears to be more favorable. Look, however, at Table 8-1, which shows the daily payments for 30 days under plan B.

Table 8-1

Day	Payment	Day	Payment	Day	Payment
1	2¢	11	\$20.48	21	\$20,971.52
2	4¢	12	\$40.96	22	\$41,943.04
3	8¢	13	\$81.92	23	\$83,886.08
4	16¢	14	\$163.84	24	\$167,772.16
5	32¢	15	\$327.68	25	\$335,544.32
6	64¢	16	\$655.36	26	\$671,088.64
7	\$1.28	17	\$1310.72	27	\$1,342,177.28
8	\$2.56	18	\$2621.44	28	\$2,684,354.56
9	\$5.12	19	\$5242.88	29	\$5,368,709.12
10	\$10.24	20	\$10,485.76	30	\$10,737,418.24

Notice that the salary on the 30th day for plan B is over \$10 million. Taking the sum of the payments, we see the total salary for the 30-day period is \$21,474,836.46.

The daily salary for plan B can be represented by the function $y = 2^x$, where x is the number of days on the job and y is the salary (in cents) for that day. An interesting feature of this function is that for every positive 1-unit change in x, the y value doubles. The function $y = 2^x$ is called an exponential function.

Definition of an Exponential Function

Let b be any real number such that $b > 0$ and $b \neq 1$. Then for any real number x, a function of the form $f(x) = b^x$ is called an **exponential function**.

An exponential function is recognized as a function with a constant base and exponent, x. Notice that the base of an exponential function must be a positive real number not equal to 1.

2. Approximating Exponential Expressions With a Calculator

Up to this point, we have evaluated exponential expressions with integer exponents and rational exponents, for example, $4^3 = 64$ and $4^{1/2} = \sqrt{4} = 2$. However, how do we evaluate an exponential expression with an irrational exponent such as 4^π? In such a case, the exponent is a nonterminating and nonrepeating decimal. The value of 4^π can be thought of as the limiting value of a sequence of approximations using rational exponents:

$$\begin{aligned} 4^{3.14} &\approx 77.7084726 \\ 4^{3.141} &\approx 77.81627412 \\ 4^{3.1415} &\approx 77.87023095 \\ &\vdots \\ 4^{\pi} &\approx 77.88023365 \end{aligned}$$

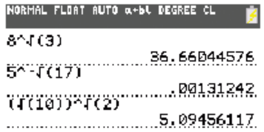
An exponential expression can be evaluated at all rational numbers and at all irrational numbers. Therefore, the domain of an exponential function is all real numbers.

Example 1 Approximating Exponential Expressions With a Calculator

Approximate the expressions. Round the answers to four decimal places.

a. $8^{\sqrt{3}}$ **b.** $5^{-\sqrt{17}}$ **c.** $\sqrt{10}^{\sqrt{2}}$

Solution:



On a calculator, use the [^], [y^x], or [x^y] key to approximate an expression with an irrational exponent.

a. $8^{\sqrt{3}} \approx 36.6604$ **b.** $5^{-\sqrt{17}} \approx 0.0013$

c. $\sqrt{10}^{\sqrt{2}} \approx 5.0946$

Skill Practice Approximate the value of the expressions. Round the answers to four decimal places.

1. 9^{π} **2.** $15^{\sqrt{5}}$ **3.** $\sqrt{7}^{\sqrt{3}}$

3. Graphs of Exponential Functions

The functions defined by $f(x) = 2^x$, $g(x) = \left(\frac{1}{2}\right)^x$, $h(x) = 3^x$, and $k(x) = 5^x$ are all examples of exponential functions. Example 2 illustrates the two general shapes of exponential functions.

Example 2 Graphing Exponential Functions

Graph the functions f and g.

a. $f(x) = 2^x$ **b.** $g(x) = \left(\frac{1}{2}\right)^x$

Solution:

Table 8-2 shows several function values for $f(x)$ and $g(x)$ for both positive and negative values of x. The graphs are shown in Figure 8-8.

Table 8-2

x	$f(x) = 2^x$	$g(x) = \left(\frac{1}{2}\right)^x$
−4	$\frac{1}{16}$	16
−3	$\frac{1}{8}$	8
−2	$\frac{1}{4}$	4
−1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
4	16	$\frac{1}{16}$

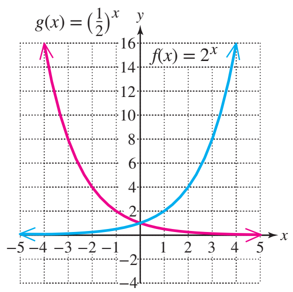


Figure 8-8

Skill Practice Graph the functions f and g on the same grid.

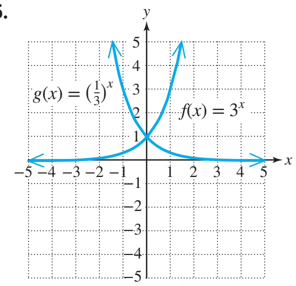
4. $f(x) = 3^x$ **5.** $g(x) = \left(\frac{1}{3}\right)^x$

FOR REVIEW

Note the difference between the polynomial function $y = x^2$ and the exponential function $y = 2^x$. The polynomial function has a variable base and constant exponent. The exponential function has a constant base and variable exponent.

Answers

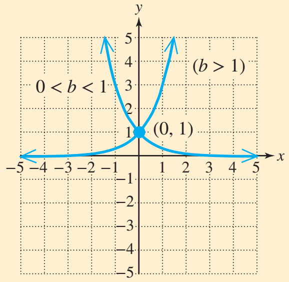
1. 995.0416 **2.** 426.4028
3. 5.3936
4–5.



The graphs in Figure 8-8 illustrate several important features of exponential functions.

Graphs of $f(x) = b^x$

The graph of an exponential function defined by $f(x) = b^x$ ($b > 0$ and $b \neq 1$) has the following properties.



1. If $b > 1$, f is an *increasing* exponential function, sometimes called an **exponential growth function**.

   If $0 < b < 1$, f is a *decreasing* exponential function, sometimes called an **exponential decay function**.

2. The domain is the set of all real numbers, $(-\infty, \infty)$.
3. The range is $(0, \infty)$.
4. The line $y = 0$ (x-axis) is a horizontal asymptote.
5. The function passes through the point $(0, 1)$ because $f(0) = b^0 = 1$.

These properties indicate that the graph of an exponential function is an increasing function if the base is greater than 1. Furthermore, the base affects the rate of increase. Consider the graphs of $f(x) = 2^x$, $h(x) = 3^x$, and $k(x) = 5^x$ (Figure 8-9). For every positive 1-unit change in x, $f(x) = 2^x$ increases by 2 times, $h(x) = 3^x$ increases by 3 times, and $k(x) = 5^x$ increases by 5 times (Table 8-3).

Table 8-3

x	$f(x) = 2^x$	$h(x) = 3^x$	$k(x) = 5^x$
-3	$\frac{1}{8}$	$\frac{1}{27}$	$\frac{1}{125}$
-2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{25}$
-1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$
0	1	1	1
1	2	3	5
2	4	9	25
3	8	27	125

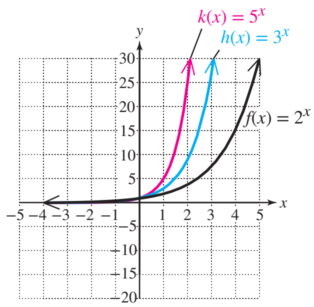


Figure 8-9

The graph of an exponential function is a *decreasing function* if the base is between 0 and 1. Consider the graphs of $g(x) = (\frac{1}{2})^x$, $m(x) = (\frac{1}{3})^x$, and $n(x) = (\frac{1}{5})^x$ (Table 8-4 and Figure 8-10).

Table 8-4

x	$g(x) = (\frac{1}{2})^x$	$m(x) = (\frac{1}{3})^x$	$n(x) = (\frac{1}{5})^x$
-3	8	27	125
-2	4	9	25
-1	2	3	5
0	1	1	1
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$
2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{25}$
3	$\frac{1}{8}$	$\frac{1}{27}$	$\frac{1}{125}$

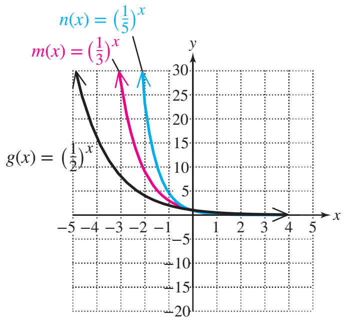


Figure 8-10

4. Applications of Exponential Functions

Exponential growth and decay can be found in a variety of real-world phenomena; for example,

- Short-term population growth can often be modeled by an exponential function.
- The growth of an investment under compound interest increases exponentially.
- The mass of a radioactive substance decreases exponentially with time.
- The temperature of a cup of coffee decreases exponentially as it approaches room temperature.

A substance that undergoes radioactive decay is said to be radioactive. The *half-life* of a radioactive substance is the amount of time it takes for one-half of the original amount of the substance to change into something else. That is, after each half-life the amount of the original substance decreases by one-half.

In 1898, Marie Curie discovered the highly radioactive element radium. She shared the 1903 Nobel Prize in Physics for her research on radioactivity and was awarded the 1911 Nobel Prize in Chemistry for her discovery of radium and polonium. Radium-226 (an isotope of radium) has a half-life of 1620 years and decays into radon-222 (a radioactive gas).

Marie and Pierre Curie

Everett Historical/Shutterstock

Example 3 **Applying an Exponential Decay Function**

In a sample originally composed of 1 g of radium-226, the amount of radium-226 present after t years is given by

$$A(t) = \left(\frac{1}{2}\right)^{t/1620}$$

where $A(t)$ is the amount of radium in grams and t is the time in years.

a. How much radium-226 will be present after 1620 years?

b. How much radium-226 will be present after 3240 years?

c. How much radium-226 will be present after 4860 years?

Solution:

$$A(t) = \left(\frac{1}{2}\right)^{t/1620}$$

a. $A(1620) = \left(\frac{1}{2}\right)^{1620/1620}$ Substitute $t = 1620$.

$= \left(\frac{1}{2}\right)^{1}$

$= 0.5$

After 1620 years (1 half-life), 0.5 g remains.

b. $A(3240) = \left(\frac{1}{2}\right)^{3240/1620}$ Substitute $t = 3240$.

$= \left(\frac{1}{2}\right)^{2}$

$= 0.25$

After 3240 years (2 half-lives), the amount of the original substance is reduced by one-half, 2 times. Thus, 0.25 g remains.

c. $A(4860) = \left(\frac{1}{2}\right)^{4860/1620}$ Substitute $t = 4860$.

$= \left(\frac{1}{2}\right)^{3}$

$= 0.125$

After 4860 years (3 half-lives), the amount of the original substance is reduced by one-half, 3 times. Thus, 0.125 g remains.

Skill Practice Cesium-137 is a radioactive metal with a short half-life of 30 years. In a sample originally composed of 1 g of cesium-137, the amount, $A(t)$ (in grams), of cesium-137 present after t years is given by

$$A(t) = \left(\frac{1}{2}\right)^{t/30}$$

6. How much cesium-137 will be present after 30 years?
7. How much cesium-137 will be present after 90 years?

Exponential functions are often used to model population growth. Suppose the initial value of a population at some time $t = 0$ is P_0. If the annual rate of increase of a population is r, then after t years, the population $P(t)$ is given by

$$P(t) = P_0(1 + r)^t$$

Answers
6. 0.5 g
7. 0.125 g

Example 4 Applying an Exponential Growth Function

The population of the Bahamas in 2008 was estimated at 321,000 with an annual rate of increase of 1.39%.

a. Find a mathematical model that relates the population of the Bahamas as a function of the number of years since 2008.

b. If the annual rate of increase remains the same, use this model to estimate the population of the Bahamas in the year 2016. Round to the nearest thousand.

Solution:

a. The initial population is $P_0 = 321{,}000$, and the rate of increase is 1.39%.

$$\begin{aligned} P(t) &= P_0(1 + r)^t && \text{Substitute } P_0 = 321{,}000 \text{ and } r = 0.0139. \\ &= 321{,}000(1 + 0.0139)^t \\ &= 321{,}000(1.0139)^t && \text{Here } t = 0 \text{ corresponds to the year 2008.} \end{aligned}$$

b. Because the initial population ($t = 0$) corresponds to the year 2008, we use $t = 8$ to find the population in the year 2016.

$$\begin{aligned} P(8) &= 321{,}000(1.0139)^8 \\ &\approx 358{,}000 && \text{The population in the year 2016 was approximately 358,000.} \end{aligned}$$

Skill Practice The population of Colorado in 2000 was approximately 3,700,000 with an annual rate of increase of 2%.

8. Find a mathematical model that relates the population of Colorado as a function of the number of years since 2000.

9. Use this model to estimate the population of Colorado in 2016. Round to the nearest thousand.

Answers

8. $P(t) = 3{,}700{,}000(1.02)^t$

9. Approximately 5,079,000

Section 8.3 Activity

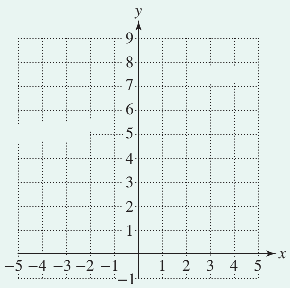
A.1. Given a positive real number b, such that $b \neq 1$, a function defined by $f(x) = b^x$ is called a(n) ________ function.

A.2. Which functions are exponential functions?

a. $f(x) = x^2$ **b.** $g(x) = 6^x$ **c.** $h(x) = \left(\frac{1}{2}\right)^x$ **d.** $k(x) = 1^x$

A.3. Consider the functions defined by $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
0		
1		
2		
−1		
−2		



a. Complete the table.

b. Refer to the values of function f in the table. How do the function values change with each 1-unit increase in x?

c. Refer to the values of function g in the table. How do the function values change with each 1-unit increase in x?

d. Graph $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$ on the same coordinate system.

e. Is function f an exponential growth function or exponential decay function?

f. Is function g an exponential growth function or exponential decay function?

g. Does the graph of either function ever meet the x-axis?

h. Write the domain and range of f and g.

A.4. A radioactive form of iodine called iodine-131 (denoted by chemists as ^{131}I) is used to treat thyroid cancer. ^{131}I undergoes radioactive decay, meaning that it spontaneously emits particles and energy. Eventually the substance changes from one form (or *isotope*) of the element into another.

The half-life of ^{131}I is roughly 8 days. This means that due to radioactive decay, half of the original amount of the substance is present after 8 days.

Suppose that a patient is given 64 units of ^{131}I. Then the amount of substance $A(t)$ still present after t days can be modeled by $A(t) = 64 \cdot \left(\frac{1}{2}\right)^{t/8}$.

a. Evaluate $A(8)$ and interpret its meaning.

b. Evaluate $A(16)$ and interpret its meaning.

c. Evaluate $A(24)$ and interpret its meaning.

d. Compare the values from parts (a)–(c) and interpret the results.

Section 8.3 Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, simplify the expressions.

R.1. **a.** 4^2 **b.** 4^0 **c.** 4^{-2}

R.2. **a.** 2^3 **b.** 2^0 **c.** 2^{-3}

R.3. **a.** $\left(\frac{1}{4}\right)^2$ **b.** $\left(\frac{1}{4}\right)^0$ **c.** $\left(\frac{1}{4}\right)^{-2}$

R.4. **a.** $\left(\frac{1}{2}\right)^3$ **b.** $\left(\frac{1}{2}\right)^0$ **c.** $\left(\frac{1}{2}\right)^{-3}$

Vocabulary and Key Concepts

1. **a.** Given a real number b, where $b > 0$ and $b \neq 1$, a function of the form $f(x) =$ ______________ is called an exponential function.

b. The function defined by $f(x) = x^4$ (is/is not) an exponential function, whereas the function defined by $f(x) = 4^x$ (is/is not) an exponential function.

c. The graph of $f(x) = \left(\frac{7}{2}\right)^x$ is (increasing/decreasing) over its domain.

d. The graph of $f(x) = \left(\frac{2}{7}\right)^x$ is (increasing/decreasing) over its domain.

e. In interval notation, the domain of an exponential function $f(x) = b^x$ is ________ and the range is ________.

f. All exponential functions $f(x) = b^x$ pass through the point ________.

g. The horizontal asymptote of an exponential function $f(x) = b^x$ is the line ________.

2. The function defined by $f(x) = 1^x$ (is/is not) an exponential function.

Concept 2: Approximating Exponential Expressions With a Calculator

For Exercises 3–10, evaluate the expression by using a calculator. Round to four decimal places. **(See Example 1.)**

3. $5^{1.1}$ **4.** $2^{\sqrt{3}}$ **5.** 10^{π} **6.** $3^{4.8}$

7. $36^{-\sqrt{2}}$ **8.** $27^{-0.5126}$ **9.** $16^{-0.04}$ **10.** $8^{-0.61}$

11. Solve for x.

a. $3^x = 9$ **b.** $3^x = 27$ **c.** Between what two consecutive integers must the solution to $3^x = 11$ lie?

12. Solve for x.

a. $5^x = 125$ **b.** $5^x = 625$ **c.** Between what two consecutive integers must the solution to $5^x = 130$ lie?

13. Solve for x.

a. $2^x = 16$ **b.** $2^x = 32$ **c.** Between what two consecutive integers must the solution to $2^x = 30$ lie?

14. Solve for x.

a. $4^x = 16$ **b.** $4^x = 64$ **c.** Between what two consecutive integers must the solution to $4^x = 20$ lie?

15. For $f(x) = \left(\frac{1}{5}\right)^x$ find $f(0)$, $f(1)$, $f(2)$, $f(-1)$, and $f(-2)$.

16. For $g(x) = \left(\frac{2}{3}\right)^x$ find $g(0)$, $g(1)$, $g(2)$, $g(-1)$, and $g(-2)$.

17. For $h(x) = 3^x$ use a calculator to find $h(0)$, $h(1)$, $h(-1)$, $h\left(\sqrt{2}\right)$, and $h(\pi)$. Round to two decimal places if necessary.

18. For $k(x) = 5^x$ use a calculator to find $k(0)$, $k(1)$, $k(-1)$, $k(\pi)$, and $k\left(\sqrt{2}\right)$. Round to two decimal places if necessary.

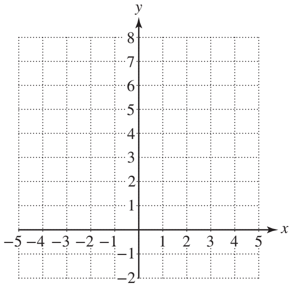
Concept 3: Graphs of Exponential Functions

19. How do you determine whether the graph of $f(x) = b^x$ $(b > 0, b \neq 1)$ is increasing or decreasing?

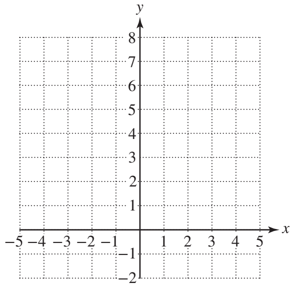
20. For $f(x) = b^x$ $(b > 0, b \neq 1)$, find $f(0)$.

Graph the functions defined in Exercises 21–28. Plot at least three points for each function. **(See Example 2.)**

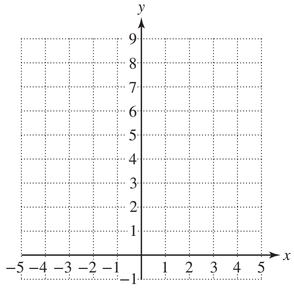
21. $f(x) = 4^x$



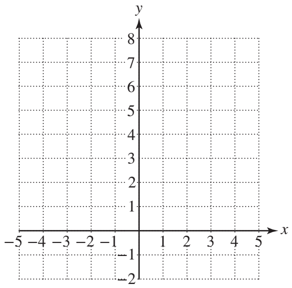
22. $g(x) = 6^x$



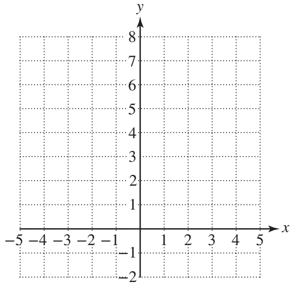
23. $m(x) = \left(\frac{1}{8}\right)^x$



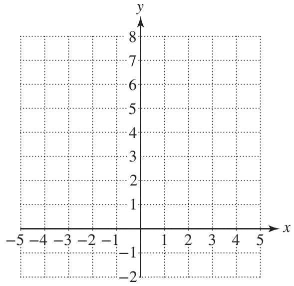
24. $n(x) = \left(\frac{1}{3}\right)^x$



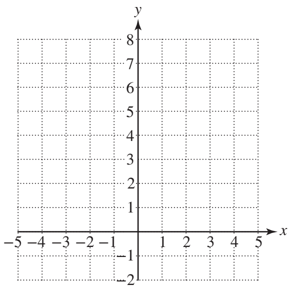
25. $h(x) = 2^{x+1}$



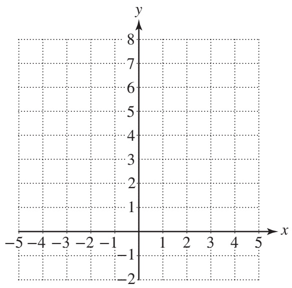
26. $k(x) = 5^{x-1}$



27. $g(x) = 5^{-x}$



28. $f(x) = 2^{-x}$



Concept 4: Applications of Exponential Functions

29. The half-life of the element radon (Rn-86) is 3.8 days. In a sample originally containing 1 g of radon, the amount left after t days is given by $A(t) = (0.5)^{t/3.8}$. (Round to two decimal places, if necessary.) **(See Example 3.)**

a. How much radon will be present after 7.6 days?

b. How much radon will be present after 10 days?

30. Nobelium, an element discovered in 1958, has a half-life of 10 min under certain conditions. In a sample containing 1 g of nobelium, the amount left after t min is given by $A(t) = (0.5)^{t/10}$. (Round to three decimal places.)

a. How much nobelium is left after 5 min?

b. How much nobelium is left after 1 hr?

31. Once an antibiotic is introduced to treat a bacterial infection, the number of bacteria decreases exponentially. For example, beginning with 1 million bacteria, the amount present t days from the time penicillin is introduced is given by $A(t) = 1{,}000{,}000(2)^{-t/5}$. Rounding to the nearest thousand, determine how many bacteria are present after

a. 2 days **b.** 1 week **c.** 2 weeks

32. Once an antibiotic is introduced to treat a bacterial infection, the number of bacteria decreases exponentially. For example, beginning with 1 million bacteria, the amount present t days from the time streptomycin is introduced is given by $A(t) = 1{,}000{,}000(2)^{-t/10}$. Rounding to the nearest thousand, determine how many bacteria are present after

a. 5 days **b.** 1 week **c.** 2 weeks

33. The population of Bangladesh was 153,000,000 in 2009 with an annual growth rate of 1.25%. **(See Example 4.)**

a. Find a mathematical model that relates the population of Bangladesh as a function of the number of years after 2009.

b. If the annual rate of increase remains the same, use this model to predict the population of Bangladesh in the year 2050. Round to the nearest million.

34. The population of Fiji was 908,000 in 2009 with an annual growth rate of 0.07%.

a. Find a mathematical model that relates the population of Fiji as a function of the number of years after 2009.

b. If the annual rate of increase remains the same, use this model to predict the population of Fiji in the year 2050. Round to the nearest thousand.

35. Suppose \$1000 is initially invested in an account and the value of the account grows exponentially. If the investment doubles in 7 years, then the amount in the account t years after the initial investment is given by $A(t)$.

$$A(t) = 1000(2)^{t/7}$$

a. Find the amount in the account after 5 years.

b. Find the amount in the account after 10 years.

c. Find $A(0)$ and $A(7)$ and interpret the answers in the context of this problem.

36. Suppose \$1500 is initially invested in an account and the value of the account grows exponentially. If the investment doubles in 8 years, then the amount in the account t years after the initial investment is given by $A(t)$.

$$A(t) = 1500(2)^{t/8}$$

a. Find the amount in the account after 5 years.

b. Find the amount in the account after 10 years.

c. Find $A(0)$ and $A(8)$ and interpret the answers in the context of this problem.

Technology Connections

For Exercises 37–44, graph the functions on your calculator to support your solutions to the indicated exercises.

37. $f(x) = 4^x$ (see Exercise 21)

38. $g(x) = 6^x$ (see Exercise 22)

39. $m(x) = (\frac{1}{8})^x$ (see Exercise 23)

40. $n(x) = (\frac{1}{3})^x$ (see Exercise 24)

41. $h(x) = 2^{x+1}$ (see Exercise 25)

42. $k(x) = 5^{x-1}$ (see Exercise 26)

43. $g(x) = 5^{-x}$ (see Exercise 27)

44. $f(x) = 2^{-x}$ (see Exercise 28)

Section 8.4 Logarithmic Functions

Concepts

1. Definition of a Logarithmic Function
2. Evaluating Logarithmic Expressions
3. The Common Logarithmic Function
4. Graphs of Logarithmic Functions
5. Applications of the Common Logarithmic Function

1. Definition of a Logarithmic Function

Consider the following equations in which the variable is located in the exponent of an expression. In some cases the solution can be found by inspection because the constant on the right-hand side of the equation is a perfect power of the base.

Equation		Solution
$5^x = 5$	$\longrightarrow$	$x = 1$
$5^x = 20$	$\longrightarrow$	$x = ?$
$5^x = 25$	$\longrightarrow$	$x = 2$
$5^x = 60$	$\longrightarrow$	$x = ?$
$5^x = 125$	$\longrightarrow$	$x = 3$

The equation $5^x = 20$ cannot be solved by inspection. However, we might suspect that x is between 1 and 2. Similarly, the solution to the equation $5^x = 60$ is between 2 and 3. To solve an exponential equation for an unknown exponent, we must use a new type of function called a logarithmic function.

Definition of a Logarithmic Function

If x and b are positive real numbers such that $b \neq 1$, then $y = \log_b x$ is called the **logarithmic function** with base b and

$$y = \log_b x \qquad \text{is equivalent to} \qquad b^y = x$$

Note: In the expression $y = \log_b x$, y is called the **logarithm**, b is called the **base**, and x is called the **argument**.

The expression $y = \log_b x$ is equivalent to $b^y = x$ and indicates that *the logarithm y is the exponent to which b must be raised to obtain x*. The expression $y = \log_b x$ is called the logarithmic form of the equation, and the expression $b^y = x$ is called the exponential form of the equation.

The definition of a logarithmic function suggests a close relationship with an exponential function of the same base. In fact, a logarithmic function is the inverse of the corresponding exponential function. For example, the following steps find the inverse of the exponential function defined by $f(x) = b^x$.

$f(x) = b^x$

$y = b^x$ Replace $f(x)$ with y.

$x = b^y$ Interchange x and y.

$y = \log_b x$ Solve for y using the definition of a logarithmic function.

$f^{-1}(x) = \log_b x$ Replace y with $f^{-1}(x)$.

The concept of a logarithm is new and unfamiliar. Therefore, it is often advantageous to rewrite a logarithm in its exponential form.

Example 1 Converting From Logarithmic Form to Exponential Form

Rewrite the logarithmic equations in exponential form.

a. $\log_2 32 = 5$ **b.** $\log_{10}\left(\dfrac{1}{1000}\right) = -3$ **c.** $\log_5 1 = 0$

Solution:

	Logarithmic Form		Exponential Form
a.	$\log_2 32 = 5$	$\Leftrightarrow$	$2^5 = 32$
b.	$\log_{10}\left(\frac{1}{1000}\right) = -3$	$\Leftrightarrow$	$10^{-3} = \frac{1}{1000}$
c.	$\log_5 1 = 0$	$\Leftrightarrow$	$5^0 = 1$

Skill Practice Rewrite the logarithmic equations in exponential form.

1. $\log_3 9 = 2$ **2.** $\log_{10}\left(\frac{1}{100}\right) = -2$ **3.** $\log_8 1 = 0$

2. Evaluating Logarithmic Expressions

Example 2 Evaluating Logarithmic Expressions

Evaluate the logarithmic expressions.

a. $\log_8 64$ **b.** $\log_5\left(\frac{1}{125}\right)$ **c.** $\log_{1/2}\left(\frac{1}{8}\right)$

Solution:

a. $\log_8 64$ is the exponent to which 8 must be raised to obtain 64.

$y = \log_8 64$ Let y represent the value of the logarithm.

$8^y = 64$ Rewrite the expression in exponential form.

$8^y = 8^2$

$y = 2$ Therefore, $\log_8 64 = 2$.

b. $\log_5\left(\frac{1}{125}\right)$ is the exponent to which 5 must be raised to obtain $\frac{1}{125}$.

$y = \log_5\left(\frac{1}{125}\right)$ Let y represent the value of the logarithm.

$5^y = \frac{1}{125}$ Rewrite the expression in exponential form.

$5^y = \frac{1}{5^3} = 5^{-3}$

$y = -3$ Therefore, $\log_5\left(\frac{1}{125}\right) = -3$.

c. $\log_{1/2}\left(\frac{1}{8}\right)$ is the exponent to which $\frac{1}{2}$ must be raised to obtain $\frac{1}{8}$.

$y = \log_{1/2}\left(\frac{1}{8}\right)$ Let y represent the value of the logarithm.

$\left(\frac{1}{2}\right)^y = \frac{1}{8}$ Rewrite the expression in exponential form.

$\left(\frac{1}{2}\right)^y = \left(\frac{1}{2}\right)^3$

$y = 3$ Therefore, $\log_{1/2}\left(\frac{1}{8}\right) = 3$.

Skill Practice Evaluate the logarithmic expressions.

4. $\log_{10} 1000$ **5.** $\log_4\left(\frac{1}{16}\right)$ **6.** $\log_{1/3} 3$

FOR REVIEW

Recall that for a nonzero real number b:

- $b^{-n} = \left(\frac{1}{b}\right)^n = \frac{1}{b^n}$
- $b^0 = 1$

For example:

$5^{-3} = \left(\frac{1}{5}\right)^3 = \frac{1}{5^3} = \frac{1}{125}$

$5^0 = 1$

Answers

1. $3^2 = 9$ **2.** $10^{-2} = \frac{1}{100}$
3. $8^0 = 1$ **4.** 3
5. −2 **6.** −1

Example 3 Evaluating Logarithmic Expressions

Evaluate the logarithmic expressions.

a. $\log_b b$ **b.** $\log_c c^7$ **c.** $\log_3 \sqrt[4]{3}$

Solution:

a. $\log_b b$ is the exponent to which b must be raised to obtain b.

$y = \log_b b$	Let y represent the value of the logarithm.
$b^y = b$	Rewrite the expression in exponential form.
$y = 1$	Therefore, $\log_b b = 1$.

b. $\log_c c^7$ is the exponent to which c must be raised to obtain c^7.

$y = \log_c c^7$	Let y represent the value of the logarithm.
$c^y = c^7$	Rewrite the expression in exponential form.
$y = 7$	Therefore, $\log_c c^7 = 7$.

c. $\log_3 \sqrt[4]{3} = \log_3 3^{1/4}$ is the exponent to which 3 must be raised to obtain $3^{1/4}$.

$y = \log_3 3^{1/4}$	Let y represent the value of the logarithm.
$3^y = 3^{1/4}$	Rewrite the expression in exponential form.
$y = \frac{1}{4}$	Therefore, $\log_3 \sqrt[4]{3} = \frac{1}{4}$.

FOR REVIEW

Recall that rational exponents can be used to represent radical expressions.

$$x^{1/n} = \sqrt[n]{x}$$
$$x^{m/n} = \sqrt[n]{x^m} \text{ or } \left(\sqrt[n]{x}\right)^m$$

provided that $\sqrt[n]{x}$ is a real number.

Skill Practice Evaluate the logarithmic expressions.

7. $\log_x x$ **8.** $\log_b b^{10}$ **9.** $\log_5 \sqrt[3]{5}$

3. The Common Logarithmic Function

The logarithmic function with base 10 is called the **common logarithmic function** and is denoted by $y = \log x$. Notice that the base is not explicitly written but is understood to be 10. That is, $y = \log_{10} x$ is written simply as $y = \log x$.

Example 4 Evaluating Common Logarithms

Evaluate the logarithmic expressions.

a. $\log 100{,}000$ **b.** $\log 0.01$

Solution:

a. $\log 100{,}000$ is the exponent to which 10 must be raised to obtain 100,000.

$y = \log 100{,}000$	Let y represent the value of the logarithm.
$10^y = 100{,}000$	Rewrite the expression in exponential form.
$10^y = 10^5$	
$y = 5$	Therefore, $\log 100{,}000 = 5$.

Answers

7. 1 **8.** 10 **9.** $\frac{1}{3}$

b. log 0.01 is the exponent to which 10 must be raised to obtain 0.01 or $\frac{1}{100}$.

$y = \log 0.01$	Let y represent the value of the logarithm.
$10^y = 0.01$	Rewrite the expression in exponential form.
$10^y = 10^{-2}$	Note that $0.01 = \frac{1}{100}$ or 10^{-2}.
$y = -2$	Therefore, $\log 0.01 = -2$.

Skill Practice Evaluate the logarithmic expressions.

10. $\log 1{,}000{,}000$ **11.** $\log 0.0001$

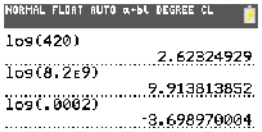
On most calculators the LOG key is used to compute logarithms with base 10. This is demonstrated in Example 5.

Example 5 **Evaluating Common Logarithms on a Calculator**

Evaluate the common logarithms. Round the answers to four decimal places.

a. $\log 420$ **b.** $\log\left(8.2 \times 10^9\right)$ **c.** $\log 0.0002$

Solution:



a. $\log 420 \approx 2.6232$

b. $\log\left(8.2 \times 10^9\right) \approx 9.9138$

c. $\log 0.0002 \approx -3.6990$

TIP: On a scientific calculator, you may need to enter the logarithm and argument in reverse order. For example: 420 log.

Skill Practice Evaluate the common logarithms. Round answers to four decimal places.

12. $\log 1200$ **13.** $\log\left(6.3 \times 10^5\right)$ **14.** $\log 0.00025$

4. Graphs of Logarithmic Functions

Recall that $f(x) = \log_b x$ is the inverse of $g(x) = b^x$. Therefore, the graph of $y = f(x)$ is symmetric to the graph of $y = g(x)$ about the line $y = x$, as shown in Figures 8-11 and 8-12.

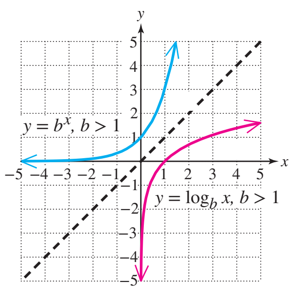


Figure 8-11

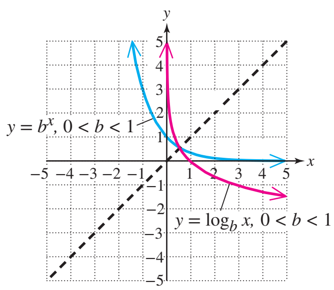


Figure 8-12

From Figures 8-11 and 8-12, we see that the range of $y = b^x$ is the set of positive real numbers. As expected, the domain of its inverse, the logarithmic function $y = \log_b x$, is also the set of positive real numbers. Therefore, the **domain of the logarithmic function** $y = \log_b x$ is the set of positive real numbers.

Answers

10. 6 **11.** −4

12. ≈ 3.0792 **13.** ≈ 5.7993

14. ≈ -3.6021

Example 6 Graphing Logarithmic Functions

Graph the logarithmic functions.

a. $y = \log_2 x$ **b.** $y = \log_{1/4} x$

Solution:

A logarithmic function is the inverse of an exponential function with the same base. For this reason, we can reverse the coordinates of the ordered pairs on an exponential function to obtain ordered pairs on the related logarithmic function. For example, to graph the function $y = \log_2 x$, reverse the coordinates of the ordered pairs from its inverse function, $y = 2^x$.

a. Exponential function

x	$y = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Logarithmic function

x	$y = \log_2 x$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Reverse the order.

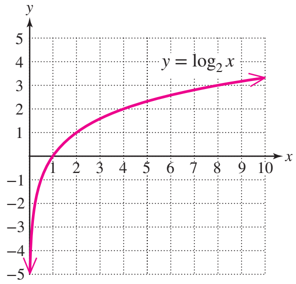


Figure 8-13

The graph of $y = \log_2 x$ is shown in Figure 8-13.

b. Exponential function

x	$y = \left(\frac{1}{4}\right)^x$
-3	64
-2	16
-1	4
0	1
1	$\frac{1}{4}$
2	$\frac{1}{16}$
3	$\frac{1}{64}$

Logarithmic function

x	$y = \log_{1/4} x$
64	-3
16	-2
4	-1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2
$\frac{1}{64}$	3

Reverse the order.

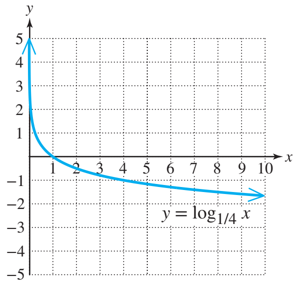


Figure 8-14

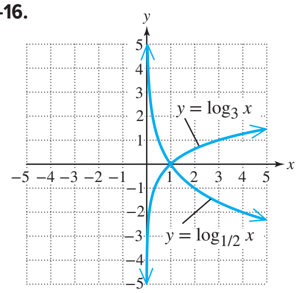
The graph of $y = \log_{1/4} x$ is shown in Figure 8-14.

Skill Practice Graph.

15. $y = \log_3 x$ **16.** $y = \log_{1/2} x$

Answers

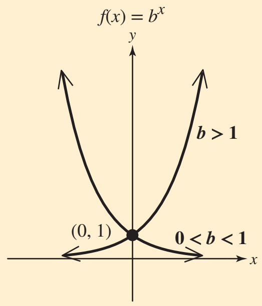
15–16.



The general shape and important features of exponential and logarithmic functions are summarized as follows:

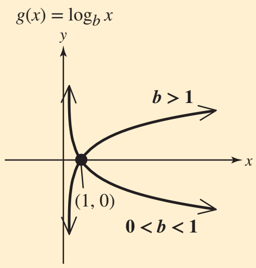
Graphs of Exponential and Logarithmic Functions

Exponential Functions



Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Horizontal asymptote: $y = 0$
Passes through $(0, 1)$
If $b > 1$, the function is increasing.
If $0 < b < 1$, the function is decreasing.

Logarithmic Functions



Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Vertical asymptote: $x = 0$
Passes through $(1, 0)$
If $b > 1$, the function is increasing.
If $0 < b < 1$, the function is decreasing.

Notice that the roles of x and y are interchanged for the functions $y = b^x$ and $b^y = x$. Therefore, it is not surprising that the domain and range are reversed between exponential and logarithmic functions. Moreover, an exponential function passes through $(0, 1)$, whereas a logarithmic function passes through $(1, 0)$. An exponential function has a horizontal asymptote at $y = 0$, whereas a logarithmic function has a vertical asymptote at $x = 0$.

When graphing a logarithmic equation, it is helpful to know its domain.

Example 7 Identifying the Domain of a Logarithmic Function

Find the domain of each function.

a. $f(x) = \log(4 - x)$ **b.** $g(x) = \log(2x + 6)$

Solution:

The domain of the function $y = \log_b x$ is the set of all positive real numbers. That is, the argument x must be greater than zero: $x > 0$.

a. $f(x) = \log(4 - x)$	The argument is $4 - x$.
$4 - x > 0$	The argument of the logarithm must be greater than zero.
$-x > -4$	Solve for x.
$x < 4$	Divide by -1 and reverse the inequality sign.

The domain is $(-\infty, 4)$.

b. $g(x) = \log(2x + 6)$ — The argument is $2x + 6$.

$2x + 6 > 0$ — The argument of the logarithm must be greater than zero.

$2x > -6$

$x > -3$

The domain is $(-3, \infty)$.

Skill Practice Find the domain of each function.

17. $f(x) = \log_3(x + 7)$ **18.** $g(x) = \log(4 - 8x)$

5. Applications of the Common Logarithmic Function

Example 8 Applying a Common Logarithm to Compute pH

The pH (hydrogen potential) of a solution is defined as

$$\text{pH} = -\log[\text{H}^+]$$

where $[\text{H}^+]$ represents the concentration of hydrogen ions in moles per liter (mol/L). The pH scale ranges from 0 to 14. The midpoint of this range, 7, represents a neutral solution. Values below 7 are progressively more acidic, and values above 7 are progressively more alkaline. Based on the equation $\text{pH} = -\log[\text{H}^+]$, a 1-unit change in pH means a 10-fold change in hydrogen ion concentration.

a. Normal rain has a pH of 5.6. However, in some areas of the northeastern United States the rainwater is more acidic. What is the pH of a rain sample for which the concentration of hydrogen ions is 0.0002 mol/L?

b. Find the pH of household ammonia if the concentration of hydrogen ions is 1.0×10^{-11} mol/L.

Solution:

a. $\text{pH} = -\log[\text{H}^+]$

$= -\log(0.0002)$ — Substitute $[\text{H}+] = 0.0002$.

≈ 3.7 — The pH of the rain sample is 3.7. (To compare this value with a familiar substance, note that the pH of orange juice is roughly 3.5.)

b. $\text{pH} = -\log[\text{H}^+]$

$= -\log(1.0 \times 10^{-11})$ — Substitute $[\text{H}^+] = 1.0 \times 10^{-11}$.

$= -\log(10^{-11})$

$= -(-11)$

$= 11$ — The pH of household ammonia is 11.

Skill Practice

19. A new all-natural shampoo on the market claims its hydrogen ion concentration is 5.88×10^{-7} mol/L. Use the formula $\text{pH} = -\log[\text{H}^+]$ to calculate the pH level of the shampoo.

Answers

17. Domain: $(-7, \infty)$

18. Domain: $\left(-\infty, \frac{1}{2}\right)$

19. $\text{pH} \approx 6.23$

Example 9 Applying a Logarithmic Function to a Memory Model

One method of measuring a student's retention of material after taking a course is to retest the student at specified time intervals after the course has been completed. A student's score on a calculus test t months after completing a course in calculus is approximated by

$$S(t) = 85 - 25 \log (t + 1)$$

where t is the time in months after completing the course and $S(t)$ is the student's score as a percent.

a. What was the student's score at $t = 0$?

b. What was the student's score after 2 months?

c. What was the student's score after 1 year?

Solution:

a. $S(t) = 85 - 25 \log (t + 1)$

$S(0) = 85 - 25 \log (0 + 1)$	Substitute $t = 0$.
$= 85 - 25 \log 1$	$\log 1 = 0$ because $10^0 = 1$.
$= 85 - 25(0)$	
$= 85 - 0$	
$= 85$	The student's score at the time the course was completed was 85%.

b. $S(t) = 85 - 25 \log (t + 1)$

$S(2) = 85 - 25 \log (2 + 1)$	
$= 85 - 25 \log 3$	Use a calculator to approximate log 3.
≈ 73.1	The student's score dropped to 73.1%.

c. $S(t) = 85 - 25 \log (t + 1)$

$S(12) = 85 - 25 \log (12 + 1)$	
$= 85 - 25 \log 13$	Use a calculator to approximate log 13.
≈ 57.2	The student's score dropped to 57.2%.

Skill Practice The memory model for a student's score on a statistics test t months after completion of the course in statistics is approximated by

$$S(t) = 92 - 28 \log (t + 1)$$

20. What was the student's score at the time the course was completed ($t = 0$)?

21. What was her score after 1 month?

22. What was the score after 2 months?

Answers

20. 92 **21.** 83.6 **22.** 78.6

Section 8.4 Activity

A.1. **a.** Consider an exponential function such as $y = 2^x$. If we input a value of x, such as 3, the y value is equal to the base, 2, raised to the third power.

$y = 2^3$ means that $y =$ ________.

b. The inverse of an exponential function reverses this process. The inverse of $y = 2^x$ is $x = 2^y$. Thus, if we input a value of x, such as 8, the y value is the exponent to which 2 is raised to make 8.

$8 = 2^y$ means that $y =$ ________.

c. The inverse of an exponential function is called a ________ function. Furthermore, $y = \log_b x$ is equivalent to ________.

For Exercises A.2–A.9, complete the table.

	Exponential Form	Logarithmic Form
A.2.		$\log_4 16 = 2$
A.3.	$3^4 = 81$	
A.4.		$\log_2 x = 7$
A.5.	$5^y = 125$	
A.6.		$\log_6 6 = 1$
A.7.	$7^0 = 1$	
A.8.		$\log_2 \dfrac{1}{16} = -4$
A.9.	$10^{-2} = \dfrac{1}{100}$	

For Exercises A.10–A.15, evaluate the logarithm.

A.10. $\log_2 32$

A.11. $\log 100$

A.12. $\log_3\left(\dfrac{1}{9}\right)$

A.13. $\log_4\left(\dfrac{1}{64}\right)$

A.14. $\log_5 1$

A.15. $\log_2 2$

A.16. **a.** Evaluate $\log 1000$.

b. Evaluate $\log 10{,}000$.

c. Approximate log 2500 between two integer values.

For Exercises A.17–A.18,

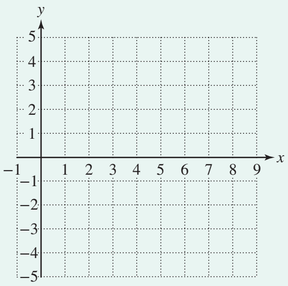
a. Write the function in exponential form.

b. Complete the table and graph the function.

A.17. **a.** $y = \log_3 x \xrightarrow{\text{exponential form}}$ ________

b.

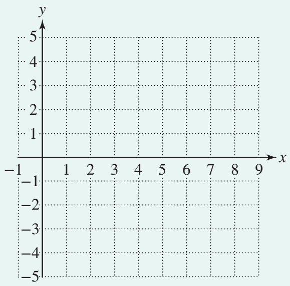
x	y
	-2
	-1
	0
	1
	2



A.18. **a.** $y = \log_{1/3} x \xrightarrow{\text{exponential form}}$ ________

b.

x	y
	-2
	-1
	0
	1
	2



For Exercises A.19–A.23, identify the similarities and differences between the graphs of $y = \log_b x$ with base b greater than 1 or with base b between 0 and 1. Refer to the graphs from Exercises A.17–A.18.

A.19. Write the domain of each function in interval notation.

A.20. Write the range of each function in interval notation.

A.21. Identify the asymptote of each function.

A.22. Identify the x-intercept of each function.

A.23. **a.** If $b > 1$, then the function (increases/decreases) from left to right.

b. If $0 < b < 1$, then the function (increases/decreases) from left to right.

For Exercises A.24–A.27, write the domain in interval notation.

A.24. $f(x) = \log_b x$

A.25. $g(x) = \log_b(x - 4)$

A.26. $h(x) = \log_b(4 - x)$

A.27. $k(x) = \log_b(2x + 1)$

Practice Exercises — Section 8.4

Prerequisite Review

For Exercises R.1–R.6, simplify the expression.

R.1. $25^{1/2}$

R.2. $1000^{1/3}$

R.3. $64^{2/3}$

R.4. $32^{3/5}$

R.5. 8^{-2}

R.6. 2^{-3}

For Exercises R.7–R.14, fill in the blank to make a true statement.

R.7. $(3)^{\square} = 9$

R.8. $(2)^{\square} = 16$

R.9. $(3)^{\square} = \frac{1}{81}$

R.10. $(2)^{\square} = \frac{1}{8}$

R.11. $\left(\frac{1}{3}\right)^{\square} = 27$

R.12. $\left(\frac{1}{2}\right)^{\square} = 4$

R.13. $\left(\frac{3}{4}\right)^{\square} = \frac{16}{9}$

R.14. $\left(\frac{4}{5}\right)^{\square} = \frac{625}{256}$

For Exercises R.15–R.18, solve the inequality. Write the solution set in interval notation.

R.15. $2x - 6 > 0$

R.16. $3x + 12 > 0$

R.17. $6 - x > 0$

R.18. $-3 - x > 0$

Vocabulary and Key Concepts

1. **a.** The function defined by $y = \log_b x$ is called the ______________ function with base ______________. The values of x and b are positive, and $b \neq 1$.

b. Given $y = \log_b x$, we call y the ______________, we call b the ______________, and we call x the ______________.

c. In interval notation, the domain of $y = \log_b x$ is ______________ and the range is ______________.

d. A logarithmic function base b is the inverse of the ______________ function base b.

e. The logarithmic function base 10 is called the ______________ logarithmic function and is denoted by $y = \log x$.

f. The graph of a logarithmic function is (increasing/decreasing) if the base $b > 1$, whereas the graph is (increasing/decreasing) if the base is between 0 and 1.

g. Which values of x are *not* in the domain of $y = \log (x - 4)$? $x = 2, x = 3, x = 4, x = 5, x = 6$

h. The graphs of a logarithmic function base b and an exponential function base b are symmetric with respect to the line ______________.

Concept 1: Definition of a Logarithmic Function

2. For the equation $y = \log_b x$, identify the base, the argument, and the logarithm.

3. Rewrite the equation in exponential form. $y = \log_b x$

4. Write the equation in logarithmic form. $b^y = x$

For Exercises 5–10, fill in the blanks.

5. a. $5^{\square} = 25$
b. $\log_5 25 = \square$

6. a. $2^{\square} = 16$
b. $\log_2 16 = \square$

7. a. $3^{\square} = 27$
b. $\log_3 27 = \square$

8. a. $10^{\square} = 100{,}000$
b. $\log_{10} 100{,}000 = \square$

9. a. $8^{\square} = 8$
b. $\log_8 8 = \square$

10. a. $4^{\square} = 1$
b. $\log_4 1 = \square$

For Exercises 11–22, write the equation in exponential form. **(See Example 1.)**

11. $\log_5 625 = 4$ **12.** $\log_{125} 25 = \frac{2}{3}$ **13.** $\log_{10}(0.0001) = -4$ **14.** $\log_{25}\left(\frac{1}{5}\right) = -\frac{1}{2}$

15. $\log_6 36 = 2$ **16.** $\log_2 128 = 7$ **17.** $\log_b 15 = x$ **18.** $\log_b 82 = y$

19. $\log_3 5 = x$ **20.** $\log_2 7 = x$ **21.** $\log_{1/4} x = 10$ **22.** $\log_{1/2} x = 6$

For Exercises 23–34, write the equation in logarithmic form.

23. $3^x = 81$ **24.** $10^3 = 1000$ **25.** $5^2 = 25$ **26.** $8^{1/3} = 2$

27. $7^{-1} = \frac{1}{7}$ **28.** $8^{-2} = \frac{1}{64}$ **29.** $b^x = y$ **30.** $b^y = x$

31. $e^x = y$ **32.** $e^y = x$ **33.** $\left(\frac{1}{3}\right)^{-2} = 9$ **34.** $\left(\frac{5}{2}\right)^{-1} = \frac{2}{5}$

Concept 2: Evaluating Logarithmic Expressions

For Exercises 35–50, evaluate the logarithm without the use of a calculator. **(See Examples 2–3.)**

35. $\log_7 49$ **36.** $\log_3 81$ **37.** $\log_{10} 0.1$ **38.** $\log_2\left(\frac{1}{16}\right)$

39. $\log_{16} 4$ **40.** $\log_8 2$ **41.** $\log_{7/2} 1$ **42.** $\log_{1/2} 2$

43. $\log_3 3^5$ **44.** $\log_9 9^3$ **45.** $\log_{10} 10$ **46.** $\log_7 1$

47. $\log_a a^3$ **48.** $\log_r r^4$ **49.** $\log_x \sqrt{x}$ **50.** $\log_y \sqrt[3]{y}$

Concept 3: The Common Logarithmic Function

For Exercises 51–58, evaluate the common logarithm without the use of a calculator. **(See Example 4.)**

51. $\log 10$ **52.** $\log 100$ **53.** $\log 1000$

54. $\log 10{,}000$ **55.** $\log(1.0 \times 10^6)$ **56.** $\log 0.1$

57. $\log 0.01$ **58.** $\log 0.001$

For Exercises 59–70, use a calculator to approximate the logarithms. Round to four decimal places. **(See Example 5.)**

59. $\log 6$

60. $\log 18$

61. $\log \pi$

62. $\log\left(\frac{1}{8}\right)$

63. $\log\left(\frac{1}{32}\right)$

64. $\log \sqrt{5}$

65. $\log 0.0054$

66. $\log 0.0000062$

67. $\log\,(3.4 \times 10^5)$

68. $\log\,(4.78 \times 10^9)$

69. $\log\,(3.8 \times 10^{-8})$

70. $\log\,(2.77 \times 10^{-4})$

71. Given: $\log 10 = 1$ and $\log 100 = 2$

a. Estimate $\log 93$.

b. Estimate $\log 12$.

c. Evaluate the logarithms in parts (a) and (b) on a calculator and compare to your estimates.

72. Given: $\log\left(\frac{1}{10}\right) = -1$ and $\log 1 = 0$

a. Estimate $\log\left(\frac{9}{10}\right)$.

b. Estimate $\log\left(\frac{1}{5}\right)$.

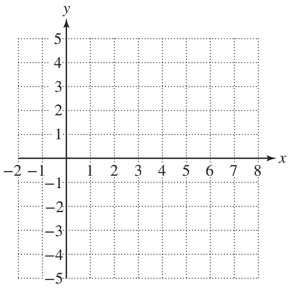
c. Evaluate the logarithms in parts (a) and (b) on a calculator and compare to your estimates.

Concept 4: Graphs of Logarithmic Functions

73. Let $f(x) = \log_4 x$.

a. Find the values of $f\left(\frac{1}{64}\right)$, $f\left(\frac{1}{16}\right)$, $f\left(\frac{1}{4}\right)$, $f(1)$, $f(4)$, $f(16)$, and $f(64)$.

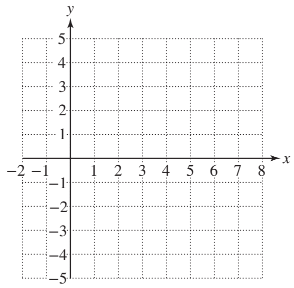
b. Graph $y = f(x)$. **(See Example 6.)**



74. Let $g(x) = \log_2 x$.

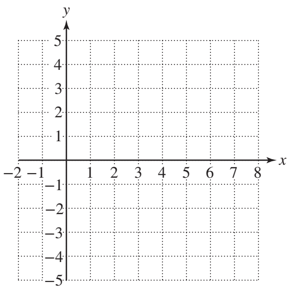
a. Find the values of $g\left(\frac{1}{8}\right)$, $g\left(\frac{1}{4}\right)$, $g\left(\frac{1}{2}\right)$, $g(1)$, $g(2)$, $g(4)$, and $g(8)$.

b. Graph $y = g(x)$.



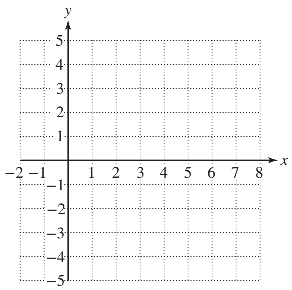
Graph the logarithmic functions in Exercises 75–78 by writing the function in exponential form and making a table of points. **(See Example 6.)**

75. $y = \log_3 x$



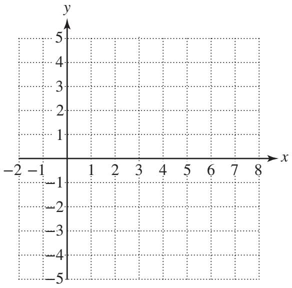
x	y

76. $y = \log_5 x$



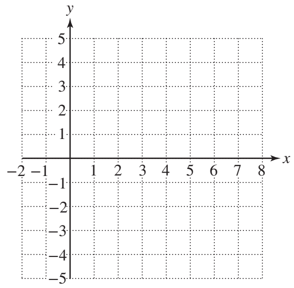
x	y

77. $y = \log_{1/2} x$ **(See Example 6.)**



x	y

78. $y = \log_{1/3} x$



x	y

For Exercises 79–90, find the domain of the function and express the domain in interval notation. **(See Example 7.)**

79. $y = \log_7 (x - 5)$

80. $y = \log (x - 4)$

81. $g(x) = \log (2 - x)$

82. $f(x) = \log (1 - x)$

83. $y = \log (3x - 1)$

84. $y = \log_3 (2x + 1)$

85. $y = \log_3 (x + 1.2)$

86. $y = \log \left(x - \frac{1}{2}\right)$

87. $k(x) = \log (4 - 2x)$

88. $h(x) = \log (6 - 2x)$

89. $y = \log x^2$

90. $y = \log (x^2 + 1)$

Concept 5: Applications of the Common Logarithmic Function

For Exercises 91–92, use the formula $\text{pH} = -\log [H^+]$, where $[H^+]$ represents the concentration of hydrogen ions in moles per liter. Round to two decimal places. **(See Example 8.)**

91. Normally, the level of hydrogen ions in the blood is approximately 4.47×10^{-8} mol/L. Find the pH level of blood.

92. The normal pH level for streams and rivers is between 6.5 and 8.5. A high level of bacteria in a particular stream caused environmentalists to test the water. The level of hydrogen ions was found to be 0.006 mol/L. What is the pH level of the stream?

93. A graduate student in education is doing research to compare the effectiveness of two different teaching techniques designed to teach vocabulary to sixth-graders. The first group of students (group 1) was taught with method I, in which the students worked individually to complete the assignments in a workbook. The second group (group 2) was taught with method II, in which the students worked cooperatively in groups of four to complete the assignments in the same workbook.

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None of the students knew any of the vocabulary words before the study began. After completing the assignments in the workbook, the students were then tested on the vocabulary at 1-month intervals to assess how much material they had retained over time. The students' average scores t months after completing the assignments are given by the following functions:

Method I: $S_1(t) = 91 - 30 \log(t + 1)$, where t is the time in months and $S_1(t)$ is the average score of students in group 1.

Method II: $S_2(t) = 88 - 15 \log(t + 1)$, where t is the time in months and $S_2(t)$ is the average score of students in group 2.

a. Complete the table to find the average scores for each group of students after the indicated number of months. Round to one decimal place. **(See Example 9.)**

b. Based on the table of values, what were the average scores for each group immediately after completion of the assigned material ($t = 0$)?

c. Based on the table of values, which teaching method helped students retain the material better for a long time?

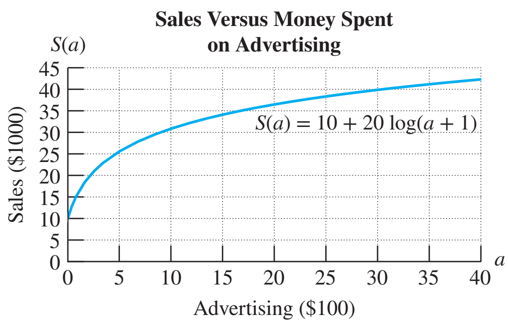
t **(months)**	0	1	2	6	12	24
$S_1(t)$						
$S_2(t)$						

94. Generally, the more money a company spends on advertising, the higher the sales. Let a represent the amount of money spent on advertising (in \$100s). Then the amount of money in sales $S(a)$ (in \$1000s) is given by

$$S(a) = 10 + 20 \log(a + 1) \text{ where } a \geq 0$$

a. The value of $S(1) \approx 16.0$ means that if \$100 is spent on advertising, \$16,000 is returned in sales. Find the values of $S(11)$, $S(21)$, and $S(31)$. Round to one decimal place. Interpret the meaning of each function value in the context of this problem.

b. The graph of $y = S(a)$ is shown here. Use the graph and your answers from part (a) to explain why the money spent in advertising becomes less "efficient" as it is used in larger amounts.



Technology Connections

For Exercises 95–100, graph the function on an appropriate viewing window. From the graph, identify the domain of the function and the location of the vertical asymptote.

95. $y = \log(x + 6)$

96. $y = \log(2x + 4)$

97. $y = \log(0.5x - 1)$

98. $y = \log(x + 8)$

99. $y = \log(2 - x)$

100. $y = \log(3 - x)$

Problem Recognition Exercises

Identifying Graphs of Functions

Match the function with the appropriate graph. Do not use a calculator.

1. $g(x) = 3^x$

2. $f(x) = \log x$

3. $h(x) = x^2$

4. $k(x) = -2x - 3$

5. $L(x) = |x|$

6. $m(x) = \sqrt{x}$

7. $B(x) = 3$

8. $A(x) = -x^2$

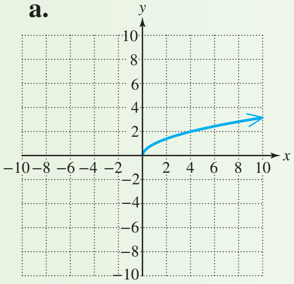
9. $n(x) = \sqrt[3]{x}$

10. $p(x) = x^3$

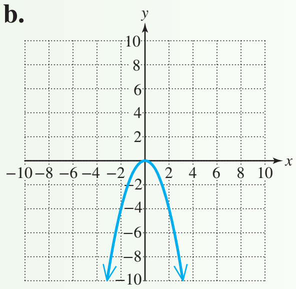
11. $q(x) = \dfrac{1}{x}$

12. $r(x) = x$

a.



b.



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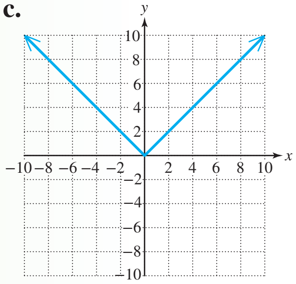

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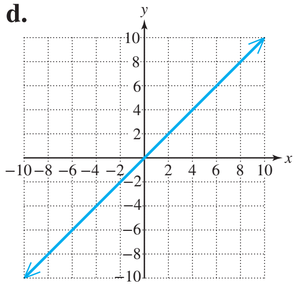

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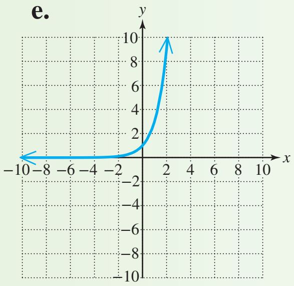

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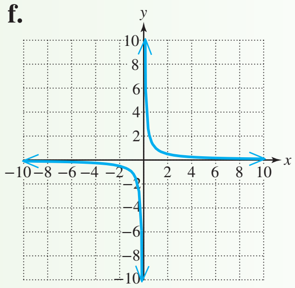
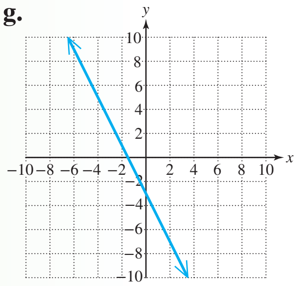


g.



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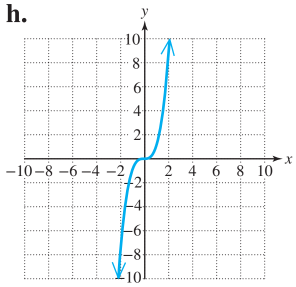
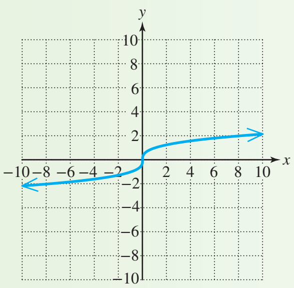


i.



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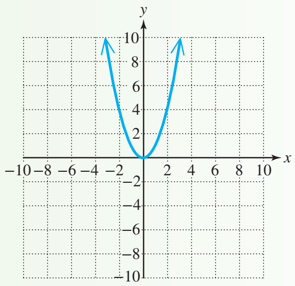
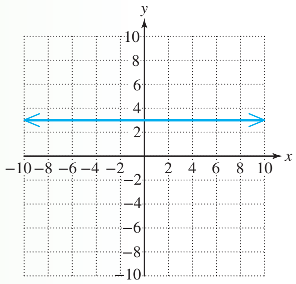
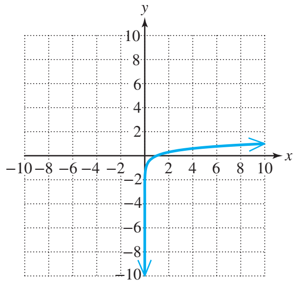


k.



l.



Properties of Logarithms

Section 8.5

Concepts

1. Properties of Logarithms
2. Expanded Logarithmic Expressions
3. Single Logarithmic Expressions

1. Properties of Logarithms

You have already been exposed to certain properties of logarithms that follow directly from the definition. Recall

$$y = \log_b x \quad \text{is equivalent to} \quad b^y = x \quad \text{for } x > 0,\ b > 0, \text{ and } b \neq 1$$

The following properties follow directly from the definition.

Property	Explanation
1. $\log_b 1 = 0$	Exponential form: $b^0 = 1$
2. $\log_b b = 1$	Exponential form: $b^1 = b$
3. $\log_b b^p = p$	Exponential form: $b^p = b^p$
4. $b^{\log_b x} = x$	Logarithmic form: $\log_b x = \log_b x$

Example 1 Applying the Properties of Logarithms to Simplify Expressions

Use the properties of logarithms to simplify the expressions. Assume that all variable expressions within the logarithms represent positive real numbers.

a. $\log_8 8 + \log_8 1$ **b.** $10^{\log(x+2)}$ **c.** $\log_{1/2}\left(\frac{1}{2}\right)^x$

Solution:

a. $\log_8 8 + \log_8 1 = 1 + 0 = 1$ Properties 2 and 1

b. $10^{\log(x+2)} = x + 2$ Property 4

c. $\log_{1/2}\left(\frac{1}{2}\right)^x = x$ Property 3

Skill Practice Use the properties of logarithms to simplify the expressions.

1. $\log_5 1 + \log_5 5$ **2.** $15^{\log_{15} 7}$ **3.** $\log_{1/3}\left(\frac{1}{3}\right)^c$

Three additional properties are useful when simplifying logarithmic expressions. The first is the product property for logarithms.

Product Property of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b (xy) = \log_b x + \log_b y$$

The logarithm of a product equals the sum of the logarithms of the factors.

Answers

1. 1 **2.** 7 **3.** c

Proof:

Let $M = \log_b x$, which implies $b^M = x$.

Let $N = \log_b y$, which implies $b^N = y$.

Then $xy = b^M b^N = b^{M+N}$.

Writing the expression $xy = b^{M+N}$ in logarithmic form, we have,

$$\log_b (xy) = M + N$$

$$\log_b (xy) = \log_b x + \log_b y \checkmark$$

To demonstrate the product property for logarithms, simplify the following expressions by using the order of operations.

$$\log_3 (3 \cdot 9) \stackrel{?}{=} \log_3 3 + \log_3 9$$

$$\log_3 27 \stackrel{?}{=} 1 + 2$$

$$3 \stackrel{?}{=} 3 \checkmark \text{ True}$$

Quotient Property of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

The logarithm of a quotient equals the difference of the logarithms of the numerator and denominator.

The proof of the quotient property for logarithms is similar to the proof of the product property and is omitted here. To demonstrate the quotient property for logarithms, simplify the following expressions by using the order of operations.

$$\log \left(\frac{1{,}000{,}000}{100}\right) \stackrel{?}{=} \log (1{,}000{,}000) - \log (100)$$

$$\log (10{,}000) \stackrel{?}{=} 6 - 2$$

$$4 \stackrel{?}{=} 4 \checkmark \text{ True}$$

Power Property of Logarithms

Let b and x be positive real numbers where $b \neq 1$. Let p be any real number. Then

$$\log_b x^p = p \log_b x$$

To demonstrate the power property for logarithms, simplify the following expressions by using the order of operations.

$$\log_4 4^2 \stackrel{?}{=} 2 \log_4 4$$

$$2 \stackrel{?}{=} 2 \cdot 1$$

$$2 \stackrel{?}{=} 2 \checkmark \text{ True}$$

The properties of logarithms are summarized in the box.

Summary of the Properties of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties of logarithms are true.

1. $\log_b 1 = 0$	**5.** $\log_b (xy) = \log_b x + \log_b y$	**Product property for logarithms**
2. $\log_b b = 1$	**6.** $\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$	**Quotient property for logarithms**
3. $\log_b b^p = p$	**7.** $\log_b x^p = p \log_b x$	**Power property for logarithms**
4. $b^{\log_b x} = x$		

2. Expanded Logarithmic Expressions

In many applications, it is advantageous to expand a logarithm into a sum or difference of simpler logarithms.

Example 2 Writing a Logarithmic Expression in Expanded Form

Write the expression as the sum or difference of logarithms of x, y, and z. Assume all variables represent positive real numbers.

$$\log_3 \left(\frac{xy^3}{z^2}\right)$$

Solution:

$$\log_3 \left(\frac{xy^3}{z^2}\right)$$

$= \log_3 (xy^3) - \log_3 z^2$ Quotient property for logarithms (property 6)

$= [\log_3 x + \log_3 y^3] - \log_3 z^2$ Product property for logarithms (property 5)

$= \log_3 x + 3 \log_3 y - 2 \log_3 z$ Power property for logarithms (property 7)

Skill Practice Write the expression as the sum or difference of logarithms of a, b, and c. Assume all variables represent positive real numbers.

4. $\log_5 \left(\dfrac{a^2b^3}{c}\right)$

Example 3 Writing a Logarithmic Expression in Expanded Form

Write the expression as the sum or difference of logarithms of x and y. Assume all variables represent positive real numbers.

$$\log \left(\frac{\sqrt{x+y}}{10}\right)$$

Answer

4. $2 \log_5 a + 3 \log_5 b - \log_5 c$

Solution:

$$\log\left(\frac{\sqrt{x+y}}{10}\right)$$

$= \log(\sqrt{x+y}) - \log 10$ — Quotient property for logarithms (property 6)

$= \log(x+y)^{1/2} - 1$ — Write $\sqrt{x+y}$ as $(x+y)^{1/2}$ and simplify $\log 10 = 1$.

$= \frac{1}{2}\log(x+y) - 1$ — Power property for logarithms (property 7)

Skill Practice Write the expression as the sum or difference of logarithms of a and b. Assume $a > b$.

5. $\log\dfrac{(a-b)^2}{\sqrt{10}}$

Example 4 Writing a Logarithmic Expression in Expanded Form

Write the expression as the sum or difference of logarithms of x, y, and z. Assume all variables represent positive real numbers.

$$\log_b \sqrt[5]{\frac{x^4}{yz^3}}$$

Solution:

$$\log_b \sqrt[5]{\frac{x^4}{yz^3}}$$

$= \log_b\left(\frac{x^4}{yz^3}\right)^{1/5}$ — Write $\sqrt[5]{\frac{x^4}{yz^3}}$ as $\left(\frac{x^4}{yz^3}\right)^{1/5}$.

$= \frac{1}{5}\log_b\left(\frac{x^4}{yz^3}\right)$ — Power property for logarithms (property 7)

$= \frac{1}{5}[\log_b x^4 - \log_b(yz^3)]$ — Quotient property for logarithms (property 6)

$= \frac{1}{5}[\log_b x^4 - (\log_b y + \log_b z^3)]$ — Product property for logarithms (property 5)

$= \frac{1}{5}[\log_b x^4 - \log_b y - \log_b z^3]$ — Distributive property

$= \frac{1}{5}[4\log_b x - \log_b y - 3\log_b z]$ — Power property for logarithms (property 7)

or $\quad \frac{4}{5}\log_b x - \frac{1}{5}\log_b y - \frac{3}{5}\log_b z$

Skill Practice Write the expression as the sum or difference of logarithms of a, b, and c. Assume all variables represent positive real numbers.

6. $\log_3 \sqrt[4]{\dfrac{a}{bc^5}}$

Answers

5. $2\log(a-b) - \frac{1}{2}$

6. $\frac{1}{4}\log_3 a - \frac{1}{4}\log_3 b - \frac{5}{4}\log_3 c$

3. Single Logarithmic Expressions

In some applications, it is necessary to write a sum or difference of logarithms as a single logarithm.

Example 5 Writing a Sum or Difference of Logarithms as a Single Logarithm

Rewrite the expression as a single logarithm, and simplify the result, if possible.

$$\log_2 560 - \log_2 7 - \log_2 5$$

Solution:

$\log_2 560 - \log_2 7 - \log_2 5$

$= \log_2 560 - (\log_2 7 + \log_2 5)$	Factor out -1 from the last two terms.
$= \log_2 560 - \log_2 (7 \cdot 5)$	Product property for logarithms (property 5)
$= \log_2\left(\dfrac{560}{7 \cdot 5}\right)$	Quotient property for logarithms (property 6)
$= \log_2 16$	Simplify inside parentheses.
$= \log_2 2^4$	Write 16 in base 2. That is, $16 = 2^4$.
$= 4$	Property 3

Skill Practice Write the expression as a single logarithm, and simplify the result, if possible.

7. $\log_3 54 + \log_3 10 - \log_3 20$

Example 6 Writing a Sum or Difference of Logarithms as a Single Logarithm

Rewrite the expression as a single logarithm, and simplify the result, if possible. Assume all variable expressions within the logarithms represent positive real numbers.

$$2 \log x - \frac{1}{2} \log y + 3 \log z$$

Solution:

$2 \log x - \dfrac{1}{2} \log y + 3 \log z$

$= \log x^2 - \log y^{1/2} + \log z^3$	Power property for logarithms (property 7)
$= \log x^2 + \log z^3 - \log y^{1/2}$	Group terms with positive coefficients.
$= \log (x^2z^3) - \log y^{1/2}$	Product property for logarithms (property 5)
$= \log\left(\dfrac{x^2z^3}{y^{1/2}}\right)$ or $\log\left(\dfrac{x^2z^3}{\sqrt{y}}\right)$	Quotient property for logarithms (property 6)

Skill Practice Write the expression as a single logarithm, and simplify, if possible.

8. $3 \log x + \dfrac{1}{3} \log y - 2 \log z$

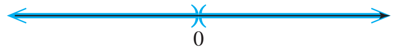
Answers

7. 3 **8.** $\log\left(\dfrac{x^3\sqrt[3]{y}}{z^2}\right)$

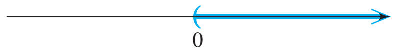
It is important to note that the properties of logarithms may be used to write a single logarithm as a sum or difference of logarithms. Furthermore, the properties may be used to write a sum or difference of logarithms as a single logarithm. In either case, these operations may change the domain.

For example, consider the function $y = \log_b x^2$. Using the power property for logarithms, we have $y = 2 \log_b x$. Consider the domain of each function:

$y = \log_b x^2$ Domain: $(-\infty, 0) \cup (0, \infty)$



$y = 2 \log_b x$ Domain: $(0, \infty)$



These two functions are equivalent only for values of x in the intersection of the two domains, that is, for $(0, \infty)$.

Section 8.5 Activity

For Exercises A.1–A.4, investigate the properties of logarithms by completing the table. Work across the table, doing one row at a time.

	Example	Example	Example	Property
A.1.	**a.** $\log_5 5 =$ ____	**b.** $\log_2 2 =$ ____	**c.** $\log_{1/2}\left(\frac{1}{2}\right) =$ ____	**d.** $\log_b b =$ ____
A.2.	**a.** $\log_5 1 =$ ____	**b.** $\log_2 1 =$ ____	**c.** $\log_{1/2} 1 =$ ____	**d.** $\log_b 1 =$ ____
A.3.	**a.** $\log_5 (5)^3 =$ ____	**b.** $\log_2 (2)^4 =$ ____	**c.** $\log_{1/2}\left(\frac{1}{2}\right)^5 =$ ____	**d.** $\log_b (b)^p =$ ____
A.4.	**a.** $5^{\log_5(25)} =$ ____	**b.** $2^{\log_2(16)} =$ ____	**c.** $\left(\frac{1}{2}\right)^{\log_{1/2}\left(\frac{1}{8}\right)} =$ ____	**d.** $b^{\log_b x} =$ ____

For Exercises A.5–A.7, investigate three more properties of logarithms. Work across the table, doing one row at a time.

	Example	Property
A.5.	$\log_2 8 = \log_2 4 + \log_2 2$ $\downarrow \quad \downarrow \quad \downarrow$ $\square = \square + \square$	**Product property of logarithms** $\log_b (x \cdot y) =$ ________________
A.6.	$\log_5\left(\frac{125}{5}\right) = \log_5 125 - \log_5 5$ $\downarrow \quad \downarrow \quad \downarrow$ $\square = \square - \square$	**Quotient property of logarithms** $\log_b\left(\frac{x}{y}\right) =$ ________________
A.7.	$\log_4 (4)^3 = 3 \cdot \log_4(4)$ $\downarrow \quad \downarrow \quad \downarrow$ $\square = \square \cdot \square$	**Power property of logarithms** $\log_b (x)^p =$ ________________

A.8. Write the expression as a single logarithm. Assume that all variables represent positive real numbers.

$$5 \log_2 x - 2 \log_2 y + \frac{1}{2}\log_2 z - \frac{1}{3}\log_2 w$$

A.9. Write the expression as a sum or difference of $\log x$, $\log y$, and $\log z$. Assume that all variables represent positive real numbers.

$$\log\left(\frac{100\sqrt{x}\,y^3}{z}\right)$$

Practice Exercises

Section 8.5

Prerequisite Review

For Exercises R.1–R.10, use the properties of exponents to simplify the expression.

R.1. $x^{-4} \cdot x^{10}$ **R.2.** $p^{13} \cdot p^{-6}$ **R.3.** $\dfrac{w^{15}}{w^4}$ **R.4.** $\dfrac{m^{20}}{m^{15}}$

R.5. $(n^3)^4$ **R.6.** $(k^2)^5$ **R.7.** t^{-6} **R.8.** a^{-4}

R.9. $(3a^{-5}b^3)^2$ **R.10.** $\left(\dfrac{2c^4}{d^{-6}}\right)^3$

For Exercises R.11–R.16, simplify the expression.

R.11. $\log_4 64$ **R.12.** $\log_2 32$ **R.13.** $\log_2 \dfrac{1}{16}$ **R.14.** $\log_4 \dfrac{1}{16}$

R.15. $\log 1000$ **R.16.** $\log \dfrac{1}{1000}$

Vocabulary and Key Concepts

1. **a.** Fill in the blanks to complete the basic properties of logarithms.
$\log_b b =$ __________, $\log_b 1 =$ __________, $\log_b b^x =$ __________, and $b^{\log_b x} =$ __________.

b. If b, x, and y are positive real numbers and $b \neq 1$, then $\log_b(xy) =$ __________ and $\log_b\left(\dfrac{x}{y}\right) =$ __________.

c. If b and x are positive real numbers and $b \neq 1$, then for any real number p, $\log_b x^p$ can be written as __________.

d. Determine if the statement is true or false: $\log_b(xy) = (\log_b x)(\log_b y)$
Use the expression $\log_2 (4 \cdot 8)$ to help you answer.

e. Determine if the statement is true or false: $\log_b\left(\dfrac{x}{y}\right) = \dfrac{\log_b x}{\log_b y}$
Use the expression $\log_3\left(\dfrac{27}{9}\right)$ to help you answer.

f. Determine if the statement is true or false: $\log_b(x)^p = (\log_b x)^p$
Use the expression $\log(1000)^2$ to help you answer.

Concept 1: Properties of Logarithms

2. Select the values that are equivalent to $\log_5 5^2$.

a. $\log_5 25$
b. $2 \log_5 5$
c. $\log_5 5 + \log_5 5$

3. Select the values that are equivalent to $\log_2 2^3$.

a. $3 \log_2 2$
b. $\log_2 8$
c. 3

4. Select the values that are equivalent to $\log 10^4$.

a. 4
b. $4 \log 10$
c. $\log 10{,}000$

For Exercises 5–28, evaluate each expression. **(See Example 1.)**

5. $\log_3 3$ **6.** $\log 10$ **7.** $\log_5 5^4$ **8.** $\log_4 4^5$

9. $6^{\log_6 11}$ **10.** $7^{\log_7 2}$ **11.** $\log 10^3$ **12.** $\log_6 6^3$

13. $\log_3 1$ **14.** $\log_8 1$ **15.** $10^{\log 9}$ **16.** $8^{\log_8 5}$

17. $\log_{1/2} 1$ **18.** $\log_{1/3}\left(\frac{1}{3}\right)$ **19.** $\log_2 1 + \log_2 2^3$ **20.** $\log 10^4 + \log 10$

21. $\log_4 4 + \log_2 1$ **22.** $\log_7 7 + \log_4 4^2$ **23.** $\log_{1/4}\left(\frac{1}{4}\right)^{2x}$ **24.** $\log_{2/3}\left(\frac{2}{3}\right)^{p}$

25. $\log_a a^4$ **26.** $\log_y y^2$ **27.** $\log 10^2 - \log_3 3^2$ **28.** $\log_6 6^4 - \log 10^4$

29. Compare the expressions by approximating their values on a calculator. Which two expressions are equivalent?

a. $\log(3 \cdot 5)$ **b.** $\log 3 \cdot \log 5$ **c.** $\log 3 + \log 5$

30. Compare the expressions by approximating their values on a calculator. Which two expressions are equivalent?

a. $\log\left(\frac{6}{5}\right)$ **b.** $\frac{\log 6}{\log 5}$ **c.** $\log 6 - \log 5$

31. Compare the expressions by approximating their values on a calculator. Which two expressions are equivalent?

a. $\log 20^2$ **b.** $[\log 20]^2$ **c.** $2 \log 20$

32. Compare the expressions by approximating their values on a calculator. Which two expressions are equivalent?

a. $\log \sqrt{4}$ **b.** $\frac{1}{2}\log 4$ **c.** $\sqrt{\log 4}$

Concept 2: Expanded Logarithmic Expressions

For Exercises 33–50, expand into sums and/or differences of logarithms. Assume all variables represent positive real numbers. **(See Examples 2–4.)**

33. $\log_3\left(\frac{x}{5}\right)$ **34.** $\log_2\left(\frac{y}{z}\right)$ **35.** $\log(2x)$

36. $\log_6(xyz)$ **37.** $\log_5 x^4$ **38.** $\log_7 z^{1/3}$

39. $\log_4\left(\frac{ab}{c}\right)$ **40.** $\log_2\left(\frac{x}{yz}\right)$ **41.** $\log_b\left(\frac{\sqrt{xy}}{z^3 w}\right)$

42. $\log\left(\frac{a\sqrt[3]{b}}{cd^2}\right)$ **43.** $\log_2\left(\frac{x+1}{y^2\sqrt{z}}\right)$ **44.** $\log\left(\frac{a+1}{b\sqrt[3]{c}}\right)$

45. $\log\left(\sqrt[3]{\frac{ab^2}{c}}\right)$ **46.** $\log_5\left(\sqrt[4]{\frac{w^3 z}{x^2}}\right)$ **47.** $\log\left(\frac{1}{w^5}\right)$

48. $\log_3\left(\frac{1}{z^4}\right)$ **49.** $\log_b\left(\frac{\sqrt{a}}{b^3 c}\right)$ **50.** $\log_x\left(\frac{x}{y\sqrt{z}}\right)$

Concept 3: Single Logarithmic Expressions

For Exercises 51–66, write the expressions as a single logarithm and simplify if possible. Assume all variable expressions represent positive real numbers. **(See Examples 5–6.)**

51. $\log_3 270 - \log_3 2 - \log_3 5$ **52.** $\log_5 8 + \log_5 50 - \log_5 16$

53. $\log_7 98 - \log_7 2$ **54.** $\log_6 24 - \log_6 4$

55. $2 \log_3 x - 3 \log_3 y + \log_3 z$

56. $\log C + \log A + \log B + \log I + \log N$

57. $2 \log_3 a - \frac{1}{4} \log_3 b + \log_3 c$

58. $\log_5 a - \frac{1}{2} \log_5 b - 3 \log_5 c$

59. $\log_b x - 3 \log_b x + 4 \log_b x$

60. $2 \log_3 z + \log_3 z - \frac{1}{2} \log_3 z$

61. $5 \log_8 a - \log_8 1 + \log_8 8$

62. $\log_2 2 + 2 \log_2 b - \log_2 1$

63. $2 \log (x + 6) + \frac{1}{3} \log y - 5 \log z$

64. $\frac{1}{4} \log (a + 1) - 2 \log b - 4 \log c$

65. $\log_b (x + 1) - \log_b (x^2 - 1)$

66. $\log_x (p^2 - 4) - \log_x (p - 2)$

Technology Connections

67. **a.** Graph $Y_1 = \log x^2$ and state its domain.

b. Graph $Y_2 = 2 \log x$ and state its domain.

c. For what values of x are the expressions $\log x^2$ and $2 \log x$ equivalent?

68. **a.** Graph $Y_1 = \log (x - 1)^2$ and state its domain.

b. Graph $Y_2 = 2 \log (x - 1)$ and state its domain.

c. For what values of x are the expressions $\log (x - 1)^2$ and $2 \log (x - 1)$ equivalent?

Expanding Your Skills

For Exercises 69–80, find the values of the logarithms given that $\log_b 2 \approx 0.693$, $\log_b 3 \approx 1.099$, and $\log_b 5 \approx 1.609$.

69. $\log_b 6$

70. $\log_b 4$

71. $\log_b 12$

72. $\log_b 25$

73. $\log_b 81$

74. $\log_b 30$

75. $\log_b\left(\frac{5}{2}\right)$

76. $\log_b\left(\frac{25}{3}\right)$

77. $\log_b 10^6$

78. $\log_b 15^3$

79. $\log_b 5^{10}$

80. $\log_b 2^{12}$

81. The intensity of sound waves is measured in decibels and is calculated by the formula

$$B = 10 \log\left(\frac{I}{I_0}\right)$$

where I_0 is the minimum detectable decibel level.

a. Expand this formula by using the properties of logarithms.

b. Let $I_0 = 10^{-16}$ W/cm^2 and simplify.

82. The Richter scale is used to measure the intensity of an earthquake and is calculated by the formula

$$R = \log\left(\frac{I}{I_0}\right)$$

where I_0 is the minimum level detectable by a seismograph.

a. Expand this formula by using the properties of logarithms.

b. Let $I_0 = 1$ and simplify.

Section 8.6 The Irrational Number e and Change of Base

Concepts

1. The Irrational Number e
2. Computing Compound Interest
3. The Natural Logarithmic Function
4. Change-of-Base Formula
5. Applications of the Natural Logarithmic Function

1. The Irrational Number e

The exponential function base 10 is particularly easy to work with because integral powers of 10 represent different place positions in the base-10 numbering system. In this section, we introduce another important exponential function whose base is an irrational number called e.

Consider the expression $(1 + \frac{1}{x})^x$. The value of the expression for increasingly large values of x approaches a constant (Table 8-5).

Table 8-5

x	$\left(1+\dfrac{1}{x}\right)^x$
100	2.70481382942
1000	2.71692393224
10,000	2.71814592683
100,000	2.71826823717
1,000,000	2.71828046932
1,000,000,000	2.71828182710

As x approaches infinity, the expression $(1 + \frac{1}{x})^x$ approaches a constant value that we call ***e***. From Table 8-5, this value is approximately 2.718281828.

$$e \approx 2.718281828$$

The value of e is an irrational number (a nonterminating, nonrepeating decimal) and like the number π, it is a universal constant.

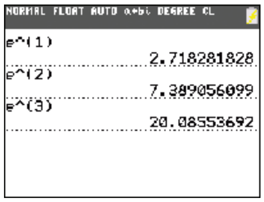
Example 1 Graphing $f(x) = e^x$

Graph the function defined by $f(x) = e^x$.

Solution:

Because the base of the function is greater than 1 ($e \approx 2.718281828$), the graph is an increasing exponential function. We can use a calculator to evaluate $f(x) = e^x$ at several values of x.

Practice using your calculator by evaluating e^x for $x = 1$, $x = 2$, and $x = 3$.



If you are using your calculator correctly, your answers should match those found in Table 8-6. Values are rounded to three decimal places. The corresponding graph of $f(x) = e^x$ is shown in Figure 8-15.

TIP: On a scientific calculator, you may need to enter the base and exponent of an exponential expression in reverse order. For example, enter e^2 as:

2 2nd LN

Table 8-6

x	$f(x) = e^x$
−3	0.050
−2	0.135
−1	0.368
0	1.000
1	2.718
2	7.389
3	20.086

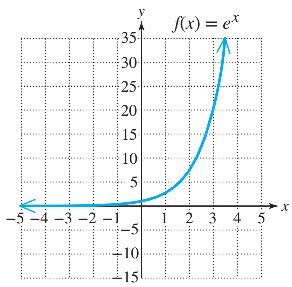


Figure 8-15

Skill Practice

1. Graph $f(x) = e^x + 1$.

2. Computing Compound Interest

One particularly interesting application of exponential functions is in computing compound interest.

1. If the number of compounding periods per year is finite, then the amount $A(t)$ in an account is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P is the initial principal, r is the annual interest rate, n is the number of times compounded per year, and t is the time in years that the money is invested.

2. If the number of compound periods per year is infinite, then interest is said to be **compounded continuously**. In such a case, the amount $A(t)$ in the account is given by

$$A(t) = Pe^{rt}$$

where P is the initial principal, r is the annual interest rate, and t is the time in years that the money is invested.

Example 2 Computing the Balance on an Account

Suppose \$5000 is invested in an account earning 6.5% interest. Find the balance in the account after 10 years under the following compounding options.

a. Compounded annually

b. Compounded quarterly

c. Compounded monthly

d. Compounded daily

e. Compounded continuously

Solution:

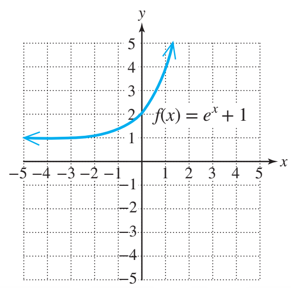
Compounding Option	n Value	Formula	Result
Annually	$n = 1$	$A(10) = 5000\left(1 + \frac{0.065}{1}\right)^{(1)(10)}$	\$9385.69
Quarterly	$n = 4$	$A(10) = 5000\left(1 + \frac{0.065}{4}\right)^{(4)(10)}$	\$9527.79
Monthly	$n = 12$	$A(10) = 5000\left(1 + \frac{0.065}{12}\right)^{(12)(10)}$	\$9560.92
Daily	$n = 365$	$A(10) = 5000\left(1 + \frac{0.065}{365}\right)^{(365)(10)}$	\$9577.15
Continuously	Not applicable	$A(10) = 5000e^{(0.065)(10)}$	\$9577.70

Notice that there is a \$191.46 difference in the account balance between annual compounding and daily compounding. However, the difference between compounding daily and compounding continuously is small, \$0.55. As n gets infinitely large, the function defined by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{converges to } A(t) = Pe^{rt}$$

Answer

1.



Skill Practice

2. Suppose \$1000 is invested at 5%. Find the balance after 8 years under the following options.

a. Compounded annually
b. Compounded quarterly
c. Compounded monthly
d. Compounded daily
e. Compounded continuously

3. The Natural Logarithmic Function

Recall that the common logarithmic function $y = \log x$ has a base of 10. Another important logarithmic function is called the **natural logarithmic function**. The natural logarithmic function has a base of e and is written as $y = \ln x$. That is,

$$y = \ln x = \log_e x$$

Example 3 Graphing $y = \ln x$

Graph $y = \ln x$.

Solution:

Because the base of the function $y = \ln x$ is e and $e > 1$, the graph is an increasing logarithmic function. We can use a calculator to find specific points on the graph of $y = \ln x$ by pressing the LN key.

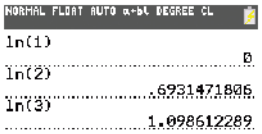
Practice using your calculator by evaluating $\ln x$ for the following values of x. If you are using your calculator correctly, your answers should match those found in Table 8-7. Values are rounded to three decimal places. The corresponding graph of $y = \ln x$ is shown in Figure 8-16.

TIP: On a scientific calculator, you may need to enter a logarithm and argument in reverse order. For example, enter ln 2 as

2 LN

Table 8-7

x	$\ln x$
0.25	−1.386
0.5	−0.693
1	0.000
2	0.693
3	1.099
4	1.386
5	1.609
6	1.792
7	1.946



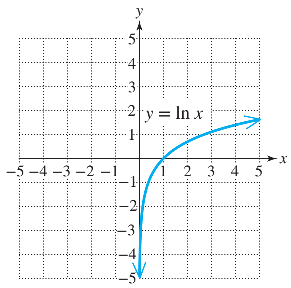


Figure 8-16

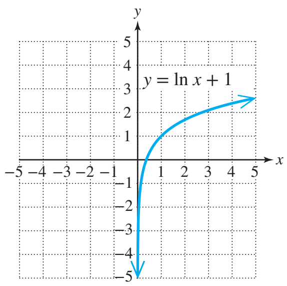
Skill Practice

3. Graph $y = \ln x + 1$.

Answers

2. a. \$1477.46 **b.** \$1488.13
c. \$1490.59 **d.** \$1491.78
e. \$1491.82

3.



The properties of logarithms learned earlier in this chapter are also true for natural logarithms.

Properties of the Natural Logarithmic Function

Let x and y be positive real numbers, and let p be a real number. Then the following properties are true.

1. $\ln 1 = 0$	**5.** $\ln(xy) = \ln x + \ln y$	**Product property for logarithms**
2. $\ln e = 1$	**6.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$	**Quotient property for logarithms**
3. $\ln e^p = p$	**7.** $\ln x^p = p \ln x$	**Power property for logarithms**
4. $e^{\ln x} = x$		

Example 4 Simplifying Expressions With Natural Logarithms

Simplify the expressions. Assume that all variable expressions within the logarithms represent positive real numbers.

a. $\ln e$ **b.** $\ln 1$ **c.** $\ln(e^{x+1})$ **d.** $e^{\ln(x+1)}$

Solution:

a. $\ln e = 1$ Property 2

b. $\ln 1 = 0$ Property 1

c. $\ln(e^{x+1}) = x + 1$ Property 3

d. $e^{\ln(x+1)} = x + 1$ Property 4

Skill Practice Simplify.

4. $\ln e^2$ **5.** $-3 \ln 1$ **6.** $\ln e^{(x+y)}$ **7.** $e^{\ln(3x)}$

Example 5 Writing a Sum or Difference of Natural Logarithms as a Single Logarithm

Write the expression as a single logarithm. Assume that all variable expressions within the logarithms represent positive real numbers.

$$\ln x - \frac{1}{3}\ln y - \ln z$$

Solution:

$\ln x - \frac{1}{3}\ln y - \ln z$

$= \ln x - \ln y^{1/3} - \ln z$ Power property for logarithms (property 7)

$= \ln x - (\ln y^{1/3} + \ln z)$ Factor out -1 from the last two terms.

$= \ln x - \ln(y^{1/3} \cdot z)$ Product property for logarithms (property 5)

Answers

4. 2 **5.** 0 **6.** $x + y$ **7.** $3x$

$= \ln\left(\dfrac{x}{y^{1/3}z}\right)$ Quotient property for logarithms (property 6)

$= \ln\left(\dfrac{x}{\sqrt[3]{y}\,z}\right)$ Write the rational exponent as a radical.

Skill Practice Write as a single logarithm.

8. $\dfrac{1}{4}\ln a - \ln b + \ln c$

Example 6 Writing a Logarithmic Expression in Expanded Form

Write the expression as a sum or difference of logarithms of x and y. Assume all variable expressions within the logarithm represent positive real numbers.

$$\ln\left(\frac{e}{x^2\sqrt{y}}\right)$$

Solution:

$\ln\left(\dfrac{e}{x^2\sqrt{y}}\right)$

$= \ln e - \ln(x^2\sqrt{y})$ Quotient property for logarithms (property 6)

$= \ln e - (\ln x^2 + \ln\sqrt{y})$ Product property for logarithms (property 5)

$= 1 - \ln x^2 - \ln y^{1/2}$ Distributive property. Also simplify $\ln e = 1$ (property 2).

$= 1 - 2\ln x - \dfrac{1}{2}\ln y$ Power property for logarithms (property 7)

Skill Practice Write as a sum or difference of logarithms of x and y.

9. $\ln\left(\dfrac{x^3\sqrt{y}}{e^2}\right)$

4. Change-of-Base Formula

A calculator can be used to approximate the value of a logarithm with a base of 10 or a base of e by using the LOG key or the LN key, respectively. However, to use a calculator to evaluate a logarithmic expression with a base other than 10 or e, we must use the **change-of-base formula**.

Change-of-Base Formula

Let a and b be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then for any positive real number x,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Answers

8. $\ln\left(\dfrac{\sqrt[4]{a}\,c}{b}\right)$

9. $3\ln x + \dfrac{1}{2}\ln y - 2$

Proof:

Let $M = \log_b x$, which implies that $b^M = x$.

Take the logarithm, base a, on both sides: $\log_a b^M = \log_a x$

Apply the power property for logarithms: $M \cdot \log_a b = \log_a x$

Divide both sides by $\log_a b$: $$\frac{M \cdot \cancel{\log_a b}}{\cancel{\log_a b}} = \frac{\log_a x}{\log_a b}$$

$$M = \frac{\log_a x}{\log_a b}$$

Because $M = \log_b x$, we have $$\log_b x = \frac{\log_a x}{\log_a b}\ \checkmark$$

The change-of-base formula converts a logarithm of one base to a ratio of logarithms of a different base. For the sake of using a calculator, we often apply the change-of-base formula with base 10 or base e.

Example 7 Using the Change-of-Base Formula

a. Use the change-of-base formula to evaluate $\log_4 80$ by using base 10. (Round to three decimal places.)

b. Use the change-of-base formula to evaluate $\log_4 80$ by using base e. (Round to three decimal places.)

Solution:

a. $$\log_4 80 = \frac{\log_{10} 80}{\log_{10} 4} = \frac{\log 80}{\log 4} \approx \frac{1.903089987}{0.6020599913} \approx 3.161$$

b. $$\log_4 80 = \frac{\log_e 80}{\log_e 4} = \frac{\ln 80}{\ln 4} \approx \frac{4.382026635}{1.386294361} \approx 3.161$$

To check the result, we see that $4^{3.161} \approx 80$.

Skill Practice

10. Use the change-of-base formula to evaluate $\log_5 95$ by using base 10. Round to three decimal places.

11. Use the change-of-base formula to evaluate $\log_5 95$ by using base e. Round to three decimal places.

5. Applications of the Natural Logarithmic Function

Plant and animal tissue contains both carbon-12 and carbon-14. Carbon-12 is a stable form of carbon, whereas carbon-14 is a radioactive isotope with a half-life of approximately 5730 years. While a plant or animal is living, it takes in carbon from the atmosphere either through photosynthesis or through its food. The ratio of carbon-14 to carbon-12 in a living organism is constant and is the same as the ratio found in the atmosphere.

When a plant or animal dies, it no longer ingests carbon from the atmosphere. The amount of stable carbon-12 remains unchanged from the time of death, but the carbon-14 begins to decay. Because the rate of decay is constant, a tissue sample can be dated by comparing the percent of carbon-14 still present to the percentage of carbon-14 assumed to be in its original living state.

Answers

10. 2.829 **11.** 2.829

The age of a tissue sample is a function of the percent of carbon-14 still present in the organism according to the following model:

$$A(p) = \frac{\ln p}{-0.000121}$$

where $A(p)$ is the age in years and p is the percentage (in decimal form) of carbon-14 still present.

Example 8 Applying the Natural Logarithmic Function to Radioactive Decay

Using the formula

$$A(p) = \frac{\ln p}{-0.000121}$$

a. Find the age of a bone that has 72% of its initial carbon-14.

b. Find the age of the Iceman, a body uncovered in the mountains of northern Italy in 1991. Samples of his hair revealed that 52.7% of the original carbon-14 was present after his death.

Solution:

a. $A(p) = \frac{\ln p}{-0.000121}$

$A(0.72) = \frac{\ln 0.72}{-0.000121}$ Substitute 0.72 for p.

≈ 2715 years

b. $A(p) = \frac{\ln p}{-0.000121}$

$A(0.527) = \frac{\ln 0.527}{-0.000121}$ Substitute 0.527 for p.

≈ 5300 years The body of the Iceman is approximately 5300 years old.

Skill Practice

12. Use the formula

$$A(p) = \frac{\ln p}{-0.000121}$$

(where $A(p)$ is the age in years and p is the percent of carbon-14 still present) to determine the age of a human skull that has 90% of its initial carbon-14.

Answer

12. ≈ 871 years

Section 8.6 Activity

A.1. Evaluate the expression $\left(1 + \frac{1}{x}\right)^x$ for the following values of x. Round to five decimal places.

a. 5000 **b.** 50,000 **c.** 500,000 **d.** 5,000,000

A.2. For larger and larger values of x, the expression $\left(1 + \frac{1}{x}\right)^x$ converges to a constant real number that mathematicians designate by e. The value of e is an irrational number (its decimal form is nonterminating and nonrepeating).

a. Based on the results of Exercise A.1, write the value of e to five decimal places.

b. Based on the value of e, is the function defined by $f(x) = e^x$ an exponential growth function or an exponential decay function?

A.3. The following three functions, A_1, A_2, and A_3, determine (in order) the amount of money in an account earning simple interest, interest compounded annually, and interest compounded continuously.

$$A_1(t) = P + Prt \qquad A_2(t) = P(1 + r)^t \qquad A_3(t) = Pe^{rt}$$

P represents the amount of principal invested, r represents the annual interest rate, and t represents the time in years of the investment. For this example, let $P = \$5000$ and $r = 0.08$.

a. Evaluate $A_1(30)$ and interpret its meaning in context.

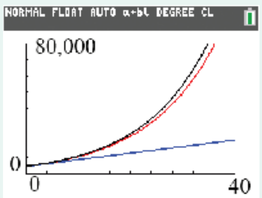
b. Evaluate $A_2(30)$ and interpret its meaning in context.

c. Evaluate $A_3(30)$ and interpret its meaning in context.

d. From parts (a)–(c), which method of compounding yields the best return?

e. How much more money is earned in 30 years for interest compounded continuously than for simple interest?

f. Match the functions A_1, A_2, and A_3 with their graphs.



A.4. The inverse of the exponential function defined by $y = e^x$ is the ________ function, base e, denoted by ________.

A.5. **a.** Write the domain and range of $f(x) = e^x$ in interval notation.

b. Write the domain and range of $g(x) = \ln x$ in interval notation.

c. How are the domain and range of functions f and g related?

The properties of logarithms learned to this point also apply to the natural logarithm function. Exercises A.6–A.8 investigate these properties.

A.6. Evaluate the logarithmic expressions.

a. $\ln 1$ **b.** $\ln e$ **c.** $\ln e^4$ **d.** $e^{\ln 6}$

A.7. Write the expression as a single logarithm. Assume that all variable expressions represent positive real numbers.

$$\ln (x^2 - y^2) - \ln(x - y) - 3 \ln z$$

A.8. Write the expression as a sum or difference of $\ln a$, $\ln b$, and $\ln c$. Assume that all variables represent positive real numbers.

$$\ln\left(\frac{a^2}{b\sqrt[3]{c}}\right)$$

A.9. **a.** Write the domain of $f(x) = \ln(3x + 12)$ in interval notation.

b. Write the domain of $g(x) = \ln(6 - x)$ in interval notation.

A.10. **a.** Complete the change-of-base formula by converting the expression into an equivalent expression using base a: $\log_b x = \boxed{\frac{\quad}{\quad}}$

b. Without using a calculator, evaluate $\log_2 16$.

c. Without using a calculator, evaluate $\log_2 32$.

d. Between which two integers is the value of $\log_2 20$?

e. Use a calculator and the change-of-base formula with common logarithms to approximate the value of $\log_2 20$. Round to four decimal places.

f. Use a calculator and the change-of-base formula with natural logarithms to approximate the value of $\log_2 20$. Round to four decimal places.

g. Compare your answers to parts (e) and (f).

Section 8.6 Practice Exercises

Prerequisite Review

R.1. Evaluate the expression $\left(1 + \frac{1}{x}\right)^x$ for the given values of x. Round to six decimal places.

a. $x = 2000$

b. $x = 2{,}000{,}000$

R.2. Evaluate the expression $P\left(1 + \frac{r}{n}\right)^{nt}$ for the given values of the variables. Round to two decimal places.

a. $P = 5000$, $r = 0.05$, $n = 12$, and $t = 10$

b. $P = 8000$, $r = 0.06$, $n = 4$, and $t = 20$

For Exercises R.3–R.10, simplify the expression. Assume that all variables represent positive real numbers.

R.3. $\log 1$

R.4. $\log_5 1$

R.5. $\log_6 6$

R.6. $\log_8 8$

R.7. $\log_2(2^5)$

R.8. $\log(10^4)$

R.9. $10^{\log(3)}$

R.10. $5^{\log_5 7}$

For Exercises R.11–R.12, write the expression as a single logarithm. Assume that all variables represent positive real numbers.

R.11. $\frac{1}{3}\log_5 x + \log_5 y - 2\log_5 z$

R.12. $\log a - 5\log c + \frac{1}{2}\log d$

For Exercises R.13–R.14, write the expression as the sum or difference of logarithms of x, y, and z. Assume that all variables represent positive real numbers.

R.13. $\log\left(\frac{\sqrt{x}}{yz^4}\right)$

R.14. $\log_3\left(\frac{xy^5}{\sqrt[3]{z}}\right)$

Vocabulary and Key Concepts

1. **a.** As x becomes increasingly large, the value of $(1 + \frac{1}{x})^x$ approaches ____________ ≈ 2.71828.

   b. The function $f(x) = e^x$ is the exponential function base ____________.

   c. The logarithmic function base e is called the ________ logarithmic function and is denoted by $y =$ ________.

   d. If x is a positive real number, then $\ln 1 =$ ______, $\ln e =$ ______, $\ln e^p =$ ______, and $e^{\ln x} =$ ______.

   e. If x and y are positive real numbers, then $\ln(xy) =$ ____________ and $\ln\left(\frac{x}{y}\right) =$ ____________.

   f. If x is a positive real number, then for any real number p, $\ln x^p$ can be written as ____________.

   g. The change-of-base formula states that $\log_b x = \frac{\log_a x}{\square}$, where a is a positive real number and $a \neq 1$.

2. $\log(x)$ is a logarithmic expression with a base of ________.

3. $\ln(x)$ is a logarithmic expression with a base of ________.

4. $f(x) = e^x$ defines a(n) (increasing/decreasing) exponential function.

5. $g(x) = \ln x$ defines a(n) (increasing/decreasing) logarithmic function.

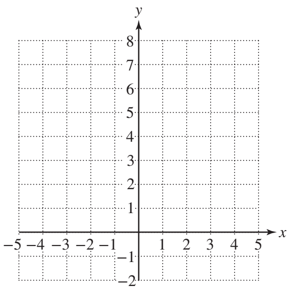
Concept 1: The Irrational Number *e*

6. From memory, write a decimal approximation of the number e, correct to three decimal places.

For Exercises 7–10, graph the equation by completing the table and plotting points. Identify the domain. Round to two decimal places when necessary. **(See Example 1.)**

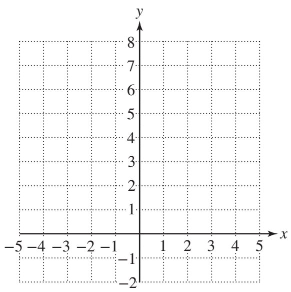
7. $y = e^{x+1}$

x	y
−4	
−3	
−2	
−1	
0	
1	



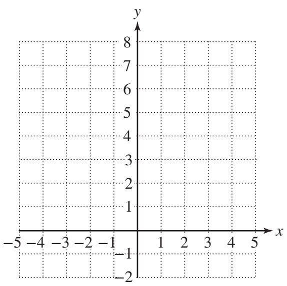
8. $y = e^{x+2}$

x	y
−5	
−4	
−3	
−2	
−1	
0	



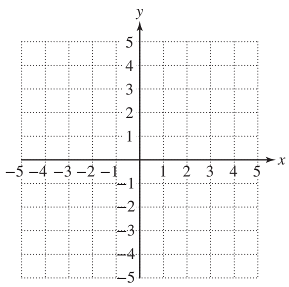
9. $y = e^x + 2$

x	y
−2	
−1	
0	
1	
2	
3	



10. $y = e^x - 1$

x	y
−4	
−3	
−2	
−1	
0	
1	



Concept 2: Computing Compound Interest

For Exercises 11–16, suppose that P dollars in principal is invested at an annual interest rate r. For interest compounded n times per year, the amount $A(t)$ in the account after t years is given by $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$. If interest is compounded continuously, the amount is given by $A(t) = Pe^{rt}$.

11. Suppose an investor deposits \$10,000 in an account for 5 years for which the interest is compounded monthly. Find the total amount of money in the account for the following interest rates. Compare your answers and comment on the effect of interest rate on an investment. **(See Example 2.)**

a. $r = 4.0\%$ **b.** $r = 6.0\%$ **c.** $r = 8.0\%$ **d.** $r = 9.5\%$

12. Suppose an investor deposits \$5000 in an account for 8 years for which the interest is compounded quarterly. Find the total amount of money in the account for the following interest rates. Compare your answers and comment on the effect of interest rate on an investment.

a. $r = 4.5\%$ **b.** $r = 5.5\%$ **c.** $r = 7.0\%$ **d.** $r = 9.0\%$

13. Suppose an investor deposits \$8000 in an account for 10 years at 4.5% interest. Find the total amount of money in the account for the following compounding options. Compare your answers. How does the number of compound periods per year affect the total investment?

a. Compounded annually
b. Compounded quarterly
c. Compounded monthly
d. Compounded daily
e. Compounded continuously

14. Suppose an investor deposits \$15,000 in an account for 8 years at 5.0% interest. Find the total amount of money in the account for the following compounding options. Compare your answers. How does the number of compound periods per year affect the total investment?

a. Compounded annually
b. Compounded quarterly
c. Compounded monthly
d. Compounded daily
e. Compounded continuously

15. Suppose an investor deposits \$5000 in an account earning 6.5% interest compounded continuously. Find the total amount in the account for the following time periods. How does the length of time affect the amount of interest earned?

a. 5 years **b.** 10 years **c.** 15 years **d.** 20 years **e.** 30 years

16. Suppose an investor deposits \$10,000 in an account earning 6.0% interest compounded continuously. Find the total amount in the account for the following time periods. How does the length of time affect the amount of interest earned?

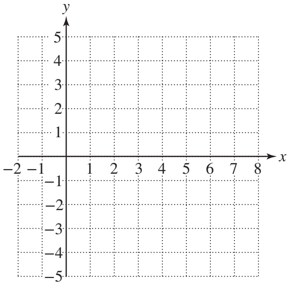
a. 5 years **b.** 10 years **c.** 15 years **d.** 20 years **e.** 30 years

Concept 3: The Natural Logarithmic Function

For Exercises 17–20, graph the equation by completing the table and plotting the points. Identify the domain. Round to two decimal places when necessary. **(See Example 3.)**

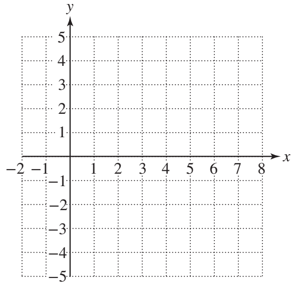
17. $y = \ln(x - 2)$

x	y
2.25	
2.50	
2.75	
3	
4	
5	
6	



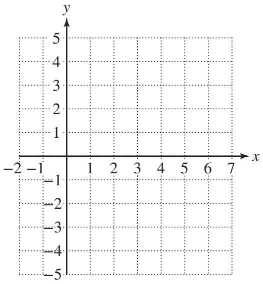
18. $y = \ln(x - 1)$

x	y
1.25	
1.50	
1.75	
2	
3	
4	
5	



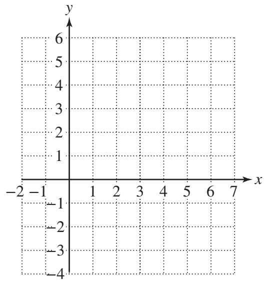
19. $y = \ln x - 1$

x	y
0.25	
0.5	
0.75	
1	
2	
3	
4	

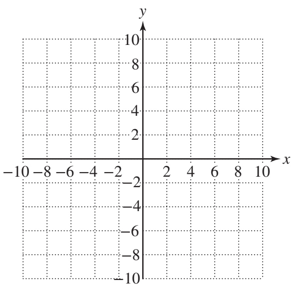


20. $y = \ln x + 2$

x	y
0.25	
0.5	
0.75	
1	
2	
3	
4	



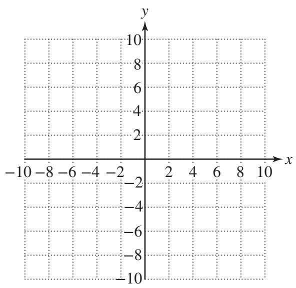
21. a. Graph $f(x) = 10^x$ and $g(x) = \log x$.



b. Identify the domain and range of f.

c. Identify the domain and range of g.

22. a. Graph $f(x) = e^x$ and $g(x) = \ln x$.



b. Identify the domain and range of f.

c. Identify the domain and range of g.

For Exercises 23–30, simplify the expressions. Assume all variables represent positive real numbers. **(See Example 4.)**

23. $\ln e$

24. $\ln e^2$

25. $\ln 1$

26. $e^{\ln x}$

27. $\ln e^{-6}$

28. $\ln e^{(5-3x)}$

29. $e^{\ln(2x+3)}$

30. $e^{\ln 4}$

For Exercises 31–38, write the expression as a single logarithm. Assume all variables represent positive real numbers. **(See Example 5.)**

31. $6 \ln p + \frac{1}{3} \ln q$

32. $2 \ln w + \ln z$

33. $\frac{1}{2} (\ln x - 3 \ln y)$

34. $\frac{1}{3}(4 \ln a - \ln b)$

35. $2 \ln a - \ln b - \frac{1}{3} \ln c$

36. $-\ln x + 3 \ln y - \ln z$

37. $4 \ln x - 3 \ln y - \ln z$

38. $\frac{1}{2} \ln c + \ln a - 2 \ln b$

For Exercises 39–46, write the expression as a sum and/or difference of ln a, ln b, and ln c. Assume all variables represent positive real numbers. **(See Example 6.)**

39. $\ln\left(\frac{a}{b}\right)^2$

40. $\ln \sqrt[3]{\frac{a}{b}}$

41. $\ln (b^2 \cdot e)$

42. $\ln (\sqrt{c} \cdot e)$

43. $\ln\left(\frac{a^4\sqrt{b}}{c}\right)$

44. $\ln\left(\frac{\sqrt{ab}}{c^3}\right)$

45. $\ln\left(\frac{ab}{c^2}\right)^{1/5}$

46. $\ln \sqrt{2ab}$

For Exercises 47–52, write the domain in interval notation.

47. $f(x) = \ln (x - 4)$

48. $g(x) = \ln (x + 3)$

49. $h(x) = \ln (2x + 5)$

50. $k(x) = \ln (4x - 1)$

51. $m(x) = \ln (7 - x)$

52. $n(x) = \ln (10 - x)$

Concept 4: Change-of-Base Formula

53. a. Evaluate $\log_6 200$ by computing $\frac{\log 200}{\log 6}$ to four decimal places.

b. Evaluate $\log_6 200$ by computing $\frac{\ln 200}{\ln 6}$ to four decimal places.

c. How do your answers to parts (a) and (b) compare?

54. a. Evaluate $\log_8 120$ by computing $\frac{\log 120}{\log 8}$ to four decimal places.

b. Evaluate $\log_8 120$ by computing $\frac{\ln 120}{\ln 8}$ to four decimal places.

c. How do your answers to parts (a) and (b) compare?

For Exercises 55–66, use the change-of-base formula to approximate the logarithms to four decimal places. Check your answers by using the exponential key on your calculator. **(See Example 7.)**

55. $\log_2 7$

56. $\log_3 5$

57. $\log_8 24$

58. $\log_4 17$

59. $\log_8 0.012$

60. $\log_7 0.251$

61. $\log_9 1$

62. $\log_2\left(\frac{1}{5}\right)$

63. $\log_4\left(\frac{1}{100}\right)$

64. $\log_5 0.0025$

65. $\log_7 0.0006$

66. $\log_2 0.24$

Concept 5: Applications of the Natural Logarithmic Function

Under continuous compounding, the amount of time t in years required for an investment to double is a function of the annual interest rate r according to the formula:

$$t = \frac{\ln 2}{r}$$

Use the formula for Exercises 67–70. **(See Example 8.)**

67. **a.** If you invest \$5000, how long will it take the investment to reach \$10,000 if the interest rate is 4.5%? Round to one decimal place.

b. If you invest \$5000, how long will it take the investment to reach \$10,000 if the interest rate is 10%? Round to one decimal place.

c. Using the doubling time found in part (b), how long would it take a \$5000 investment to reach \$20,000 if the interest rate is 10%?

68. **a.** If you invest \$3000, how long will it take the investment to reach \$6000 if the interest rate is 5.5%? Round to one decimal place.

b. If you invest \$3000, how long will it take the investment to reach \$6000 if the interest rate is 8%? Round to one decimal place.

c. Using the doubling time found in part (b), how long would it take a \$3000 investment to reach \$12,000 if the interest rate is 8%?

69. **a.** If you invest \$4000, how long will it take the investment to reach \$8000 if the interest rate is 3.5%? Round to one decimal place.

b. If you invest \$4000, how long will it take the investment to reach \$8000 if the interest rate is 5%? Round to one decimal place.

c. Using the doubling time found in part (b), how long would it take a \$4000 investment to reach \$16,000 if the interest rate is 5%?

70. On August 31, 1854, an epidemic of cholera was discovered in London, England, resulting from a contaminated community water pump at Broad Street. By the end of September more than 600 citizens who drank water from the pump had died.

The cumulative number of deaths from cholera in the 1854 London epidemic can be approximated by

$$D(t) = 91 + 160 \ln (t + 1)$$

where t is the number of days after the start of the epidemic ($t = 0$ corresponds to September 1, 1854).

a. Approximate the total number of deaths as of September 1 ($t = 0$).

b. Approximate the total number of deaths as of September 5, September 10, and September 20.

Technology Connections

71. **a.** Graph the function defined by $f(x) = \log_3 x$ by graphing $Y_1 = \dfrac{\log x}{\log 3}$.

b. Graph the function defined by $f(x) = \log_3 x$ by graphing $Y_2 = \dfrac{\ln x}{\ln 3}$.

c. Does it appear that $Y_1 = Y_2$?

72. **a.** Graph the function defined by $f(x) = \log_7 x$ by graphing $Y_1 = \dfrac{\log x}{\log 7}$.

b. Graph the function defined by $f(x) = \log_7 x$ by graphing $Y_2 = \dfrac{\ln x}{\ln 7}$.

c. Does it appear that $Y_1 = Y_2$?

For Exercises 73–75, graph the functions on a graphing calculator.

73. Graph $s(x) = \log_{1/2} x$

74. Graph $y = e^{x-2}$

75. Graph $y = e^x - 4$

Problem Recognition Exercises

Logarithmic and Exponential Forms

Fill out the table by writing the exponential expressions in logarithmic form, and the logarithmic expressions in exponential form. Use the fact that:

$$y = \log_b x \quad \text{is equivalent to} \quad b^y = x$$

	Exponential Form	Logarithmic Form
1.	$2^5 = 32$	
2.		$\log_3 81 = 4$
3.	$z^y = x$	
4.		$\log_b a = c$
5.	$10^3 = 1000$	
6.		$\log 10 = 1$
7.	$e^a = b$	
8.		$\ln p = q$
9.	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	
10.		$\log_{1/3} 9 = -2$
11.	$10^{-2} = 0.01$	
12.		$\log 4 = x$
13.	$e^0 = 1$	
14.		$\ln e = 1$
15.	$25^{1/2} = 5$	
16.		$\log_{16} 2 = \frac{1}{4}$
17.	$e^t = s$	
18.		$\ln w = r$
19.	$15^{-2} = \frac{1}{225}$	
20.		$\log_3 p = -1$

Logarithmic and Exponential Equations and Applications

Section 8.7

Concepts

1. Solving Logarithmic Equations

Equations containing one or more logarithms are called **logarithmic equations**. For example, the following are logarithmic equations.

$$\log_3(2x+5)=1 \qquad \text{and} \qquad \log_4 x = 1-\log_4(x-3)$$

To solve equations involving logarithms of first degree, we use the following guidelines.

Solving Logarithmic Equations

Step 1 Isolate the logarithms on one side of the equation.

Step 2 Write a sum or difference of logarithms as a single logarithm.

Step 3 Rewrite the equation in step 2 in exponential form.

Step 4 Solve the resulting equation from step 3.

Step 5 Check all solutions to verify that they are within the domain of the logarithmic expressions in the original equation.

Example 1 Solving a Logarithmic Equation

Solve the equation. $\log_3(2x+5)=1$

Solution:

$\log_3(2x+5)=1$ This equation has a single logarithm isolated on the left-hand side.

$2x+5=3^1$ Write the logarithm in its equivalent exponential form.

$2x+5=3$ Solve the resulting equation.

$2x=-2$

$x=-1$ Check: $x=-1$

$\log_3(2x+5)=1$

$\log_3[2(-1)+5]\stackrel{?}{=}1$

$\log_3(3)\stackrel{?}{=}1$ ✓

The solution set is $\{-1\}$.

Skill Practice Solve the equation.

1. $\log_6(5x-4)=2$

Answer

1. $\{8\}$

Example 2 **Solving a Logarithmic Equation**

Solve the equation. $\log_4 x = 1 - \log_4 (x - 3)$

Solution:

$$\log_4 x = 1 - \log_4 (x - 3)$$

$\log_4 x + \log_4 (x - 3) = 1$ Isolate the logarithms on one side of the equation.

$\log_4 [x(x - 3)] = 1$ Write as a single logarithm.

$\log_4 (x^2 - 3x) = 1$ Simplify inside the parentheses.

$x^2 - 3x = 4^1$ Write the equation in exponential form.

$x^2 - 3x - 4 = 0$ The resulting equation is quadratic.

$(x - 4)(x + 1) = 0$ Factor.

$x = 4$ or $x = -1$ Apply the zero product rule.

Notice that -1 is *not* a solution because $\log_4 x$ is not defined at $x = -1$. However, $x = 4$ *is* defined in both expressions $\log_4 x$ and $\log_4 (x - 3)$. We can substitute $x = 4$ into the original equation to show that it checks.

Check: $x = 4$

$$\log_4 x = 1 - \log_4 (x - 3)$$

$$\log_4 4 \stackrel{?}{=} 1 - \log_4 (4 - 3)$$

$$1 \stackrel{?}{=} 1 - \log_4 1$$

$$1 \stackrel{?}{=} 1 - 0 \checkmark \text{ True}$$

The solution set is $\{4\}$. (The value -1 does not check.)

Skill Practice Solve the equation.

2. $\log_3 (x - 8) = 2 - \log_3 x$

FOR REVIEW

Recall:

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

Example 3 **Solving a Logarithmic Equation**

Solve the equation. $\log (x + 300) = 3.7$

Solution:

$\log (x + 300) = 3.7$ The equation has a single logarithm that is already isolated.

$10^{3.7} = x + 300$ Write the equation in exponential form.

$10^{3.7} - 300 = x$ Solve for x.

$$x = 10^{3.7} - 300$$

$$\approx 4711.87$$

Answer

2. $\{9\}$ (The value -1 does not check.)

Check: $x = 10^{3.7} - 300$ — Check the exact value of x in the original equation.

$$\log(x + 300) = 3.7$$
$$\log[(10^{3.7} - 300) + 300] \stackrel{?}{=} 3.7$$
$$\log(10^{3.7} - 300 + 300) \stackrel{?}{=} 3.7$$
$$\log 10^{3.7} \stackrel{?}{=} 3.7$$ Property 3 of logarithms: $\log_b b^p = p$
$$3.7 \stackrel{?}{=} 3.7 \checkmark \text{ True}$$

The solution $10^{3.7} - 300$ checks.

The solution set is $\{10^{3.7} - 300\}$.

Skill Practice Solve the equation.

3. $\log(p + 6) = 1.3$

The following property is another useful tool to solve a logarithmic equation. This is demonstrated in Example 4.

Equivalence of Logarithmic Expressions

Let x, y, and b represent real numbers such that $x > 0$, $y > 0$, $b > 0$, and $b \neq 1$. Then

$$\log_b x = \log_b y \quad \text{implies} \quad x = y$$

The equivalence property of logarithmic expressions indicates that if two logarithms of the same base are equal, then their arguments are equal.

Example 4 Solving a Logarithmic Equation

Solve the equation.

$$\ln(x + 2) + \ln(x - 1) = \ln(9x - 17)$$

Solution:

$$\ln(x + 2) + \ln(x - 1) = \ln(9x - 17)$$

$$\ln[(x + 2)(x - 1)] = \ln(9x - 17)$$ Write the logarithms on the left-hand side as a single logarithm.

$$\ln(x^2 + x - 2) = \ln(9x - 17)$$ Simplify.

$$x^2 + x - 2 = 9x - 17$$ Use the equivalence property of logarithms. That is, equate the arguments of each log.

$$x^2 - 8x + 15 = 0$$ Solve the resulting quadratic equation.

$$(x - 5)(x - 3) = 0$$

$$x = 5 \quad \text{or} \quad x = 3$$

Answer

3. $\{10^{1.3} - 6\}$

The solutions 5 and 3 are both within the domain of the logarithmic functions in the original equation. Both solutions check.

The solution set is $\{5, 3\}$.

Skill Practice Solve the equation.

4. $\ln(t-3) + \ln(t-1) = \ln(2t-5)$

2. Solving Exponential Equations

An equation with one or more exponential expressions is called an **exponential equation**. The following property is often useful in solving exponential equations.

Equivalence of Exponential Expressions

Let x, y, and b be real numbers such that $b > 0$ and $b \neq 1$. Then

$$b^x = b^y \quad \text{implies} \quad x = y$$

The equivalence property of exponential expressions indicates that if two exponential expressions of the same base are equal, then their exponents must be equal.

Example 5 Solving an Exponential Equation

Solve the equation. $4^{2x-9} = 64$

Solution:

$4^{2x-9} = 64$

$4^{2x-9} = 4^3$ Write both sides with a common base.

$2x - 9 = 3$ If $b^x = b^y$, then $x = y$.

$2x = 12$ Solve for x.

$x = 6$

To check, substitute $x = 6$ into the original equation.

$$4^{2(6)-9} \stackrel{?}{=} 64$$

$$4^{12-9} \stackrel{?}{=} 64$$

$$4^3 \stackrel{?}{=} 64 \checkmark \text{ True}$$

The solution set is $\{6\}$.

Skill Practice Solve the equation.

5. $2^{3x+1} = 8$

Answers

4. $\{4\}$ (The value 2 does not check.)

5. $\left\{\frac{2}{3}\right\}$

Example 6 Solving an Exponential Equation

Solve the equation. $(2^x)^{x+3} = \frac{1}{4}$

Solution:

$$(2^x)^{x+3} = \frac{1}{4}$$

$$2^{x^2+3x} = 2^{-2}$$ Apply the multiplication property of exponents. Write both sides of the equation with a common base.

$$x^2 + 3x = -2$$ If $b^x = b^y$, then $x = y$.

$$x^2 + 3x + 2 = 0$$ The resulting equation is quadratic.

$$(x + 2)(x + 1) = 0$$ Solve for x.

$$x = -2 \quad \text{or} \quad x = -1$$ Both solutions check.

The solution set is $\{-2, -1\}$.

Skill Practice Solve the equation.

6. $(3^x)^{x-5} = \frac{1}{81}$

Example 7 Solving an Exponential Equation

Solve the equation. $4^x = 25$

Solution:

Because 25 cannot be written as a recognizable power of 4, we cannot easily use the property that if $b^x = b^y$, then $x = y$. Instead, we can take a logarithm of any base on both sides of the equation. Then by applying the power property of logarithms, the unknown exponent can be written as a factor.

$$4^x = 25$$

$$\log 4^x = \log 25$$ Take the common logarithm of both sides.

$$x \log 4 = \log 25$$ Apply the power property of logarithms to express the exponent as a factor. This is now a linear equation in x.

$$\frac{x \cancel{\log 4}}{\cancel{\log 4}} = \frac{\log 25}{\log 4}$$ Solve for x.

$$x = \frac{\log 25}{\log 4}$$ or approximately 2.322.

The solution set is $\left\{\frac{\log 25}{\log 4}\right\}$ or equivalently $\left\{\frac{\ln 25}{\ln 4}\right\}$.

Skill Practice Solve the equation.

7. $5^x = 32$

FOR REVIEW

Recall:

$$\log_b x^p = p \log_b x$$

For example,

$$\log 4^x = x \cdot \log 4$$

Answers

6. $\{1, 4\}$

7. $\left\{\frac{\log 32}{\log 5}\right\}$ or $\left\{\frac{\ln 32}{\ln 5}\right\}$

TIP: The equation from Example 7 is written in the form $b^x = \text{constant}$. In this special case, we also have the option of rewriting the equation in its corresponding logarithmic form to solve for x.

$$4^x = 25$$
$$x = \log_4 25$$
$$= \frac{\ln 25}{\ln 4} \qquad \text{(Change-of-base formula)}$$

Solving Exponential Equations

Step 1 Isolate one of the exponential expressions in the equation.

Step 2 Take a logarithm on both sides of the equation. (The natural logarithmic function or the common logarithmic function is often used so that the final answer can be approximated with a calculator.)

Step 3 Use the power property of logarithms (property 7) to write exponents as factors. Recall: $\log_b x^p = p \log_b x$.

Step 4 Solve the resulting equation from step 3.

Example 8 Solving an Exponential Equation by Taking a Logarithm on Both Sides

Solve the equation. $e^{-3.6x} - 2 = 7.74$

Solution:

$e^{-3.6x} - 2 = 7.74$

$e^{-3.6x} = 9.74$ Add 2 to both sides to isolate the exponential expression.

$\ln e^{-3.6x} = \ln 9.74$ The exponential expression has a base of e, so it is convenient to take the natural logarithm of both sides.

$(-3.6x)\ln e = \ln 9.74$ Use the power property of logarithms.

$-3.6x = \ln 9.74$ Simplify (recall that $\ln e = 1$).

$x = \dfrac{\ln 9.74}{-3.6} \approx -0.632$ Divide both sides by -3.6.

The solution set is $\left\{-\dfrac{\ln 9.74}{3.6}\right\}$.

Skill Practice Solve the equation.

8. $e^{-0.2t} + 1 = 8.52$

FOR REVIEW

Recall:

$$\log_b(b^x) = x$$

For example,

$$\ln\left(e^{-3.6x}\right) = -3.6x$$

This illustrates a function composed with its inverse.

Answer

8. $\left\{\dfrac{\ln 7.52}{-0.2}\right\}$

Example 9 **Solving an Exponential Equation by Taking a Logarithm on Both Sides**

Solve the equation. $2^{x+3} = 7^x$

Solution:

$$2^{x+3} = 7^x$$

$$\ln 2^{(x+3)} = \ln 7^x$$ Take the natural logarithm of both sides.

$$(x + 3)\ln 2 = x \ln 7$$ Use the power property of logarithms.

$$x(\ln 2) + 3(\ln 2) = x \ln 7$$ Apply the distributive property.

$$x(\ln 2) - x(\ln 7) = -3 \ln 2$$ Collect x terms on one side.

$$x(\ln 2 - \ln 7) = -3 \ln 2$$ Factor out x.

$$\frac{x\cancel{(\ln 2 - \ln 7)}}{\cancel{(\ln 2 - \ln 7)}} = \frac{-3 \ln 2}{\ln 2 - \ln 7}$$ Solve for x.

$$x = \frac{-3 \ln 2}{\ln 2 - \ln 7} \approx 1.66$$

The solution set is $\left\{-\dfrac{3 \ln 2}{\ln 2 - \ln 7}\right\}$.

TIP: The exponential equation $2^{x+3} = 7^x$ could have been solved by taking a logarithm of *any* base on both sides of the equation.

TIP: Using the properties of logarithms, we can write the solution to Example 9 in other forms, such as:

$x = -\dfrac{\ln 8}{\ln 2 - \ln 7}$ and

$x = \dfrac{\ln 8}{\ln 3.5}$

Skill Practice Solve the equation.

9. $3^x = 8^{x+2}$

3. Applications

Example 10 **Applying an Exponential Function to World Population**

The population of the world was estimated to have reached 6.5 billion in April 2006. The population growth rate for the world is estimated to be 1.4% (*source:* U.S. Census Bureau). The function defined by

$$P(t) = 6.5(1.014)^t$$

represents the world population $P(t)$ in billions as a function of the number of years after April 2006 ($t = 0$ represents April 2006).

a. Use the function to estimate the world population in April 2010.

b. Use the function to predict the amount of time after April 2006 required for the world population to reach 13 billion if this trend continues.

Solution:

a. $P(t) = 6.5(1.014)^t$

$P(4) = 6.5(1.014)^4$ The year 2010 corresponds to $t = 4$.

≈ 6.87 In 2010, the world's population was approximately 6.87 billion.

Answer

9. $\left\{\dfrac{2 \ln 8}{\ln 3 - \ln 8}\right\}$

b. $P(t) = 6.5(1.014)^t$

$13 = 6.5(1.014)^t$ Substitute $P(t) = 13$ and solve for t.

$\frac{13}{6.5} = \frac{\cancel{6.5}(1.014)^t}{\cancel{6.5}}$ Isolate the exponential expression on one side of the equation.

$2 = 1.014^t$

$\ln 2 = \ln 1.014^t$ Take the natural logarithm of both sides.

$\ln 2 = t \ln 1.014$ Use the power property of logarithms.

$\frac{\ln 2}{\ln 1.014} = \frac{t \cancel{\ln 1.014}}{\cancel{\ln 1.014}}$ Solve for t.

$t = \frac{\ln 2}{\ln 1.014} \approx 50$ The population will reach 13 billion (double the April 2006 value) approximately 50 years after 2006.

Note: It has taken thousands of years for the world's population to reach 6.5 billion. However, with a growth rate of 1.4%, it will take only 50 years to gain an additional 6.5 billion, if this trend continues.

Skill Practice Use the population function from Example 10.

10. Predict the world population in April 2020.

11. Predict the year in which the world population will reach 9 billion.

On Friday, April 25, 1986, a nuclear accident occurred at the Chernobyl nuclear reactor, resulting in radioactive contaminates being released into the atmosphere. The most hazardous isotopes released in this accident were ^{137}Cs (cesium-137), ^{131}I (iodine-131), and ^{90}Sr (strontium-90). People living close to Chernobyl (in Ukraine) were at risk of radiation exposure from inhalation, from absorption through the skin, and from food contamination. Years after the incident, scientists have seen an increase in the incidence of thyroid disease among children living in the contaminated areas. Because iodine is readily absorbed in the thyroid gland, scientists suspect that radiation from iodine-131 is the cause.

Example 11 Applying an Exponential Equation to Radioactive Decay

The half-life of radioactive iodine, ^{131}I, is 8.04 days. If 10 g of iodine-131 is initially present, then the amount $A(t)$ (in grams) of radioactive iodine still present after t days is approximated by

$$A(t) = 10e^{-0.0862t}$$

where t is the time in days.

a. Use the model to approximate the amount of ^{131}I still present after 2 weeks. Round to the nearest 0.1 g.

b. How long will it take for the amount of ^{131}I to decay to 0.5 g? Round to the nearest 0.1 day.

Answers

10. Approximately 7.9 billion

11. The year 2029

Solution:

a. $A(t) = 10e^{-0.0862t}$

$A(14) = 10e^{-0.0862(14)}$ Substitute $t = 14$ (2 weeks).

≈ 3.0 g There was 3.0 g still present after 2 weeks.

b. $A(t) = 10e^{-0.0862t}$

$0.5 = 10e^{-0.0862t}$ Substitute $A = 0.5$.

$\frac{0.5}{10} = \frac{\cancel{10}e^{-0.0862t}}{\cancel{10}}$ Isolate the exponential expression.

$0.05 = e^{-0.0862t}$

$\ln 0.05 = \ln e^{-0.0862t}$ Take the natural logarithm of both sides.

$\ln 0.05 = -0.0862t$ The resulting equation is linear.

$\frac{\ln 0.05}{-0.0862} = \frac{\cancel{-0.0862}t}{\cancel{-0.0862}}$ Solve for t.

$t = \frac{\ln 0.05}{-0.0862} \approx 34.8$ days It will take 34.8 days for 10 g of ^{131}I to decay to 0.5 g.

Skill Practice Radioactive strontium-90 (^{90}Sr) has a half-life of 28 years. If 100 g of strontium-90 is initially present, the amount left after t years is approximated by

$$A(t) = 100e^{-0.0248t}$$

12. Find the amount of ^{90}Sr present after 85 years.

13. How long will it take for the amount of ^{90}Sr to decay to 40 g?

Answers

12. 12.1 g **13.** 36.9 years

Section 8.7 Activity

A.1. **a.** The equivalence property of exponential expressions states that if $b^x = b^y$, then ________.

b. Consider the equation $5^{x+1} = 125$. This can be written as $5^{x+1} = 5^{\square}$.

c. Use the equivalence property of exponential expressions to solve $5^{x+1} = 125$.

d. Solve the equation $3^{2x-6} = 81$.

A.2. If the two sides of an exponential equation cannot be easily written with a common base, we must use logarithms to solve the equation. Solve the equation $e^{2x+1} - 7 = 3$ by following these steps.

a. Isolate the exponential expression on one side of the equation. In this case, isolate e^{2x+1}.

b. Take a logarithm of the same base on each side of the equation. In this case, because one of the terms in the equation involves a base of e, take the natural logarithm of the expressions on each side.

c. Apply the power property of logarithms to write the exponents as factors. (Recall $\log_b x^p = p \cdot \log_b x$.)

d. Simplify both sides of the equation and solve for x. Write the exact solution and approximate the solution to four decimal places.

A.3. Solve the equation $3^{x-4} = 8^x$ by following these steps.

a. Take a logarithm of each side (usually the common logarithm or the natural logarithm is used so that a calculator can be used to easily approximate the final answer). In this case, take the common logarithm of each side.

b. Apply the power property of logarithms on each side.

c. The equation in part (b) may look daunting, but it is a linear equation. Clear parentheses and collect all variable terms on one side of the equation. (Note that log 3 and log 8 are simply constants.) Solve the resulting equation.

d. Factor out x from the variable terms and divide by the coefficient on x.

e. Write the exact solution and approximate the solution to four decimal places.

A.4. **a.** The equivalence property of logarithmic expressions states that if $\log_b x = \log_b y$, then ________.

b. Use the equivalence property of logarithms to solve the equation $\log_4(-3x) = \log_4(2x - 15)$ for x. Do not write the solution set yet.

c. Check the potential solutions from part (b) and write the solution set.

d. Explain why it is necessary to check the potential solutions to a logarithmic equation.

A.5. If the two sides of a logarithmic equation cannot be written easily with a common base, we must follow a different approach. Solve $\log_5 x = 3 - \log_5(x - 20)$ using these steps.

a. Collect all logarithmic terms on one side of the equation.

b. Write the logarithms as a single logarithm.

c. Write the equation in exponential form.

d. Solve the resulting equation from part (c).

e. Check the potential solutions from part (d) and write the solution set to the original equation.

A.6. Radioactive iodine (^{131}I) is used to treat patients with thyroid cancer. Patients with this condition may have symptoms that include rapid weight loss, heart palpitations, and high blood pressure. Because iodine is readily absorbed in the thyroid gland, the radiation is localized and will reduce the size of the thyroid while minimizing damage to surrounding tissues. If a patient is given an initial dose of 2 μg (micrograms) of ^{131}I, the amount of ^{131}I remaining after t days is approximated by $A(t) = 2e^{-0.0866t}$.

a. How much ^{131}I remains after 5 days? Round to one decimal place.

b. How long will it take for the amount of ^{131}I to reach 0.2 μg? Round to the nearest day.

Section 8.7 Practice Exercises

Prerequisite Review

For Exercises R.1–R.4, write the expression as a single logarithm. Assume that all variable expressions represent positive real numbers.

R.1. $\log_2(x - 1) + \log_2(x + 2)$

R.2. $\log x + \log(2x + 3)$

R.3. $\log x - \log(1 - x)$

R.4. $\log_4(x + 2) - \log_4(3x - 5)$

For Exercises R.5–R.8, simplify the logarithmic expression.

R.5. $\log 10$

R.6. $\log \dfrac{1}{10}$

R.7. $\ln e^6$

R.8. $\ln \sqrt{e}$

For Exercises R.9–R.10, apply the power property of logarithms.

R.9. $\log 5^{2x+1}$

R.10. $\ln 2^{x-6}$

For Exercises R.11–R.14, simplify the expression.

R.11. $(4^2)^x$

R.12. $(5^4)^x$

R.13. $(2^5)^{x-3}$

R.14. $(3^2)^{4x+1}$

For Exercises R.15–R.20, solve the equation.

R.15. $3x - 4 = 5x + 6$

R.16. $4t + 7 = 3t + 1$

R.17. $\dfrac{2x - 1}{x + 5} = 3$

R.18. $\dfrac{x - 7}{2x + 1} = 3$

R.19. $x^2 + 8x = -12$

R.20. $x^2 - 7x = 18$

Vocabulary and Key Concepts

1. **a.** The equivalence property of logarithmic expressions states that if $\log_b x = \log_b y$, then ______________.

b. The equivalence property of exponential expressions states that if $b^x = b^y$, then ______________.

For Exercises 2–6, solve the equations.

2. **a.** $2^x = 8$ **b.** $\log_2 x = 3$

3. **a.** $5^x = 25$ **b.** $\log_5 x = 2$

4. **a.** $10^x = 1{,}000{,}000$ **b.** $\log x = 6$

5. **a.** $10^x = \dfrac{1}{10}$ **b.** $\log x = -1$

6. **a.** $6^x = \dfrac{1}{36}$ **b.** $\log_6 x = -2$

Concept 1: Solving Logarithmic Equations

For Exercises 7–38, solve the logarithmic equation. **(See Examples 1–4.)**

7. $\log_3 x = 2$

8. $\log_4 x = 9$

9. $\log p = 42$

10. $\log q = \dfrac{1}{2}$

11. $\ln x = 0.08$

12. $\ln x = 19$

13. $\log (x + 40) = -9.2$

14. $\log (z - 3) = -6.7$

15. $\log_x 25 = 2 \quad (x > 0)$

16. $\log_x 100 = 2 \quad (x > 0)$

17. $\log_b 10{,}000 = 4 \quad (b > 0)$

18. $\log_b e^3 = 3 \quad (b > 0)$

19. $\log_y 5 = \dfrac{1}{2} \quad (y > 0)$

20. $\log_b 8 = \dfrac{1}{2} \quad (b > 0)$

21. $\log_4 (c + 5) = 3$

22. $\log_5 (a - 4) = 2$

23. $\log_5 (4y + 1) = 1$

24. $\log_6 (5t - 2) = 1$

25. $\ln (1 - x) = 0$

26. $\log_4 (2 - x) = 1$

27. $\log_3 8 - \log_3 (x + 5) = 2$

28. $\log_2 (x + 3) - \log_2 (x + 2) = 1$

29. $\log_2 (h - 1) + \log_2 (h + 1) = 3$

30. $\log_3 k + \log_3 (2k + 3) = 2$

31. $\log (x + 2) = \log (3x - 6)$

32. $\log x = \log (1 - x)$

33. $\ln x - \ln (4x - 9) = 0$

34. $\ln (x + 5) - \ln x = \ln (4x)$

35. $\log_5 (3t + 2) - \log_5 t = \log_5 4$

36. $\log (6y - 7) + \log y = \log 5$

37. $\log (4m) = \log 2 + \log (m - 3)$

38. $\log (-h) + \log 3 = \log (2h - 15)$

Concept 2: Solving Exponential Equations

For Exercises 39–54, solve the exponential equation by using the property that $b^x = b^y$ implies $x = y$, for $b > 0$ and $b \neq 1$. **(See Examples 5–6.)**

39. $5^x = 625$

40. $3^x = 81$

41. $2^{-x} = 64$

42. $6^{-x} = 216$

43. $36^x = 6$

44. $343^x = 7$

45. $4^{2x-1} = 64$

46. $5^{3x-1} = 125$

47. $81^{3x-4} = \dfrac{1}{243}$

48. $4^{2x-7} = \dfrac{1}{128}$

49. $\left(\dfrac{2}{3}\right)^{-x+4} = \dfrac{8}{27}$

50. $\left(\dfrac{1}{4}\right)^{3x+2} = \dfrac{1}{64}$

51. $16^{-x+1} = 8^{5x}$

52. $27^{(1/3)x} = \left(\dfrac{1}{9}\right)^{2x-1}$

53. $(4^x)^{x+1} = 16$

54. $(3^x)^{x+2} = \dfrac{1}{3}$

For Exercises 55–74, solve the exponential equation by taking a logarithm of both sides. **(See Examples 7–9.)**

55. $8^a = 21$

56. $6^y = 39$

57. $e^x = 8.1254$

58. $e^x = 0.3151$

59. $10^t = 0.0138$

60. $10^p = 16.8125$

61. $e^{0.07h} - 6 = 9$

62. $e^{0.03k} + 7 = 11$

63. $e^{1.2t} = 3$

64. $e^{1.5t} = 7$

65. $3^{x+1} = 5^x$

66. $2^{x-1} = 7^x$

67. $2^{x+2} = 6^x$

68. $5^{x-2} = 3^x$

69. $32e^{0.04m} = 128$

70. $8e^{0.05n} = 160$

71. $6e^{x/3} = 125$

72. $8e^{x/5} = 155$

73. $5^{x-2} - 4 = 16$

74. $2^{x+4} + 17 = 50$

Concept 3: Applications

75. The population of China can be modeled by

$$P(t) = 1237(1.0095)^t$$

where $P(t)$ is in millions and t is the number of years since 1998. **(See Example 10.)**

a. Using this model, what was the population in the year 2002?

b. Estimate the population in the year 2016.

c. If this growth rate continues, in what year will the population reach 2 billion people (2 billion is 2000 million)?

76. The population of Delhi, India, can be modeled by

$$P(t) = 9817(1.031)^t$$

where $P(t)$ is in thousands and t is the number of years since 2001.

a. Using this model, estimate the population in the year 2018.

b. In what year did the population reach 15 million (15 million is 15,000 thousand)?

77. The growth of certain bacteria in a culture is given by the model

$$A(t) = 500e^{0.0277t}$$

where $A(t)$ is the number of bacteria and t is time in minutes.

a. What is the initial number of bacteria?

b. What is the population after 10 min?

c. How long will it take for the population to double (i.e., reach 1000)?

78. The population of the bacteria *Salmonella typhimurium* is given by the model

$$A(t) = 300e^{0.01733t}$$

where $A(t)$ is the number of bacteria and t is time in minutes.

a. What is the initial number of bacteria?

b. What is the population after 10 min?

c. How long will it take for the population to double?

79. Suppose \$5000 is invested at 7% interest compounded continuously. How long will it take for the investment to grow to \$10,000? Use the model $A(t) = Pe^{rt}$ and round to the nearest tenth of a year.

80. Suppose \$2000 is invested at 10% interest compounded continuously. How long will it take for the investment to triple? Use the model $A(t) = Pe^{rt}$ and round to the nearest year.

81. Suppose that \$8000 is invested at 4.5% interest compounded monthly. How long will it take the investment to reach \$10,000? Use the model $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ and round to the nearest year.

82. Suppose that a couple invests \$30,000 in a bond fund that pays 6.5% interest compounded quarterly. How long will it take the investment to reach \$250,000? Use the model $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ and round to the nearest year.

83. Phosphorus-32 (^{32}P) has a half-life of approximately 14 days. If 10 g of ^{32}P is present initially, then the amount $A(t)$ (in grams) of phosphorus-32 still present after t days is given by $A(t) = 10(0.5)^{t/14}$. **(See Example 11.)**

a. Find the amount of phosphorus-32 still present after 5 days. Round to the nearest tenth of a gram.

b. Find the amount of time necessary for the amount of ^{32}P to decay to 4 g. Round to the nearest tenth of a day.

84. Polonium-210 (^{210}Po) has a half-life of approximately 138.6 days. If 4 g of ^{210}Po is present initially, then the amount $A(t)$ (in grams) of polonium-210 still present after t days is given by $A(t) = 4e^{-0.005t}$.

a. Find the amount of polonium-210 still present after 50 days. Round to the nearest tenth of a gram.

b. Find the amount of time necessary for the amount of ^{210}P to decay to 0.5 g. Round to the nearest tenth of a day.

The decibel level of sound can be found by the equation $D = 10 \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound and I_0 is the intensity of the least audible sound that an average person can hear. Generally I_0 is found as 10^{-12} watt per square meter (W/m^2). Use this information to answer Exercises 85–86.

85. Given that heavy traffic has a decibel level of 89.3 and that $I_0 = 10^{-12}$, find the intensity of the sound of heavy traffic.

86. Given that normal conversation has a decibel level of 65 and $I_0 = 10^{-12}$, find the intensity of the sound of normal conversation.

87. Suppose you save \$10,000 from working an extra job. Rather than spending the money, you decide to save the money for retirement by investing in a mutual fund that averages 12% per year. How long will it take for this money to grow to \$1,000,000? Use the model $A(t) = Pe^{rt}$ and round to the nearest tenth of a year.

88. The model $A = Pe^{rt}$ is used to compute the total amount of money in an account after t years at an interest rate r, compounded continuously. The value P is the initial principal. Find the amount of time required for the investment to double as a function of the interest rate. (*Hint:* Substitute $A = 2P$ and solve for t.)

Technology Connections

89. Graph $Y_1 = 8^x$ and $Y_2 = 21$ on a window where $0 \le x \le 5$ and $0 \le y \le 40$. Use the graph and an *Intersect* feature to support your answer to Exercise 55.

90. Graph $Y_1 = 6^x$ and $Y_2 = 39$ on a window where $0 \le x \le 5$ and $0 \le y \le 50$. Use the graph and an *Intersect* feature to support your answer to Exercise 56.

Expanding Your Skills

91. The isotope of plutonium of mass 238 (written ^{238}Pu) is used to make thermoelectric power sources for spacecraft. The heat and electric power derived from such units have made the Voyager, Galileo, and Cassini missions to the outer reaches of our solar system possible. The half-life of ^{238}Pu is 87.7 years.

Suppose a space probe was launched in the year 2002 with 2.0 kg of ^{238}Pu. Then the amount of ^{238}Pu available to power the spacecraft decays over time according to

$$P(t) = 2e^{-0.0079t}$$

where $t \ge 0$ is the number of years since 2002 and $P(t)$ is the amount of plutonium still present (in kilograms).

a. Suppose the space probe is due to arrive at Pluto in the year 2045. How much plutonium will remain when the spacecraft reaches Pluto? Round to two decimal places.

b. If 1.5 kg of ^{238}Pu is required to power the spacecraft's data transmitter, will there be enough power in the year 2045 for us to receive close-up images of Pluto?

92. ^{99m}Tc is a radionuclide of technetium that is widely used in nuclear medicine. Although its half-life is only 6 hr, the isotope is continuously produced via the decay of its longer-lived parent ^{99}Mo (molybdenum-99), whose half-life is approximately 3 days. The ^{99}Mo generators (or "cows") are sold to hospitals in which the ^{99m}Tc can be "milked" as needed over a period of a few weeks. Once separated from its parent, the ^{99m}Tc may be chemically incorporated into a variety of imaging agents, each of which is designed to be taken up by a specific target organ within the body. Special cameras, sensitive to the gamma rays emitted by the technetium, are then used to record a "picture" (similar in appearance to an X-ray film) of the selected organ.

Suppose a technician prepares a sample of ^{99m}Tc-pyrophosphate to image the heart of a patient suspected of having had a mild heart attack. If the injection contains 10 millicuries (mCi) of ^{99m}Tc at 1:00 P.M., then the amount of technetium still present is given by

$$T(t) = 10e^{-0.1155t}$$

where $t > 0$ represents the time in hours after 1:00 P.M. and $T(t)$ represents the amount of ^{99m}Tc (in millicuries) still present.

a. How many millicuries of ^{99m}Tc will remain at 4:20 P.M. when the image is recorded? Round to the nearest tenth of a millicurie.

b. How long will it take for the radioactive level of the ^{99m}Tc to reach 2 mCi? Round to the nearest tenth of an hour.

For Exercises 93–96, solve the equations.

93. $(\log x)^2 - 2 \log x - 15 = 0$
(*Hint:* Let $u = \log x$.)

94. $(\log_2 z)^2 - 3 \log_2 z - 4 = 0$

95. $(\log_3 w)^2 + 5 \log_3 w + 6 = 0$

96. $(\ln x)^2 - 2 \ln x = 0$

Chapter 8 Summary

Section 8.1 Algebra of Functions and Composition

Key Concepts

The Algebra of Functions

Given two functions f and g, the functions

$f+g, f-g, f\cdot g$, and $\dfrac{f}{g}$ are defined as

$(f+g)(x)=f(x)+g(x)$

$(f-g)(x)=f(x)-g(x)$

$(f\cdot g)(x)=f(x)\cdot g(x)$

$\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$ provided $g(x)\neq 0$

Composition of Functions

The **composition** of f and g, denoted $f\circ g$, is defined by the rule

$(f\circ g)(x)=f(g(x))$ provided that $g(x)$ is in the domain of f

Examples

Example 1

Let $g(x)=5x+1$ and $h(x)=x^3$. Find:

1. $(g+h)(3)=g(3)+h(3)=16+27=43$
2. $(g\cdot h)(-1)=g(-1)\cdot h(-1)=(-4)\cdot(-1)=4$
3. $(g-h)(x)=5x+1-x^3$
4. $\left(\dfrac{g}{h}\right)(x)=\dfrac{5x+1}{x^3}, x\neq 0$

Example 2

Find $(f\circ g)(x)$ given the functions defined by $f(x)=4x+3$ and $g(x)=7x$.

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f(7x)\\&=4(7x)+3\\&=28x+3\end{aligned}$$

Section 8.2 Inverse Functions

Key Concepts

Horizontal Line Test

Consider a function defined by a set of points (x, y) in a rectangular coordinate system. Then y is a one-to-one function of x if no horizontal line intersects the graph in more than one point.

Finding an Equation of the Inverse of a Function

For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows:

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve for y.
4. Replace y with $f^{-1}(x)$.

The graphs defined by $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.

Definition of an Inverse Function

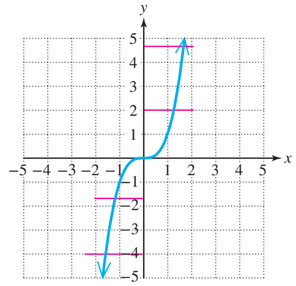
If f is a one-to-one function represented by ordered pairs of the form (x, y), then the inverse function, denoted f^{-1}, is the set of ordered pairs (y, x).

Inverse Function Property

If f is a one-to-one function, then g is the inverse of f if and only if $(f \circ g)(x) = x$ for all x in the domain of g, and $(g \circ f)(x) = x$ for all x in the domain of f.

Examples

Example 1



The function is one-to-one because it passes the horizontal line test.

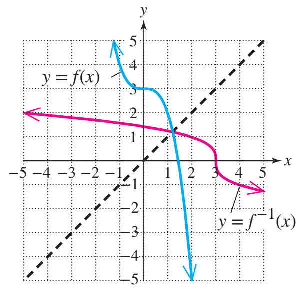
Example 2

Find the inverse of the one-to-one function defined by $f(x) = 3 - x^3$.

1. $y = 3 - x^3$
2. $x = 3 - y^3$
3. $x - 3 = -y^3$

   $-x + 3 = y^3$

   $\sqrt[3]{-x + 3} = y$
4. $f^{-1}(x) = \sqrt[3]{-x + 3}$



Example 3

Verify that the functions defined by $f(x) = x - 1$ and $g(x) = x + 1$ are inverses.

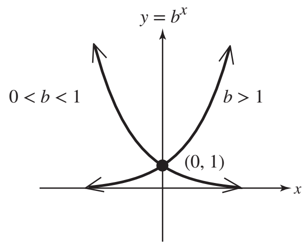
$(f \circ g)(x) = f(x + 1) = (x + 1) - 1 = x$

$(g \circ f)(x) = g(x - 1) = (x - 1) + 1 = x$

Section 8.3 Exponential Functions

Key Concepts

A function $f(x) = b^x$ $(b > 0, b \neq 1)$ is an **exponential function**.



The domain is $(-\infty, \infty)$.

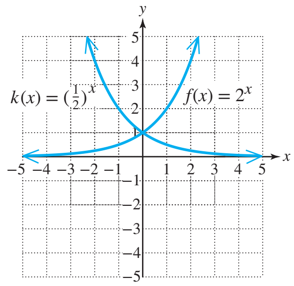
The range is $(0, \infty)$.

The line $y = 0$ (x-axis) is a horizontal asymptote.

The y-intercept is (0, 1).

Example

Example 1

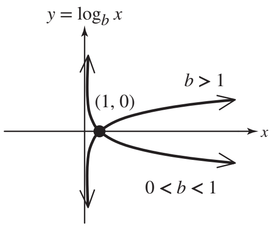


Section 8.4 Logarithmic Functions

Key Concepts

The function $y = \log_b x$ is a **logarithmic function**.

$y = \log_b x \Leftrightarrow b^y = x \quad (x > 0, b > 0, b \neq 1)$



For $y = \log_b x$, the domain is $(0, \infty)$.

The range is $(-\infty, \infty)$.

The line $x = 0$ (y-axis) is a vertical asymptote.

The x-intercept is (1, 0).

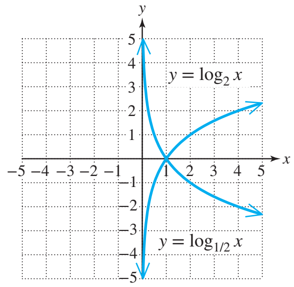
The function $f(x) = \log x$ is the **common logarithmic function** (base 10).

Examples

Example 1

$\log_4 64 = 3$ because $4^3 = 64$

Example 2



Example 3

$\log 10{,}000 = 4$ because $10^4 = 10{,}000$

Section 8.5 Properties of Logarithms

Key Concepts

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties are true.

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^p = p$
4. $b^{\log_b x} = x$
5. $\log_b (xy) = \log_b x + \log_b y$
6. $\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
7. $\log_b x^p = p \log_b x$

The **properties of logarithms** can be used to write multiple logarithms as a single logarithm.

The **properties of logarithms** can be used to write a single logarithm as a sum or difference of logarithms.

Examples

Example 1

1. $\log_5 1 = 0$
2. $\log_6 6 = 1$
3. $\log_4 4^7 = 7$
4. $2^{\log_2 5} = 5$
5. $\log (5x) = \log 5 + \log x$
6. $\log_7 \left(\dfrac{z}{10}\right) = \log_7 z - \log_7 10$
7. $\log x^5 = 5 \log x$

Example 2

$$\log x - \frac{1}{2}\log y - 3 \log z$$
$$= \log x - \log y^{1/2} - \log z^3$$
$$= \log x - (\log y^{1/2} + \log z^3)$$
$$= \log x - \log (\sqrt{y}z^3)$$
$$= \log\left(\frac{x}{\sqrt{y}z^3}\right)$$

Example 3

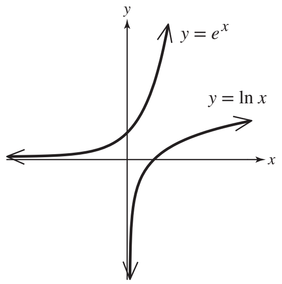
$$\log \sqrt[3]{\frac{x}{y^2}}$$
$$= \log \left(\frac{x}{y^2}\right)^{1/3}$$
$$= \frac{1}{3}\log \left(\frac{x}{y^2}\right)$$
$$= \frac{1}{3}(\log x - \log y^2)$$
$$= \frac{1}{3}(\log x - 2 \log y)$$
$$= \frac{1}{3}\log x - \frac{2}{3}\log y$$

Section 8.6 The Irrational Number e and Change of Base

Key Concepts

The function $y = e^x$ is the exponential function with base e.

The **natural logarithm function** $y = \ln x$ is the logarithm function with base e.



Change-of-Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b} \qquad a > 0, a \neq 1, b > 0, b \neq 1$$

Examples

Example 1

Use a calculator to approximate the value of the expressions.

$e^{7.5} \approx 1808.0424$

$e^{-\pi} \approx 0.0432$

$\ln 107 \approx 4.6728$

$\ln\left(\frac{1}{\sqrt{2}}\right) \approx -0.3466$

Example 2

$$\log_3 59 = \frac{\log 59}{\log 3} \approx 3.7115$$

Section 8.7 Logarithmic and Exponential Equations and Applications

Key Concepts

Guidelines to Solve Logarithmic Equations

1. Isolate the logarithms on one side of the equation.
2. Write a sum or difference of logarithms as a single logarithm.
3. Rewrite the equation in step 2 in exponential form.
4. Solve the resulting equation from step 3.
5. Check all solutions to verify that they are within the domain of the logarithmic expressions in the equation.

Equivalence Property of Logarithmic Expressions

If two logarithms with the same base are equal, then their arguments are equal. That is,

If $\log_b x = \log_b y$, then $x = y$.

Example 2

$$\begin{aligned} \ln(x-4) + \ln(3) &= \ln(2x+3) \\ \ln[3(x-4)] &= \ln(2x+3) \\ \ln(3x-12) &= \ln(2x+3) \\ 3x - 12 &= 2x + 3 && \text{Arguments are equal.} \\ x &= 15 && \text{The solution checks.} \end{aligned}$$

The solution set is $\{15\}$.

Examples

Example 1

$\log(3x-1) + 1 = \log(2x+1)$

Step 1: $\log(3x-1) - \log(2x+1) = -1$

Step 2: $\log\left(\frac{3x-1}{2x+1}\right) = -1$

Step 3: $10^{-1} = \frac{3x-1}{2x+1}$

Step 4: $\frac{1}{10} = \frac{3x-1}{2x+1}$

$$\begin{aligned} 2x + 1 &= 10(3x-1) \\ 2x + 1 &= 30x - 10 \\ -28x &= -11 \\ x &= \frac{11}{28} && \text{The solution checks.} \end{aligned}$$

Step 5: The solution set is $\left\{\frac{11}{28}\right\}$.

Equivalence Property of Exponential Expressions

If two exponential expressions with the same base are equal, then their exponents are equal. That is,

If $b^x = b^y$, then $x = y$.

Example 3

$$5^{2x} = 125$$

$$5^{2x} = 5^3 \quad \text{implies that} \quad 2x = 3$$

$$x = \frac{3}{2}$$

The solution set is $\left\{\frac{3}{2}\right\}$.

Guidelines to Solve Exponential Equations

1. Isolate one of the exponential expressions in the equation.
2. Take a logarithm of both sides of the equation.
3. Use the power property of logarithms to write exponents as factors.
4. Solve the resulting equation from step 3.

Example 4

$$4^{x+1} - 2 = 1055$$

Step 1: $4^{x+1} = 1057$

Step 2: $\ln 4^{x+1} = \ln 1057$

Step 3: $(x + 1) \ln 4 = \ln 1057$

Step 4: $x + 1 = \dfrac{\ln 1057}{\ln 4}$

$$x = \frac{\ln 1057}{\ln 4} - 1 \approx 4.023$$

The solution set is $\left\{\dfrac{\ln 1057}{\ln 4} - 1\right\}$.

Chapter 8 Review Exercises

Section 8.1

For Exercises 1–8, refer to the functions defined here.

$$f(x) = x - 7 \qquad g(x) = -2x^3 - 8x$$

$$m(x) = x^2 \qquad n(x) = \frac{1}{x - 2}$$

Find the indicated functions.

1. $(f - g)(x)$

2. $(f + g)(x)$

3. $(f \cdot n)(x)$

4. $(f \cdot m)(x)$

5. $\left(\dfrac{f}{g}\right)(x)$

6. $\left(\dfrac{g}{f}\right)(x)$

7. $(m \circ f)(x)$

8. $(n \circ f)(x)$

For Exercises 9–12, refer to the functions defined for Exercises 1–8. Find the function values, if possible.

9. $(m \circ g)(-1)$

10. $(n \circ g)(-1)$

11. $(f \circ g)(4)$

12. $(g \circ f)(8)$

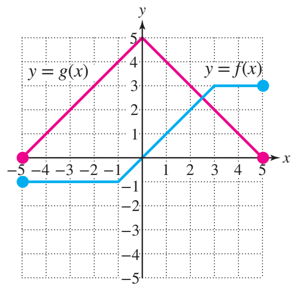
13. Given: $f(x) = 2x + 1$ and $g(x) = x^2$

a. Find $(g \circ f)(x)$.

b. Find $(f \circ g)(x)$.

c. Based on your answers to part (a), is $f \circ g$ equal to $g \circ f$?

For Exercises 14–19, refer to the graph. Approximate the function values, if possible.



14. $\left(\dfrac{f}{g}\right)(1)$

15. $(f \cdot g)(-2)$

16. $(f + g)(-4)$

17. $(f - g)(2)$

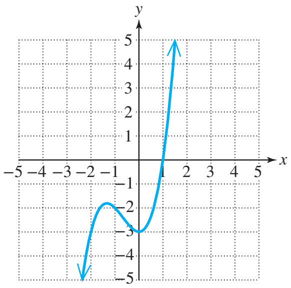
18. $(g \circ f)(-3)$

19. $(f \circ g)(4)$

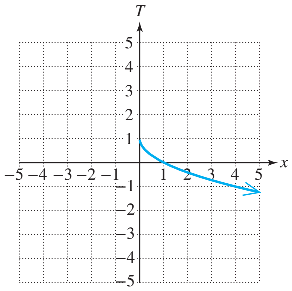
Section 8.2

For Exercises 20–21, determine if the function is one-to-one by using the horizontal line test.

20.



21.



For Exercises 22–26, write the inverse for each one-to-one function.

22. $\{(3, 5), (2, 9), (0, -1), (4, 1)\}$

23. $q(x) = \dfrac{3}{4}x - 2$

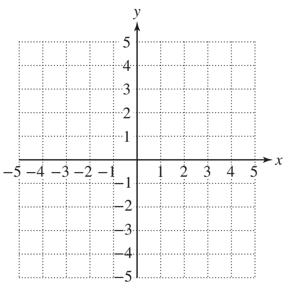
24. $g(x) = \sqrt[5]{x} + 3$

25. $f(x) = (x - 1)^3$

26. $n(x) = \dfrac{4}{x - 2}$

27. Verify that the functions defined by $f(x) = 5x - 2$ and $g(x) = \frac{1}{5}x + \frac{2}{5}$ are inverses by showing that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

28. Graph the functions q and q^{-1} from Exercise 23 on the same grid. What can you say about the relationship between these two graphs?



29. **a.** Find the domain and range of the function defined by $h(x) = \sqrt{x + 1}$.

b. Find the domain and range of the function defined by $k(x) = x^2 - 1, x \geq 0$.

30. Determine the inverse of the function $p(x) = \sqrt{x} + 2$.

Section 8.3

For Exercises 31–38, evaluate the exponential expressions. Use a calculator and round to three decimal places, if necessary.

31. 4^5

32. 6^{-2}

33. $8^{1/3}$

34. $\left(\dfrac{1}{100}\right)^{-1/2}$

35. 2^{π}

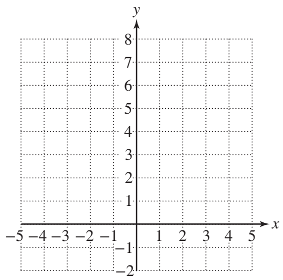
36. $5^{\sqrt{3}}$

37. $(\sqrt{7})^{1/2}$

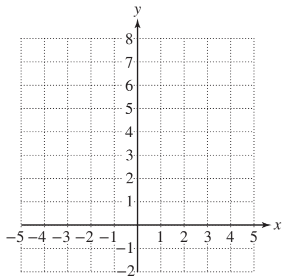
38. $\left(\dfrac{3}{4}\right)^{4/3}$

For Exercises 39–42, graph the functions.

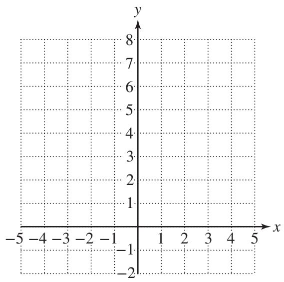
39. $f(x) = 3^x$



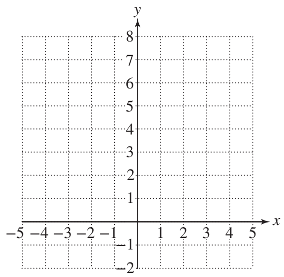
40. $g(x) = \left(\dfrac{1}{4}\right)^x$



41. $h(x) = 5^{-x}$



42. $k(x) = \left(\dfrac{2}{5}\right)^{-x}$



43. **a.** Does the graph of $y = b^x, b > 0, b \neq 1$, have a vertical or a horizontal asymptote?

b. Write an equation of the asymptote.

44. Background radiation is radiation that we are exposed to from naturally occurring sources including the soil, the foods we eat, and the Sun. Background radiation varies depending on where we live. A typical background radiation level is 150 millirems (mrem) per year. (A rem is a measure of energy produced from radiation.) Suppose a substance emits

30,000 mrem per year and has a half-life of 5 years. The function defined by

$$A(t) = 30{,}000\left(\frac{1}{2}\right)^{t/5}$$

gives the radiation level (in millirems) of this substance after t years.

a. What is the radiation level after 5 years?

b. What is the radiation level after 15 years?

c. Will the radiation level of this substance be below the background level of 150 mrem after 50 years?

Section 8.4

For Exercises 45–52, evaluate the logarithms without using a calculator.

45. $\log_3\left(\frac{1}{27}\right)$

46. $\log_5 1$

47. $\log_7 7$

48. $\log_2 2^8$

49. $\log_2 16$

50. $\log_3 81$

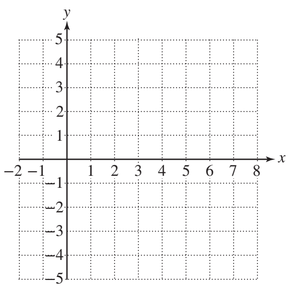
51. $\log 100{,}000$

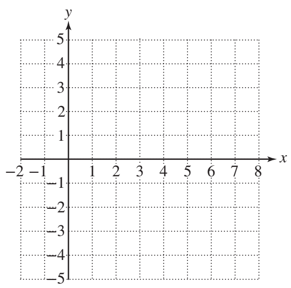
52. $\log_8\left(\frac{1}{8}\right)$

For Exercises 53–54, graph the logarithmic functions.

53. $q(x) = \log_3 x$

54. $r(x) = \log_{1/2} x$





55. a. Does the graph of $y = \log_b x$ have a vertical or a horizontal asymptote?

b. Write an equation of the asymptote.

56. Acidity of a substance is measured by its pH. The pH can be calculated by the formula $\text{pH} = -\log\,[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration.

a. What is the pH of a fruit with a hydrogen ion concentration of 0.00316 mol/L? Round to one decimal place.

b. What is the pH of an antacid tablet with $[\text{H}^+] = 3.16 \times 10^{-10}$? Round to one decimal place.

Section 8.5

For Exercises 57–60, evaluate the logarithms without using a calculator.

57. $\log_8 8$

58. $\log_{11} 11^6$

59. $\log_{1/2} 1$

60. $12^{\log_{12} 7}$

61. Complete the properties Assume x, y, and b are positive real numbers such that $b \neq 1$.

a. $\log_b (xy) =$

b. $\log_b x - \log_b y =$

c. $\log_b x^p =$

For Exercises 62–65, write the logarithmic expressions as a single logarithm and simplify if possible.

62. $\frac{1}{4}(\log_b y - 4\log_b z + 3\log_b x)$

63. $\frac{1}{2}\log_3 a + \frac{1}{2}\log_3 b - 2\log_3 c - 4\log_3 d$

64. $\log 540 - 3\log 3 - 2\log 2$

65. $-\log_4 18 + \log_4 6 + \log_4 3 - \log_4 1$

66. Which of the following is equivalent to $\dfrac{2\log 7}{\log 7 + \log 6}$?

a. $\dfrac{\log 7}{\log 6}$ **b.** $\dfrac{\log 49}{\log 42}$ **c.** $\log\left(\dfrac{7}{6}\right)$

67. Which of the following is equivalent to $\dfrac{\log 8^{-3}}{\log 2 + \log 4}$?

a. -3 **b.** $-3\log\left(\dfrac{4}{3}\right)$ **c.** $\dfrac{-3\log 4}{\log 3}$

Section 8.6

For Exercises 68–75, use a calculator to approximate the expressions to four decimal places.

68. e^5

69. $e^{\sqrt{7}}$

70. $32e^{0.008}$

71. $58e^{-0.0125}$

72. $\ln 6$

73. $\ln\left(\frac{1}{9}\right)$

74. $\log 22$

75. $\log e^3$

For Exercises 76–79, use the change-of-base formula to approximate the logarithms to four decimal places.

76. $\log_2 10$

77. $\log_9 80$

78. $\log_5 0.26$

79. $\log_4 0.0062$

80. An investor wants to deposit \$20,000 in an account for 10 years at 5.25% interest. Compare the amount $A(t)$ she would have if her money were invested with the following different compounding options. Use

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

for interest compounded n times per year and $A(t) = Pe^{rt}$ for interest compounded continuously.

a. Compounded annually

b. Compounded quarterly

c. Compounded monthly

d. Compounded continuously

81. To measure a student's retention of material at the end of a course, researchers give the student a test on the material every month for 24 months after the course is over. The student's average score t months after completing the course is given by

$$S(t) = 75e^{-0.5t} + 20$$

where $S(t)$ is the test score.

a. Find $S(0)$ and interpret the result.

b. Find $S(6)$ and interpret the result.

c. Find $S(12)$ and interpret the result.

For Exercises 82–89, identify the domain. Write the answer in interval notation.

82. $f(x) = e^x$

83. $g(x) = e^{x+6}$

84. $h(x) = e^{x-3}$

85. $k(x) = \ln x$

86. $q(x) = \ln(x + 5)$

87. $p(x) = \ln(x - 7)$

88. $r(x) = \ln(3x - 4)$

89. $w(x) = \ln(5 - x)$

Section 8.7

For Exercises 90–107, solve the equations.

90. $\log_5 x = 3$

91. $\log_7 x = -2$

92. $\log_6 y = 3$

93. $\log_3 y = \frac{1}{12}$

94. $\log(2w - 1) = 3$

95. $\log_2(3w + 5) = 5$

96. $-1 + \log p = -\log(p - 3)$

97. $\log_4(2 + t) - 3 = \log_4(3 - 5t)$

98. $4^{3x+5} = 16$

99. $5^{7x} = 625$

100. $4^a = 21$

101. $5^a = 18$

102. $e^{-x} = 0.1$

103. $e^{-2x} = 0.06$

104. $10^{2n} = 1512$

105. $10^{-3m} = \frac{1}{821}$

106. $2^{x+3} = 7^x$

107. $14^{x-5} = 6^x$

108. Radioactive iodine (^{131}I) is used to treat patients with a hyperactive (overactive) thyroid. The half-life of radioactive iodine is 8.04 days. If a patient is given an initial dose of 2 μg, then the amount of iodine in the body after t days is approximated by

$$A(t) = 2e^{-0.0862t}$$

where t is the time in days and $A(t)$ is the amount (in micrograms) of ^{131}I remaining.

a. How much radioactive iodine is present after a week? Round to two decimal places.

b. How much radioactive iodine is present after 30 days? Round to two decimal places.

c. How long will it take for the level of radioactive iodine to reach 0.5 μg?

109. The growth of certain bacteria in a culture is given by the model $A(t) = 150e^{0.007t}$, where $A(t)$ is the number of bacteria and t is time in minutes.

a. What is the initial number of bacteria?

b. What is the population after $\frac{1}{2}$ hr?

c. How long will it take for the population to double?

110. The value of a car is depreciated with time according to

$$V(t) = 15{,}000e^{-0.15t}$$

where $V(t)$ is the value in dollars and t is the time in years after purchase.

a. Find $V(0)$ and interpret the result in the context of this problem.

b. Find $V(10)$ and interpret the result in the context of this problem. Round to the nearest dollar.

c. Find the time required for the value of the car to drop to \$5000. Round to the nearest tenth of a year.

Chapter 8 Test

Study Skills Exercise

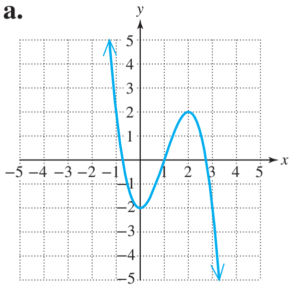
The final exam is just around the corner. Use your old tests to make a list of the chapters that need extra attention. The Review Exercises and the Chapter Test at the end of each chapter are excellent tools to help you prepare.

- Make a one-page summary sheet of all key information you need to memorize.
- Ask for help if there are still concepts that you do not understand.
- On the day of the final exam, sit where you are comfortable, take a deep breath, and relax. Visualize success.
- Immediately write down the key concepts contained on your summary sheet.
- Include a positive statement to yourself to build your confidence.
- First go through the test and complete all of the problems that you know. Then go back and work on the problems that are more difficult.
- Give yourself a time limit for each problem.
- Write out all the steps required for solving the problem and answer all the problems on the test.
- Review your solutions before turning in the test.

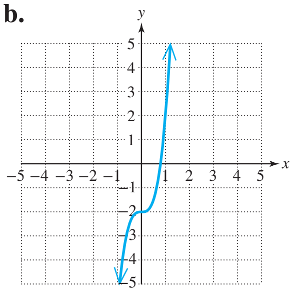
1. Explain how to determine graphically if a function is one-to-one.

2. Which of the functions is one-to-one?

a.



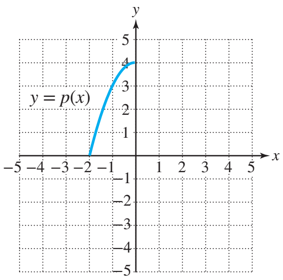
b.



3. Write an equation of the inverse of the one-to-one function defined by $f(x) = \frac{1}{4}x + 3$.

4. Write an equation of the inverse of the function defined by $g(x) = (x - 1)^2, \quad x \geq 1$.

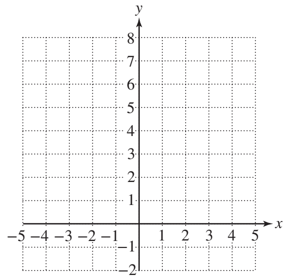
5. Given the graph of the function $y = p(x)$, graph its inverse $p^{-1}(x)$.



6. Use a calculator to approximate the expression to four decimal places.

a. $10^{2/3}$ **b.** $3^{\sqrt{10}}$ **c.** 8^{π}

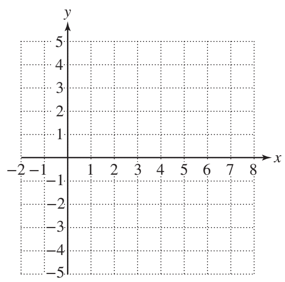
7. Graph $f(x) = 4^{x-1}$.



8. a. Write in logarithmic form. $16^{3/4} = 8$

b. Write in exponential form. $\log_x 31 = 5$

9. Graph $g(x) = \log_3 x$.



10. Complete the change-of-base formula:
$\log_b n =$ ______

11. Use a calculator to approximate the expression to four decimal places.

a. $\log 21$ **b.** $\log_4 13$ **c.** $\log_{1/2} 6$

12. Using the properties of logarithms, expand and simplify. Assume all variables represent positive real numbers.

a. $-\log_3\left(\frac{3}{9x}\right)$ **b.** $\log\left(\frac{1}{10^5}\right)$

13. Write as a single logarithm. Assume all variables represent positive real numbers.

a. $\frac{1}{2}\log_b x + 3\log_b y$ **b.** $\log a - 4\log a$

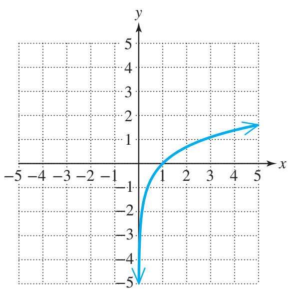
14. Use a calculator to approximate the expression to four decimal places, if necessary.

a. $e^{1/2}$ **b.** e^{-3}

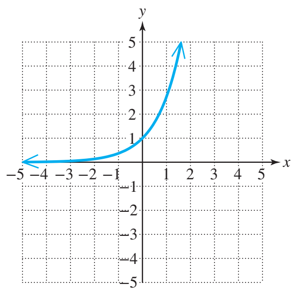
c. $\ln\left(\frac{1}{3}\right)$ **d.** $\ln e$

15. Identify the graphs as $y = e^x$ or $y = \ln x$.

a.



b.



16. Researchers found that t months after taking a course, students remembered $p\%$ of the material according to

$$p(t) = 92 - 20\ln(t + 1)$$

where $0 \le t \le 24$ is the time in months.

a. Find $p(4)$ and interpret the results.

b. Find $p(12)$ and interpret the results.

c. Find $p(0)$ and interpret the results.

17. The population of New York City has a 2% growth rate and can be modeled by the function $P(t) = 8008(1.02)^t$, where $P(t)$ is in thousands and t is in years ($t = 0$ corresponds to the year 2000).

a. Using this model, estimate the population in the year 2010.

b. In what year will the population reach 12 million (12 million is 12,000 thousand)?

18. A certain bacterial culture grows according to

$$P(t) = \frac{1{,}500{,}000}{1 + 5000e^{-0.8t}}$$

where P is the population of the bacteria and t is the time in hours.

a. Find $P(0)$ and interpret the result. Round to the nearest whole number.

b. How many bacteria will be present after 6 hr?

c. How many bacteria will be present after 12 hr?

d. How many bacteria will be present after 18 hr?

For Exercises 19–26, solve the exponential and logarithmic equations.

19. $\log x + \log(x - 21) = 2$

20. $\log_{1/2} x = -5$

21. $\ln(x + 7) = 2.4$

22. $3^{x+4} = \frac{1}{27}$

23. $4^x = 50$

24. $e^{2.4x} = 250$

25. $68 = 10 + 2^{x-3}$

26. $4^{x+7} = 5^x$

27. Atmospheric pressure P decreases exponentially with altitude x according to

$$P(x) = 760e^{-0.000122x}$$

where $P(x)$ is the pressure measured in millimeters of mercury (mm Hg) and x is the altitude measured in meters.

a. Find $P(2500)$ and interpret the result. Round to one decimal place.

b. Find the pressure at sea level.

c. Find the altitude at which the pressure is 633 mm Hg.

28. Use the formula $A(t) = Pe^{rt}$ to compute the value of an investment under continuous compounding.

a. If \$2000 is invested at 7.5% compounded continuously, find the value of the investment after 5 years.

b. How long will it take the investment to double? Round to two decimal places.

For Exercises 29–37, refer to functions f, g, and h.

$f(x) = x - 4 \qquad g(x) = x^2 + 2 \qquad h(x) = \dfrac{1}{x}$

Find the indicated functions or function values.

29. $\left(\dfrac{f}{g}\right)(x)$

30. $(h \cdot g)(x)$

31. $(g \circ f)(x)$

32. $(h \circ f)(x)$

33. $(f - g)(7)$

34. $(h + f)(2)$

35. $(h \circ g)(4)$

36. $(g \circ f)(0)$

37. $\left(\dfrac{g}{f}\right)(x)$

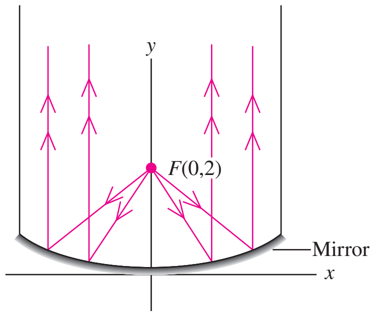
Conic Sections

9

CHAPTER OUTLINE

Mathematics in Engineering

In this chapter, we will revisit our study of parabolas as well as two new curves called *ellipses* and *hyperbolas*. These curves are called **conic sections**. Conic sections and their three-dimensional counterparts have numerous applications in engineering. For example, the mirror in a reflecting telescope has cross sections in the shape of a parabola. This is important because the curved surface of the mirror focuses light to a common point called the **focus**. This targeted focusing of light enables astronomers to view an image from a distant star with better clarity.



Aaron Roeth Photography

The cooling towers for nuclear power plants have cross sections in the shape of a hyperbola. One advantage of a hyperbolic cooling tower is that air accelerates as it rises inside, toward the narrow portion of the tower. This provides for an efficient, nonturbulent flow. Then, above the narrow part, the tower flares out for efficient dispersal of warm air.

The ellipse is an oval-shaped curve that also has numerous applications such as in architectural design and for describing the orbits of planets. For example, the Roman Coliseum is an elliptical stone and concrete amphitheater in the center of Rome, built between 70 A.D. and 80 A.D. The Coliseum seated approximately 50,000 spectators and was used for gladiatorial contests among other things.

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Section 9.1 Distance Formula, Midpoint Formula, and Circles

Concepts

1. Distance Formula
2. Circles
3. Writing an Equation of a Circle
4. The Midpoint Formula

1. Distance Formula

Suppose we are given two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate system. The distance between the two points can be found by using the Pythagorean theorem (Figure 9-1).

First draw a right triangle with the distance d as the hypotenuse. The length of the horizontal leg a is $|x_2 - x_1|$, and the length of the vertical leg b is $|y_2 - y_1|$. From the Pythagorean theorem we have

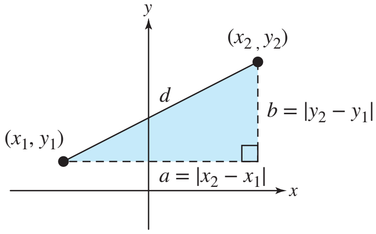


Figure 9-1

$$d^2 = a^2 + b^2$$
$$= |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Because distance is positive, reject the negative value.

TIP: Squaring any real-valued quantity results in a nonnegative real number. Therefore, the absolute value bars can be dropped.

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1 Finding the Distance Between Two Points

Find the distance between the points $(-2, 3)$ and $(4, -1)$ (Figure 9-2).

Solution:

$(-2, 3)$ and $(4, -1)$
(x_1, y_1) (x_2, y_2) Label the points.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[(4) - (-2)]^2 + [(-1) - (3)]^2}$$ Apply the distance formula.
$$= \sqrt{(6)^2 + (-4)^2}$$
$$= \sqrt{36 + 16}$$
$$= \sqrt{52}$$
$$= \sqrt{4 \cdot 13}$$
$$= 2\sqrt{13}$$

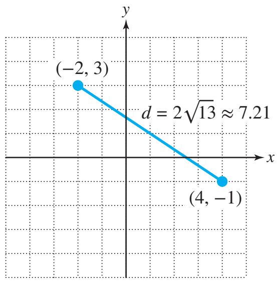


Figure 9-2

FOR REVIEW

Recall that to simplify $\sqrt{52}$, write the radicand as the product of the largest perfect square and a "leftover" factor. Then apply the product property of radicals.

$$\sqrt{52} = \sqrt{2^2 \cdot 13}$$
$$= \sqrt{2^2} \cdot \sqrt{13}$$
$$= 2\sqrt{13}$$

Skill Practice

1. Find the distance between the points $(-4, -2)$ and $(2, -5)$.

Answer

1. $3\sqrt{5}$

TIP: The order in which the points are labeled does not affect the result of the distance formula. For example, if the points in Example 1 had been labeled in reverse, the distance formula would still yield the same result:

$(-2, 3)$ and $(4, -1)$
(x_2, y_2) $\quad$ (x_1, y_1)

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[(-2) - (4)]^2 + [3 - (-1)]^2} \\ &= \sqrt{(-6)^2 + (4)^2} \\ &= \sqrt{36 + 16} \\ &= \sqrt{52} \\ &= 2\sqrt{13} \end{aligned}$$

2. Circles

A **circle** is defined as the set of all points in a plane that are equidistant from a fixed point called the **center**. The fixed distance from the center is called the **radius** and is denoted by r, where $r > 0$.

Suppose a circle is centered at the point (h, k) and has radius, r (Figure 9-3). The distance formula can be used to derive an equation of the circle.

Let (x, y) be any arbitrary point on the circle. Then, by definition, the distance between (h, k) and (x, y) must be r.

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Square both sides.}$$

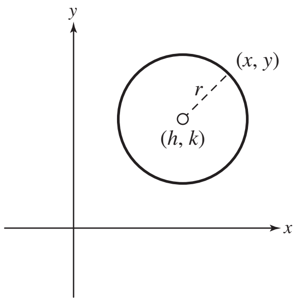


Figure 9-3

Standard Equation of a Circle

The **standard equation of a circle**, centered at (h, k) with radius r, is given by

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{where } r > 0.$$

Note: If a circle is centered at the origin $(0, 0)$, then $h = 0$ and $k = 0$, and the equation simplifies to $x^2 + y^2 = r^2$.

Example 2 Graphing a Circle

Find the center and radius of the circle. Then graph the circle.

$$(x - 3)^2 + (y + 4)^2 = 36$$

Solution:

$$(x - 3)^2 + (y + 4)^2 = 36$$

$$(x - 3)^2 + [y - (-4)]^2 = (6)^2$$

$$h = 3, k = -4, \text{ and } r = 6$$

The equation is in standard form $(x - h)^2 + (y - k)^2 = r^2$. The center is $(3, -4)$, and the radius is $r = 6$. To graph the circle, first locate the center. From the center, mark points 6 units to the right, left, above, and below the center. Then sketch the circle through these four points (Figure 9-4).

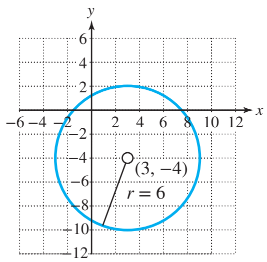


Figure 9-4

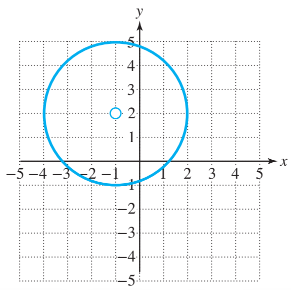
Note that the center of the circle is not actually part of the circle. It is drawn as an open dot for reference.

Skill Practice Find the center and radius of the circle. Then graph the circle.

2. $(x + 1)^2 + (y - 2)^2 = 9$

Answer

2. Center: $(-1, 2)$; radius: 3



Example 3 Graphing a Circle

Find the center and radius of each circle. Then graph the circle.

a. $x^2 + \left(y - \frac{10}{3}\right)^2 = \frac{25}{9}$ **b.** $x^2 + y^2 = 10$

Solution:

a. $x^2 + \left(y - \frac{10}{3}\right)^2 = \frac{25}{9}$

$$(x - 0)^2 + \left(y - \frac{10}{3}\right)^2 = \left(\frac{5}{3}\right)^2$$

The equation is now in standard form $(x - h)^2 + (y - k)^2 = r^2$.

The center is $(0, \frac{10}{3})$ and the radius is $\frac{5}{3}$ (Figure 9-5).

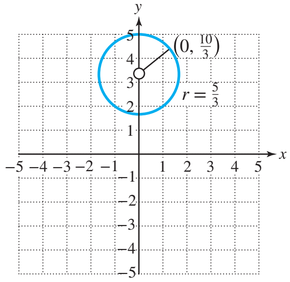


Figure 9-5

b. $x^2 + y^2 = 10$

$$(x - 0)^2 + (y - 0)^2 = (\sqrt{10})^2$$

$$(x - h)^2 + (y - k)^2 = r^2$$

The center is $(0, 0)$ and the radius is $\sqrt{10} \approx 3.16$ (Figure 9-6).

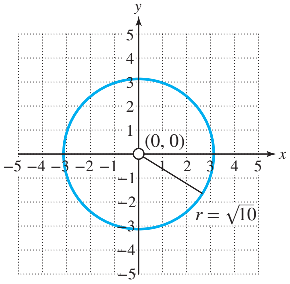


Figure 9-6

FOR REVIEW

From the vertical line test, we determine that a circle does not define y as a function of x.

Skill Practice Find the center and radius of the circle. Then graph the circle.

3. $\left(x + \frac{7}{2}\right)^2 + y^2 = \frac{9}{4}$

Sometimes it is necessary to complete the square to write an equation of a circle in standard form.

Example 4 Writing an Equation of a Circle in the Form $(x - h)^2 + (y - k)^2 = r^2$

Identify the center and radius of the circle given by the equation.

$$x^2 + y^2 + 2x - 16y + 61 = 0$$

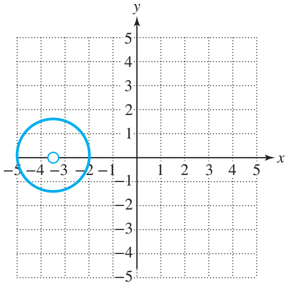
Solution:

$x^2 + y^2 + 2x - 16y + 61 = 0$ To identify the center and radius, write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$.

$(x^2 + 2x \quad) + (y^2 - 16y \quad) = -61$ Group the x terms and group the y terms. Move the constant to the right-hand side.

Answer

3. Center: $\left(-\frac{7}{2}, 0\right)$; radius: $\frac{3}{2}$



$(x^2 + 2x + 1) + (y^2 - 16y + 64) = -61 + 1 + 64$

- Complete the square on x. Add $\left[\frac{1}{2}(2)\right]^2 = 1$ to both sides of the equation.
- Complete the square on y. Add $\left[\frac{1}{2}(-16)\right]^2 = 64$ to both sides of the equation.

$(x + 1)^2 + (y - 8)^2 = 4$ Factor and simplify.

$[x - (-1)]^2 + (y - 8)^2 = 2^2$ Standard form: $(x - h)^2 + (y - k)^2 = r^2$

The center is $(-1, 8)$ and the radius is 2.

Skill Practice Identify the center and radius of the circle given by the equation.

4. $x^2 + y^2 - 10x + 4y - 7 = 0$

FOR REVIEW

Given $x^2 + bx + \square$, complete the square by adding the square of one-half the linear term coefficient. Thus,

$$x^2 + bx + \left(\frac{1}{2}b\right)^2.$$

3. Writing an Equation of a Circle

Example 5 Writing an Equation of a Circle

Write an equation of the circle shown in Figure 9-7.

Solution:

The center is $(-3, 2)$; therefore, $h = -3$ and $k = 2$.
From the graph, $r = 2$.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$[x - (-3)]^2 + (y - 2)^2 = (2)^2$$

$$(x + 3)^2 + (y - 2)^2 = 4$$

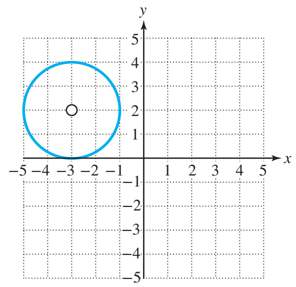


Figure 9-7

Avoiding Mistakes

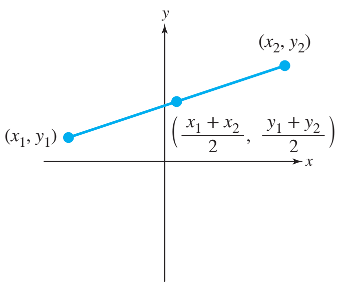
Be sure that you *subtract* both coordinates of the center, when substituting the coordinates into the standard form of a circle.

Skill Practice

5. Write an equation for a circle whose center is $(6, -1)$ and whose radius is 8.

4. The Midpoint Formula

Consider two points in the coordinate plane and the line segment determined by the points. It is sometimes necessary to determine the point that is halfway between the endpoints of the segment. This point is called the *midpoint*.



TIP: The midpoint of a line segment is found by taking the *average* of the x-coordinates and the *average* of the y-coordinates of the endpoints.

Midpoint Formula

Given two points (x_1, y_1) and (x_2, y_2), the midpoint of the line segment between the two points is given by

$$\textbf{Midpoint: } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Answers

4. Center: $(5, -2)$; radius: 6

5. $(x - 6)^2 + (y + 1)^2 = 64$

Example 6 Finding the Midpoint of a Segment

Find the midpoint of the line segment with the given endpoints.

a. $(-4, 6)$ and $(8, 1)$ **b.** $(-1.2, -3.1)$ and $(-6.6, 1.2)$

Solution:

a. $(-4, 6)$ and $(8, 1)$
(x_1, y_1) (x_2, y_2)

$$\left(\frac{-4+8}{2}, \frac{6+1}{2}\right)$$ Apply the midpoint formula: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$$\left(2, \frac{7}{2}\right)$$ Simplify.

The midpoint of the segment is $(2, \frac{7}{2})$.

b. $(-1.2, -3.1)$ and $(-6.6, 1.2)$
(x_1, y_1) (x_2, y_2)

$$\left(\frac{-1.2+(-6.6)}{2}, \frac{-3.1+1.2}{2}\right)$$ Apply the midpoint formula.

$(-3.9, -0.95)$ Simplify.

Avoiding Mistakes

Always remember that the midpoint of a line segment is a *point*. The answer should be an ordered pair, (x, y).

Skill Practice Find the midpoint of the line segment with the given endpoints.

6. $(5, 6)$ and $(-10, 4)$ **7.** $(-2.6, -6.3)$ and $(1.2, 4.1)$

Example 7 Applying the Midpoint Formula

Suppose that $(-2, 3)$ and $(4, 1)$ are endpoints of a diameter of a circle.

a. Find the center of the circle. **b.** Write an equation of the circle.

Solution:

a. Because the midpoint of a diameter of a circle is the center of the circle, apply the midpoint formula. See Figure 9-8.

$(-2, 3)$ and $(4, 1)$
(x_1, y_1) (x_2, y_2)

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\left(\frac{-2+4}{2}, \frac{3+1}{2}\right)$$ Apply the midpoint formula.

$(1, 2)$ Simplify.

The center of the circle is $(1, 2)$.

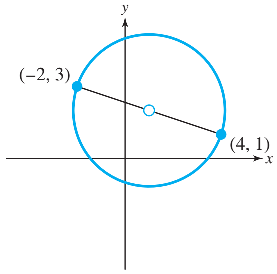


Figure 9-8

Answers

6. $\left(-\frac{5}{2}, 5\right)$ **7.** $(-0.7, -1.1)$

b. The radius can be determined by finding the distance between the center and either endpoint of the diameter.

$(1, 2)$ and $(4, 1)$
(x_1, y_1) (x_2, y_2)

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Apply the distance formula.

$d = \sqrt{(4 - 1)^2 + (1 - 2)^2}$ Substitute (1, 2) and (4, 1) for (x_1, y_1) and (x_2, y_2).

$= \sqrt{(3)^2 + (-1)^2}$ Simplify.

$= \sqrt{10}$

The circle is represented by $(x - 1)^2 + (y - 2)^2 = (\sqrt{10})^2$

or $(x - 1)^2 + (y - 2)^2 = 10$.

Skill Practice

8. Suppose that (3, 2) and (7, 10) are endpoints of a diameter of a circle.

a. Find the center of the circle.

b. Write an equation of the circle.

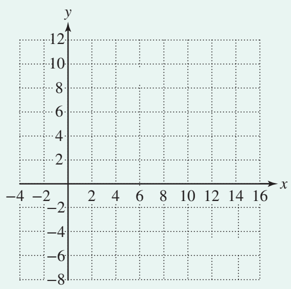
Answer

8. a. (5, 6)

b. $(x - 5)^2 + (y - 6)^2 = 20$

Section 9.1 Activity

A.1. Two bicyclists start out from the same point of origin. One rides east for $\frac{1}{4}$ hr and then north for $\frac{1}{2}$ hr at a rate of 16 mph. The other cyclist rides east for 1 hr and then south for $\frac{1}{3}$ hr at a rate of 12 mph.



a. Find the ordered pairs representing the positions of the cyclists at their respective destinations. Use the point of origin as (0, 0).

b. Draw a right triangle using the distance between the cyclists as the hypotenuse.

c. Find the horizontal distance between the two cyclists. Find the vertical distance between the two cyclists.

d. Use the Pythagorean theorem to find the exact distance between the two cyclists at their destinations. Then approximate the distance to the nearest tenth of a mile.

A.2. Refer to Exercise A.1. Label the point (4, 8) as (x_1, y_1) and label the point (12, −4) as (x_2, y_2).

a. Write the generic distance formula to find the distance d between the points (x_1, y_1) and (x_2, y_2).

b. Find the exact distance between the points (4, 8) and (12, −4).

c. In the distance formula, the term $(x_2 - x_1)^2$ represents the square of the (choose one: horizontal/vertical) distance between the two points. The term $(y_2 - y_1)^2$ represents the square of the (choose one: horizontal/vertical) distance between the two points.

d. Apply the distance formula with the point (4, 8) labeled as (x_2, y_2) and the point (12, −4) labeled as (x_1, y_1). Does the manner in which the points are labeled affect the result?

A.3. a. Explain how to find the average of two numbers.

b. The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) has coordinates given by $\left(\boxed{\begin{matrix}\text{Average of}\\ x \text{ values}\end{matrix}}, \boxed{\begin{matrix}\text{Average of}\\ y \text{ values}\end{matrix}}\right)$. Write the formula for the midpoint between the points (x_1, y_1) and (x_2, y_2).

c. Refer to Exercise A.1. The bicyclists are at points (4, 8) and (12, −4) and plan to meet for lunch at the midpoint between them. Find the midpoint.

A.4. **a.** By definition, a circle is the set of points equidistant from a fixed point. The fixed point is called the ______ of the circle, and the common distance from the center to any point on the circle is called the ______.

b. Suppose that (x, y) is a point on a circle with center (h, k) and radius r. Write the standard equation for the circle.

For Exercises A.5–A.6, identify the center and radius of the circle.

A.5. $(x - 5)^2 + (y + 2)^2 = 4$

A.6. $\left(x + \frac{3}{2}\right)^2 + y^2 = 20$

A.7. The equation $x^2 + y^2 + 6x - 8y + 16 = 0$ is an equation of a circle in expanded form. To identify the center and radius, it is necessary to complete the square so that the equation is expressed as $(x - h)^2 + (y - k)^2 = r^2$. Follow these steps.

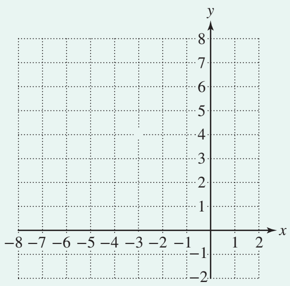
a. Group the x terms and group the y terms on the left side of the equation and move the constant term to the right. Fill in the blanks to complete the squares.

$$x^2 + 6x + \_\_\_\_\_\_ + y^2 - 8y + \_\_\_\_\_\_ = -16 + \_\_\_\_\_\_ + \_\_\_\_\_\_$$

b. Factor on the left and simplify on the right.

c. Identify the center and radius.

d. Graph the circle.



A.8. Write an equation of a circle with center $(0, -4)$ and radius 7.

Section 9.1 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, simplify the radical.

R.1. $\sqrt{50}$

R.2. $\sqrt{48}$

For Exercises R.3–R.6, simplify the expression.

R.3. $\dfrac{36}{48}$

R.4. $\dfrac{108}{72}$

R.5. $\dfrac{\frac{1}{2} + \frac{5}{4}}{2}$

R.6. $\dfrac{\frac{2}{3} + \frac{4}{5}}{2}$

For Exercises R.7–R.8, square the binomial.

R.7. $(x + 8)^2$

R.8. $(x - 9)^2$

For Exercises R.9–R.10, find the value of n that will make the trinomial a perfect square trinomial. Then factor.

R.9. $x^2 - 12x + n$

R.10. $x^2 + 20x + n$

Vocabulary and Key Concepts

1. **a.** The distance between two distinct points (x_1, y_1) and (x_2, y_2) is given by $d =$ ______________.

b. A ______________ is the set of all points equidistant from a fixed point called the ______________.

c. The distance from the center of a circle to any point on the circle is called the ______________ and is often denoted by r.

d. The standard equation of a circle with center (h, k) and radius r is given by ______________.

e. The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is given by ______________.

Concept 1: Distance Formula

For Exercises 2–16, use the distance formula to find the distance between the two points. **(See Example 1.)**

2. $(-2, 7)$ and $(4, -5)$

3. $(1, 10)$ and $(-2, 4)$

4. $(0, 5)$ and $(-3, 8)$

5. $(6, 7)$ and $(3, 2)$

6. $\left(\frac{2}{3}, \frac{1}{5}\right)$ and $\left(-\frac{5}{6}, \frac{3}{10}\right)$

7. $\left(-\frac{1}{2}, \frac{5}{8}\right)$ and $\left(-\frac{3}{2}, \frac{1}{4}\right)$

8. $(4, 13)$ and $(4, -6)$

9. $(-2, 5)$ and $(-2, 9)$

10. $(8, -6)$ and $(-2, -6)$

11. $(7, 2)$ and $(15, 2)$

12. $(-6, -2)$ and $(-3, -5)$

13. $(-1, -5)$ and $(-5, -9)$

14. $(3\sqrt{5}, 2\sqrt{7})$ and $(-\sqrt{5}, -3\sqrt{7})$

15. $(4\sqrt{6}, -2\sqrt{2})$ and $(2\sqrt{6}, \sqrt{2})$

16. $(6, 0)$ and $(0, -1)$

17. Explain how to find the distance between 5 and -7 on the y-axis.

18. Explain how to find the distance between 15 and -37 on the x-axis.

19. Find the values of y such that the distance between the points $(4, 7)$ and $(-4, y)$ is 10 units.

20. Find the values of x such that the distance between the points $(-4, -2)$ and $(x, 3)$ is 13 units.

21. Find the values of x such that the distance between the points $(x, 2)$ and $(4, -1)$ is 5 units.

22. Find the values of y such that the distance between the points $(-5, 2)$ and $(3, y)$ is 10 units.

For Exercises 23–26, determine if the three points define the vertices of a right triangle.

23. $(-3, 2)$, $(-2, -4)$, and $(3, 3)$

24. $(1, -2)$, $(-2, 4)$, and $(7, 1)$

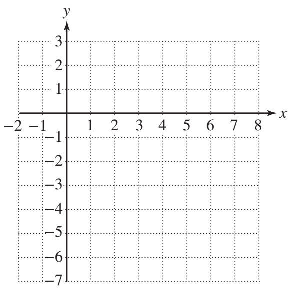
25. $(-3, -2)$, $(4, -3)$, and $(1, 5)$

26. $(1, 4)$, $(5, 3)$, and $(2, 0)$

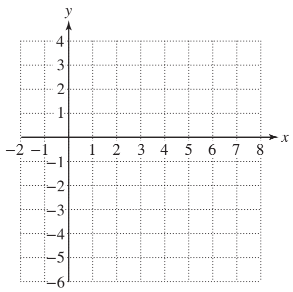
Concept 2: Circles

For Exercises 27–47, identify the center and radius of the circle and then graph the circle. Complete the square, if necessary. **(See Examples 2–4.)**

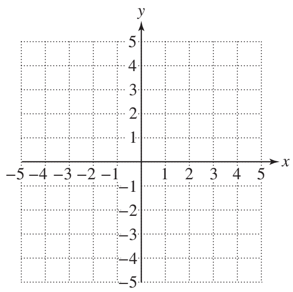
27. $(x - 4)^2 + (y + 2)^2 = 9$



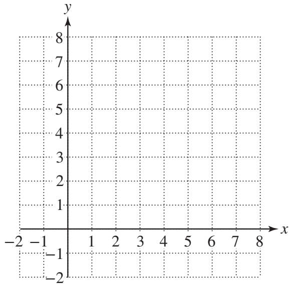
28. $(x - 3)^2 + (y + 1)^2 = 16$



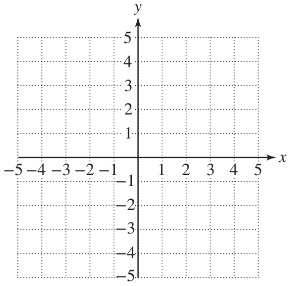
29. $(x + 1)^2 + (y + 1)^2 = 1$



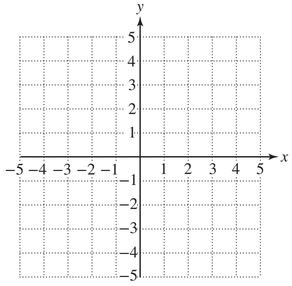
30. $(x-4)^2+(y-4)^2=4$



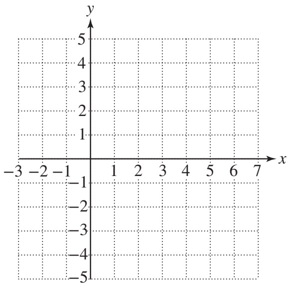
31. $x^2+(y-2)^2=4$



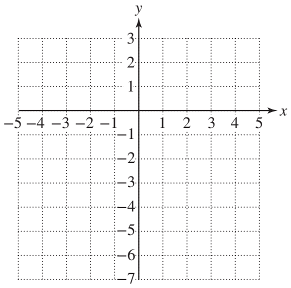
32. $(x+1)^2+y^2=1$



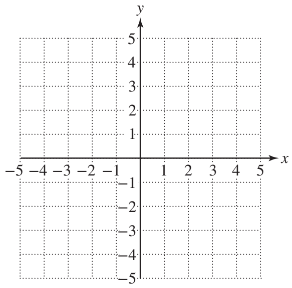
33. $(x-3)^2+y^2=8$



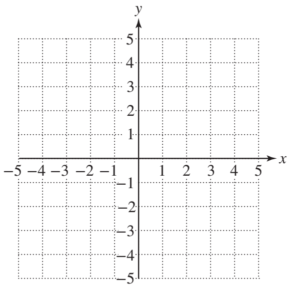
34. $x^2+(y+2)^2=20$



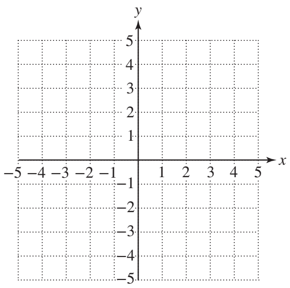
35. $x^2+y^2=6$



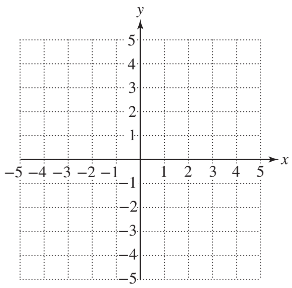
36. $x^2+y^2=15$



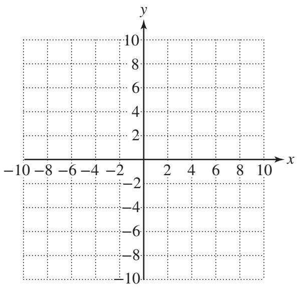
37. $\left(x+\frac{4}{5}\right)^2+y^2=\frac{64}{25}$



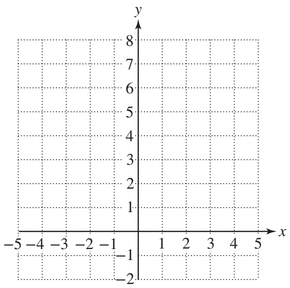
38. $x^2+\left(y-\frac{5}{2}\right)^2=\frac{9}{4}$



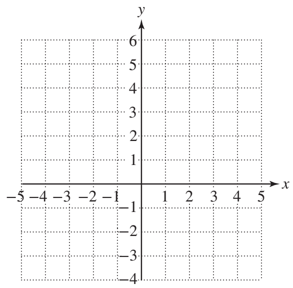
39. $x^2+y^2-2x-6y-26=0$



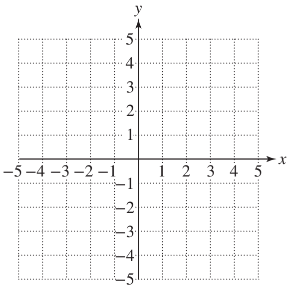
40. $x^2+y^2+4x-8y+16=0$



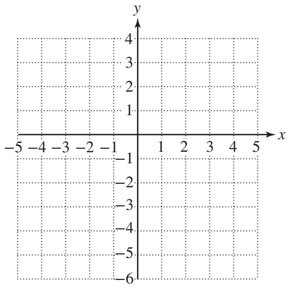
41. $x^2+y^2-6y+5=0$



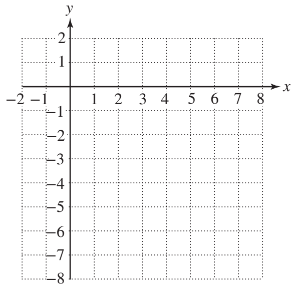
42. $x^2 + 2x + y^2 - 15 = 0$



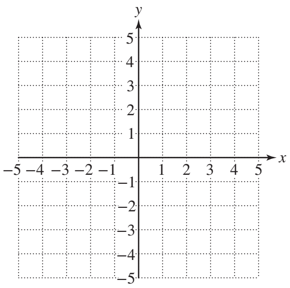
43. $x^2 + y^2 + 6y + \dfrac{65}{9} = 0$



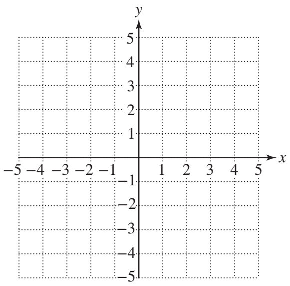
44. $x^2 + y^2 - 12x + 12y + 71 = 0$



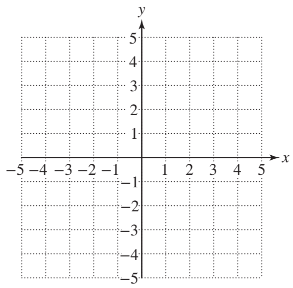
45. $x^2 + y^2 + 2x + 4y - 4 = 0$



46. $2x^2 + 2y^2 = 32$



47. $3x^2 + 3y^2 = 3$

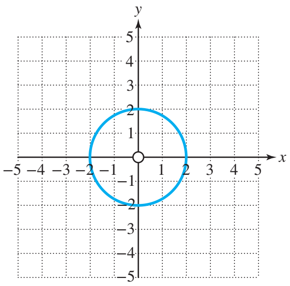


Concept 3: Writing an Equation of a Circle

48. If the diameter of a circle is 10 ft, what is the radius?

For Exercises 49–54, write an equation that represents the graph of the circle. **(See Example 5.)**

49.



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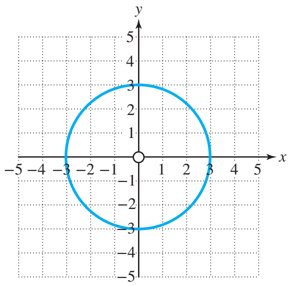

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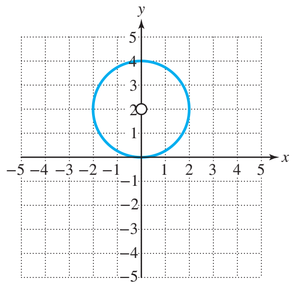

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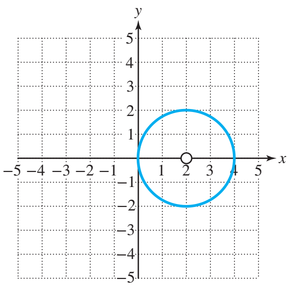

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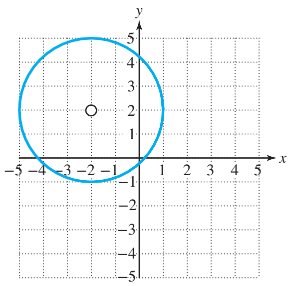

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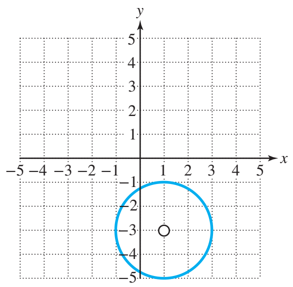


55. Write an equation of a circle centered at the origin with a radius of 7.

56. Write an equation of a circle centered at the origin with a radius of 12.

57. Write an equation of a circle centered at $(-3, -4)$ with a diameter of 12.

58. Write an equation of a circle centered at $(5, -1)$ with a diameter of 8.

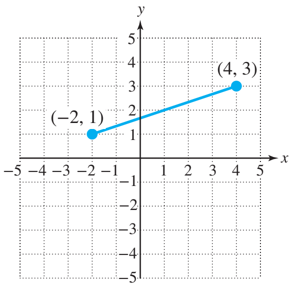
59. A cell tower is a site where antennas, transmitters, and receivers are placed to create a cellular network. Suppose that a cell tower is located at a point $A(5, 3)$ on a map and its range is 1.5 mi. Write an equation that represents the boundary of the area that can receive a signal from the tower. Assume that all distances are in miles.

60. A radar transmitter on a ship has a range of 20 nautical miles. If the ship is located at a point $(-28, 32)$ on a map, write an equation for the boundary of the area within the range of the ship's radar. Assume that all distances on the map are represented in nautical miles.

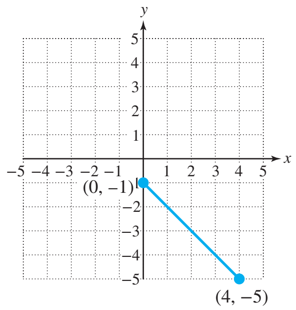
Concept 4: The Midpoint Formula

For Exercises 61–64, find the midpoint of the line segment. Check your answers by plotting the midpoint on the graph.

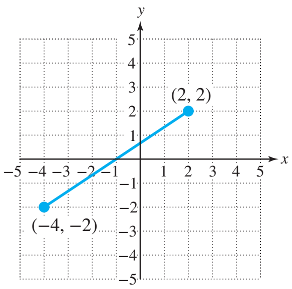
61.



62.



63.



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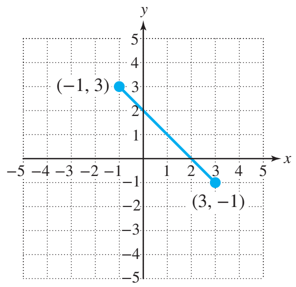


For Exercises 65–72, find the midpoint of the line segment between the two given points. **(See Example 6.)**

65. $(4, 0)$ and $(-6, 12)$

66. $(-7, 2)$ and $(-3, -2)$

67. $(-3, 8)$ and $(3, -2)$

68. $(0, 5)$ and $(4, -5)$

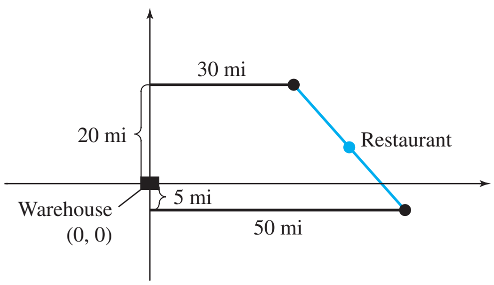
69. $(5, 2)$ and $(-6, 1)$

70. $(-9, 3)$ and $(0, -4)$

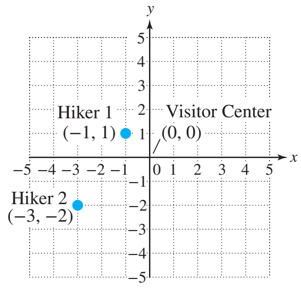
71. $(-2.4, -3.1)$ and $(1.6, 1.1)$

72. $(0.8, 5.3)$ and $(-4.2, 7.1)$

73. Two courier trucks leave the warehouse to make deliveries. One travels 20 mi north and 30 mi east. The other truck travels 5 mi south and 50 mi east. If the two drivers want to meet for lunch at a restaurant at a point halfway between them, where should they meet relative to the warehouse? (*Hint:* Label the warehouse as the origin, and find the coordinates of the restaurant. See the figure.)



74. A map of a hiking area is drawn so that the Visitor Center is at the origin of a rectangular grid. Two hikers are located at positions $(-1, 1)$ and $(-3, -2)$ with respect to the Visitor Center where all units are in miles. A campground is located exactly halfway between the hikers. What are the coordinates of the campground?



For Exercises 75–78, the two given points are endpoints of a diameter of a circle.

a. Find the center of the circle.

b. Write an equation of the circle. **(See Example 7.)**

75. $(-1, 2)$ and $(3, 4)$ **76.** $(-3, 3)$ and $(7, -1)$ **77.** $(-2, 3)$ and $(2, 3)$ **78.** $(-1, 3)$ and $(-1, -3)$

Technology Connections

For Exercises 79–84, graph the circle from the indicated exercise on a square viewing window, and approximate the center and the radius from the graph.

79. $(x - 4)^2 + (y + 2)^2 = 9$ (Exercise 27)

80. $(x - 3)^2 + (y + 1)^2 = 16$ (Exercise 28)

81. $x^2 + (y - 2)^2 = 4$ (Exercise 31)

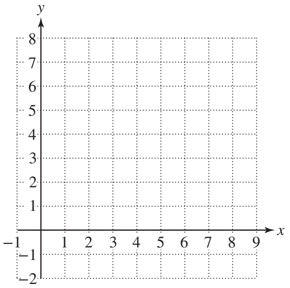
82. $(x + 1)^2 + y^2 = 1$ (Exercise 32)

83. $x^2 + y^2 = 6$ (Exercise 35)

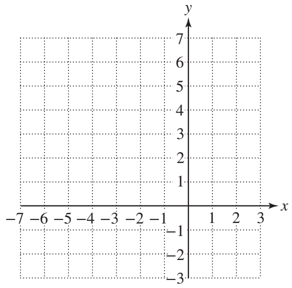
84. $x^2 + y^2 = 15$ (Exercise 36)

Expanding Your Skills

85. Write an equation of a circle whose center is $(4, 4)$ and is tangent to the x- and y-axes. (*Hint:* Sketch the circle first.)



86. Write an equation of a circle whose center is $(-3, 3)$ and is tangent to the x- and y-axes. (*Hint:* Sketch the circle first.)



87. Write an equation of a circle whose center is $(1, 1)$ and that passes through the point $(-4, 3)$.

88. Write an equation of a circle whose center is $(-3, -1)$ and that passes through the point $(5, -2)$.

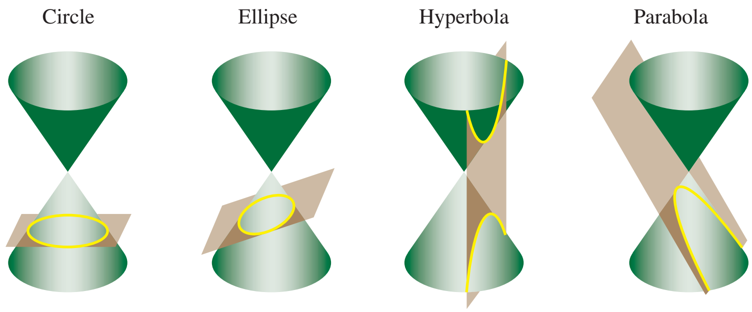
Section 9.2 More on the Parabola

Concepts

1. Introduction to Conic Sections
2. Parabola—Vertical Axis of Symmetry
3. Parabola—Horizontal Axis of Symmetry
4. Vertex Formula

1. Introduction to Conic Sections

The graph of a second-degree equation of the form $y = ax^2 + bx + c$ $(a \neq 0)$ is a parabola and the graph of $(x - h)^2 + (y - k)^2 = r^2$ is a circle. These and two other types of figures called ellipses and hyperbolas are called **conic sections**. Conic sections derive their names because each is the intersection of a plane and a double-napped cone.



2. Parabola—Vertical Axis of Symmetry

A **parabola** is defined by a set of points in a plane that are equidistant from a fixed line (called the directrix) and a fixed point (called the focus) not on the directrix (Figure 9-9). Parabolas have numerous real-world applications. For example, a flashlight has a mirror with the cross section in the shape of a parabola. The bulb is located at the focus. The light hits the mirror and is reflected outward parallel to the sides of the cylinder. This forms a beam of light (Figure 9-10).

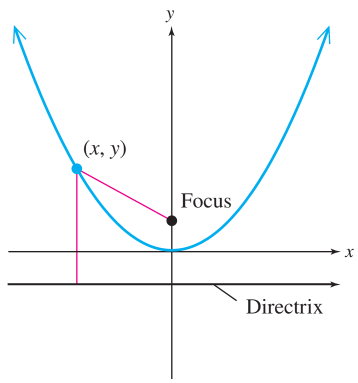


Figure 9-9

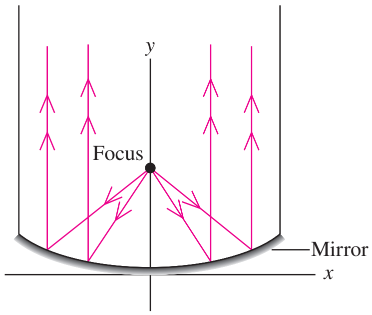


Figure 9-10

The graph of the solutions to the quadratic equation $y = ax^2 + bx + c$ is a parabola. Recall that the **vertex** is the highest or the lowest point of a parabola. The **axis of symmetry** of the parabola is a line that passes through the vertex and is perpendicular to the directrix (Figure 9-11).

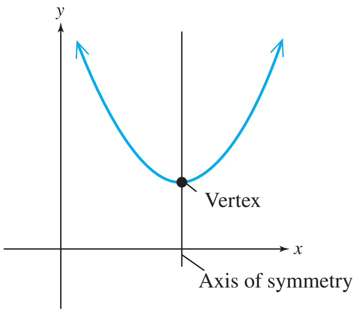


Figure 9-11

Standard Form of the Equation of a Parabola—Vertical Axis of Symmetry

The standard form of the equation of a parabola with vertex (h, k) and vertical axis of symmetry is

$$y = a(x - h)^2 + k \qquad \text{where } a \neq 0$$

If $a > 0$, the parabola opens upward; and if $a < 0$, the parabola opens downward.

The axis of symmetry is given by $x = h$.

It is sometimes necessary to complete the square to write an equation of a parabola in standard form.

Example 1 Graphing a Parabola by First Completing the Square

Given the equation of the parabola $y = -2x^2 + 4x + 1$,

a. Write the equation in standard form $y = a(x - h)^2 + k$.

b. Identify the vertex and axis of symmetry. Determine if the parabola opens upward or downward.

c. Graph the parabola.

Solution:

a. Complete the square to write the equation in the form $y = a(x - h)^2 + k$.

$y = -2x^2 + 4x + 1$	
$y = -2(x^2 - 2x) + 1$	Factor out -2 from the variable terms.
$y = -2(x^2 - 2x + 1 - 1) + 1$	Add and subtract the quantity $\left[\frac{1}{2}(-2)\right]^2 = 1$.
$y = -2(x^2 - 2x + 1) + (-2)(-1) + 1$	Remove the -1 term from within the parentheses by first applying the distributive property. When -1 is removed from the parentheses, it carries with it the factor of -2 from outside the parentheses.
$y = -2(x - 1)^2 + 2 + 1$	
$y = -2(x - 1)^2 + 3$	

The equation is in the form $y = a(x - h)^2 + k$ where $a = -2$, $h = 1$, and $k = 3$.

b. The vertex is $(1, 3)$. Because a is negative, the parabola opens downward. The axis of symmetry is $x = 1$.

FOR REVIEW

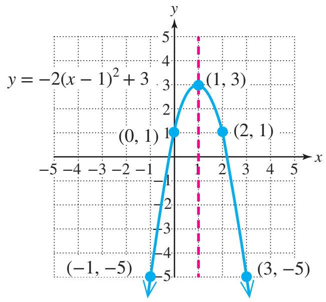
Recall that a perfect square trinomial factors as the square of a binomial. For example:

$$x^2 - 2x + 1 = (x - 1)^2$$

c. To graph the parabola, we know that its orientation is downward because the leading term is negative. Furthermore, we know the vertex is (1, 3). To find other points on the parabola, select several values of x and solve for y. Recall that the y-intercept is found by substituting $x = 0$ and solving for y.

x	y	
1	3	← Vertex
0	1	← y-intercept
−1	−5	
2	1	
3	−5	

↑ Choose x. ↑ Solve for y.



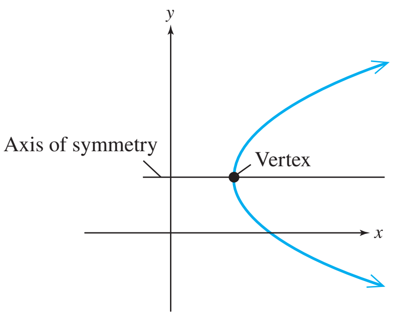
Notice that the points (2, 1) and (3, −5) are the mirror images of the points (0, 1) and (−1, −5) around the axis of symmetry, $x = 1$. For this reason, once we plot several points on one side of the axis of symmetry, the points on the other side automatically follow as a mirror image.

Skill Practice Given the equation of the parabola $y = 3x^2 + 6x + 4$,

1. Write the equation in standard form.
2. Identify the vertex and axis of symmetry.
3. Graph the parabola.

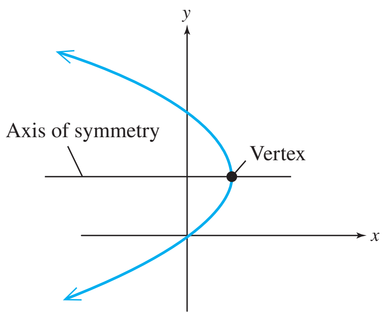
3. Parabola—Horizontal Axis of Symmetry

We have seen that the graph of a parabola $y = ax^2 + bx + c$ opens upward if $a > 0$ and downward if $a < 0$. A parabola can also open to the left or right. In such a case, the "roles" of x and y are essentially interchanged in the equation. Thus, the graph of $x = ay^2 + by + c$ opens to the right if $a > 0$ (Figure 9-12) and to the left if $a < 0$ (Figure 9-13).



$x = ay^2 + by + c, a > 0$

Figure 9-12

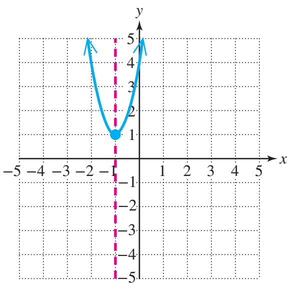


$x = ay^2 + by + c, a < 0$

Figure 9-13

Answers

1. $y = 3(x + 1)^2 + 1$
2. Vertex: (−1, 1); axis of symmetry: $x = -1$
3.



Standard Form of the Equation of a Parabola—Horizontal Axis of Symmetry

The standard form of an equation of a parabola with vertex (h, k) and horizontal axis of symmetry is

$$x = a(y - k)^2 + h \qquad \text{where } a \neq 0$$

If $a > 0$, the parabola opens to the right and if $a < 0$, the parabola opens to the left.

The axis of symmetry is given by $y = k$.

Example 2 Graphing a Parabola With a Horizontal Axis of Symmetry

Given the equation of the parabola $x = 4y^2$,

a. Determine the coordinates of the vertex and the equation of the axis of symmetry.

b. Use the value of a to determine if the parabola opens to the right or left.

c. Plot several points and graph the parabola.

Solution:

a. The equation can be written in the form $x = a(y - k)^2 + h$:

$$x = 4(y - 0)^2 + 0$$

Therefore, $h = 0$ and $k = 0$.

The vertex is $(0, 0)$. The axis of symmetry is $y = 0$ (the x-axis).

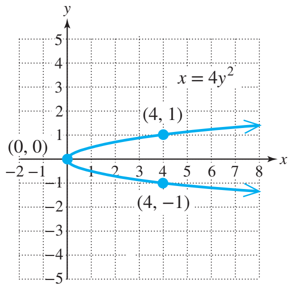
b. Because a is positive, the parabola opens to the right.

c. The vertex of the parabola is $(0, 0)$. To find other points on the graph, select values for y and solve for x.

x	y
0	0
4	1
4	−1
16	2
16	−2

← Vertex

Solve for x. Choose y.



TIP: For Example 2, the x-axis is the axis of symmetry. Therefore, once we find points on the parabola above the axis of symmetry, their mirror image will be points on the parabola below the axis of symmetry.

Skill Practice Given the equation $x = -y^2 + 1$,

4. Identify the vertex and the axis of symmetry.

5. Determine if the parabola opens to the right or to the left.

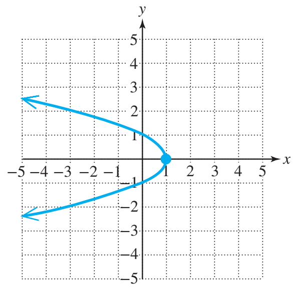
6. Graph the parabola.

Answers

4. Vertex: (1, 0); axis of symmetry: $y = 0$

5. Parabola opens left.

6.



Example 3 Graphing a Parabola by First Completing the Square

Given the equation of the parabola $x = -y^2 + 8y - 14$,

a. Write the equation in standard form $x = a(y - k)^2 + h$.

b. Identify the vertex and axis of symmetry. Determine if the parabola opens to the right or left.

c. Graph the parabola.

Solution:

a. Complete the square to write the equation in the form $x = a(y - k)^2 + h$.

$x = -y^2 + 8y - 14$

$x = -1(y^2 - 8y) - 14$ — Factor out -1 from the variable terms.

$x = -1(y^2 - 8y + 16 - 16) - 14$ — Add and subtract the quantity $[\frac{1}{2}(-8)]^2 = 16$.

$x = -1(y^2 - 8y + 16) + (-1)(-16) - 14$ — Remove the -16 term from within the parentheses by first applying the distributive property. When -16 is removed from the parentheses, it carries with it the factor of -1 from outside the parentheses.

$x = -1(y - 4)^2 + 16 - 14$

$x = -1(y - 4)^2 + 2$

The equation is in the form $x = a(y - k)^2 + h$, where $a = -1$, $h = 2$, and $k = 4$.

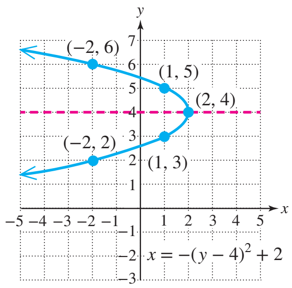
b. The vertex is (2, 4). Because a is negative, the parabola opens to the left. The axis of symmetry is $y = 4$.

c.

x	y
2	4
1	5
1	3
−2	6
−2	2

← Vertex

Solve for x. Choose y.



Skill Practice Given the equation of the parabola $x = y^2 + 4y + 7$,

7. Identify the vertex and axis of symmetry.

8. Determine if the parabola opens to the right or to the left.

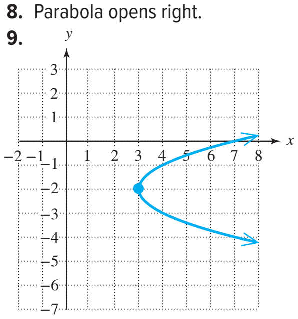
9. Graph the parabola.

Answers

7. Vertex: (3, −2); Axis of symmetry: $y = -2$

8. Parabola opens right.

9.



4. Vertex Formula

The vertex formula can also be used to find the vertex of a parabola.

Vertex Formula

For a parabola defined by $y = ax^2 + bx + c$,

- The x-coordinate of the vertex is given by $x = \frac{-b}{2a}$.
- The y-coordinate of the vertex is found by substituting this value for x into the original equation and solving for y.

For a parabola defined by $x = ay^2 + by + c$,

- The y-coordinate of the vertex is given by $y = \frac{-b}{2a}$.
- The x-coordinate of the vertex is found by substituting this value for y into the original equation and solving for x.

Example 4 Finding the Vertex of a Parabola by Using the Vertex Formula

Find the vertex by using the vertex formula. $x = y^2 + 4y + 5$

Solution:

$x = y^2 + 4y + 5$ The parabola is in the form $x = ay^2 + by + c$.

$a = 1$ $b = 4$ $c = 5$ Identify a, b, and c.

The y-coordinate of the vertex is given by $y = \dfrac{-b}{2a} = \dfrac{-(4)}{2(1)} = -2$.

The x-coordinate of the vertex is found by substitution:

$$x = (-2)^2 + 4(-2) + 5 = 1$$

The vertex is $(1, -2)$.

Skill Practice Find the vertex by using the vertex formula.

10. $x = -4y^2 + 12y$

Example 5 Find the Vertex of a Parabola by Using the Vertex Formula

Find the vertex by using the vertex formula. $y = \dfrac{1}{2}x^2 - 3x + \dfrac{5}{2}$

Solution:

$y = \dfrac{1}{2}x^2 - 3x + \dfrac{5}{2}$ The parabola is in the form $y = ax^2 + bx + c$.

$a = \dfrac{1}{2}$ $b = -3$ $c = \dfrac{5}{2}$ Identify a, b, and c.

The x-coordinate of the vertex is given by $x = \dfrac{-b}{2a} = \dfrac{-(-3)}{2\left(\frac{1}{2}\right)} = 3$.

Answer

10. Vertex: $\left(9, \dfrac{3}{2}\right)$

The y-coordinate of the vertex is found by substitution.

$$y = \frac{1}{2}(3)^2 - 3(3) + \frac{5}{2}$$

$$= \frac{9}{2} - 9 + \frac{5}{2}$$

$$= \frac{14}{2} - 9$$

$$= 7 - 9$$

$$= -2$$

The vertex is $(3, -2)$.

Skill Practice Find the vertex by using the vertex formula.

11. $y = \frac{3}{4}x^2 + 3x + 5$

Answer

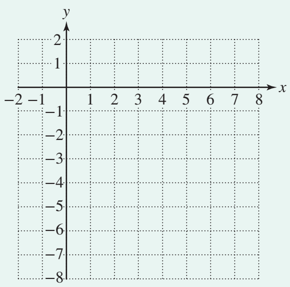
11. $(-2, 2)$

Section 9.2 Activity

A.1. **a.** Describe the graph of $y = a(x - h)^2 + k$.

b. Describe the graph of $x = a(y - k)^2 + h$.

A.2. Graph $y = -(x - 4)^2 + 1$.



A.3. Refer to Exercise A.2. How would the graph change for the equation $x = -(y - 1)^2 + 4$?

A.4. The equation $x = y^2 + 4y + 1$ is of the form $x = ay^2 + by + c$.

a. Identify the values of a, b, and c.

b. Does the parabola open upward, downward, to the left, or to the right?

c. Write the equation in the form $x = a(y - k)^2 + h$ by completing the square.

d. Identify the vertex.

e. Identify the axis of symmetry.

Practice Exercises

Section 9.2

Prerequisite Review

For Exercises R.1–R.4,

a. Identify the vertex of the parabola defined by the quadratic function.
b. Determine the y-intercept.
c. Determine the x-intercepts (if any exist).

R.1. $y = -x^2 + 2x + 3$

R.2. $y = x^2 + 6x + 8$

R.3. $y = x^2 + 4x + 4$

R.4. $y = x^2 - 10x + 25$

For Exercises R.5–R.6, evaluate the expression for the given values of the variables.

R.5. $\dfrac{-b}{2a}$ for $a = 6$ and $b = 24$

R.6. $\dfrac{-b}{2a}$ for $a = 5$ and $b = -12$

Vocabulary and Key Concepts

1. **a.** A circle, a parabola, an ellipse, and a hyperbola are curves that collectively are called __________ sections. These types of curves derive their names because each is the intersection of a __________ and a double-napped cone.

b. A __________ is defined by a set of points in a plane that are equidistant from a fixed line (called the __________) and a fixed point (called the __________).

c. Given a parabola defined by $y = a(x - h)^2 + k$ $(a \neq 0)$, the __________ is (h, k) and is the highest or lowest point on the graph. The axis of __________ is the vertical line through the vertex whose equation is $x = h$.

d. Given a parabola defined by $x = a(y - k)^2 + h$ $(a \neq 0)$, the axis of symmetry is the horizontal line whose equation is __________.

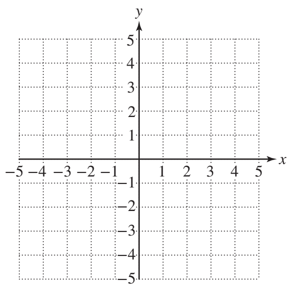
2. Determine whether the parabola defined by the given function opens upward, downward, to the right, or to the left.

a. $y = -x^2$ **b.** $x = y^2$ **c.** $x = -y^2$ **d.** $y = x^2$

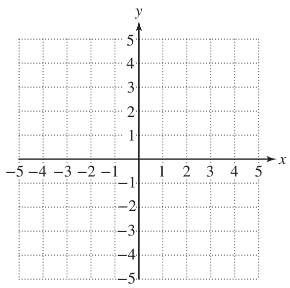
Concept 2: Parabola—Vertical Axis of Symmetry

For Exercises 3–10, use the equation of the parabola in standard form $y = a(x - h)^2 + k$ to determine the coordinates of the vertex and the equation of the axis of symmetry (complete the square if necessary). Then graph the parabola.
(See Example 1.)

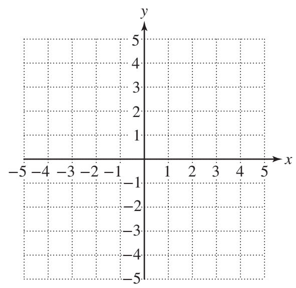
3. $y = (x + 2)^2 + 1$



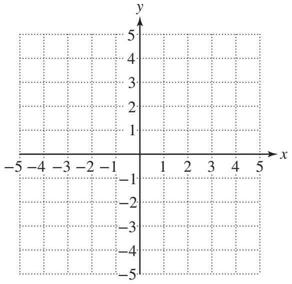
4. $y = (x - 1)^2 + 1$



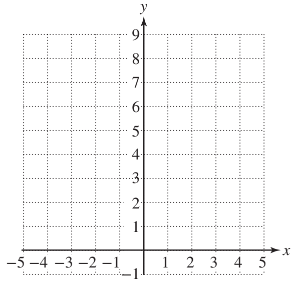
5. $y = x^2 - 4x + 3$



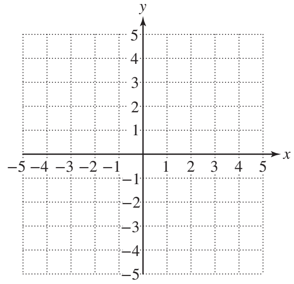
6. $y = x^2 + 6x + 5$



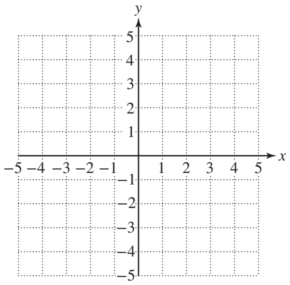
7. $y = -2x^2 + 8x$



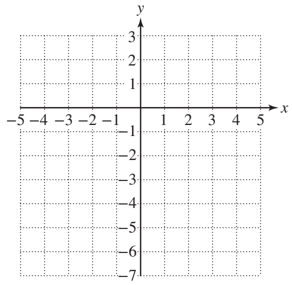
8. $y = -3x^2 - 6x$



9. $y = -x^2 - 3x + 2$



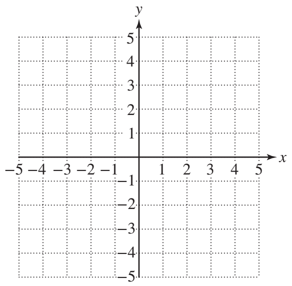
10. $y = -x^2 + x - 4$



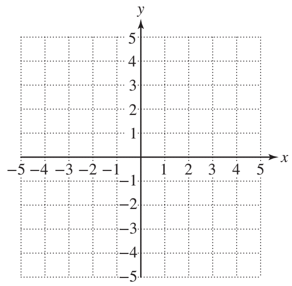
Concept 3: Parabola—Horizontal Axis of Symmetry

For Exercises 11–18, use the equation of the parabola in standard form $x = a(y - k)^2 + h$ to determine the coordinates of the vertex and the axis of symmetry (complete the square if necessary). Then graph the parabola. **(See Examples 2–3.)**

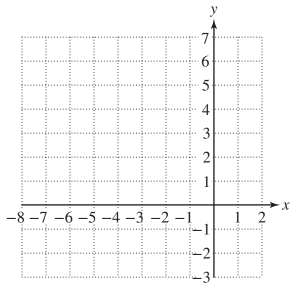
11. $x = y^2 - 3$



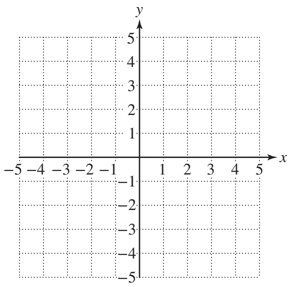
12. $x = y^2 + 1$



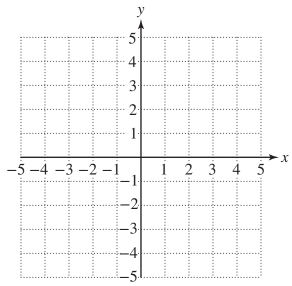
13. $x = -(y - 3)^2 - 3$



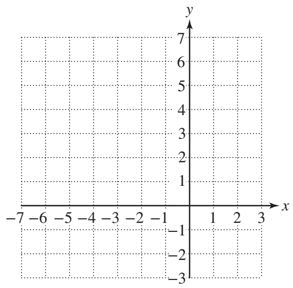
14. $x = -2(y + 2)^2 + 1$



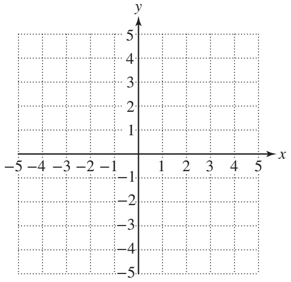
15. $x = -y^2 + 4y - 4$



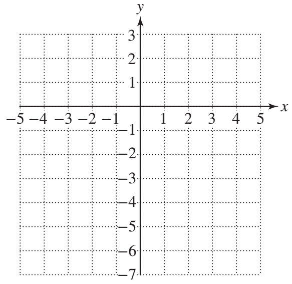
16. $x = -4y^2 - 4y - 2$



17. $x = y^2 - 2y + 2$



18. $x = y^2 + 4y + 1$



Concept 4: Vertex Formula

For Exercises 19–27, determine the vertex by using the vertex formula. **(See Examples 4–5.)**

19. $y = x^2 - 4x + 3$

20. $y = x^2 + 6x - 2$

21. $x = y^2 + 2y + 6$

22. $x = y^2 - 8y + 3$

23. $y = -\frac{1}{4}x^2 + x + \frac{3}{4}$

24. $y = -\frac{1}{2}x^2 - x + \frac{1}{2}$

25. $y = x^2 - 3x + 2$

26. $x = -2y^2 + 16y + 1$

27. $x = -3y^2 - 6y + 7$

28. Ricardo has a satellite dish for his television. The cross sections of the satellite dish are parabolic in shape. A cross section through the vertex can be described by the equation

$$d(x) = \frac{2}{25}x^2 - \frac{2}{5}x$$

where $d(x)$ is the depth of the dish (in feet) at a distance of x ft from the edge of the dish. How deep is the satellite dish?

Ryan McVay/Photodisc/Getty Images

29. Water from an outdoor fountain is projected outward from a jet on the side wall of the fountain. The water follows a parabolic path and can be modeled by $h(x) = -x^2 + 10x - 3$. In this function, $h(x)$ represents the height of the water, x ft from the jet. Find the maximum height of the water.

rudi1976/123RF

Mixed Exercises

30. Explain how to determine whether a parabola opens upward, downward, left, or right.

31. Explain how to determine whether a parabola has a vertical or horizontal axis of symmetry.

For Exercises 32–43, use the equation of the parabola first to determine whether the axis of symmetry is vertical or horizontal. Then determine if the parabola opens upward, downward, left, or right.

32. $y = (x - 2)^2 + 3$

33. $y = (x - 4)^2 + 2$

34. $y = -2(x + 1)^2 - 4$

35. $y = -3(x + 2)^2 - 1$

36. $x = y^2 + 4$

37. $x = y^2 - 2$

38. $x = -(y + 3)^2 + 2$

39. $x = -2(y - 1)^2 - 3$

40. $y = -2x^2 - 5$

41. $y = -x^2 + 3$

42. $x = 2y^2 + 3y - 2$

43. $x = y^2 - 5y + 1$

Section 9.3 The Ellipse and Hyperbola

Concepts

1. Standard Form of an Equation of an Ellipse
2. Standard Form of an Equation of a Hyperbola

1. Standard Form of an Equation of an Ellipse

In this section, we will study the two remaining conic sections: the ellipse and the hyperbola. An **ellipse** is the set of all points (x, y) such that the sum of the distances between (x, y) and two distinct points is a constant. The fixed points are called the foci (plural of *focus*) of the ellipse.

To visualize an ellipse, consider the following application. Suppose Sonya wants to cut an elliptical rug from a rectangular rug to avoid a stain made by the family dog. She places two tacks along the center horizontal line. Then she ties the ends of a slack piece of rope to each tack. With the rope pulled tight, she traces out a curve. This curve is an ellipse, and the tacks are located at the foci of the ellipse (Figure 9-14).

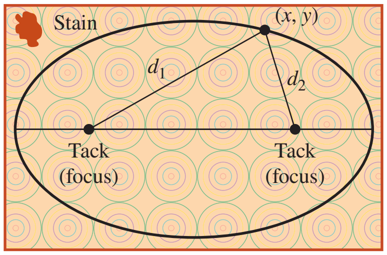


Figure 9-14

We will first graph ellipses that are centered at the origin.

Standard Form of an Equation of an Ellipse Centered at the Origin

An ellipse with the center at the origin has the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are positive real numbers. In the standard form of the equation, the right side must equal 1.

To graph an ellipse centered at the origin, find the x- and y-intercepts.

To find the x-intercepts, let $y = 0$.

$$\frac{x^2}{a^2} + \frac{0}{b^2} = 1$$

$$\frac{x^2}{a^2} = 1$$

$$x^2 = a^2$$

$$x = \pm\sqrt{a^2}$$

$$x = \pm a$$

The x-intercepts are $(a, 0)$ and $(-a, 0)$. See Figure 9-15.

To find the y-intercepts, let $x = 0$.

$$\frac{0}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1$$

$$y^2 = b^2$$

$$y = \pm\sqrt{b^2}$$

$$y = \pm b$$

The y-intercepts are $(0, b)$ and $(0, -b)$. See Figure 9-15.

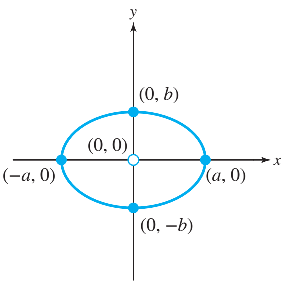


Figure 9-15

Example 1 Graphing an Ellipse

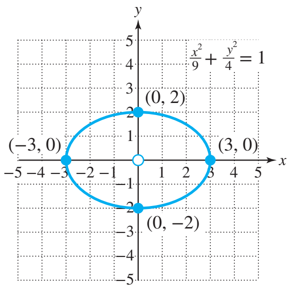
Graph the ellipse given by the equation. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Solution:

The equation can be written as $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$.

The equation is in standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, therefore, $a = 3$ and $b = 2$.

Graph the intercepts $(3, 0)$, $(-3, 0)$, $(0, 2)$, $(0, -2)$ and sketch the ellipse.



Skill Practice

1. Graph the ellipse given by the equation. $\frac{x^2}{4} + \frac{y^2}{16} = 1$

Example 2 Graphing an Ellipse

Graph the ellipse given by the equation. $25x^2 + y^2 = 25$

Solution:

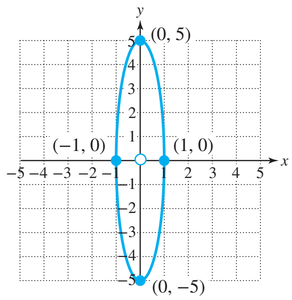
$25x^2 + y^2 = 25$ Divide both sides by 25.

$$\frac{25x^2}{25} + \frac{y^2}{25} = \frac{25}{25}$$

$$x^2 + \frac{y^2}{25} = 1$$

The equation can then be written as $\frac{x^2}{1^2} + \frac{y^2}{5^2} = 1$.

The equation is in standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, therefore, $a = 1$ and $b = 5$.



Graph the intercepts $(1, 0)$, $(-1, 0)$, $(0, 5)$, and $(0, -5)$ and sketch the ellipse.

Skill Practice

2. Graph the ellipse given by the equation. $x^2 + 9y^2 = 9$

A circle is a special case of an ellipse where $a = b$. Therefore, it is not surprising that we graph an ellipse centered at the point (h, k) in much the same way we graph a circle.

Example 3 Graphing an Ellipse Whose Center Is Not at the Origin

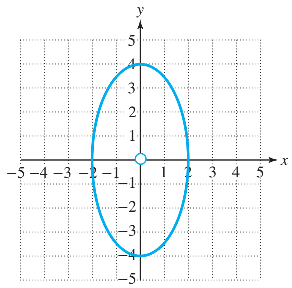
Graph the ellipse given by the equation. $\frac{(x-1)^2}{16} + \frac{(y+3)^2}{4} = 1$

Solution:

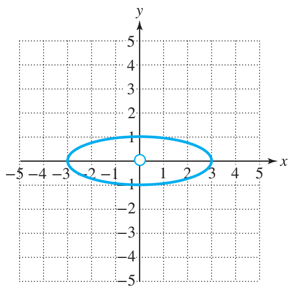
Just as we would find the center of a circle, we see that the center of the ellipse is $(1, -3)$. Now we can use the values of a and b to help us plot four strategic points to define the curve.

Answers

1.



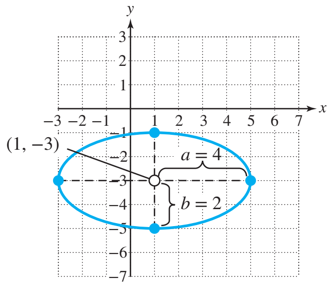
2.



The equation can be written as

$$\frac{(x-1)^2}{(4)^2}+\frac{(y+3)^2}{(2)^2}=1$$

From this, we have $a = 4$ and $b = 2$. To sketch the curve, locate the center at $(1, -3)$. Then move $a = 4$ units to the left and right of the center and plot two points. Similarly, move $b = 2$ units up and down from the center and plot two points.



Skill Practice

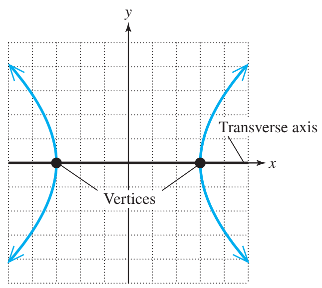
3. Graph the ellipse. $\frac{(x+1)^2}{4}+\frac{(y-4)^2}{9}=1$

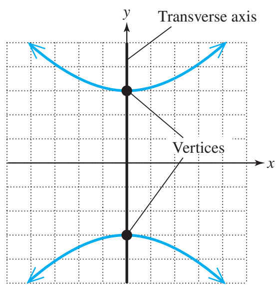
2. Standard Form of an Equation of a Hyperbola

A **hyperbola** is the set of all points (x, y) such that the absolute value of the *difference* of the distances between (x, y) and two distinct points is a constant. The fixed points are called the foci of the hyperbola. The graph of a hyperbola has two parts, called branches. A hyperbola has two vertices that lie on an axis of symmetry called the **transverse axis**. For the hyperbolas studied here, the transverse axis is either horizontal or vertical.

FOR REVIEW

From the vertical line test, neither the graph of an ellipse nor the graph of a hyperbola defines y as a function of x.





Standard Forms of an Equation of a Hyperbola With Center at the Origin

Let a and b represent positive real numbers.

Horizontal transverse axis:
The standard form of an equation of a hyperbola with a *horizontal transverse axis* and center at the origin is given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Note: The x term is positive. The branches of the hyperbola open left and right.

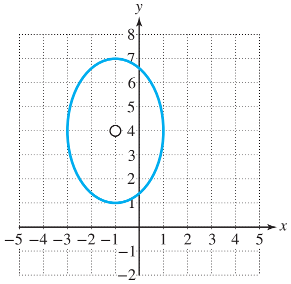
Vertical transverse axis:
The standard form of an equation of a hyperbola with a *vertical transverse axis* and center at the origin is given by $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$.

Note: The y term is positive. The branches of the hyperbola open up and down.

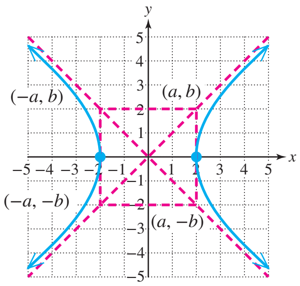
In the standard forms of an equation of a hyperbola, the right side must equal 1.

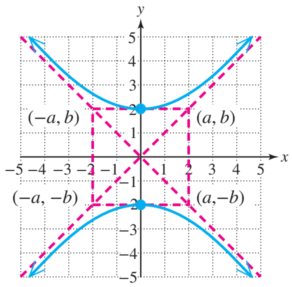
Answer

3.



To graph a hyperbola centered at the origin, first construct a reference rectangle. Draw this rectangle by using the points (a, b), $(-a, b)$, $(a, -b)$, and $(-a, -b)$. Asymptotes lie on the diagonals of the rectangle. The branches of the hyperbola are drawn to approach the asymptotes.





Example 4 **Graphing a Hyperbola**

Graph the hyperbola given by the equation. $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$

a. Determine whether the transverse axis is horizontal or vertical.

b. Draw the reference rectangle and asymptotes.

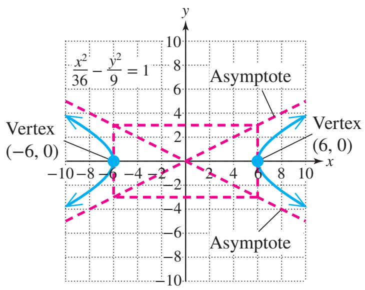
c. Graph the hyperbola and label the vertices.

Solution:

a. Since the x term is positive, the transverse axis is horizontal.

b. The equation can be written $\dfrac{x^2}{6^2} - \dfrac{y^2}{3^2} = 1$; therefore, $a = 6$ and $b = 3$. Graph the reference rectangle from the points $(6, 3)$, $(6, -3)$, $(-6, 3)$, $(-6, -3)$.

c. The vertices are $(-6, 0)$ and $(6, 0)$.



Skill Practice

4. Graph the hyperbola. $x^2 - \dfrac{y^2}{9} = 1$

Example 5 **Graphing a Hyperbola**

Graph the hyperbola given by the equation. $y^2 - 4x^2 - 16 = 0$

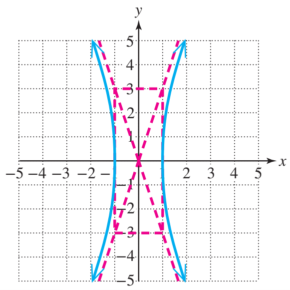
a. Write the equation in standard form to determine whether the transverse axis is horizontal or vertical.

b. Draw the reference rectangle and asymptotes.

c. Graph the hyperbola and label the vertices.

Answer

4.



Solution:

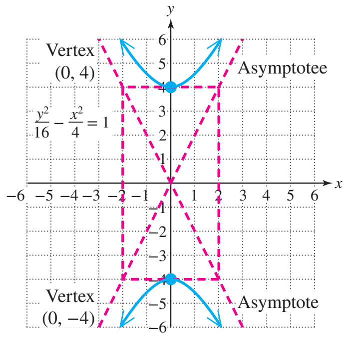
a. $y^2 - 4x^2 - 16 = 0$

$y^2 - 4x^2 = 16$ Isolate the constant term on the right.

$\frac{y^2}{16} - \frac{4x^2}{16} = \frac{16}{16}$ Divide by 16 to make the constant term 1.

$$\frac{y^2}{16} - \frac{x^2}{4} = 1$$

Since the y term is positive, the transverse axis is vertical.



b. The equation can be written $\frac{y^2}{4^2} - \frac{x^2}{2^2} = 1$; therefore, $a = 2$ and $b = 4$. Graph the reference rectangle from the points (2, 4), (2, −4), (−2, 4), (−2, −4).

c. The vertices are (0, 4) and (0, −4).

Skill Practice

5. Graph the hyperbola. $9y^2 - 4x^2 = 36$

The following summarizes the steps for graphing ellipses and hyperbolas.

Graphing an Ellipse With Center at the Origin

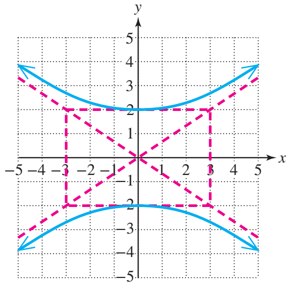
1. Write the equation in standard form:
   $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
2. Plot the intercepts.
   x-intercepts: $(a, 0)$ and $(-a, 0)$
   y-intercepts: $(0, b)$ and $(0, -b)$
3. Sketch the ellipse through the intercepts.

Graphing a Hyperbola With Center at the Origin

1. Write the equation in standard form:
   - Horizontal transverse axis:
     $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Branches open left and right
   - Vertical transverse axis:
     $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ Branches open up and down
2. Draw the reference rectangle through the points (a, b), $(a, -b)$, $(-a, b)$, and $(-a, -b)$.
3. Draw the asymptotes.
4. Plot the vertices and sketch the hyperbola.

Answer

5.

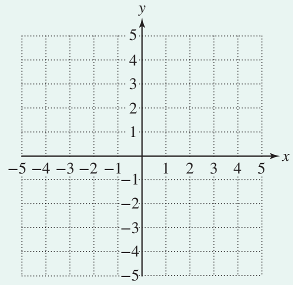


Section 9.3 Activity

A.1. **a.** Describe the graph of $x^2 + y^2 = 4$.

b. Describe the graph of $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

c. Graph $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ from part (b).



A.2. **a.** Refer to Exercise A.1(c). How is the graph of $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$ different from the graph of $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$?

A.3. **a.** Describe the graph of $\dfrac{x^2}{25} + \dfrac{y^2}{49} = 1$.

b. Describe the graph of $\dfrac{x^2}{25} - \dfrac{y^2}{49} = 1$, including the vertices of the reference rectangle.

c. Describe the graph of $\dfrac{y^2}{49} - \dfrac{x^2}{25} = 1$, including the vertices of the reference rectangle.

Practice Exercises — Section 9.3

Prerequisite Review

For Exercises R.1–R.4, determine the center and radius of the circle.

R.1. $x^2 + y^2 = 16$

R.2. $x^2 + y^2 = 36$

R.3. $(x-3)^2 + (y+2)^2 = 12$

R.4. $(x+5)^2 + (y-1)^2 = 20$

Vocabulary and Key Concepts

1. **a.** A(n) __________ is the set of all points (x, y) such that the sum of the distances between (x, y) and two distinct points (called __________) is constant.

b. The standard form of an equation of an __________ with center at the origin is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

c. A(n) __________ is the set of all points (x, y) such that the absolute value of the difference in distances between (x, y) and two distinct points (called __________) is a constant.

2. **a.** A hyperbola has two vertices that lie on an axis of symmetry called the __________ axis.

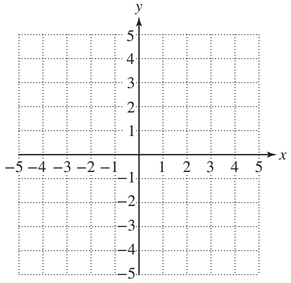
b. The standard form of an equation of a hyperbola with a *horizontal* transverse axis and center at (0, 0), is given by __________.

c. The standard form of an equation of a hyperbola with a *vertical* transverse axis and center at (0, 0), is given by __________.

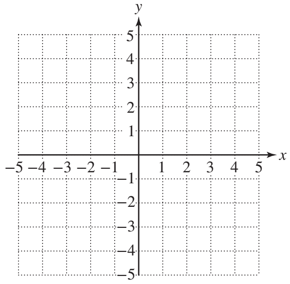
Concept 1: Standard Form of an Equation of an Ellipse

For Exercises 3–10, find the x- and y-intercepts and graph the ellipse. **(See Examples 1–2.)**

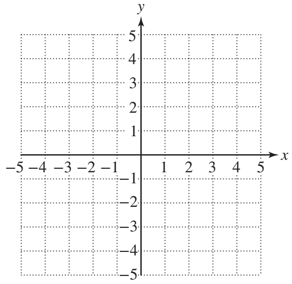
3. $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$



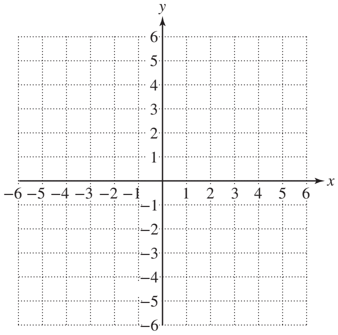
4. $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$



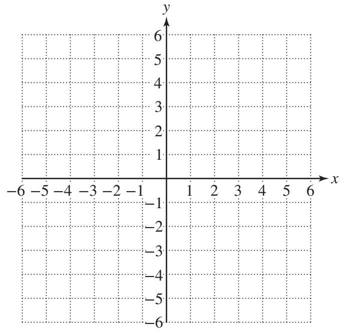
5. $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$



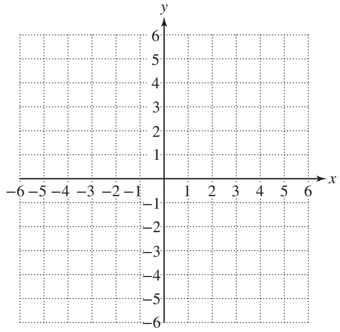
6. $\dfrac{x^2}{36}+\dfrac{y^2}{4}=1$



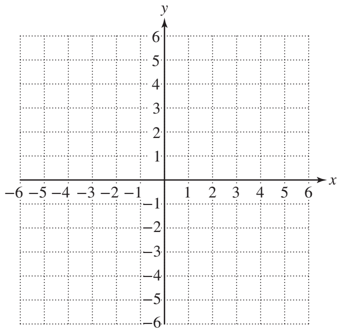
7. $4x^2+y^2=4$



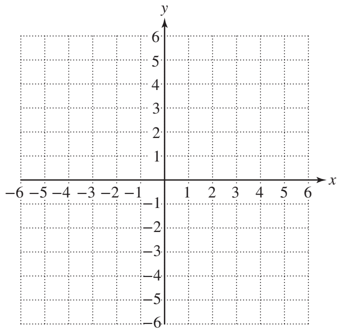
8. $9x^2+y^2=36$



9. $x^2+25y^2-25=0$

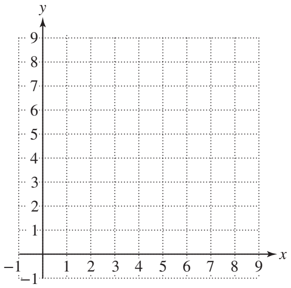


10. $4x^2+9y^2=144$

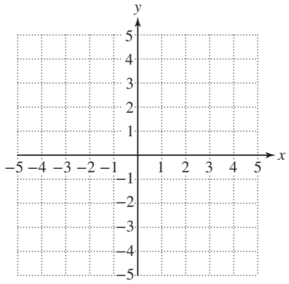


For Exercises 11–18, identify the center (h, k) of the ellipse. Then graph the ellipse. **(See Example 3.)**

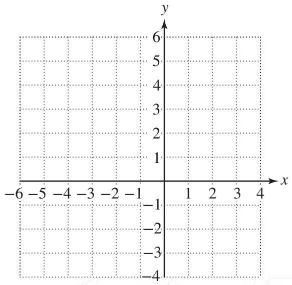
11. $\dfrac{(x-4)^2}{4}+\dfrac{(y-5)^2}{9}=1$



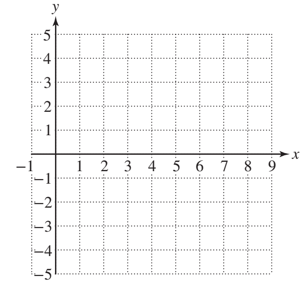
12. $\dfrac{(x+2)^2}{9}+\dfrac{(y-1)^2}{16}=1$



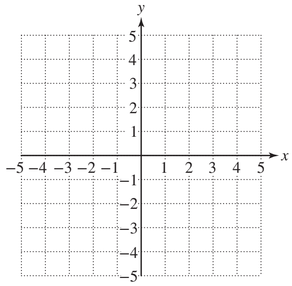
13. $\dfrac{(x+1)^2}{25}+\dfrac{(y-2)^2}{9}=1$



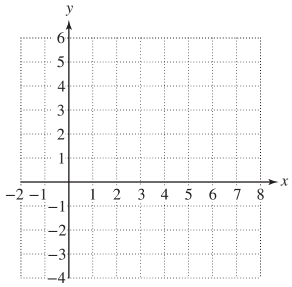
14. $\dfrac{(x-4)^2}{25} + \dfrac{(y+1)^2}{16} = 1$



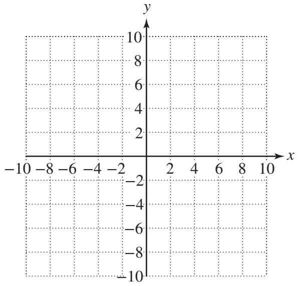
15. $\dfrac{(x-2)^2}{9} + (y+3)^2 = 1$



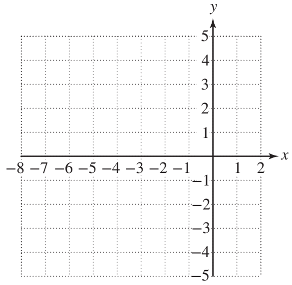
16. $(x-5)^2 + \dfrac{(y-3)^2}{4} = 1$



17. $\dfrac{x^2}{36} + \dfrac{(y-1)^2}{25} = 1$



18. $\dfrac{(x+5)^2}{4} + \dfrac{y^2}{16} = 1$



Concept 2: Standard Form of an Equation of a Hyperbola

For Exercises 19–26, determine whether the transverse axis is horizontal or vertical.

19. $\dfrac{y^2}{6} - \dfrac{x^2}{18} = 1$

20. $\dfrac{y^2}{10} - \dfrac{x^2}{4} = 1$

21. $\dfrac{x^2}{20} - \dfrac{y^2}{15} = 1$

22. $\dfrac{x^2}{12} - \dfrac{y^2}{9} = 1$

23. $x^2 - y^2 = 12$

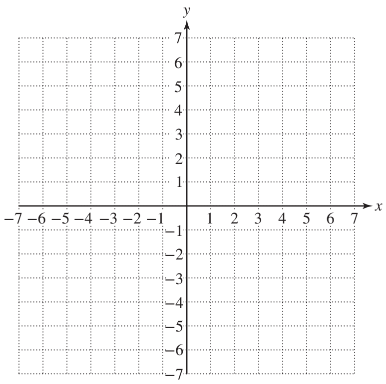
24. $x^2 - y^2 = 15$

25. $x^2 - 3y^2 = -9$

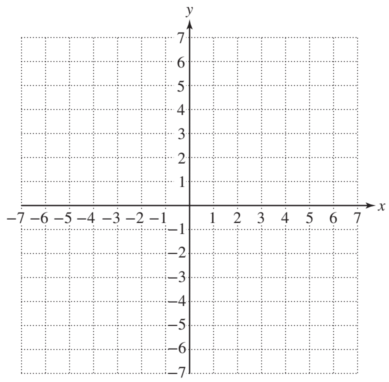
26. $2x^2 - y^2 = -10$

For Exercises 27–34, use the equation in standard form to graph the hyperbola. Label the vertices of the hyperbola. **(See Examples 4–5.)**

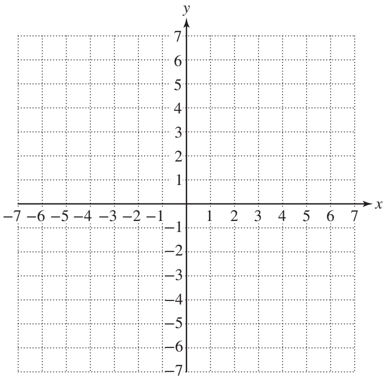
27. $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$



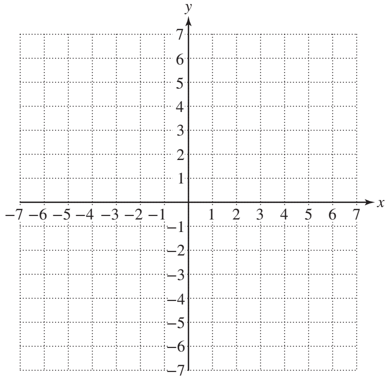
28. $\dfrac{x^2}{9} - \dfrac{y^2}{36} = 1$



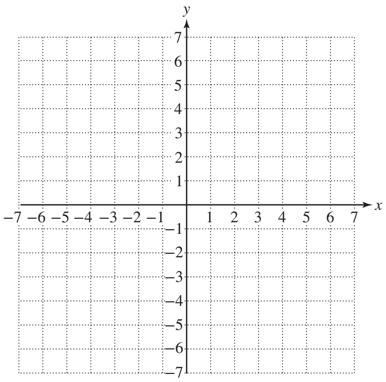
29. $\dfrac{y^2}{4} - \dfrac{x^2}{4} = 1$



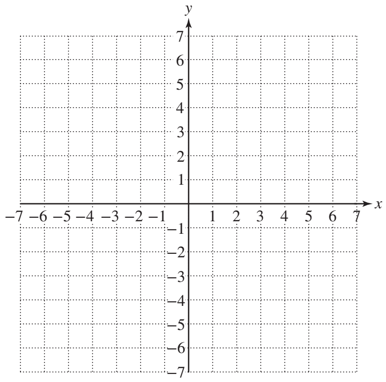
30. $\dfrac{y^2}{9} - \dfrac{x^2}{9} = 1$



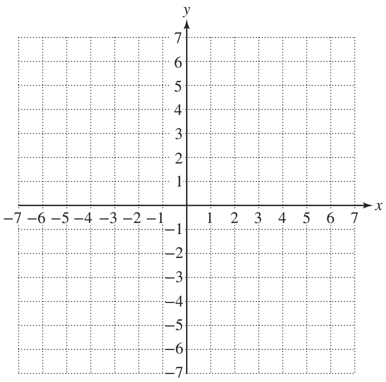
31. $36x^2 - y^2 = 36$



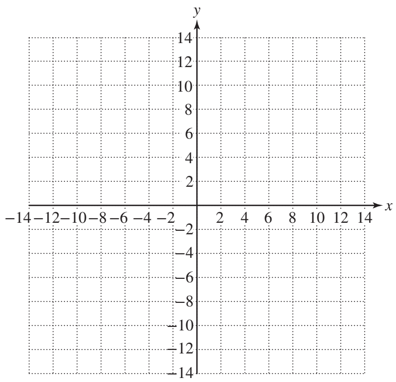
32. $x^2 - 25y^2 = 25$



33. $y^2 - 4x^2 - 16 = 0$



34. $y^2 - 4x^2 - 36 = 0$



Mixed Exercises

For Exercises 35–46, determine if the equation represents an ellipse or a hyperbola.

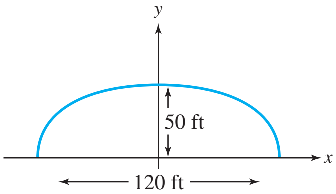
35. $\dfrac{x^2}{6} - \dfrac{y^2}{10} = 1$ **36.** $\dfrac{x^2}{14} + \dfrac{y^2}{2} = 1$ **37.** $\dfrac{y^2}{4} + \dfrac{x^2}{16} = 1$ **38.** $\dfrac{x^2}{5} + \dfrac{y^2}{10} = 1$

39. $4x^2 + y^2 = 16$ **40.** $-3x^2 - 4y^2 = -36$ **41.** $-y^2 + 2x^2 = -10$ **42.** $x^2 - y^2 = -1$

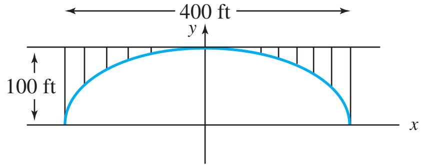
43. $5x^2 + y^2 - 10 = 0$ **44.** $5x^2 - 3y^2 = 15$ **45.** $y^2 - 6x^2 = 6$ **46.** $3x^2 + 5y^2 = 15$

Expanding Your Skills

47. An arch for a tunnel is in the shape of a semiellipse. The distance between vertices is 120 ft, and the height to the top of the arch is 50 ft. Find the height of the arch 10 ft from the center. Round to the nearest foot.

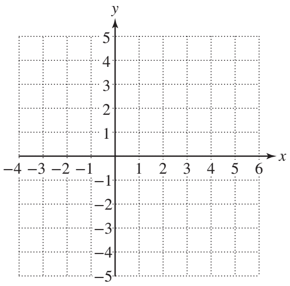


48. A bridge over a gorge is supported by an arch in the shape of a semiellipse. The length of the bridge is 400 ft, and the height is 100 ft. Find the height of the arch 50 ft from the center. Round to the nearest foot.

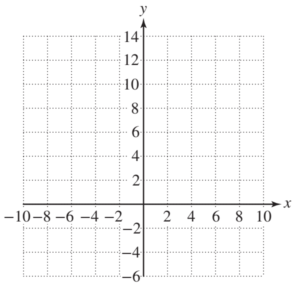


In Exercises 49–52, graph the hyperbola centered at (h, k) by following these guidelines. First locate the center. Then draw the reference rectangle relative to the center (h, k). Using the reference rectangle as a guide, sketch the hyperbola. Label the center and vertices.

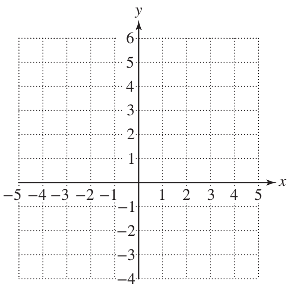
49. $\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{4} = 1$



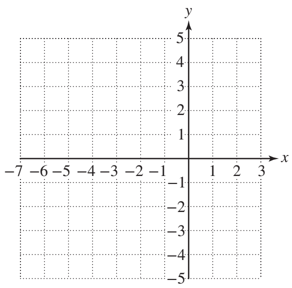
50. $\dfrac{(x+2)^2}{16} - \dfrac{(y-3)^2}{4} = 1$



51. $\dfrac{(y-1)^2}{4} - (x+3)^2 = 1$



52. $(y+2)^2 - \dfrac{(x+3)^2}{4} = 1$



Problem Recognition Exercises

Formulas and Conic Sections

For Exercises 1–8, identify the formula.

1. $(x-h)^2 + (y-k)^2 = r^2$

2. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

3. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

4. $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

5. $y = a(x-h)^2 + k$

6. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

7. $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

8. $x = a(y-k)^2 + h$

For Exercises 9–30, identify the equation as representing a circle, parabola, ellipse, hyperbola, or none of these.

9. $y = -2(x - 3)^2 + 4$

10. $\dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$

11. $(x + 3)^2 + (y + 2)^2 = 4$

12. $(x - 2)^2 + (y - 4)^2 = 9$

13. $\dfrac{x^2}{9} - \dfrac{y^2}{9} = 1$

14. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

15. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 0$

16. $x^2 + y^2 - 2x + 4y - 4 = 0$

17. $y = \dfrac{1}{2}(x + 2)^2 - 3$

18. $\dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$

19. $x = (y + 2)^2 - 4$

20. $x^2 + y^2 + 6x + 8 = 0$

21. $(x - 1)^2 + (y + 1)^2 = 0$

22. $x = -(y - 2)^2 - 1$

23. $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$

24. $x^2 + y^2 = 15$

25. $y = (x - 6)^2 + 4$

26. $\dfrac{(x + 1)^2}{2} + \dfrac{(y + 1)^2}{5} = 1$

27. $\dfrac{y^2}{3} - \dfrac{x^2}{3} = 1$

28. $3x^2 + 3y^2 = 1$

29. $\dfrac{x^2}{9} + \dfrac{y^2}{12} = 1$

30. $x = (y + 2)^2 - 5$

Section 9.4 Nonlinear Systems of Equations in Two Variables

Concepts

1. Solving Nonlinear Systems of Equations by the Substitution Method
2. Solving Nonlinear Systems of Equations by the Addition Method

1. Solving Nonlinear Systems of Equations by the Substitution Method

Recall that a linear equation in two variables x and y is an equation that can be written in the form $Ax + By = C$, where A and B are not both zero. We have solved systems of linear equations in two variables by using the graphing method, the substitution method, and the addition method. In this section, we will solve *nonlinear* systems of equations by using the same methods. A **nonlinear system of equations** is a system in which at least one of the equations is nonlinear.

Graphing the equations in a nonlinear system helps to determine the number of solutions and to approximate the coordinates of the solutions. The substitution method is used most often to solve a nonlinear system of equations analytically.

Example 1 Solving a Nonlinear System of Equations

Given the system $\quad x - 7y = -25$

$\qquad\qquad\qquad\quad x^2 + y^2 = 25$

a. Solve the system by graphing.

b. Solve the system by the substitution method.

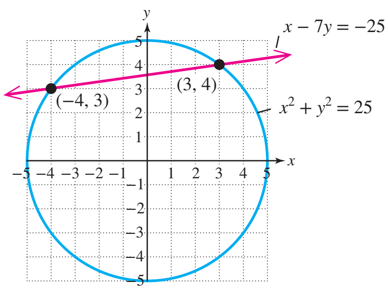


Figure 9-16

Solution:

a. $x - 7y = -25$ is a line $\left(\text{the slope-intercept form is } y = \frac{1}{7}x + \frac{25}{7}\right)$.

$x^2 + y^2 = 25$ is a circle centered at the origin with radius 5.

From Figure 9-16, we appear to have two solutions $(-4, 3)$ and $(3, 4)$.

b. To use the substitution method, isolate one of the variables from one of the equations. We will solve for x in the first equation.

$$\boxed{A} \quad x - 7y = -25 \xrightarrow{\text{Solve for } x.} x = 7y - 25$$

$$\boxed{B} \quad x^2 + y^2 = 25$$

$\boxed{B}$ $(7y - 25)^2 + y^2 = 25$ Substitute $(7y - 25)$ for x in the second equation.

$49y^2 - 350y + 625 + y^2 = 25$ The resulting equation is quadratic in y.

$50y^2 - 350y + 600 = 0$ Set the equation equal to zero.

$50(y^2 - 7y + 12) = 0$ Factor.

$50(y - 3)(y - 4) = 0$

$y = 3$ or $y = 4$

For each value of y, find the corresponding x value from the equation $x = 7y - 25$.

$y = 3$: $x = 7(3) - 25 = -4$ The solution point is $(-4, 3)$.

$y = 4$: $x = 7(4) - 25 = 3$ The solution point is $(3, 4)$. (See Figure 9-16.)

The solution set is $\{(-4, 3), (3, 4)\}$.

Skill Practice Given the system

$$2x + y = 5$$
$$x^2 + y^2 = 50$$

1. Solve the system by graphing.
2. Solve the system by the substitution method.

FOR REVIEW

Remember to use parentheses when substituting an expression for a variable. For example, using parentheses for the expression $(7y - 5)^2$ helps us remember to square the binomial, rather than the individual terms.

Example 2 Solving a Nonlinear System by the Substitution Method

Given the system
$$y = \sqrt{x}$$
$$x^2 + y^2 = 20$$

a. Sketch the graphs.

b. Solve the system by the substitution method.

Solution:

a. $y = \sqrt{x}$ is the basic square root function.

$x^2 + y^2 = 20$ is a circle centered at the origin with radius $\sqrt{20} \approx 4.5$.

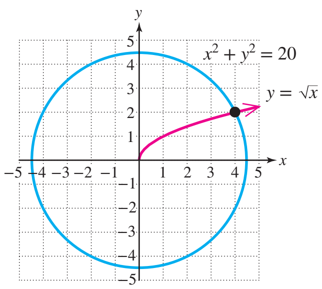
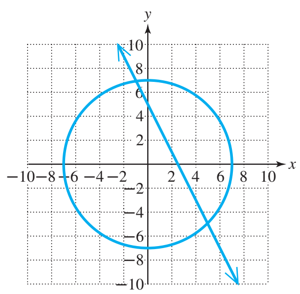


Figure 9-17

From Figure 9-17, we see that there is one solution.

Answers

1.



The solutions appear to be $(-1, 7)$ and $(5, -5)$.

2. $\{(-1, 7), (5, -5)\}$

b. To use the substitution method, we will substitute $y = \sqrt{x}$ into equation $\boxed{B}$.

$\boxed{A}$ $y = \sqrt{x}$ Substitute $y = \sqrt{x}$ into the second equation.

$\boxed{B}$ $x^2 + y^2 = 20$

$\boxed{B}$ $x^2 + (\sqrt{x})^2 = 20$

$x^2 + x = 20$

$x^2 + x - 20 = 0$ Set the second equation equal to zero.

$(x + 5)(x - 4) = 0$ Factor.

$\cancel{x = -5}$ or $x = 4$ Reject $x = -5$ because it is not in the domain of $y = \sqrt{x}$.

Substitute $x = 4$ into the equation $y = \sqrt{x}$.

If $x = 4$, then $y = \sqrt{4} = 2$. The solution set is $\{(4, 2)\}$.

Skill Practice Given the system

$$y = \sqrt{2x}$$
$$x^2 + y^2 = 8$$

3. Sketch the graphs.

4. Solve the system by using the substitution method.

Example 3 Solving a Nonlinear System by the Substitution Method

Solve the system by using the substitution method.

$$y = \sqrt[3]{x}$$
$$y = x$$

Solution:

$\boxed{A}$ $y = \sqrt[3]{x}$

$\boxed{B}$ $y = x$

$\sqrt[3]{x} = x$ Because y is isolated in both equations, we can equate the expressions for y.

$(\sqrt[3]{x})^3 = (x)^3$ To solve the radical equation, raise both sides to the third power.

$x = x^3$ This is a third-degree polynomial equation.

$0 = x^3 - x$ Set the equation equal to zero.

$0 = x(x^2 - 1)$ Factor out the GCF.

$0 = x(x + 1)(x - 1)$ Factor completely.

$x = 0$ or $x = -1$ or $x = 1$

For each value of x, find the corresponding y value from either original equation. We will use equation $\boxed{B}$: $y = x$.

If $x = 0$, then $y = 0$. The solution point is $(0, 0)$.

If $x = -1$, then $y = -1$. The solution point is $(-1, -1)$.

If $x = 1$, then $y = 1$. The solution point is $(1, 1)$.

The solution set is $\{(0, 0), (-1, -1), (1, 1)\}$. See Figure 9-18.

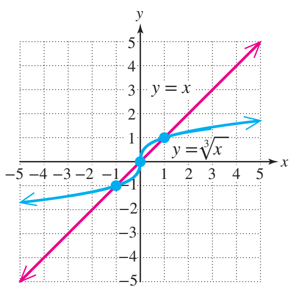
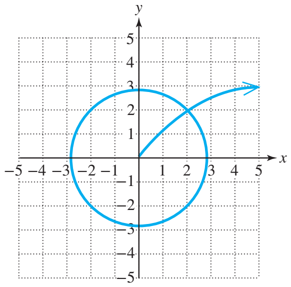


Figure 9-18

Answers

3.



4. $\{(2, 2)\}$

Skill Practice Solve the system by using the substitution method.

5. $y = \sqrt[3]{9x}$
$y = x$

2. Solving Nonlinear Systems of Equations by the Addition Method

The substitution method is used most often to solve a system of nonlinear equations. In some situations, however, the addition method offers an efficient means of finding a solution. Example 4 demonstrates that we can eliminate a variable from both equations provided the terms containing the corresponding variables are *like* terms.

Example 4 Solving a Nonlinear System of Equations by the Addition Method

Solve the system.

$$2x^2 + y^2 = 17$$
$$x^2 + 2y^2 = 22$$

Solution:

$\boxed{A}$ $2x^2 + y^2 = 17$ Notice that the y^2 terms are *like* in each equation.

$\boxed{B}$ $x^2 + 2y^2 = 22$ To eliminate the y^2 terms, multiply the first equation by -2.

$$\boxed{A}\quad 2x^2 + y^2 = 17 \xrightarrow{\text{Multiply by } -2.} -4x^2 - 2y^2 = -34$$
$$\boxed{B}\quad x^2 + 2y^2 = 22 \longrightarrow \underline{x^2 + 2y^2 = 22}$$
$$-3x^2 = -12 \quad \text{Eliminate the } y^2 \text{ term.}$$

$$\frac{-3x^2}{-3} = \frac{-12}{-3}$$
$$x^2 = 4$$
$$x = \pm 2 \quad \text{Use the square root property.}$$

Substitute each value of x into one of the original equations to solve for y. We will use equation $\boxed{A}$: $2x^2 + y^2 = 17$.

$x = 2$: $\boxed{A}$
$$2(2)^2 + y^2 = 17$$
$$8 + y^2 = 17$$
$$y^2 = 9$$
$$y = \pm 3 \quad \text{The solution points are } (2, 3) \text{ and } (2, -3).$$

$x = -2$: $\boxed{A}$
$$2(-2)^2 + y^2 = 17$$
$$8 + y^2 = 17$$
$$y^2 = 9$$
$$y = \pm 3 \quad \text{The solution points are } (-2, 3) \text{ and } (-2, -3).$$

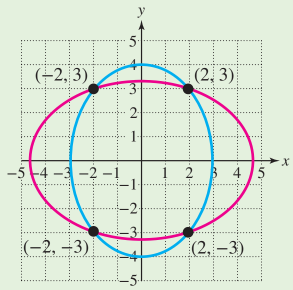
The solution set is $\{(2, 3), (2, -3), (-2, 3), (-2, -3)\}$.

Skill Practice Solve the system by using the addition method.

6. $x^2 - y^2 = 24$
$3x^2 + y^2 = 76$

TIP: In Example 4, the x^2 terms are also *like* in both equations. We could have eliminated the x^2 terms by multiplying equation $\boxed{B}$ by -2.

TIP: The two equations in Example 4 are ellipses, but an exact graph is difficult to render by hand.



To check the solutions to the system, verify that each ordered pair satisfies both equations.

Answers

5. $\{(0, 0), (3, 3), (-3, -3)\}$
6. $\{(5, 1), (5, -1), (-5, 1), (-5, -1)\}$

TIP: It is important to note that the addition method can be used only if two equations share a pair of *like* terms. The substitution method is effective in solving a wider range of systems of equations. The system in Example 4 could also have been solved by using substitution.

$\boxed{A}$ $2x^2 + y^2 = 17 \xrightarrow{\text{Solve for } y^2.} y^2 = 17 - 2x^2$

$\boxed{B}$ $x^2 + 2y^2 = 22$

$\boxed{B}$ $x^2 + 2(17 - 2x^2) = 22$

$x^2 + 34 - 4x^2 = 22$

$-3x^2 = -12$

$x^2 = 4$

$x = \pm 2$

$x = 2$: $y^2 = 17 - 2(2)^2$

$y^2 = 9$

$y = \pm 3$ The solutions are (2, 3) and (−2, −3).

$x = -2$: $y^2 = 17 - 2(-2)^2$

$y^2 = 9$

$y = \pm 3$ The solutions are (−2, 3) and (−2, −3).

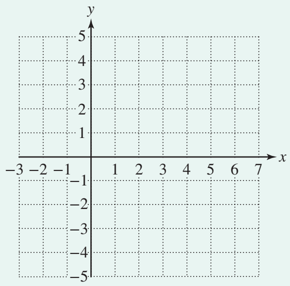
Section 9.4 Activity

A.1. **a.** Graph the two equations on the same coordinate system.

$$(x - 2)^2 + y^2 = 9$$
$$y = x + 1$$

b. From the graph, determine the points of intersection.

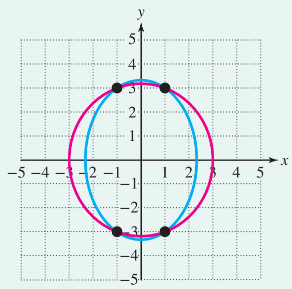
c. Solve the system of nonlinear equations from part (a) using the substitution method.



A.2. **a.** Use the addition method to solve the system of nonlinear equations.

$$x^2 + y^2 = 10$$
$$2x^2 + y^2 = 11$$

b. The graph of the first equation from part (a) is a circle, and the graph of the second equation is an ellipse. Verify your answers from part (a) from the graph.



Section 9.4 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, solve the system using the addition method.

R.1. $2x - 3y = 7$
$-5x + 2y = -23$

R.2. $4x + 2y = 8$
$3x - 7y = 23$

For Exercises R.3–R.4, solve the system using the substitution method.

R.3. $7x - 2y = 3$
$x + 5y = 11$

R.4. $8x - y = -4$
$3x + 5y = -23$

For Exercises R.5–R.10, identify the type of curve defined by the equation. Choose from: line, parabola, circle, ellipse, or hyperbola.

R.5. $x^2 + (y - 4)^2 = 9$

R.6. $4x + 2y = 8$

R.7. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

R.8. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

R.9. $x = y^2 + 4$

R.10. $y = -x^2 - 6$

Vocabulary and Key Concepts

1. a. A ____________ system of equations in two variables is a system in which at least one of the equations is nonlinear.

b. Graphically, the solution set to a nonlinear system of equations is the set of ordered pairs representing the points of ____________ of the graphs of the two equations.

Concept 1: Solving Nonlinear Systems of Equations by the Substitution Method

For Exercises 2–8, use sketches to explain.

2. How many points of intersection are possible between a line and a parabola?

3. How many points of intersection are possible between a line and a circle?

4. How many points of intersection are possible between two distinct circles?

5. How many points of intersection are possible between two distinct parabolas of the form $y = ax^2 + bx + c, a \neq 0$?

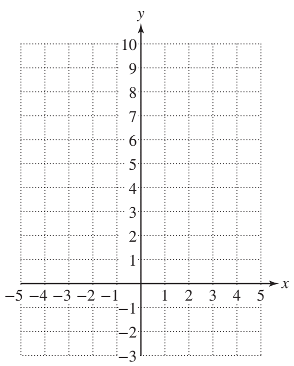
6. How many points of intersection are possible between a circle and a parabola?

7. How many points of intersection are possible between an ellipse and a hyperbola?

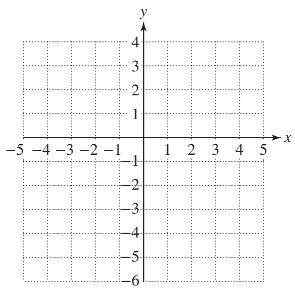
8. How many points of intersection are possible between an ellipse and a parabola?

For Exercises 9–14, sketch each system of equations. Then solve the system by the substitution method. **(See Example 1.)**

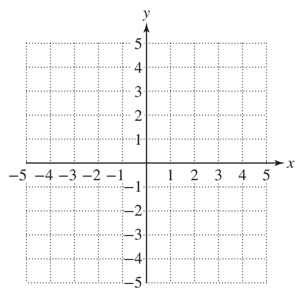
9. $y = x + 3$
$x^2 + y = 9$



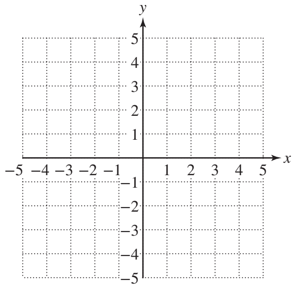
10. $y = x - 2$
$x^2 + y = 4$



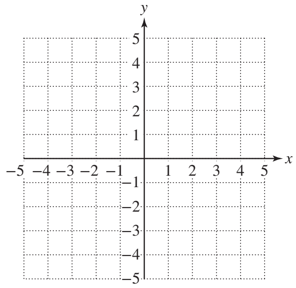
11. $x^2 + y^2 = 1$
$y = x + 1$



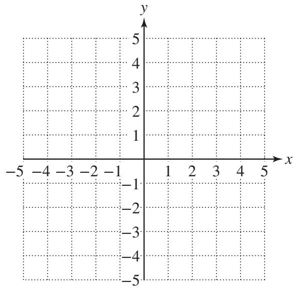
12. $x^2 + y^2 = 25$
$y = 2x$



13. $x^2 + y^2 = 6$
$y = x^2$



14. $x^2 + y^2 = 12$
$y = x^2$



For Exercises 15–23, solve the system by the substitution method. **(See Examples 2–3.)**

15. $y = \sqrt{x}$
$2x^2 - y^2 = 1$

16. $x^2 + y^2 = 30$
$y = \sqrt{x}$

17. $y = x^2$
$y = -\sqrt{x}$

18. $y = -x^2$
$y = -\sqrt{x}$

19. $y = x^2$
$y = (x - 3)^2$

20. $y = (x + 4)^2$
$y = x^2$

21. $y = x^2 + 6x$
$y = 4x$

22. $y = 3x^2 - 6x$
$y = 3x$

23. $x^2 - 5x + y = 0$
$y = 3x + 1$

Concept 2: Solving Nonlinear Systems of Equations by the Addition Method

For Exercises 24–40, solve the system of nonlinear equations by the addition method if appropriate. **(See Example 4.)**

24. $x^2 + y^2 = 13$
$x^2 - y^2 = 5$

25. $4x^2 - y^2 = 4$
$4x^2 + y^2 = 4$

26. $9x^2 + 4y^2 = 36$
$x^2 + y^2 = 9$

27. $x^2 + y^2 = 4$
$2x^2 + y^2 = 8$

28. $3x^2 + 4y^2 = 16$
$2x^2 - 3y^2 = 5$

29. $2x^2 - 5y^2 = -2$
$3x^2 + 2y^2 = 35$

30. $x^2 + y^2 = 169$
$12x - 5y = 0$

31. $x^2 + y^2 = 100$
$4y - 3x = 0$

32. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$
$x^2 + y^2 = 4$

33. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$
$x^2 + y^2 = 4$

34. $\dfrac{x^2}{10} + \dfrac{y^2}{10} = 1$
$2x^2 + y^2 = 11$

35. $2y = -x + 2$
$x^2 + 4y^2 = 4$

36. $2y = -3x - 6$
$9x^2 + 4y^2 = 36$

37. $x^2 - xy = -4$
$2x^2 - xy = 12$

38. $x^2 - xy = 3$
$2x^2 + xy = 6$

39. $3x^2 + 2xy = 4$
$x^2 - xy = 3$

40. $x^2 - 3xy = 8$
$x^2 + xy = 4$

Technology Connections

For Exercises 41–44, use the *Intersect* feature to approximate the solutions to the system.

41. $y = x + 3$ (Exercise 9)
$x^2 + y = 9$

42. $y = x - 2$ (Exercise 10)
$x^2 + y = 4$

43. $y = x^2$ (Exercise 17)
$y = -\sqrt{x}$

44. $y = -x^2$ (Exercise 18)
$y = -\sqrt{x}$

For Exercises 45–46, graph the system on a square viewing window. What can be said about the solution to the system?

45. $x^2 + y^2 = 4$
$y = x^2 + 3$

46. $x^2 + y^2 = 16$
$y = -x^2 - 5$

Expanding Your Skills

47. The sum of two numbers is 7. The sum of the squares of the numbers is 25. Find the numbers.

48. The sum of the squares of two numbers is 100. The sum of the numbers is 2. Find the numbers.

49. The sum of the squares of two numbers is 32. The difference of the squares of the numbers is 18. Find the numbers.

50. The sum of the squares of two numbers is 24. The difference of the squares of the numbers is 8. Find the numbers.

Nonlinear Inequalities and Systems of Inequalities in Two Variables

Section 9.5

1. Nonlinear Inequalities in Two Variables

Concepts

In our earlier study of linear inequalities in two variables, we used the test point method to graph the solution set. For example, to graph the solution set to $y \le 2x + 1$, first graph the related equation $y = 2x + 1$. This is the line shown in Figure 9-19. Then using test points, we see that points on and below the line make up the solution set to the inequality.

Test Point Above: (−2, 2)

$y \le 2x + 1$

$2 \overset{?}{\le} 2(-2) + 1$

$2 \overset{?}{\le} -3$ False

Test Point Below: (0, 0)

$y \le 2x + 1$

$0 \overset{?}{\le} 2(0) + 1$

$0 \overset{?}{\le} 1$ ✓True

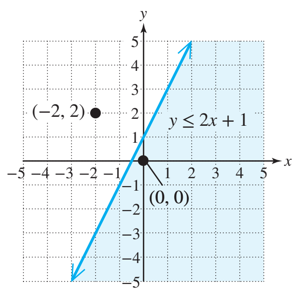


Figure 9-19

Example 1 Graphing a Nonlinear Inequality in Two Variables

Graph the solution set of the inequality $x^2 + y^2 < 16$.

Solution:

Avoiding Mistakes

The first step in solving an inequality in two variables is to graph the related equation.

The related equation $x^2 + y^2 = 16$ is a circle of radius 4, centered at the origin. Graph the related equation by using a dashed curve because the points satisfying the equation $x^2 + y^2 = 16$ are not part of the solution to the strict inequality $x^2 + y^2 < 16$. See Figure 9-20.

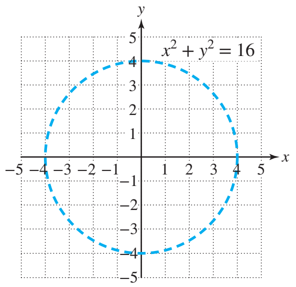


Figure 9-20

Notice that the dashed curve separates the xy-plane into two regions, one "inside" the circle, the other "outside" the circle. Select a test point from each region and test the point in the original inequality.

Test Point "Inside": (0, 0)	**Test Point "Outside": (4, 4)**
$x^2 + y^2 < 16$	$x^2 + y^2 < 16$
$(0)^2 + (0)^2 \overset{?}{<} 16$	$(4)^2 + (4)^2 \overset{?}{<} 16$
$0 \overset{?}{<} 16$ True	$32 \overset{?}{<} 16$ False

The inequality $x^2 + y^2 < 16$ is true at the test point (0, 0). Therefore, the solution set is the region "inside" the circle. See Figure 9-21.

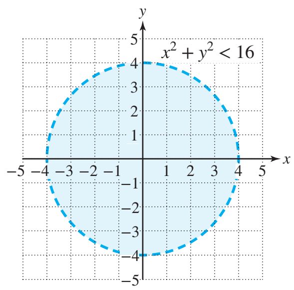


Figure 9-21

FOR REVIEW

Recall that when graphing an inequality in two variables, use a dashed curve to indicate that the curve or "boundary" is not part of the solution set. Use a solid curve to indicate that the curve is part of the solution set.

Skill Practice

1. Graph the solution set of the inequality. $x^2 + y^2 \geq 9$

Example 2 Graphing a Nonlinear Inequality in Two Variables

Graph the solution set of the inequality $9y^2 \geq 36 + 4x^2$.

Solution:

First graph the related equation $9y^2 = 36 + 4x^2$. Notice that the equation can be written in the standard form of a hyperbola.

$$9y^2 = 36 + 4x^2$$

$$9y^2 - 4x^2 = 36 \quad \text{Subtract } 4x^2 \text{ from both sides.}$$

$$\frac{9y^2}{36} - \frac{4x^2}{36} = \frac{36}{36} \quad \text{Divide both sides by 36.}$$

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

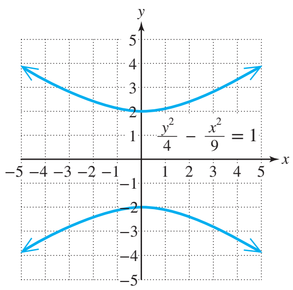
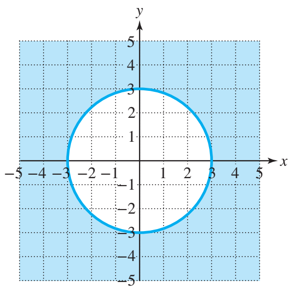


Figure 9-22

Graph the hyperbola as a solid curve, because the original inequality includes equality. See Figure 9-22.

Answer

1.



The hyperbola divides the xy-plane into three regions: a region above the upper branch, a region between the branches, and a region below the lower branch. Select a test point from each region.

$$9y^2 \geq 36 + 4x^2$$

Test: (0, 3)	**Test: (0, 0)**	**Test: (0, −3)**
$9(3)^2 \overset{?}{\geq} 36 + 4(0)^2$	$9(0)^2 \overset{?}{\geq} 36 + 4(0)^2$	$9(-3)^2 \overset{?}{\geq} 36 + 4(0)^2$
$81 \overset{?}{\geq} 36$ True	$0 \overset{?}{\geq} 36$ False	$81 \overset{?}{\geq} 36$ True

Shade the regions above the top branch and below the bottom branch of the hyperbola. See Figure 9-23.

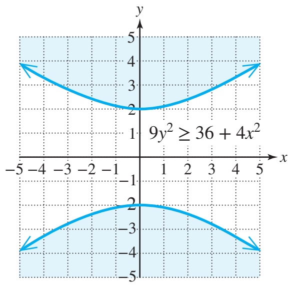


Figure 9-23

Skill Practice

2. Graph the solution set of the inequality. $9x^2 < 144 - 16y^2$

2. Systems of Nonlinear Inequalities in Two Variables

The solution set for a system of nonlinear equations in two variables is a set of ordered pairs that satisfy both equations simultaneously. We will now solve systems of nonlinear inequalities in two variables. Similarly, the solution set is the set of all ordered pairs that simultaneously satisfy each inequality. To solve a system of inequalities, we will graph the solution to each individual inequality and then take the intersection of the solution sets.

Example 3 Graphing a System of Nonlinear Inequalities in Two Variables

Graph the solution set.
$$y > \frac{1}{3}x^2$$
$$y < -x^2 + 2$$

Solution:

The solution to $y > \frac{1}{3}x^2$ is the set of points above the parabola $y = \frac{1}{3}x^2$. See Figure 9-24. The solution to $y < -x^2 + 2$ is the set of points below the parabola $y = -x^2 + 2$. See Figure 9-25.

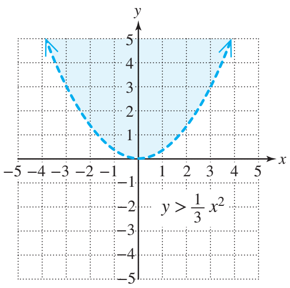


Figure 9-24

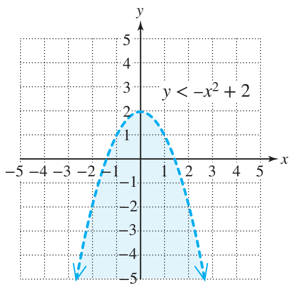
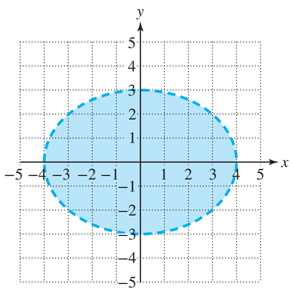


Figure 9-25

Answer

2.



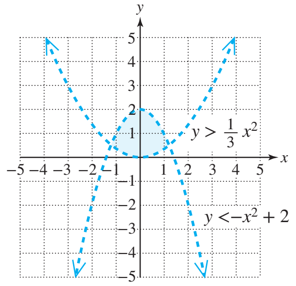


Figure 9-26

The solution to the system of inequalities is the intersection of the solution sets of the individual inequalities. See Figure 9-26.

Skill Practice

3. Graph the solution set. $\quad \dfrac{x^2}{4} + \dfrac{y^2}{9} < 1$

$x > y^2$

Example 4 Graphing a System of Nonlinear Inequalities in Two Variables

Graph the solution set. $\quad y > e^x$

$y < -x^2 + 4$

Solution:

The solution to $y > e^x$ is the set of points above the curve $y = e^x$. See Figure 9-27. The solution to $y < -x^2 + 4$ is the set of points below the parabola $y = -x^2 + 4$. See Figure 9-28.

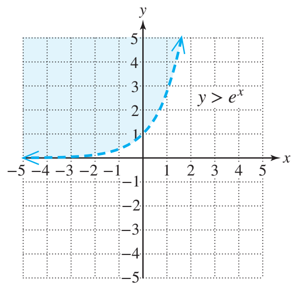


Figure 9-27

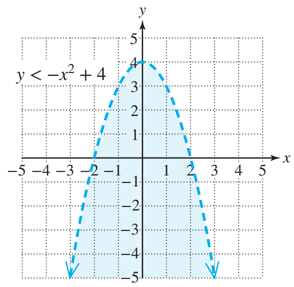


Figure 9-28

The solution to the system of inequalities is the intersection of the solution sets of the individual inequalities. See Figure 9-29.

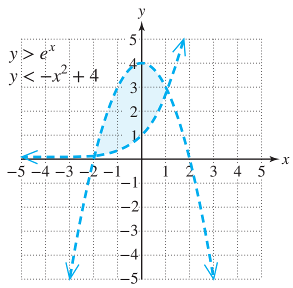


Figure 9-29

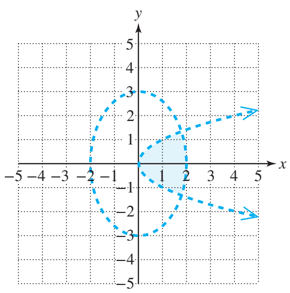
Skill Practice

4. Graph the solution set. $\quad y > \dfrac{1}{2}x^2$

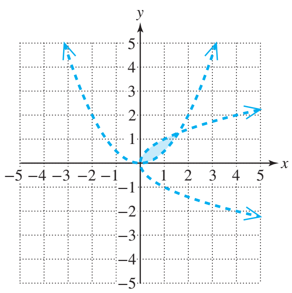
$x > y^2$

Answers

3.

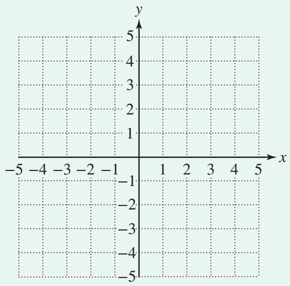


4.

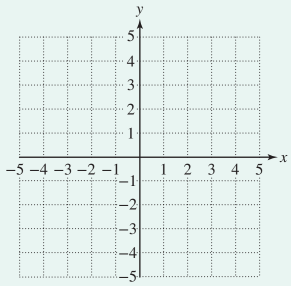


Section 9.5 Activity

A.1. **a.** Graph the equation $y = -x^2 + 4$.



b. On the same graph as part (a), graph the inequality $y \leq -x^2 + 4$.

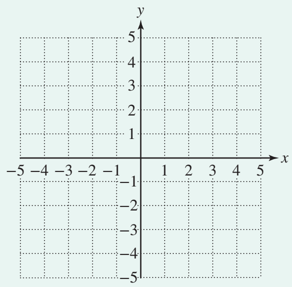


c. How would the graph from part (b) have changed if the inequality were $y < -x^2 + 4$?

d. How would the graph from part (b) have changed for the inequality $y \geq -x^2 + 4$?

A.2. Earlier in this text, we learned how to graph a system of linear inequalities. Use the same procedure to graph the system of *nonlinear* inequalities.

$$\begin{aligned} y &\leq -x^2 + 4 \\ y &\geq x + 2 \end{aligned}$$

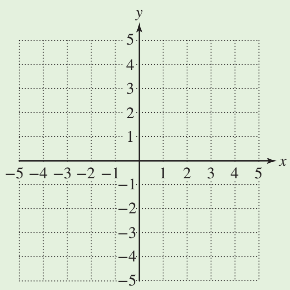


Practice Exercises — Section 9.5

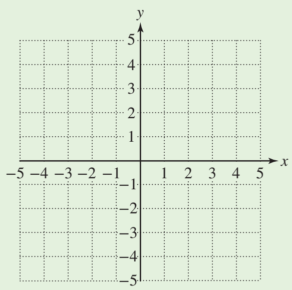
Prerequisite Review

For Exercises R.1–R.2, graph the solution set to the inequality.

R.1. $3x + y \geq 2$



R.2. $4x - 2y < 6$



R.3. Refer to Exercise R.1.

a. How would the graph of the solution set change for the inequality $3x + y > 2$?

b. How would the graph of the solution set change for the inequality $3x + y \le 2$?

R.4. Refer to Exercise R.2.

a. How would the graph of the solution set change for the inequality $4x - 2y \le 6$?

b. How would the graph of the solution set change for the inequality $4x - 2y > 6$?

For Exercises R.5–R.10, identify the type of curve defined by the equation. Choose from: line, parabola, circle, ellipse, or hyperbola.

R.5. $x = y^2 - 1$

R.6. $\dfrac{x^2}{4} + y^2 = 4$

R.7. $3x + 2y = 8$

R.8. $(x - 4)^2 + (y + 2)^2 = 1$

R.9. $-\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

R.10. $y = 3 - x^2$

Concept 1: Nonlinear Inequalities in Two Variables

1. True or false: The point (2, 3) satisfies the inequality $-x^2 + y^3 > 1$.

2. True or false: The point (4, −2) satisfies the inequality $4x^2 - 2x + 1 + y^2 < 3$.

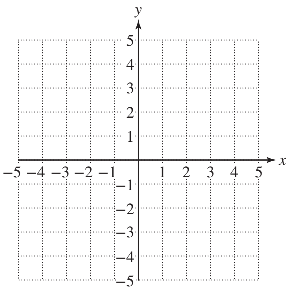
3. True or false: The point (5, 4) satisfies the system of inequalities.

$$\frac{x^2}{36} + \frac{y^2}{25} < 1$$
$$x^2 + y^2 \ge 4$$

4. True or false: The point (1, −2) satisfies the system of inequalities.

$$y < x^2$$
$$y > x^2 - 4$$

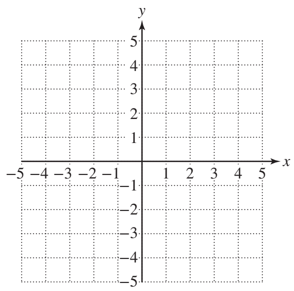
5. a. Graph the solution set for $x^2 + y^2 \le 9$.



b. Describe the solution set for the inequality $x^2 + y^2 \ge 9$.

c. Describe the solution set for the equation $x^2 + y^2 = 9$.

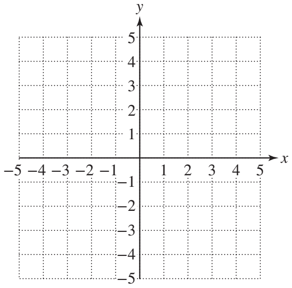
6. a. Graph the solution set for $\dfrac{x^2}{4} + \dfrac{y^2}{9} \ge 1$.



b. Describe the solution set for the inequality $\dfrac{x^2}{4} + \dfrac{y^2}{9} \le 1$.

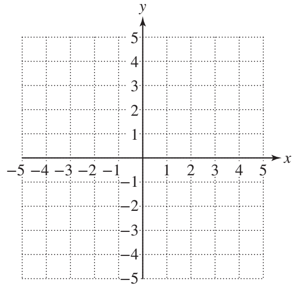
c. Describe the solution set for the equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

7. a. Graph the solution set for $y \geq x^2 + 1$.



b. How would the solution change for the strict inequality $y > x^2 + 1$?

8. a. Graph the solution set for $\dfrac{x^2}{4} - \dfrac{y^2}{9} \leq 1$.



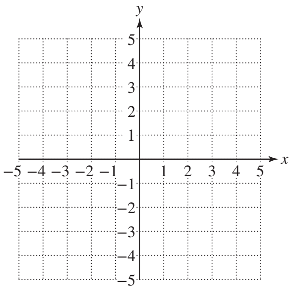
b. How would the solution change for the strict inequality $\dfrac{x^2}{4} - \dfrac{y^2}{9} < 1$?

9. A weak earthquake occurred in northern California roughly 4 mi south and 3 mi east of Sunol, California. The quake could be felt 25 mi away. Suppose the origin of a map is placed at the center of Sunol with the positive x-axis pointing east and the positive y-axis pointing north. Find an inequality that describes the points on the map for which the earthquake could be felt.

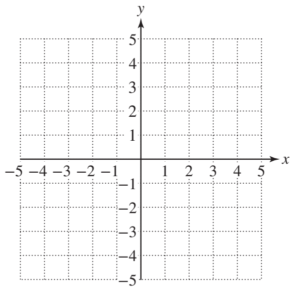
10. A coordinate system is placed at the center of a town with the positive x-axis pointing east and the positive y-axis pointing north. A cell tower is located 2 mi west and 4 mi north of the center of town. If the tower has a 30-mi range, write an inequality that represents the points on the map serviced by this tower.

For Exercises 11–25, graph the solution set. **(See Examples 1–2.)**

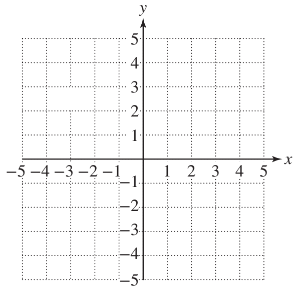
11. $2x + y \geq 1$



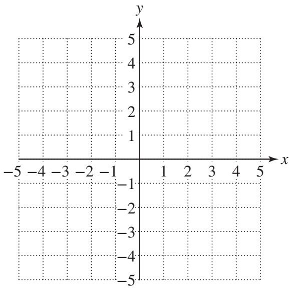
12. $3x + 2y \geq 6$



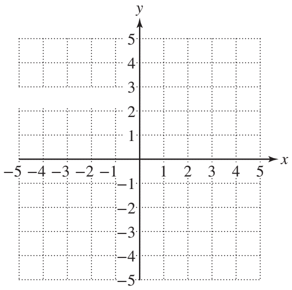
13. $x \leq y^2$



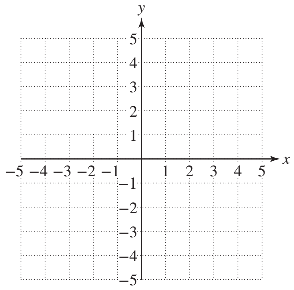
14. $y \leq -x^2$



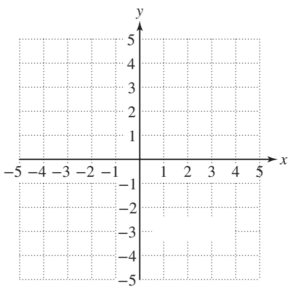
15. $(x - 1)^2 + (y + 2)^2 > 9$



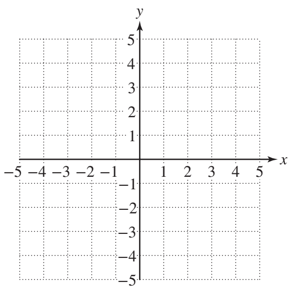
16. $(x + 1)^2 + (y - 4)^2 > 1$



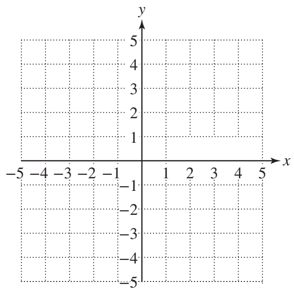
17. $x + y^2 \geq 4$



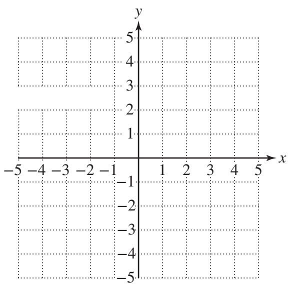
18. $x^2 + 2x + y - 1 \leq 0$



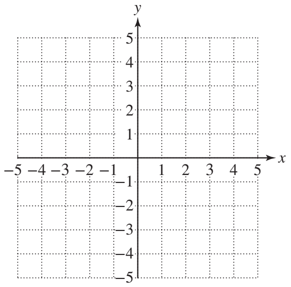
19. $9x^2 - y^2 > 9$



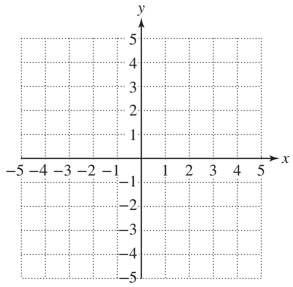
20. $y^2 - 4x^2 \leq 4$



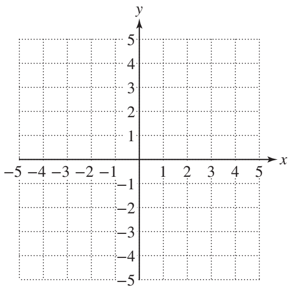
21. $x^2 + 16y^2 \leq 16$



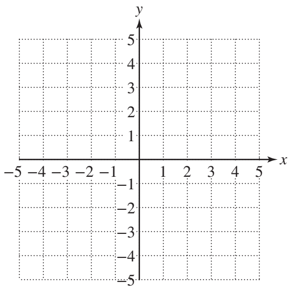
22. $4x^2 + y^2 \leq 4$



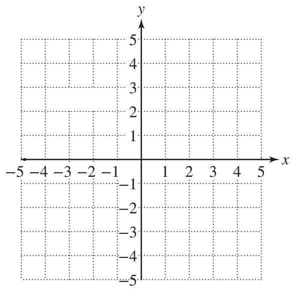
23. $y \leq \ln x$



24. $y \leq \log x$



25. $y > 5^x$

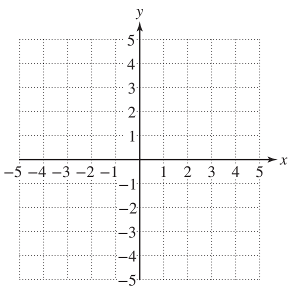


Concept 2: Systems of Nonlinear Inequalities in Two Variables

For Exercises 26–39, graph the solution set to the system of inequalities. **(See Examples 3–4.)**

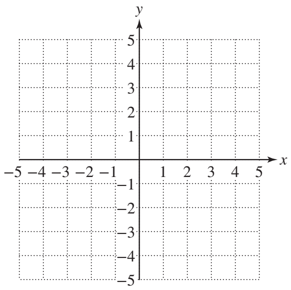
26. $y < \sqrt{x}$

$x > 1$



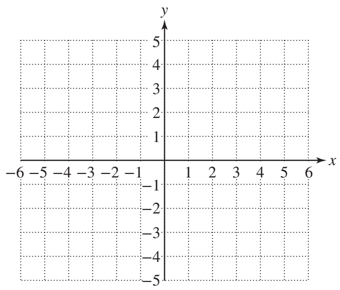
27. $y \geq \sqrt{x}$

$x \geq 0$



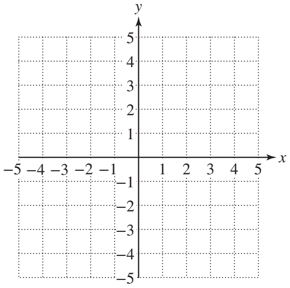
28. $\dfrac{x^2}{36} + \dfrac{y^2}{25} < 1$

$x^2 + y^2 \geq 4$



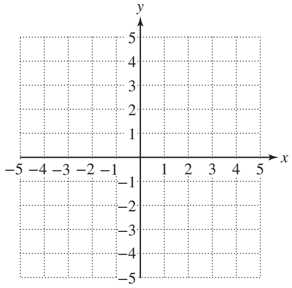
29. $x^2 - y^2 \geq 1$

$x \leq 0$



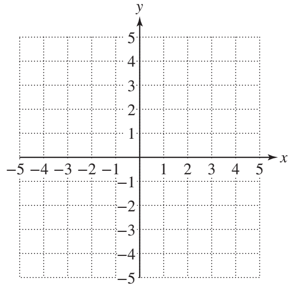
30. $y < x^2$

$y > x^2 - 4$



31. $y^2 - x^2 \geq 1$

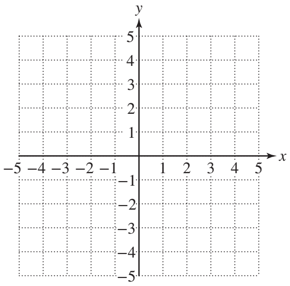
$y \geq 0$



32. $y < \dfrac{1}{x}$

$y > 0$

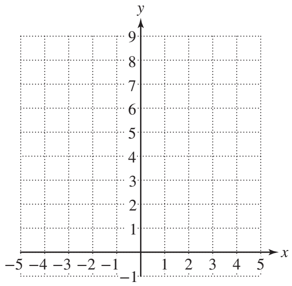
$y < x$



33. $y > x^3$

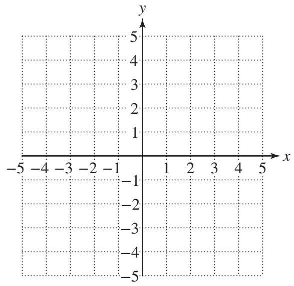
$y < 8$

$x > 0$



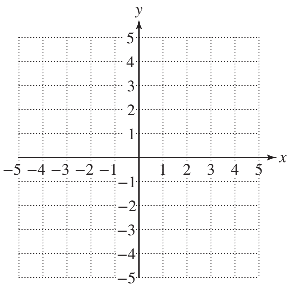
34. $x^2 + y^2 \geq 25$

$x^2 + y^2 \leq 9$



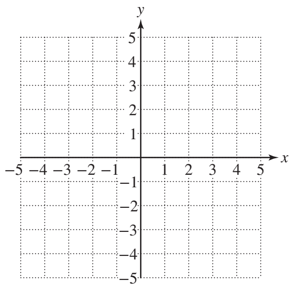
35. $\dfrac{x^2}{4} + \dfrac{y^2}{25} \geq 1$

$x^2 + \dfrac{y^2}{4} \leq 1$



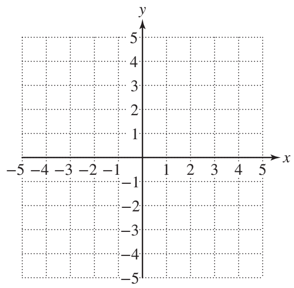
36. $x < -(y - 1)^2 + 3$

$x + y > 2$



37. $x > (y - 2)^2 + 1$

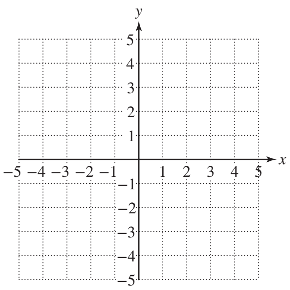
$x - y < 1$



38. $x^2 + y^2 < 25$

$y < \dfrac{4}{3}x$

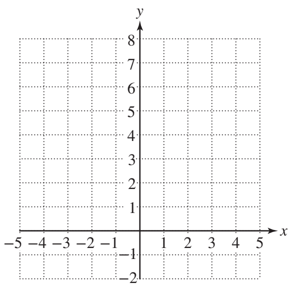
$y > -\dfrac{4}{3}x$



39. $y < e^x$

$y > 1$

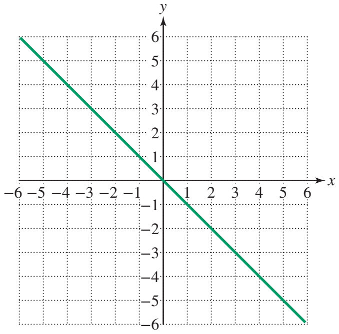
$x < 2$



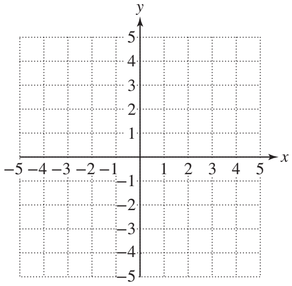
Expanding Your Skills

For Exercises 40–43, graph the compound inequalities.

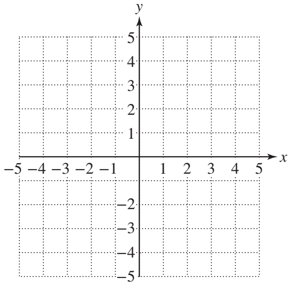
40. $x^2 + y^2 \le 36$ or $x + y \ge 0$



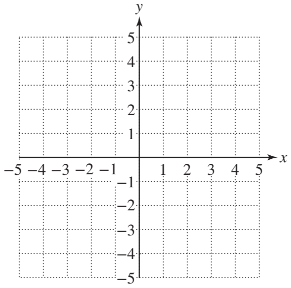
41. $y \le -x^2 + 4$ or $y \ge x^2 - 4$



42. $y + 1 \ge x^2$ or $y + 1 \le -x^2$



43. $(x + 2)^2 + (y + 3)^2 \le 4$ or $x \ge y^2$



Chapter 9 Summary

Section 9.1 Distance Formula, Midpoint Formula, and Circles

Key Concepts

The distance between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The standard equation of a **circle** with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

The midpoint between two points is found by using the formula:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Examples

Example 1

Find the distance between two points.
$(5, -2)$ and $(-1, -6)$

$$\begin{aligned} d &= \sqrt{(-1 - 5)^2 + [-6 - (-2)]^2} \\ &= \sqrt{(-6)^2 + (-4)^2} \\ &= \sqrt{36 + 16} \\ &= \sqrt{52} = 2\sqrt{13} \end{aligned}$$

Example 2

Find the center and radius of the circle.

$$x^2 + y^2 - 8x + 6y = 0$$

$$(x^2 - 8x + 16) + (y^2 + 6y + 9) = 16 + 9$$

$$(x - 4)^2 + (y + 3)^2 = 25$$

The center is $(4, -3)$ and the radius is 5.

Example 3

Find the midpoint between $(-3, 1)$ and $(5, 7)$.

$$\left(\frac{-3 + 5}{2}, \frac{1 + 7}{2}\right) = (1, 4)$$

Section 9.2 More on the Parabola

Key Concepts

A **parabola** is the set of points in a plane that are equidistant from a fixed line (called the directrix) and a fixed point (called the focus) not on the directrix.

The standard form of an equation of a parabola with **vertex** (h, k) and vertical **axis of symmetry** is

$$y = a(x - h)^2 + k \qquad \text{where } a \neq 0$$

- The equation of the axis of symmetry is $x = h$.
- If $a > 0$, the parabola opens upward.
- If $a < 0$, the parabola opens downward.

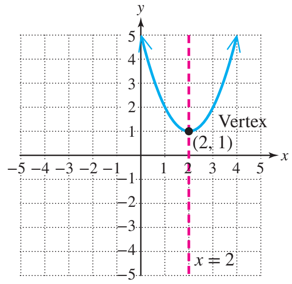
Examples

Example 1

Given the parabola $y = (x - 2)^2 + 1$,

The vertex is (2, 1).

The axis of symmetry is $x = 2$.



The standard form of an equation of a parabola with vertex (h, k) and horizontal axis of symmetry is

$$x = a(y - k)^2 + h \qquad \text{where } a \neq 0$$

- The equation of the axis of symmetry is $y = k$.
- If $a > 0$, the parabola opens to the right.
- If $a < 0$, the parabola opens to the left.

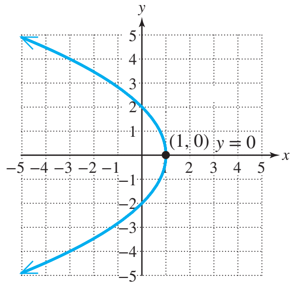
Example 2

Given the parabola $x = -\frac{1}{4}y^2 + 1$,
determine the coordinates of the vertex and the equation of the axis of symmetry.

$x = -\frac{1}{4}(y - 0)^2 + 1$

The vertex is (1, 0).

The axis of symmetry is $y = 0$.



Section 9.3 The Ellipse and Hyperbola

Key Concepts

An **ellipse** is the set of all points (x, y) such that the sum of the distances between (x, y) and two distinct points (called foci) is constant.

Standard Form of an Ellipse With Center at the Origin

An ellipse with the center at the origin has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive real numbers.

For an ellipse centered at the origin, the x-intercepts are given by $(a, 0)$ and $(-a, 0)$, and the y-intercepts are given by $(0, b)$ and $(0, -b)$.

A **hyperbola** is the set of all points (x, y) such that the absolute value of the difference of the distances between (x, y) and two distinct points is a constant. The fixed points are called the foci of the hyperbola.

Standard Forms of an Equation of a Hyperbola

Let a and b represent positive real numbers.

Horizontal Transverse Axis. The standard form of an equation of a hyperbola with a horizontal transverse axis and center at the origin is given by

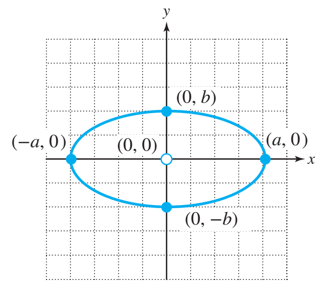
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Vertical Transverse Axis. The standard form of an equation of a hyperbola with a vertical transverse axis and center at the origin is given by

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

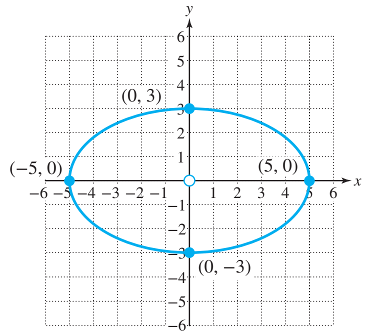
Examples

Example 1



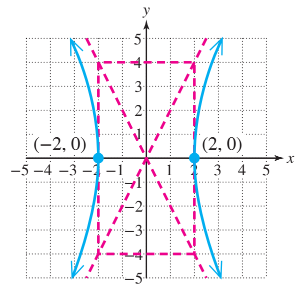
Example 2

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$



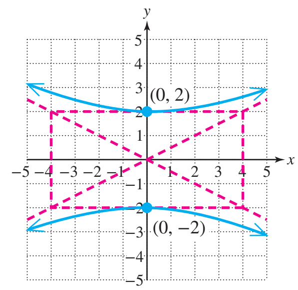
Example 3

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$



Example 4

$$\frac{y^2}{4} - \frac{x^2}{16} = 1$$

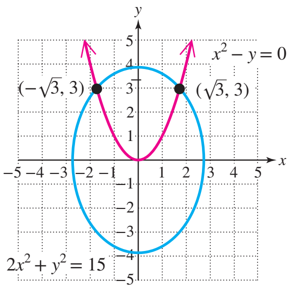


Section 9.4 Nonlinear Systems of Equations in Two Variables

Key Concepts

A **nonlinear system of equations** can be solved by graphing or by the substitution method.

$$2x^2 + y^2 = 15$$
$$x^2 - y = 0$$



Examples

Example 1

$\boxed{A}$ $2x^2 + y^2 = 15$

$\boxed{B}$ $x^2 - y = 0$ Solve for y. $y = x^2$

$\boxed{A}$ $2x^2 + (x^2)^2 = 15$ Substitute in first equation.

$$2x^2 + x^4 = 15$$
$$x^4 + 2x^2 - 15 = 0$$
$$(x^2 + 5)(x^2 - 3) = 0$$

$x^2 + 5 = 0$ or $x^2 - 3 = 0$

$\cancel{x^2 = -5}$ or $x^2 = 3$

$x = \pm\sqrt{3}$

If $x = \sqrt{3}$ then $y = (\sqrt{3})^2 = 3$.

If $x = -\sqrt{3}$ then $y = (-\sqrt{3})^2 = 3$.

Points of intersection are $(\sqrt{3}, 3)$ and $(-\sqrt{3}, 3)$.

The solution set is $\{(\sqrt{3}, 3), (-\sqrt{3}, 3)\}$.

Section 9.5 Nonlinear Inequalities and Systems of Inequalities in Two Variables

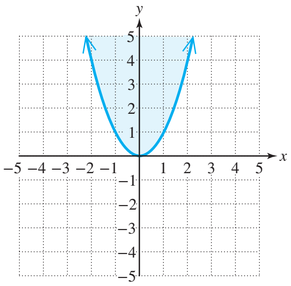
Key Concepts

Graph a nonlinear inequality by using the test point method. That is, graph the related equation. Then choose test points in each region to determine where the inequality is true.

Examples

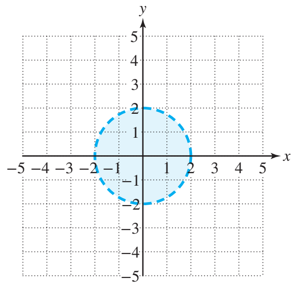
Example 1

$y \geq x^2$



Example 2

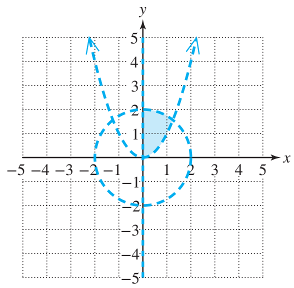
$x^2 + y^2 < 4$



Graph a system of nonlinear inequalities by finding the intersection of the solution sets. That is, graph the solution set for each individual inequality, then take the intersection.

Example 3

$x \geq 0, \quad y > x^2, \quad \text{and} \quad x^2 + y^2 < 4$



Chapter 9 Review Exercises

Section 9.1

For Exercises 1–2, find the distance between the two points by using the distance formula.

1. $(-6, 3)$ and $(0, 1)$

2. $(4, 13)$ and $(-1, 5)$

3. Find x such that $(x, 5)$ is 5 units from $(2, 9)$.

4. Find x such that $(-3, 4)$ is 3 units from $(x, 1)$.

For Exercises 5–8, find the center and the radius of the circle.

5. $(x - 12)^2 + (y - 3)^2 = 16$

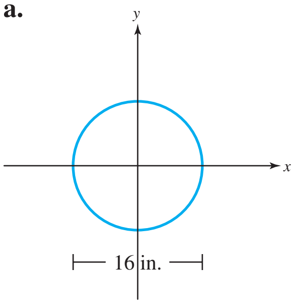
6. $(x + 7)^2 + (y - 5)^2 = 81$

7. $(x + 3)^2 + (y + 8)^2 = 20$

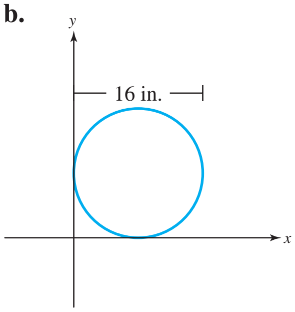
8. $(x - 1)^2 + (y + 6)^2 = 32$

9. A stained glass window is in the shape of a circle with a 16-in. diameter. Find an equation of the circle relative to the origin for each of the following graphs.

a.



b.



For Exercises 10–13, write the equation of the circle in standard form by completing the square.

10. $x^2 + y^2 + 12x - 10y + 51 = 0$

11. $x^2 + y^2 + 4x + 16y + 60 = 0$

12. $x^2 + y^2 - x - 4y + \frac{1}{4} = 0$

13. $x^2 + y^2 - 6x - \frac{2}{3}y + \frac{1}{9} = 0$

14. Write an equation of a circle with center at the origin and a diameter of 7 m.

15. Write an equation of a circle with center $(0, 2)$ and radius 3 m.

For Exercises 16–17, find the midpoint of the segment with the given endpoints.

16. $(-3, 1)$ and $(-5, -5)$

17. $(0, 9)$ and $(-2, 7)$

Section 9.2

For Exercises 18–21, determine whether the axis of symmetry is vertical or horizontal and if the parabola opens upward, downward, left, or right.

18. $y = -2(x - 3)^2 + 2$

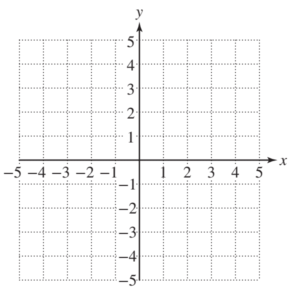
19. $x = 3(y - 9)^2 + 1$

20. $x = -(y + 4)^2 - 8$

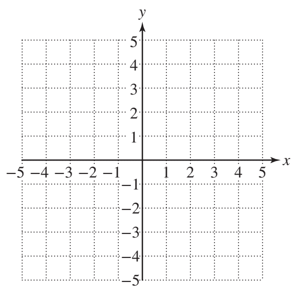
21. $y = (x + 3)^2 - 10$

For Exercises 22–25, determine the coordinates of the vertex and the equation of the axis of symmetry. Then use this information to graph the parabola.

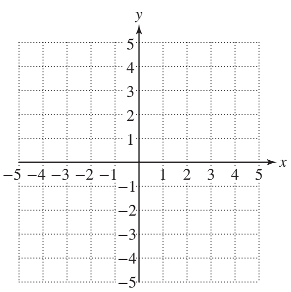
22. $x = -(y - 1)^2$



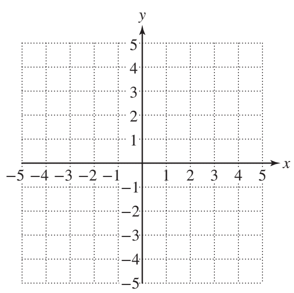
23. $y = (x + 2)^2$



24. $y = -\frac{1}{4}x^2$



25. $x = 2y^2 - 1$



For Exercises 26–29, write the equation in standard form $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$. Then identify the vertex and axis of symmetry.

26. $y = x^2 - 6x + 5$

27. $x = y^2 + 4y + 2$

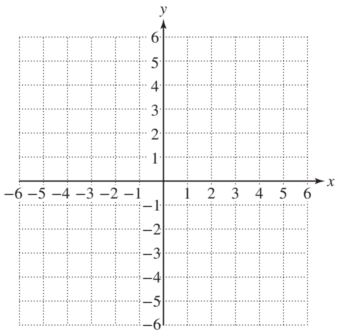
28. $x = -4y^2 + 4y$

29. $y = -2x^2 - 2x$

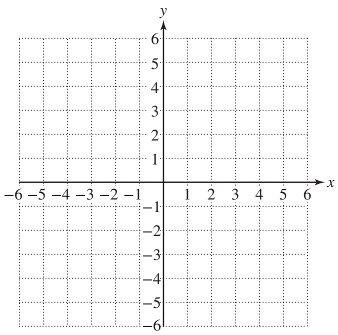
Section 9.3

For Exercises 30–31, identify the x- and y-intercepts. Then graph the ellipse.

30. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

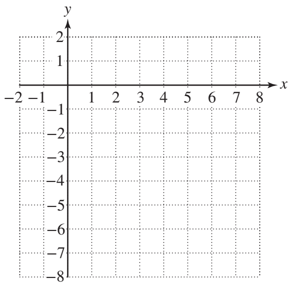


31. $x^2 + 4y^2 = 36$

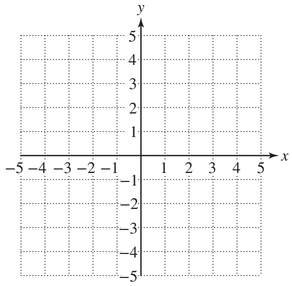


For Exercises 32–33, identify the center of the ellipse and graph the ellipse.

32. $\dfrac{(x - 5)^2}{4} + \dfrac{(y + 3)^2}{16} = 1$



33. $\dfrac{x^2}{25} + \dfrac{(y - 2)^2}{9} = 1$



For Exercises 34–37, determine whether the transverse axis of the hyperbola is horizontal or vertical.

34. $\dfrac{x^2}{12} - \dfrac{y^2}{16} = 1$

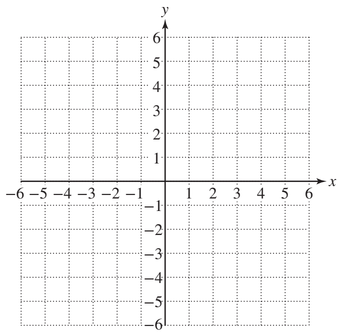
35. $\dfrac{y^2}{9} - \dfrac{x^2}{9} = 1$

36. $y^2 - 8x^2 = 16$

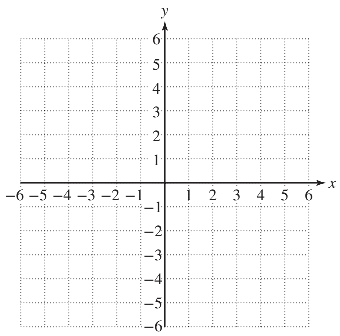
37. $3x^2 - y^2 = 18$

For Exercises 38–39, graph the hyperbola by first drawing the reference rectangle and the asymptotes. Label the vertices.

38. $\dfrac{x^2}{4} - y^2 = 1$



39. $y^2 - x^2 = 16$



For Exercises 40–43, identify the equations as representing an ellipse or a hyperbola.

40. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

41. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

42. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

43. $\dfrac{y^2}{1} - \dfrac{x^2}{16} = 1$

Section 9.4

For Exercises 44–47,

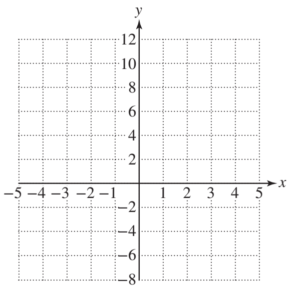
a. Identify each equation as representing a line, a parabola, a circle, an ellipse, or a hyperbola.

b. Graph both equations on the same coordinate system.

c. Solve the system analytically and verify the answers from the graph.

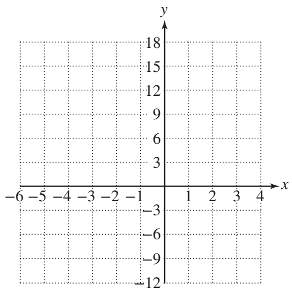
44. $3x + 2y = 10$

$y = x^2 - 5$



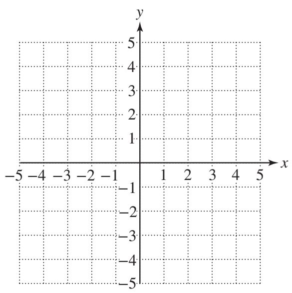
45. $4x + 2y = 10$

$y = x^2 - 10$



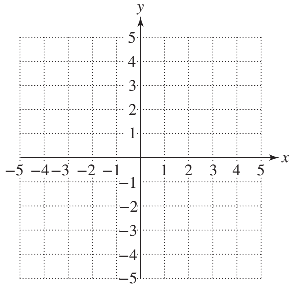
46. $x^2 + y^2 = 9$

$2x + y = 3$



47. $x^2 + y^2 = 16$

$x - 2y = 8$



For Exercises 48–53, solve the system of nonlinear equations by using either the substitution method or the addition method.

48. $x^2 + 2y^2 = 8$

$2x - y = 2$

49. $x^2 + 4y^2 = 29$

$x - y = -4$

50. $x - y = 4$

$y^2 = 2x$

51. $y = x^2$

$6x^2 - y^2 = 8$

52. $x^2 + y^2 = 10$

$x^2 + 9y^2 = 18$

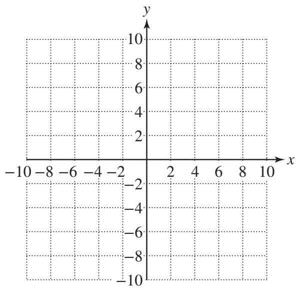
53. $x^2 + y^2 = 61$

$x^2 - y^2 = 11$

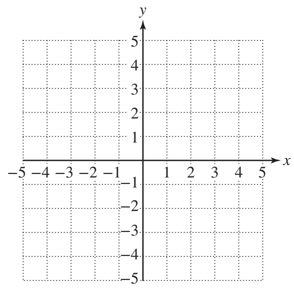
Section 9.5

For Exercises 54–59, graph the solution set to the inequality.

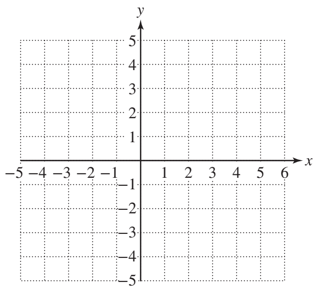
54. $\dfrac{x^2}{16} + \dfrac{y^2}{81} < 1$



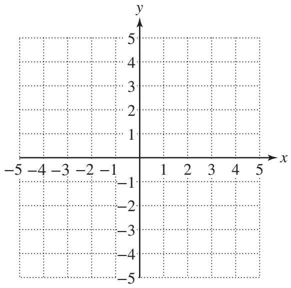
55. $\dfrac{x^2}{25} + \dfrac{y^2}{4} > 1$



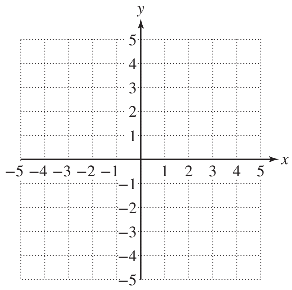
56. $(x - 3)^2 + (y + 1)^2 \geq 9$



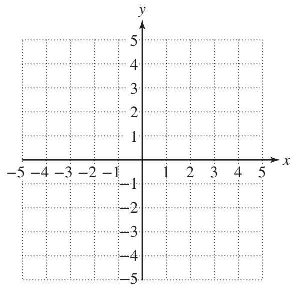
57. $(x + 2)^2 + (y + 1)^2 \leq 4$



58. $y > (x - 1)^2$

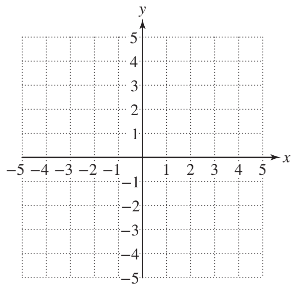


59. $x^2 - \dfrac{y^2}{4} \leq 1$

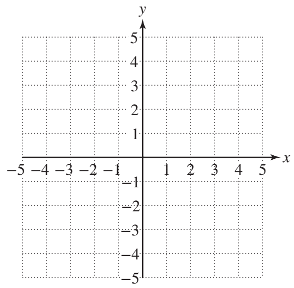


For Exercises 60–61, graph the solution set to the system of nonlinear inequalities.

60. $y > 2^x$
$x^2 + y^2 < 4$



61. $y < x^2$
$x^2 + y^2 < 9$



Chapter 9 Test

1. Use the distance formula to find the distance between the two points $(-5, 2)$ and $(-1, 6)$.

2. Determine the center and radius of the circle.

$$\left(x - \frac{5}{6}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{25}{49}$$

3. Determine the center and radius of the circle.

$$x^2 + y^2 - 4y - 5 = 0$$

4. Let $(0, 4)$ be the center of a circle that passes through the point $(-2, 5)$.

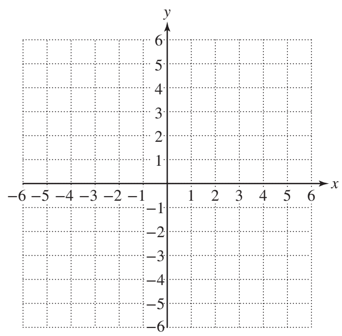
a. What is the radius of the circle?

b. Write the equation of the circle in standard form.

5. Find the center of the circle that has a diameter with endpoints $(7.3, -1.2)$ and $(0.3, 5.1)$.

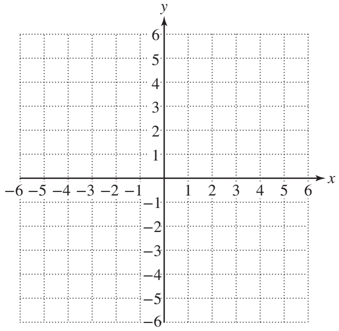
6. Determine the vertex and the equation of the axis of symmetry. Then graph the parabola.

$$x = -(y - 2)^2 + 3$$



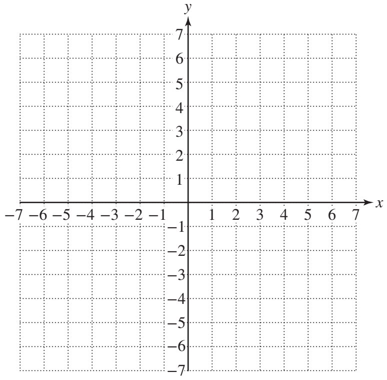
7. Write the equation in standard form $y = a(x - h)^2 + k$, and graph the parabola.

$y = x^2 + 4x + 5$



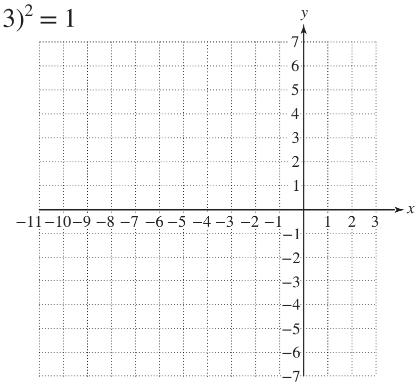
8. Graph the ellipse.

$\dfrac{x^2}{16} + \dfrac{y^2}{49} = 1$



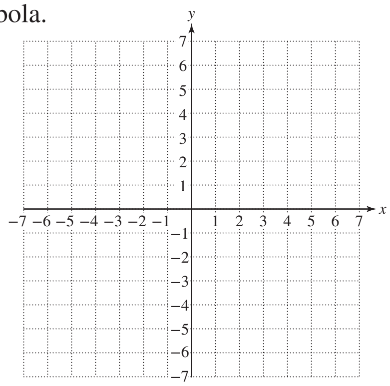
9. Graph the ellipse.

$\dfrac{(x + 4)^2}{25} + (y - 3)^2 = 1$



10. Graph the hyperbola.

$y^2 - \dfrac{x^2}{4} = 1$



For Exercises 11–12, solve the system and identify the correct graph.

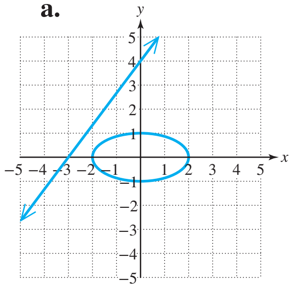
11. $16x^2 + 9y^2 = 144$

$4x - 3y = -12$

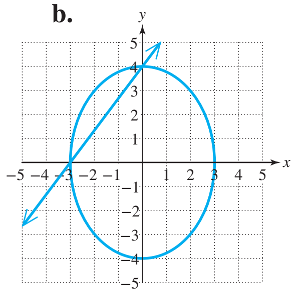
12. $x^2 + 4y^2 = 4$

$4x - 3y = -12$

a.



b.



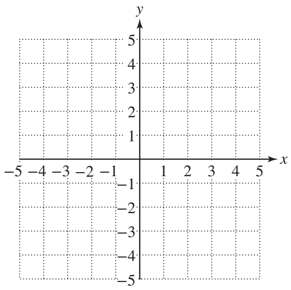
13. Describe the circumstances in which a nonlinear system of equations can be solved by using the addition method.

14. Solve the system by using either the substitution method or the addition method.

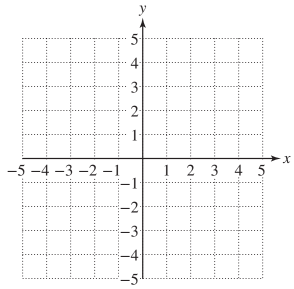
$$25x^2 + 4y^2 = 100$$
$$25x^2 - 4y^2 = 100$$

For Exercises 15–18, graph the solution set.

15. $x \le y^2 + 1$

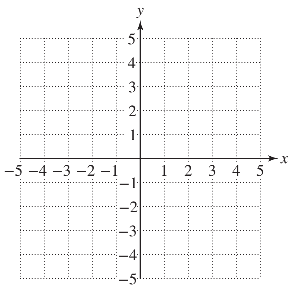


16. $y \ge -\dfrac{1}{3}x + 1$



17. $x < y^2 + 1$

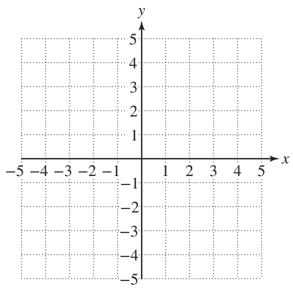
$y > -\dfrac{1}{3}x + 1$



18. $y < \sqrt{x}$

$y > x - 2$

$x > 0$



Binomial Expansions, Sequences, and Series

10

CHAPTER OUTLINE

Mathematics in Business

In this chapter we present applications involving sequences and series. A **sequence** is a function whose domain is the set of positive integers. While infinitely many sequences can be defined, two specific categories are presented here: **arithmetic** and **geometric sequences**. An arithmetic sequence is a progression of numbers that increase or decrease by a fixed amount, whereas the numbers in a geometric sequence increase or decrease by a fixed ratio. A **series** is a sum of the terms in a sequence.

To illustrate the difference between an arithmetic and a geometric sequence, suppose that you have two different job offers. The first pays \$60,000 per year with a \$2000 raise at the start of each new year. The second pays \$58,000 with a 4% raise at the start of each new year. The salaries for each job for each year can be written as the terms of a sequence. For example, the terms of the sequence of salaries for Job 1 are \$60,000, 62,000, 64,000, and so on.

Year	Job 1	Job 2		Total Job 1	Total Job 2
1	\$60,000	\$58,000		\$60,000	\$58,000
2	\$62,000	\$60,320		\$122,000	\$118,320
3	\$64,000	\$62,733		\$186,000	\$181,053
4	\$66,000	\$65,242		\$252,000	\$246,295
5	\$68,000	\$67,852		\$320,000	\$314,147
6	\$70,000	\$70,566		\$390,000	\$384,713
7	\$72,000	\$73,389		\$462,000	\$458,101
8	\$74,000	\$76,324		\$536,000	\$534,425
9	\$76,000	\$79,377		\$612,000	\$613,802
10	\$78,000	\$82,552		\$690,000	\$696,354

Notice that the yearly salary for Job 1 is initially more, but the salary increases by a fixed amount each year. The initial salary for Job 2 is less, but it grows exponentially and eventually overtakes the salary for Job 1. The sum of the salaries for the first 6 years can be added directly from the given data, or can be computed using formulas presented in this chapter. The total income over 6 years for Job 1 is \$390,000, and the total income for Job 2 over 6 years is \$384,713. However, because the yearly salary for Job 2 grows at a faster rate, by year 9 the cumulative income for Job 2 will be greater. The total income for Job 2 for the first 9 years will be \$613,802, whereas for Job 1, the total income will be \$612,000.

Our study of sequences and series in this chapter will help you learn to analyze numerical patterns and to make informed financial decisions.

Section 10.1 Binomial Expansions

Concepts

1. **Binomial Expansions and Pascal's Triangle**
2. **Factorial Notation**
3. **The Binomial Theorem**

1. Binomial Expansions and Pascal's Triangle

In this section, we will learn how to raise binomials to positive integer powers. First recall the formula to square a binomial.

$$(a+b)^2 = a^2 + 2ab + b^2$$

The expression $a^2 + 2ab + b^2$ is called the **binomial expansion** of $(a+b)^2$. To expand $(a+b)^3$, we can find the product $(a+b)(a+b)^2$.

$$\begin{aligned}(a+b)(a+b)^2 &= (a+b)(a^2+2ab+b^2)\\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Similarly, to expand $(a+b)^4$, we can multiply $(a+b)$ by $(a+b)^3$. Using this method, we can expand several powers of $(a+b)$ to find the following pattern:

FOR REVIEW

Recall that for any nonzero real number x, we have $x^0 = 1$.

$$\begin{aligned}(a+b)^0 &= 1\\ (a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\ (a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\end{aligned}$$

Notice that the exponents on a decrease from left to right, while the exponents on b increase from left to right. Also observe that for each term, the sum of the exponents is equal to the exponent to which $(a+b)$ is raised. Finally, notice that the number of terms in the expansion is exactly 1 more than the power to which $(a+b)$ is raised. For example, the expansion of $(a+b)^4$ has five terms, and the expansion of $(a+b)^5$ has six terms.

With these guidelines in mind, we know that $(a+b)^6$ will contain seven terms involving

$$a^6 \qquad a^5b \qquad a^4b^2 \qquad a^3b^3 \qquad a^2b^4 \qquad ab^5 \qquad b^6$$

We can complete the expansion of $(a+b)^6$ if we can determine the correct coefficients of each term.

If we write the coefficients for several expansions of $(a+b)^n$, where $n \ge 0$, we have a triangular array of numbers.

$(a+b)^0 = 1$	1
$(a+b)^1 = 1a + 1b$	1 1
$(a+b)^2 = 1a^2 + 2ab + 1b^2$	1 2 1
$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1
$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$	1 4 6 4 1
$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$	1 5 10 10 5 1

Each row begins and ends with a 1, and each entry in between is the sum of the two entries from the line above. For example, in row six, $5 = 1 + 4$, $10 = 4 + 6$, and so on. This triangular array of coefficients for a binomial expansion is called **Pascal's triangle**, named after French mathematician Blaise Pascal (1623–1662).

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}
$$

By using the pattern shown in Pascal's triangle, the coefficients corresponding to $(a + b)^6$ would be 1, 6, 15, 20, 15, 6, 1. By inserting the coefficients, the sum becomes

$$(a + b)^6 = 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$$

2. Factorial Notation

Although Pascal's triangle provides an easy method to find the coefficients of $(a + b)^n$, it is impractical for large values of n. A more efficient method to find the coefficients of a binomial expansion involves **factorial notation**.

Definition of $n!$

Let n be a positive integer. Then $n!$ (read as "n factorial") is defined as the product of integers from 1 through n. That is,

$$n! = n(n - 1)(n - 2) \cdots (2)(1)$$

Note: We define $0! = 1$.

Example 1 Evaluating Factorial Notation

Evaluate the expressions.

a. $4!$ **b.** $10!$ **c.** $0!$

Solution:

a. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

b. $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}628{,}800$

c. $0! = 1$ by definition

Skill Practice Evaluate the expressions.

1. $3!$ **2.** $8!$ **3.** $1!$

Answers

1. 6 **2.** 40,320 **3.** 1

Sometimes factorial notation is used with other operations such as multiplication and division.

FOR REVIEW

Recall that to simplify a fraction, "cancel" common factors that form a ratio of 1. When working with factorial notation, sometimes it is easier to see the cancellation when the factors are listed. For example,

$$\frac{4!}{3!\cdot 1!} = \frac{4\cdot \cancel{3\cdot 2\cdot 1}}{(\cancel{3\cdot 2\cdot 1})\cdot 1} = \frac{4}{1} = 4$$

Example 2 Evaluating Expressions with Factorials

Evaluate the expressions.

a. $\dfrac{4!}{4!\cdot 0!}$ **b.** $\dfrac{4!}{3!\cdot 1!}$ **c.** $\dfrac{4!}{2!\cdot 2!}$ **d.** $\dfrac{4!}{1!\cdot 3!}$ **e.** $\dfrac{4!}{0!\cdot 4!}$

Solution:

a. $\dfrac{4!}{4!\cdot 0!} = \dfrac{\cancel{4!}}{\cancel{4!}\cdot 1} = 1$

b. $\dfrac{4!}{3!\cdot 1!} = \dfrac{4\cdot \cancel{3!}}{\cancel{3!}\cdot 1} = 4$ Note that $4! = 4\cdot(3\cdot 2\cdot 1) = 4\cdot 3!$

c. $\dfrac{4!}{2!\cdot 2!} = \dfrac{4\cdot 3\cdot \cancel{2!}}{\cancel{2!}\cdot 2\cdot 1} = \dfrac{12}{2} = 6$

d. $\dfrac{4!}{1!\cdot 3!} = \dfrac{4\cdot \cancel{3!}}{1\cdot \cancel{3!}} = 4$

e. $\dfrac{4!}{0!\cdot 4!} = \dfrac{\cancel{4!}}{1\cdot \cancel{4!}} = 1$

Skill Practice Evaluate.

4. $\dfrac{5!}{5!\cdot 0!}$ **5.** $\dfrac{5!}{4!\cdot 1!}$ **6.** $\dfrac{5!}{3!\cdot 2!}$

3. The Binomial Theorem

Notice from Example 2 that the values of

$$\frac{4!}{4!\cdot 0!} \quad \frac{4!}{3!\cdot 1!} \quad \frac{4!}{2!\cdot 2!} \quad \frac{4!}{1!\cdot 3!} \quad \text{and} \quad \frac{4!}{0!\cdot 4!}$$

correspond to the values 1, 4, 6, 4, 1, which are the coefficients for the expansion of $(a + b)^4$. Generalizing this pattern, we see the coefficients for the terms in the expansion of $(a + b)^n$ are given by

$$\frac{n!}{r!\cdot(n-r)!}$$

where r corresponds to the exponent on the factor of a and $(n - r)$ corresponds to the exponent on the factor of b. Using this formula for the coefficients in a binomial expansion results in the **binomial theorem**.

Answers

4. 1 **5.** 5 **6.** 10

The Binomial Theorem

For any positive integer n,

$$(a+b)^n = \frac{n!}{n!\cdot 0!}a^n + \frac{n!}{(n-1)!\cdot 1!}a^{(n-1)}b + \frac{n!}{(n-2)!\cdot 2!}a^{(n-2)}b^2 + \cdots + \frac{n!}{0!\cdot n!}b^n$$

Example 3 Applying the Binomial Theorem

Write out the first three terms of the expansion of $(a+b)^{10}$.

Solution:

The first three terms of $(a+b)^{10}$ are

$$\frac{10!}{10!\cdot 0!}a^{10} + \frac{10!}{9!\cdot 1!}a^9b + \frac{10!}{8!\cdot 2!}a^8b^2$$

$$= \frac{\cancel{10!}}{\cancel{10!}\cdot 1}a^{10} + \frac{10\cdot \cancel{9!}}{\cancel{9!}\cdot 1!}a^9b + \frac{10\cdot 9\cdot \cancel{8!}}{\cancel{8!}\cdot 2\cdot 1}a^8b^2$$

$$= a^{10} + 10a^9b + 45a^8b^2$$

TIP: When applying the binomial theorem, exponents on a and b must add up to n.

Skill Practice

7. Write out the first three terms of $(x+y)^5$.

Example 4 Applying the Binomial Theorem

Use the binomial theorem to expand $(2m+7)^3$.

Solution:

This expression is of the form $(a+b)^3$, where $a = 2m$ and $b = 7$. Applying the binomial theorem, we have

$$(a+b)^3 = \frac{3!}{3!\cdot 0!}a^3 + \frac{3!}{2!\cdot 1!}a^2b + \frac{3!}{1!\cdot 2!}ab^2 + \frac{3!}{0!\cdot 3!}b^3$$

$$= \frac{3!}{3!\cdot 0!}(2m)^3 + \frac{3!}{2!\cdot 1!}(2m)^2(7) + \frac{3!}{1!\cdot 2!}(2m)(7)^2 + \frac{3!}{0!\cdot 3!}(7)^3$$

$$= 1\cdot(8m^3) + 3\cdot(4m^2)(7) + 3\cdot(2m)(49) + 1\cdot(343)$$

$$= 8m^3 + 84m^2 + 294m + 343$$

TIP: The values of $\frac{n!}{r!\cdot(n-r)!}$ can also be found by using Pascal's triangle.

1

1 1

1 2 1

1 3 3 1

Skill Practice

8. Use the binomial theorem to expand $(3y+4)^3$.

Answers

7. $x^5 + 5x^4y + 10x^3y^2$

8. $27y^3 + 108y^2 + 144y + 64$

Example 5 Applying the Binomial Theorem

Use the binomial theorem to expand $(3x^2 - 5y)^4$.

Solution:

Write $(3x^2 - 5y)^4$ as $[(3x^2) + (-5y)]^4$. In this case, the expressions $3x^2$ and $-5y$ may be substituted for a and b in the expansion of $(a + b)^4$.

$$(a+b)^4 = \frac{4!}{4! \cdot 0!}a^4 + \frac{4!}{3! \cdot 1!}a^3b + \frac{4!}{2! \cdot 2!}a^2b^2 + \frac{4!}{1! \cdot 3!}ab^3 + \frac{4!}{0! \cdot 4!}b^4$$

$$= \frac{4!}{4! \cdot 0!}(3x^2)^4 + \frac{4!}{3! \cdot 1!}(3x^2)^3(-5y) + \frac{4!}{2! \cdot 2!}(3x^2)^2(-5y)^2$$

$$+ \frac{4!}{1! \cdot 3!}(3x^2)(-5y)^3 + \frac{4!}{0! \cdot 4!}(-5y)^4$$

$$= 1 \cdot (81x^8) + 4 \cdot (27x^6)(-5y) + 6 \cdot (9x^4)(25y^2) + 4 \cdot (3x^2)(-125y^3)$$
$$+ 1 \cdot (625y^4)$$

$$= 81x^8 - 540x^6y + 1350x^4y^2 - 1500x^2y^3 + 625y^4$$

TIP: The values of $\frac{n!}{r! \cdot (n-r)!}$ can also be found by using Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Skill Practice

9. Use the binomial theorem to expand $(2a - 3b^2)^4$.

The binomial theorem may also be used to find a specific term in a binomial expansion.

Example 6 Finding a Specific Term in a Binomial Expansion

Find the fourth term of the expansion $(a + b)^{13}$.

Solution:

There are 14 terms in the expansion of $(a + b)^{13}$. The first term will have variable factors $a^{13}b^0$. The second term will have variable factors $a^{12}b^1$. The third term will have $a^{11}b^2$, and the fourth term will have $a^{10}b^3$. Hence, the fourth term is

$$\frac{13!}{10! \cdot 3!}a^{10}b^3 = 286a^{10}b^3$$

Skill Practice

10. Find the fourth term of $(x + y)^8$.

From Example 6, we see that for the kth term in the expansion $(a + b)^n$, where k is an integer greater than zero, the variable factors are $a^{n-(k-1)}$ and b^{k-1}. Therefore, to find the kth term of $(a + b)^n$, we can make the following generalization.

Finding a Specific Term in a Binomial Expansion

Let n and k be positive integers such that $k \leq n$. Then the kth term in the expansion of $(a + b)^n$ is given by

$$\frac{n!}{[n-(k-1)]! \cdot (k-1)!} \cdot a^{n-(k-1)} \cdot b^{k-1}$$

Answers

9. $16a^4 - 96a^3b^2 + 216a^2b^4 - 216ab^6 + 81b^8$
10. $56x^5y^3$

Example 7 Finding a Specific Term in a Binomial Expansion

Find the sixth term of $(p^3 + 2w)^8$.

Solution:

Apply the formula.

$$\frac{n!}{[n-(k-1)]!\cdot(k-1)!}\cdot a^{n-(k-1)}\cdot b^{k-1} \quad \text{with } n = 8, k = 6, a = p^3, \text{and } b = 2w$$

$$\frac{8!}{[8-(6-1)]!\cdot(6-1)!}\cdot(p^3)^{8-(6-1)}\cdot(2w)^{6-1}$$

$$= \frac{8!}{(3)!\cdot(5)!}\cdot(p^3)^3\cdot(2w)^5$$

$$= 56\cdot(p^9)\cdot(32w^5)$$

$$= 1792p^9w^5$$

Skill Practice

11. Find the fifth term of $(t - m^2)^8$.

Answer

11. $70t^4m^8$

Section 10.1 Activity

For Exercises A.1–A.4, expand the binomial.

A.1. $(a + b)^0$ **A.2.** $(a + b)^1$ **A.3.** $(a + b)^2$

A.4. $(a + b)^3$ *Hint:* Multiply the result of Exercise A.3 by $(a + b)$.

A.5. **a.** Fill in the fifth row of Pascal's triangle.

b. Write the expansion of $(a + b)^4$ by using the coefficients from Pascal's triangle and the pattern from Exercises A.1–A.4.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ \square & & \square & & \square & & \square & & \square \end{array} \qquad \begin{array}{c} 1 \\ a + b \\ a^2 + 2ab + b^2 \\ a^3 + 3a^2b + 3ab^2 + b^3 \\ \square + \square + \square + \square + \square \end{array}$$

A.6. Evaluate the factorial expressions and compare the results to the fifth row of Pascal's triangle from Exercise A.5(a).

a. $\dfrac{4!}{4!\cdot 0!}$ **b.** $\dfrac{4!}{3!\cdot 1!}$ **c.** $\dfrac{4!}{2!\cdot 2!}$ **d.** $\dfrac{4!}{1!\cdot 3!}$ **e.** $\dfrac{4!}{0!\cdot 4!}$

A.7. Simplify $(5x - y^2)^4$ by following these steps.

a. Write the expression in parentheses as a sum.

$$(5x - y^2)^4 = [5x + (\square)]^4$$

b. Use the expansion of $(a + b)^4$ from Exercise A.5(b) to expand $(5x - y^2)^4$.

Section 10.1 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, simplify the expression.

R.1. $\dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)}$

R.2. $\dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}$

For Exercises R.3–R.4, multiply the expression.

R.3. **a.** $(4x - 3)^2$ **b.** $(4x - 3)^3$

R.4. **a.** $(2x + 5)^2$ **b.** $(2x + 5)^3$

Vocabulary and Key Concepts

1. **a.** The expanded form of $(a + b)^2 =$ ____________. The expanded form of $(a + b)^3 =$ ______________. These are both called ____________ expansions.

b. Given a positive integer n, the value of $n!$ is ______________. Furthermore, $n!$ is read as "n ____________."

c. $3! =$ ____________, $2! =$ ____________, $1! =$ ____________, and $0! =$ ____________.

d. The ____________ theorem states that for any positive integer n,

$$(a + b)^n = \frac{n!}{n! \cdot 0!}a^n + \frac{n!}{(n-1)! \cdot 1!}a^{(n-1)}b + \frac{n!}{(n-2)! \cdot 2!}a^{(n-2)}b^2 + \cdots + \frac{n!}{0! \cdot n!}b^n.$$

e. The coefficients of the binomial expansion of $(a + b)^n$ can also be found by using ____________ triangle.

Concept 1: Binomial Expansions and Pascal's Triangle

For Exercises 2–7, expand the binomials. Use Pascal's triangle to find the coefficients.

2. $(x + y)^4$ **3.** $(a + b)^3$ **4.** $(4 + p)^3$

5. $(1 + g)^4$ **6.** $(a^2 + b)^6$ **7.** $(p + q^2)^7$

For Exercises 8–13, rewrite each binomial of the form $(a - b)^n$ as $[a + (-b)]^n$. Then expand the binomials. Use Pascal's triangle to find the coefficients.

8. $(t - 2)^5$ **9.** $(s - t)^5$

10. $(p^2 - w)^3$ **11.** $(5 - u^3)^4$

12. $(a - b^2)^4$ **13.** $(x^2 - 4)^3$

14. For $a > 0$ and $b > 0$, what happens to the signs of the terms when expanding the binomial $(a - b)^n$ compared with $(a + b)^n$?

Concept 2: Factorial Notation

For Exercises 15–18, evaluate the expression. **(See Example 1.)**

15. $5!$ **16.** $3!$ **17.** $0!$ **18.** $1!$

19. True or false: $0! \neq 1!$

20. True or false: $n!$ is defined for negative integers.

21. True or false: $n! = n$ for $n = 1$ and 2.

22. Show that $9! = 9 \cdot 8!$

23. Show that $6! = 6 \cdot 5!$

24. Show that $8! = 8 \cdot 7!$

For Exercises 25–32, evaluate the expression. **(See Example 2.)**

25. $\dfrac{8!}{4!}$ **26.** $\dfrac{7!}{5!}$ **27.** $\dfrac{3!}{0!}$ **28.** $\dfrac{4!}{0!}$

29. $\dfrac{8!}{3! \cdot 5!}$ **30.** $\dfrac{6!}{2! \cdot 4!}$ **31.** $\dfrac{4!}{0! \cdot 4!}$ **32.** $\dfrac{6!}{0! \cdot 6!}$

Concept 3: The Binomial Theorem

For Exercises 33–36, find the first three terms of the expansion. **(See Example 3.)**

33. $(m + n)^{11}$ **34.** $(p + q)^9$ **35.** $(u^2 - v)^{12}$ **36.** $(r - s^2)^8$

37. How many terms are in the expansion of $(a + b)^8$?

38. How many terms are in the expansion of $(x + y)^{13}$?

For Exercises 39–50, use the binomial theorem to expand the binomials. **(See Examples 4–5.)**

39. $(s + t)^6$ **40.** $(h + k)^4$ **41.** $(b - 3)^3$ **42.** $(c - 2)^5$

43. $(2x + y)^4$ **44.** $(p + 3q)^3$ **45.** $(c^2 - d)^7$ **46.** $(u - v^3)^6$

47. $\left(\dfrac{a}{2} - b\right)^5$ **48.** $\left(\dfrac{s}{3} + t\right)^5$ **49.** $(x + 4y)^4$ **50.** $(3y - w)^3$

For Exercises 51–56, find the indicated term of the binomial expansion. **(See Examples 6–7.)**

51. $(m - n)^{11}$; sixth term **52.** $(p - q)^9$; fourth term **53.** $(u^2 - v)^{12}$; fifth term

54. $(2r - s^2)^8$; sixth term **55.** $(5f + g)^9$; 10th term **56.** $(4m + n)^{10}$; 11th term

Sequences and Series

Section 10.2

Concepts

1. Finite and Infinite Sequences
2. Series

1. Finite and Infinite Sequences

In day-to-day life, we think of a sequence as a set of items with some order or pattern. In mathematics, a sequence is a list of terms that correspond to the set of positive integers.

For example, suppose Justine is allowed to choose between two salary plans. Plan A offers \$48,000 as a starting salary with a 5% raise each year. Plan B offers

\$50,000 with a \$2000 raise each year. Then the following sequences represent the salaries for each plan for the first 5 years.

Plan A:	\$48,000	\$50,400	\$52,920	\$55,566	\$58,344.30
Plan B:	\$50,000	\$52,000	\$54,000	\$56,000	\$58,000

By studying sequences we can see that Plan A yields a better salary if Justine plans to stay in the job for at least 5 years.

These sequences are called *finite sequences* because they have a finite number of terms. The sequence 1, 4, 9, 16, 25, . . . is called an *infinite sequence* because it continues indefinitely.

Because the terms in a sequence are related to the set of positive integers, we give a formal definition of finite and infinite sequences, using the language of functions.

Definition of Finite and Infinite Sequences

An **infinite sequence** is a function whose domain is the set of positive integers.
A **finite sequence** is a function whose domain is the set of the first n positive integers.

For any positive integer n, the value of the sequence is denoted by a_n (read as "a sub n"). The values a_1, a_2, a_3, . . . are called the **terms of the sequence**. The expression a_n defines the ***n*th term** (or general term) **of the sequence**.

Example 1 Listing the Terms of a Sequence

List the terms of the following sequences.

a. $a_n = 3n^2 - 4, \quad 1 \le n \le 4$ **b.** $a_n = 3 \cdot 2^n$

Solution:

a. The domain is restricted to the first four positive integers, indicating that the sequence is finite.

n	a_n
1	$3(1)^2 - 4 = -1$
2	$3(2)^2 - 4 = 8$
3	$3(3)^2 - 4 = 23$
4	$3(4)^2 - 4 = 44$

The sequence is -1, 8, 23, 44.

FOR REVIEW

Substituting values of n into a sequence defined by a_n is similar to substituting values of x into a function defined by $y = f(x)$.

b. The sequence $a_n = 3 \cdot 2^n$ has no restrictions on its domain; therefore, it is an infinite sequence.

n	a_n
1	$3 \cdot 2^1 = 6$
2	$3 \cdot 2^2 = 12$
3	$3 \cdot 2^3 = 24$
4	$3 \cdot 2^4 = 48$

. . .

The sequence is 6, 12, 24, 48, . . .

Skill Practice List the terms of the sequences.

1. $a_n = n^3 - 2, \quad 1 \le n \le 3$ **2.** $a_n = \left(\frac{2}{3}\right)^n$

Answers

1. $-1, 6, 25$ **2.** $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$

Sometimes the terms of a sequence may have alternating signs. Such a sequence is called an **alternating sequence**.

Example 2 Listing the Terms of an Alternating Sequence

List the first four terms of each alternating sequence.

a. $a_n = (-1)^n \cdot \frac{1}{n}$ **b.** $a_n = (-1)^{n+1} \cdot \left(\frac{2}{3}\right)^n$

Solution:

a.

n	a_n
1	$(-1)^1 \cdot \frac{1}{1} = -1$
2	$(-1)^2 \cdot \frac{1}{2} = \frac{1}{2}$
3	$(-1)^3 \cdot \frac{1}{3} = -\frac{1}{3}$
4	$(-1)^4 \cdot \frac{1}{4} = \frac{1}{4}$

The first four terms are $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}$.

TIP: Notice that the factor $(-1)^n$ makes the even-numbered terms positive and the odd-numbered terms negative.

b.

n	a_n
1	$(-1)^{1+1} \cdot \left(\frac{2}{3}\right)^1 = (-1)^2 \cdot \left(\frac{2}{3}\right) = \frac{2}{3}$
2	$(-1)^{2+1} \cdot \left(\frac{2}{3}\right)^2 = (-1)^3 \cdot \left(\frac{4}{9}\right) = -\frac{4}{9}$
3	$(-1)^{3+1} \cdot \left(\frac{2}{3}\right)^3 = (-1)^4 \cdot \left(\frac{8}{27}\right) = \frac{8}{27}$
4	$(-1)^{4+1} \cdot \left(\frac{2}{3}\right)^4 = (-1)^5 \cdot \left(\frac{16}{81}\right) = -\frac{16}{81}$

The first four terms are $\frac{2}{3}, -\frac{4}{9}, \frac{8}{27}, -\frac{16}{81}$.

TIP: Notice that the factor $(-1)^{n+1}$ makes the odd-numbered terms positive and the even-numbered terms negative.

Skill Practice List the first four terms of each alternating sequence.

3. $a_n = (-1)^n\left(\frac{1}{2}\right)^n$ **4.** $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$

In Examples 1 and 2, we were given the formula for the nth term of a sequence and asked to list several terms of the sequence. We now consider the reverse process. Given several terms of the sequence, we will find a formula for the nth term. To do so, look for a pattern that establishes each term as a function of the term number.

Answers

3. $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$

4. $2, -1, \frac{2}{3}, -\frac{1}{2}$

Example 3 Finding the *n*th Term of a Sequence

Find a formula for the *n*th term of the sequence.

a. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **b.** $-2, 4, -6, 8, -10, \ldots$ **c.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Solution:

a. For each term in the sequence, the numerator is equal to the term number, and the denominator is equal to 1 more than the term number. Therefore, the *n*th term may be given by

$$a_n = \frac{n}{n+1}$$

b. The odd-numbered terms are negative, and the even-numbered terms are positive. The factor $(-1)^n$ will produce the required alternation of signs. The numbers 2, 4, 6, 8, 10, . . . are equal to 2(1), 2(2), 2(3), 2(4), 2(5), ... Therefore, the *n*th term may be given by

$$a_n = (-1)^n \cdot 2n$$

c. The denominators are consecutive powers of 2. The sequence can be written as

$$\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots$$

Therefore, the *n*th term may be given by

$$a_n = \frac{1}{2^n}$$

Skill Practice Find a formula for the *n*th term of the sequence.

5. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ **6.** $-1, 4, -9, 16, -25, \ldots$ **7.** $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \ldots$

TIP: The first few terms of a sequence are not sufficient to define the sequence uniquely. For example, consider the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

The following formulas both produce the first three terms, but differ at the fourth term:

$$a_n = \frac{1}{2^n} \quad \text{and} \quad b_n = \frac{1}{n^2 - n + 2}$$

$$a_4 = \frac{1}{16} \quad \text{whereas} \quad b_4 = \frac{1}{14}$$

To define a sequence uniquely, the *n*th term must be provided.

Answers

5. $a_n = \frac{1}{3^n}$

6. $a_n = (-1)^n n^2$

7. $a_n = \frac{n+2}{n+1}$

Example 4 Using a Sequence in an Application

A child drops a ball from a height of 4 ft. With each bounce, the ball rebounds to 50% of its height. Write a sequence whose terms represent the heights from which the ball falls (begin with the initial height from which the ball was dropped).

Solution:

The ball first drops 4 ft and then rebounds to a new height of 0.50(4 ft) = 2 ft. Similarly, it falls from 2 ft and rebounds 0.50(2 ft) = 1 ft. Repeating this process, we have

$$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

The nth term can be represented by $a_n = 4 \cdot (0.50)^{n-1}$.

Skill Practice

8. The first swing of a pendulum measures 30°. If each swing that follows is 25% less, list the first three terms of this sequence.

2. Series

In many mathematical applications it is important to find the sum of the terms of a sequence. For example, suppose that the yearly interest earned in an account over a 4-year period is given by the sequence

$$\$250, \quad \$265, \quad \$278.25, \quad \$292.16$$

The sum of the terms gives the total interest earned

$$\$250 + \$265 + \$278.25 + \$292.16 = \$1085.41$$

By adding the terms of a sequence, we obtain a series.

Definition of a Series

The indicated sum of the terms of a sequence is called a **series**.

As with a sequence, a series may be a finite or an infinite sum of terms.

A convenient notation used to denote the sum of a set of terms is called **summation notation**, or sigma notation. The Greek letter Σ (sigma) is used to indicate sums. For example, the sum of the first four terms of the sequence defined by $a_n = n^3$ are denoted by

$$\sum_{n=1}^{4} n^3$$

This is read as "the sum from n equals 1 to 4 of n^3" and is simplified as

$$\begin{aligned}\sum_{n=1}^{4} n^3 &= (1)^3 + (2)^3 + (3)^3 + (4)^3 \\ &= 1 + 8 + 27 + 64 \\ &= 100\end{aligned}$$

In this example, the letter n is called the **index of summation**. Many times, the letters i, j, and k are also used for the index of summation.

Answer

8. 30°, 22.5°, 16.875°

Example 5 Finding a Sum from Summation Notation

Find the sum. $\sum_{i=1}^{4} 2^i$

Solution:

$$\sum_{i=1}^{4} 2^i = 2^1 + 2^2 + 2^3 + 2^4$$
$$= 2 + 4 + 8 + 16$$
$$= 30$$

Skill Practice Find the sum.

9. $\sum_{i=1}^{4} i(i+4)$

Example 6 Finding a Sum from Summation Notation

Find the sum. $\sum_{i=1}^{3} (-1)^{i+1} \cdot (3i+4)$

Solution:

Avoiding Mistakes

The letter i is used as a variable for the index of summation. In this context, it does not represent an imaginary number.

$$\sum_{i=1}^{3} (-1)^{i+1} \cdot (3i+4) = (-1)^{1+1} \cdot [3(1)+4] + (-1)^{2+1} \cdot [3(2)+4]$$
$$+ (-1)^{3+1} \cdot [3(3)+4]$$
$$= (-1)^2 \cdot (7) + (-1)^3 \cdot (10) + (-1)^4 \cdot (13)$$
$$= 7 - 10 + 13$$
$$= 10$$

Skill Practice Find the sum.

10. $\sum_{i=1}^{4} (-1)^i(4i-3)$

Example 7 Converting to Summation Notation

Write the series in summation notation.

a. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81}$ Use n as the index of summation.

b. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$ Use j as the index of summation.

Answers

9. 70 **10.** 8

Solution:

a. The sum can be written as

$$\left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4$$

Taking n from 1 to 4, we have

$$\sum_{n=1}^{4}\left(\frac{2}{3}\right)^n$$

b. The even-numbered terms are negative. The factor $(-1)^{j+1}$ is negative for even values of j. Therefore, the series can be written as

$$\sum_{j=1}^{6}(-1)^{j+1}\cdot\frac{1}{j}$$

Skill Practice Write the series in summation notation.

11. $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$ **12.** $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}$

Answers

11. $\sum_{i=1}^{4}\frac{1}{4^i}$

12. $\sum_{i=1}^{5}(-1)^{i+1}\frac{1}{2i}$

Section 10.2 Activity

A.1. List the terms of the sequence.

a. $a_n = \frac{2}{n+1}$ for $1 \le n \le 5$

b. $b_n = (-1)^n \cdot \frac{2}{n+1}$ for $1 \le n \le 5$

c. $c_n = (-1)^{n+1} \cdot \frac{2}{n+1}$ for $1 \le n \le 5$

A.2. Find the value of the sum.

a. $\sum_{n=1}^{5}\frac{2}{n+1}$

b. $\sum_{n=1}^{5}(-1)^n \cdot \frac{2}{n+1}$

c. $\sum_{n=1}^{5}(-1)^{n+1} \cdot \frac{2}{n+1}$

A.3. Suppose that an Internet video channel has 100 subscribers in its first year. Every year thereafter, the number of subscriptions doubles.

a. List the number of subscriptions for the first 5 years.

b. Write a formula representing the nth term of the sequence representing the number of subscriptions s_n by year number, n.

c. Suppose that each subscriber brings in an average of \$10 in advertising revenue per year. Write a formula representing the revenue r_n brought in for year n.

d. Write a series using summation notation that represents the total revenue brought in for 5 years. Use i as the index of summation.

e. Write the sum from part (d) in expanded form.

f. Determine the total revenue brought in for 5 years.

Section 10.2 Practice Exercises

Prerequisite Review

For Exercises R.1–R.6, evaluate the expression for the given value of n.

R.1. $(-1)^n$

a. $n = 1$ **b.** $n = 2$ **c.** $n = 3$

d. $n = 4$ **e.** $n = 5$ **f.** $n = 6$

R.2. $(-1)^{n+1}$

a. $n = 1$ **b.** $n = 2$ **c.** $n = 3$

d. $n = 4$ **e.** $n = 5$ **f.** $n = 6$

R.3. $-3 + (n - 1) \cdot 5$ for $n = 11$

R.4. $2 + 5n$ for $n = 21$

R.5. $10\left(\frac{2}{5}\right)^{n-1}$ for $n = 4$

R.6. $27\left(\frac{1}{3}\right)^{n-1}$ for $n = 5$

For Exercises R.7–R.10, simplify the expression.

R.7. $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}$

R.8. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81}$

R.9. $\frac{8!}{6!}$

R.10. $\frac{10!}{7!}$

Vocabulary and Key Concepts

1. a. A(n) (finite/infinite) sequence is a function whose domain is the set of positive integers. A(n) (finite/infinite) sequence is a function whose domain is the set of the first n positive integers.

b. Given a sequence, the values a_1, a_2, a_3, and so on are called the ______________ of the sequence. The expression a_n defines the ______________ term (or general term) of the sequence.

c. In an ______________ sequence, consecutive terms alternate in sign.

d. A sum of a sequence is called a ______________.

e. To represent a sum of terms, ______________ notation is often used and is represented by the Greek letter $\sum$.

f. Given the sum $\sum_{n=1}^{5} n^2$, the letter n is called the ______________ of summation. In expanded form, this series is $(1)^2 + (2)^2 +$ ______________ $= 55$.

Concept 1: Finite and Infinite Sequences

For Exercises 2–6, look at the pattern shown in the sequence and fill in the next three terms.

2. $a_n = -1, 4, 9, 14, 19, \square, \square, \square$

3. $b_n = 7, 10, 13, 16, 19, \square, \square, \square$

4. $c_n = -2, 4, -6, 8, -10, \square, \square, \square$

5. $d_n = 3, -6, 9, -12, 15, \square, \square, \square$

6. $t_n = 1, 4, 9, 16, 25, \square, \square, \square$

For Exercises 7–18, list the terms of each sequence. **(See Examples 1–2.)**

7. $a_n = 3n + 1, \quad 1 \le n \le 5$

8. $a_n = -2n + 3, \quad 1 \le n \le 5$

9. $a_n = \sqrt{n+2}, \quad 1 \le n \le 4$

10. $a_n = \sqrt{n-1}, \quad 1 \le n \le 4$

11. $a_n = (-1)^n \dfrac{n+1}{n+2}, \quad 1 \le n \le 4$

12. $a_n = (-1)^n \dfrac{n-1}{n+2}, \quad 1 \le n \le 4$

13. $a_n = (-1)^{n+1}(n^2 - 1), \quad 1 \le n \le 3$

14. $a_n = (-1)^{n+1}(n^2), \quad 1 \le n \le 3$

15. $a_n = n^2 - n, \quad 1 \le n \le 6$

16. $a_n = n(n^2 - 1), \quad 1 \le n \le 6$

17. $a_n = (-1)^n 3^n, \quad 1 \le n \le 4$

18. $a_n = (-1)^n n, \quad 1 \le n \le 4$

19. If the nth term of a sequence is $a_n = (-1)^n \cdot n^2$, which terms are positive and which are negative?

20. If the nth term of a sequence is $a_n = (-1)^{n-1} \cdot \frac{1}{n}$, which terms are positive and which are negative?

For Exercises 21–32, find a formula for the nth term of the sequence. Answers may vary. **(See Example 3.)**

21. 2, 4, 6, 8, . . .

22. 3, 6, 9, 12, . . .

23. 1, 3, 5, 7, . . .

24. 3, 5, 7, 9, . . .

25. $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots$

26. $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$

27. 1, −1, 1, −1, . . .

28. −1, 1 −1, 1, . . .

29. −2, 4, −8, 16, . . .

30. 3, −9, 27, −81, . . .

31. $\dfrac{3}{5}, \dfrac{3}{25}, \dfrac{3}{125}, \dfrac{3}{625}, \ldots$

32. $\dfrac{1}{4}, \dfrac{1}{16}, \dfrac{1}{64}, \dfrac{1}{256}, \ldots$

33. Edmond borrowed \$500. To pay off the loan, he agreed to pay 2% of the balance plus \$50 each month. Write a sequence representing the amount Edmond will pay each month for the next 4 months. Round each term to the nearest cent.

34. Janice deposited \$1000 in an account that pays 3% interest compounded annually. Write a sequence representing the amount Janice receives in interest each year for the first 4 years. Round each term to the nearest cent.

35. A certain bacteria culture doubles its size each day. If there are 25,000 bacteria on the first day, write a sequence representing the population each day for the first week (7 days). **(See Example 4.)**

36. A radioactive chemical decays by one-half of its amount each week. If there is 16 g of the chemical in week 1, write a sequence representing the amount present each week for 2 months (8 weeks).

Concept 2: Series

37. What is the difference between a sequence and a series?

38. **a.** Identify the index of summation for the series. $\displaystyle\sum_{k=1}^{7}(k^2 + 1)$

b. How many terms does this series have?

For Exercises 39–54, find the sums. **(See Examples 5–6.)**

39. $\sum_{i=1}^{4}(3i)^2$

40. $\sum_{i=1}^{4}(2i^2)$

41. $\sum_{j=0}^{4}\left(\frac{1}{2}\right)^j$

42. $\sum_{j=0}^{4}\left(\frac{1}{3}\right)^j$

43. $\sum_{i=1}^{6}5$

44. $\sum_{i=1}^{7}3$

45. $\sum_{j=1}^{4}(-1)^j(5j)$

46. $\sum_{j=1}^{4}(-1)^j(4j)$

47. $\sum_{i=1}^{4}\frac{i+1}{i}$

48. $\sum_{i=2}^{5}\frac{i-1}{i}$

49. $\sum_{j=1}^{3}(j+1)(j+2)$

50. $\sum_{j=1}^{3}j(j+2)$

51. $\sum_{k=1}^{7}(-1)^k$

52. $\sum_{k=0}^{5}(-1)^{k+1}$

53. $\sum_{k=1}^{5}k^2$

54. $\sum_{k=1}^{5}2^k$

For Exercises 55–66,write the series in summation notation. **(See Example 7.)**

55. $1+2+3+4+5+6$

56. $1-2+3-4+5-6$

57. $4+4+4+4+4$

58. $8+8+8+8+8$

59. $4+8+12+16+20$

60. $3+6+9+12+15$

61. $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}$

62. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}$

63. $\frac{5}{11}+\frac{6}{22}+\frac{7}{33}+\frac{8}{44}+\frac{9}{55}+\frac{10}{66}$

64. $-\frac{5}{7}-\frac{10}{8}-\frac{15}{9}-\frac{20}{10}-\frac{25}{11}-\frac{30}{12}-\frac{35}{13}$

65. $x+x^2+x^3+x^4+x^5$

66. $y^2+y^4+y^6+y^8+y^{10}$

Expanding Your Skills

Some sequences are defined by a recursive formula, which defines each term of a sequence in terms of one or more of its preceding terms. For example, if $a_1=5$ and $a_n=2a_{n-1}+1$ for $n>1$, then the terms of the sequence are 5, 11, 23, 47, ... In this case, each term after the first is one more than twice the term before it.

For Exercises 67–70, list the first five terms of the sequence.

67. $a_1=-3,\ a_n=a_{n-1}+5 \quad \text{for } n>1$

68. $a_1=10,\ a_n=a_{n-1}-3 \quad \text{for } n>1$

69. $a_1=5,\ a_n=4a_{n-1}+1 \quad \text{for } n>1$

70. $a_1=-2,\ a_n=-3a_{n-1}+4 \quad \text{for } n>1$

71. A famous sequence in mathematics is called the Fibonacci sequence, named after the Italian mathematician Leonardo Fibonacci of the thirteenth century. The Fibonacci sequence is defined by

$$a_1=1$$

$$a_2=1$$

$$a_n=a_{n-1}+a_{n-2} \qquad \text{for } n>2$$

This definition implies that beginning with the third term, each term is the sum of the preceding two terms. Write out the first 10 terms of the Fibonacci sequence.

Arithmetic Sequences and Series

Section 10.3

Concepts

1. Arithmetic Sequences
2. Arithmetic Series

1. Arithmetic Sequences

In this section, we present a special type of sequence called an arithmetic sequence, and then we will study its related sum called an arithmetic series. The sequence 4, 7, 10, 13, 16, . . . is an example of an arithmetic sequence. Note the characteristic that each successive term after the first is a fixed value more than the previous term (in this case the terms differ by 3).

Definition of an Arithmetic Sequence

An **arithmetic sequence** is a sequence in which the difference between consecutive terms is constant.

The fixed difference between a term and its predecessor is called the **common difference** and is denoted by the letter d. That is,

$$d = a_{n+1} - a_n$$

Furthermore, if a_1 is the first term, then

$$\begin{aligned}
&a_1 = a_1 + 0d \\
&a_2 = a_1 + 1d && \text{is the second term,} \\
&a_3 = a_1 + 2d && \text{is the third term,} \\
&a_4 = a_1 + 3d && \text{is the fourth term, and so on.} \\
&\ldots
\end{aligned}$$

In general, $a_n = a_1 + (n - 1)d$.

*n*th Term of an Arithmetic Sequence

The nth term of an arithmetic sequence is given by

$$a_n = a_1 + (n - 1)d$$

where a_1 is the first term of the sequence and d is the common difference.

Example 1 **Writing the *n*th Term of an Arithmetic Sequence**

Write the nth term of the sequence. $9, 2, -5, -12, \ldots$

Solution:

$$9, \quad 2, \quad -5, \quad -12, \ldots$$
$$-7 \quad -7 \quad -7$$

The common difference can be found by subtracting a term from its predecessor: $2 - 9 = -7$

TIP: Notice that the terms of an arithmetic sequence increase or decrease linearly. In Example 1, the terms decrease by 7 for each subsequent term. This is similar to a line with a slope of -7.

FOR REVIEW

Notice the similarity between an arithmetic sequence such as $a_n = -7n + 16$ and a linear function such as $y = -7x + 16$. However, the domain of the sequence is the set of positive integers.

With $a_1 = 9$ and $d = -7$, we have

$$\begin{aligned} a_n &= 9 + (n-1)(-7) \\ &= 9 - 7n + 7 \\ &= -7n + 16 \end{aligned}$$

Skill Practice Write the nth term of the sequence.

1. $-3, -1, 1, 3, 5, \ldots$

In Example 1, the common difference between terms is -7. Accordingly, each term of the sequence *decreases* by 7.

The formula $a_n = a_1 + (n-1)d$ contains four variables: a_n, a_1, n, and d. Consequently, if we know the value of three of the four variables, we can solve for the fourth.

Example 2 Finding a Specified Term of an Arithmetic Sequence

Find the ninth term of the arithmetic sequence in which $a_1 = -4$ and $a_{22} = 164$.

Solution:

To find the value of the ninth term a_9, we need to determine the value of d. To find d, substitute $a_1 = -4$, $n = 22$, and $a_{22} = 164$ into the formula $a_n = a_1 + (n-1)d$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 164 &= -4 + (22-1)d \\ 164 &= -4 + 21d \\ 168 &= 21d \\ d &= 8 \end{aligned}$$

Therefore, $a_n = -4 + (n-1)(8)$

$$\begin{aligned} a_9 &= -4 + (9-1)(8) \\ &= -4 + (8)(8) \\ &= 60 \end{aligned}$$

Skill Practice

2. Find the tenth term of the arithmetic sequence in which $a_1 = 14$ and $a_{15} = -28$. (*Hint:* First find d.)

Example 3 Finding the Number of Terms in an Arithmetic Sequence

Find the number of terms of the sequence $7, 3, -1, -5, \ldots, -113$.

Answers

1. $2n - 5$ **2.** $d = -3$, $a_{10} = -13$

Solution:

To find the number of terms n, we can substitute $a_1 = 7$, $d = -4$, and $a_n = -113$ into the formula for the nth term.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ -113 &= 7 + (n-1)(-4) \\ -113 &= 7 - 4n + 4 \\ -113 &= 11 - 4n \\ -124 &= -4n \\ n &= 31 \end{aligned}$$

Skill Practice Find the number of terms of the sequence.

3. $-15, -11, -7, -3, \ldots, 81$

2. Arithmetic Series

The indicated sum of an arithmetic sequence is called an **arithmetic series**. For example, the series

$$3 + 7 + 11 + 15 + 19 + 23$$

is an arithmetic series because the common difference between terms is constant (4). Adding the terms in a lengthy sum is cumbersome, so we offer the following "shortcut," which is developed here. Let S represent the sum of the terms in the series.

$$\begin{aligned} S &= 3 + 7 + 11 + 15 + 19 + 23 && \text{Add the terms in ascending order.} \\ S &= 23 + 19 + 15 + 11 + 7 + 3 && \text{Add the terms in descending order.} \\ \hline 2S &= 26 + 26 + 26 + 26 + 26 + 26 && \text{Adding the two series produces six terms of 26.} \\ 2S &= 6 \cdot 26 \\ S &= \frac{6 \cdot 26}{2} \\ &= 78 \end{aligned}$$

By adding the terms in ascending and descending order, we double the sum but create a pattern that is easily added. This is true in general. To find the sum, S_n, of the first n terms of the arithmetic series $a_1 + a_2 + a_3 + \cdots + a_n$, we have

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + a_n && \text{Ascending order} \\ S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + a_1 && \text{Descending order} \\ \hline 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) \\ 2S_n &= n(a_1 + a_n) \\ S_n &= \frac{n}{2}(a_1 + a_n) \end{aligned}$$

Answer

3. 25

Sum of an Arithmetic Series

The sum S_n of the first n terms of an arithmetic series is given by

$$S_n = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term of the series and a_n is the nth term of the series.

Example 4 Finding the Sum of an Arithmetic Series

Find the sum of the series. $\sum_{i=1}^{25}(2i+3)$

Avoiding Mistakes

When we apply the sum formula $S_n = \frac{n}{2}(a_1 + a_n)$ to find the sum $\sum_{i=1}^{n} a_n$, the index of summation i must begin at 1.

Solution:

In this series, $n = 25$. Furthermore, $a_1 = 2(1) + 3 = 5$ and $a_{25} = 2(25) + 3 = 53$. Therefore,

$$S_{25} = \frac{25}{2}(5+53) \quad \text{Apply the formula } S_n = \frac{n}{2}(a_1 + a_n).$$

$$= \frac{25}{2}(58) \quad \text{Simplify.}$$

$$= 725$$

Skill Practice Find the sum of the series.

4. $\sum_{i=1}^{20}(3i-2)$

Example 5 Finding the Sum of an Arithmetic Series

Find the sum of the series.

$$-3 + (-5) + (-7) + \cdots + (-127)$$

Solution:

For this series, $a_1 = -3$ and $a_n = -127$. However, to determine the sum, we also need to find the value of n. The difference between the second term and its predecessor is $-5 - (-3) = -2$. Thus, $d = -2$. We have

$$-127 = -3 + (n-1)(-2) \quad \text{Apply the formula } a_n = a_1 + (n-1)d.$$

$$-127 = -3 - 2n + 2 \quad \text{Apply the distributive property.}$$

$$-127 = -1 - 2n \quad \text{Combine } \textit{like} \text{ terms.}$$

$$-126 = -2n \quad \text{Solve for } n.$$

$$n = 63$$

Using $n = 63$, $a_1 = -3$, and $a_{63} = -127$, we have

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{63}{2}[-3 + (-127)] = \frac{63}{2}(-130) = -4095$$

Skill Practice Find the sum of the series.

5. $8 + 3 + (-2) + \cdots + (-137)$

Answers

4. 590 **5.** −1935

Section 10.3 Activity

A.1. Joyce earns a salary of \$55,000 during her first year of work. She receives a raise of \$2000 per year thereafter.

a. Fill in the table representing the sequence for Joyce's salary for year n.

Year	Base Salary Plus Raise(s)	Total Salary
Year 1	\$55,000 + 0(\$2000)	\$55,000
Year 2	\$55,000 + 1(\$2000) (Joyce has had 1 raise.)	\$57,000
Year 3	\$55,000 + 2(\$2000) (Joyce has had 2 raises.)	\$59,000
Year 4		
Year 5		
Year n	\$55,000 +	

b. Use the formula from the last row in the table to determine Joyce's salary in year 25.

A.2. a. The sequence from Exercise A.1 has the characteristic that each successive term differs by a constant, d. (In this case, the constant is the \$2000 raise that Joyce earns each year.) This type of sequence is called a(n) ________ sequence.

b. If a_1 is the first term of an arithmetic sequence, and d is the common difference, use the pattern below to write a formula for the nth term of the sequence.

$$a_1 = a_1 + 0d$$
$$a_2 = a_1 + 1d$$
$$a_3 = a_1 + 2d$$
$$\vdots$$
$$a_n = \underline{\hspace{3cm}}$$

A.3. Consider the arithmetic sequence

$$17, 11, 5, -1, \ldots, -301$$

a. What is the first term of the sequence?

b. What is the common difference?

c. Write a formula for the nth term of the sequence.

d. How many terms are there in this sequence?

A.4. Refer to Exercise A.1.

a. Write a series using summation notation that represents Joyce's total earnings over 25 years.

b. Find Joyce's total earnings over 25 years.

Practice Exercises — Section 10.3

Prerequisite Review

For Exercises R.1–R.4, solve the equation.

R.1. $62 = 2 + (n - 1) \cdot 3$

R.2. $110 = 5 + (n - 1) \cdot 7$

R.3. $-95 = -15 + 40d$

R.4. $-142 = 14 + 52d$

For Exercises R.5–R.6, find the function values.

R.5. Find the function value for the function defined by $f(x) = 4x - 3$.

a. $f(1)$ **b.** $f(2)$ **c.** $f(3)$ **d.** $f(4)$

R.6. Find the function value for the function defined by $g(x) = -2x + 5$.

a. $g(1)$ **b.** $g(2)$ **c.** $g(3)$ **d.** $g(4)$

For Exercises R.7–R.8, write the sum in expanded form and then simplify.

R.7. $\sum_{n=1}^{5}(1 - 4n)$

R.8. $\sum_{n=1}^{6}(5n - 3)$

Vocabulary and Key Concepts

1. a. An ____________ sequence is a sequence in which the difference between consecutive terms is constant.

b. The common ____________ between a term and its predecessor in an arithmetic sequence is often denoted by the letter d.

c. The nth term of an arithmetic sequence is given by $a_n =$ ____________, where a_n is the nth term, a_1 is the first term, n is the number of terms, and ____________ is the common difference between terms.

d. An indicated sum of an arithmetic sequence is called an arithmetic ____________.

e. The sum S_n of the first n terms of an arithmetic series is given by $S_n =$ ____________, where a_1 is the first term and a_n is the nth term.

Concept 1: Arithmetic Sequences

2. Explain how to determine if a sequence is arithmetic.

For Exercises 3–6, write the first four terms of the sequence.

3. $a_n = -5 + 4n$

4. $b_n = 8 + (-5)n$

5. $c_n = \frac{1}{2} + (n - 1)\frac{3}{2}$

6. $d_n = \frac{1}{4} + (n - 1)\frac{3}{4}$

For Exercises 7–12, the first term of an arithmetic sequence is given along with its common difference. Write the first five terms of the sequence.

7. $a_1 = 3, d = 8$

8. $a_1 = -5, d = 2$

9. $a_1 = 80, d = -20$

10. $a_1 = -4, d = -5$

11. $a_1 = 3, d = \frac{3}{4}$

12. $a_1 = 1, d = \frac{2}{3}$

For Exercises 13–18, find the common difference d for each arithmetic sequence.

13. 1, 3, 5, 7, 9, . . .

14. 2, 8, 14, 20, 26, . . .

15. 6, 3, 0, −3, −6, . . .

16. 8, 3, −2, −7, −12, . . .

17. −7, −9, −11, −13, −15, . . .

18. −15, −11, −7, −3, 1, . . .

For Exercises 19–24, write the first five terms of the arithmetic sequence.

19. $a_1 = 3, d = 5$

20. $a_1 = -3, d = 2$

21. $a_1 = 2, d = \frac{1}{2}$

22. $a_1 = 0, d = \frac{1}{3}$

23. $a_1 = 2, d = -4$

24. $a_1 = 10, d = -6$

For Exercises 25–33, write the nth term of the sequence. **(See Example 1.)**

25. $0, 5, 10, 15, 20, \ldots$

26. $7, 12, 17, 22, 27, \ldots$

27. $-2, -4, -6, -8, -10, \ldots$

28. $1, -3, -7, -11, -15, \ldots$

29. $2, \frac{5}{2}, 3, \frac{7}{2}, 4, \ldots$

30. $1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \ldots$

31. $21, 17, 13, 9, 5, \ldots$

32. $9, 6, 3, 0, -3, \ldots$

33. $-8, -2, 4, 10, 16, \ldots$

For Exercises 34–41, find the indicated term of each arithmetic sequence. **(See Example 2.)**

34. Find the eighth term given $a_1 = -6$ and $d = -3$.

35. Find the sixth term given $a_1 = -3$ and $d = 4$.

36. Find the 12th term given $a_1 = -8$ and $d = -2$.

37. Find the ninth term given $a_1 = -1$ and $d = 6$.

38. Find the tenth term given $a_1 = 1$ and $a_7 = 31$.

39. Find the seventh term given $a_1 = 0$ and $a_{10} = -45$.

40. Find the sixth term given $a_1 = 3$ and $a_{13} = 39$.

41. Find the 11th term given $a_1 = 12$ and $a_6 = -18$.

For Exercises 42–49, find the number of terms, n, of each arithmetic sequence. **(See Example 3.)**

42. $2, 0, -2, \ldots, -56$

43. $8, 13, 18, \ldots, 98$

44. $1, -3, -7, \ldots, -67$

45. $1, 5, 9, \ldots, 85$

46. $1, \frac{3}{4}, \frac{1}{2}, \ldots, -4$

47. $2, \frac{5}{2}, 3, \ldots, 13$

48. $-\frac{5}{3}, -1, -\frac{1}{3}, \ldots, \frac{37}{3}$

49. $\frac{13}{3}, \frac{19}{3}, \frac{25}{3}, \ldots, \frac{73}{3}$

50. If the third and fourth terms of an arithmetic sequence are 18 and 21, what are the first and second terms?

51. If the third and fourth terms of an arithmetic sequence are -8 and -11, what are the first and second terms?

Concept 2: Arithmetic Series

52. Explain the difference between an arithmetic sequence and an arithmetic series.

For Exercises 53–66, find the sum of the arithmetic series. **(See Examples 4–5.)**

53. $\sum_{i=1}^{20} (3i + 2)$

54. $\sum_{i=1}^{15} (2i - 3)$

55. $\sum_{i=1}^{20} (i + 4)$

56. $\sum_{i=1}^{25} (i - 3)$

57. $\sum_{j=1}^{10} (4 - j)$

58. $\sum_{j=1}^{10} (6 - j)$

59. $\sum_{j=1}^{15} \left(\frac{2}{3}j + 1\right)$

60. $\sum_{j=1}^{15} \left(\frac{1}{2}j - 2\right)$

61. $4 + 8 + 12 + \cdots + 84$

62. $4 + 9 + 14 + \cdots + 49$

63. $6 + 8 + 10 + \cdots + 34$

64. $-4 + (-3) + (-2) + \cdots + 12$

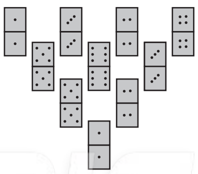
65. $-3 + (-7) + (-11) + \cdots + (-39)$

66. $2 + 5 + 8 + \cdots + 53$

67. Find the sum of the first 100 positive integers.

68. Find the sum of the first 50 positive even integers.

69. The seating in a certain theater is arranged so that there are 30 seats in row 1, 32 in row 2, 34 in row 3, and so on. If there are 20 rows, how many total seats are there? What is the total revenue if the average ticket price is \$15 per seat and the theater is sold out?



70. A triangular array of dominoes has one domino in the first row, two dominoes in the second row, three dominoes in the third row, and so on. If there are 15 rows, how many dominoes are in the array?

Section 10.4 Geometric Sequences and Series

Concepts

1. Geometric Sequences
2. Geometric Series

1. Geometric Sequences

The sequence 2, 4, 8, 16, 32, . . . is not an arithmetic sequence because the difference between terms is not constant. However, a different pattern exists. Notice that each term after the first is 2 times the preceding term. This sequence is called a geometric sequence.

Definition of a Geometric Sequence

A **geometric sequence** is a sequence in which the ratio between each term and its predecessor is constant.

The constant quotient of a term and its predecessor is called the **common ratio** and is denoted by r. The common ratio is found by dividing a term by the preceding term. That is,

$$r = \frac{a_{n+1}}{a_n}$$

For the sequence 2, 4, 8, 16, 32, . . . we have $r = \frac{4}{2} = \frac{8}{4} = \frac{16}{8} = \frac{32}{16} = 2$.

If a_1 denotes the first term of a geometric sequence, then

$a_2 = a_1 r$ is the second term,

$a_3 = a_1 r^2$ is the third term,

$a_4 = a_1 r^3$ is the fourth term, and so on.

. . .

This pattern gives $a_n = a_1 r^{n-1}$.

*n*th Term of a Geometric Sequence

The nth term of a geometric sequence is given by

$$a_n = a_1 r^{n-1}$$

where a_1 is the first term and r is the common ratio.

Example 1 Finding the *n*th Term of a Geometric Sequence

Find the nth term of the sequence.

a. $-1, 4, -16, 64, \ldots$ **b.** $12, 8, \frac{16}{3}, \frac{32}{9}, \ldots$

Solution:

a. The common ratio is found by dividing any term (after the first) by its predecessor.

$$r = \frac{4}{-1} = -4$$

With $r = -4$ and $a_1 = -1$, we have $a_n = -1(-4)^{n-1}$.

b. The common ratio is $r = \frac{8}{12} = \frac{2}{3}$. With $a_1 = 12$ and $r = \frac{2}{3}$, we have

$$a_n = 12\left(\frac{2}{3}\right)^{n-1}$$

Skill Practice Find the nth term of the geometric sequence.

1. 6, 12, 24, 48, . . . **2.** $6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$

The formula $a_n = a_1 r^{n-1}$ contains the variables a_n, a_1, n, and r. If we know the value of three of the four variables, we can find the fourth.

Example 2 Finding a Specified Term of a Geometric Sequence

Given $a_n = 6\left(\frac{1}{2}\right)^{n-1}$, find a_5.

Solution:

$$a_n = 6\left(\frac{1}{2}\right)^{n-1}$$

$$a_5 = 6\left(\frac{1}{2}\right)^{5-1} = 6\left(\frac{1}{2}\right)^4 = 6\left(\frac{1}{16}\right) = \frac{3}{8}$$

Skill Practice

3. Given $a_n = 3(-2)^{n-1}$, find a_6.

Example 3 Finding a Specified Term of a Geometric Sequence

Find the first term of the geometric sequence where $a_5 = -162$ and $r = 3$.

Solution:

$-162 = a_1(3)^{5-1}$ Substitute $a_5 = -162$, $n = 5$, and $r = 3$ into the formula $a_n = a_1 r^{n-1}$.

$-162 = a_1(3)^4$ Simplify and solve for a_1.

$-162 = a_1(81)$

$a_1 = -2$

Skill Practice

4. Find the first term of the geometric sequence where $a_4 = \frac{25}{32}$ and $r = \frac{1}{4}$.

2. Geometric Series

The indicated sum of a geometric sequence is called a **geometric series**. For example,

$$1 + 3 + 9 + 27 + 81 + 243$$

is a geometric series. To find the sum, consider the following procedure. Let S represent the sum

$$S = 1 + 3 + 9 + 27 + 81 + 243$$

Answers

1. $a_n = 6(2)^{n-1}$ **2.** $a_n = 6\left(\frac{1}{3}\right)^{n-1}$

3. -96 **4.** 50

Now multiply S by the common ratio, which in this case is 3.

$$3S = 3 + 9 + 27 + 81 + 243 + 729$$

Then

$$\begin{aligned} 3S - S &= (3 + 9 + 27 + 81 + 243 + 729) - (1 + 3 + 9 + 27 + 81 + 243) \\ 2S &= 3 + 9 + 27 + 81 + 243 + 729 - 1 - 3 - 9 - 27 - 81 - 243 \\ 2S &= 729 - 1 \qquad \text{The terms in red form a sum of zero.} \\ 2S &= 728 \\ S &= 364 \end{aligned}$$

A similar procedure can be used to find the sum S_n of the first n terms of any geometric series. Subtract rS_n from S_n.

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} \\ rS_n &= a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^n \\ \hline S_n - rS_n &= (a_1 - a_1 r) + (a_1 r - a_1 r^2) + (a_1 r^2 - a_1 r^3) + \cdots + (a_1 r^{n-1} - a_1 r^n) \end{aligned}$$

$$\begin{aligned} S_n - rS_n &= a_1 - a_1 r^n \qquad \text{The terms in red form a sum of zero.} \\ S_n(1 - r) &= a_1(1 - r^n) \qquad \text{Factor each side of the equation.} \\ S_n &= \frac{a_1(1 - r^n)}{1 - r} \qquad \text{Divide by } (1 - r). \end{aligned}$$

Sum of a Geometric Series

The sum, S_n, of the first n terms of a geometric series $\sum_{i=1}^{n} a_1 r^{i-1}$ is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

where a_1 is the first term of the series, r is the common ratio, and $r \neq 1$.

Example 4 Finding the Sum of a Geometric Series

Find the sum of the series. $\sum_{i=1}^{6} 4\left(\frac{1}{2}\right)^{i-1}$

Solution:

By expanding the terms of this series, we see that the series is geometric.

$$\sum_{i=1}^{6} 4\left(\frac{1}{2}\right)^{i-1} = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

We have: $a_1 = 4$, $r = \frac{1}{2}$, and $n = 6$,

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{4[1 - (\frac{1}{2})^6]}{1 - \frac{1}{2}} = \frac{4(1 - \frac{1}{64})}{\frac{1}{2}} = 8\left(\frac{63}{64}\right) = \frac{63}{8}$$

Skill Practice Find the sum of the geometric series.

5. $\frac{1}{9} + \frac{1}{3} + 1 + 3 + 9 + 27$

Answer

5. $\frac{364}{9}$

Example 5 Finding the Sum of a Geometric Series

Find the sum of the series $5 + 10 + 20 + \cdots + 5120$.

Solution:

The common ratio is 2 and $a_1 = 5$. The nth term of the sequence can be written as $a_n = 5(2)^{n-1}$. To find the value of n, substitute 5120 for a_n.

$$5120 = 5(2)^{n-1}$$

$$\frac{5120}{5} = \frac{5(2)^{n-1}}{5} \qquad \text{Divide both sides by 5.}$$

$$1024 = 2^{n-1} \qquad \text{To solve the exponential equation, write each side as a power of 2.}$$

$$2^{10} = 2^{n-1}$$

$$10 = n - 1 \qquad \text{Recall that if } b^x = b^y\text{, then } x = y.$$

$$n = 11$$

With $a_1 = 5$, $r = 2$, and $n = 11$, we have

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{5(1 - 2^{11})}{1 - 2} = \frac{5(1 - 2048)}{-1} = \frac{5(-2047)}{-1} = 10{,}235$$

Skill Practice Find the sum of the geometric series.

6. $3 + 6 + 12 + \cdots + 768$

Consider a geometric series where $|r| < 1$. For increasing values of n, r^n decreases. For example,

$$\left(\frac{1}{2}\right)^5 = 0.03125 \qquad \left(\frac{1}{2}\right)^{10} \approx 0.00097656 \qquad \left(\frac{1}{2}\right)^{15} \approx 0.00003052$$

For $|r| < 1$, r^n approaches 0 as n gets larger and larger. As n approaches infinity, the sum

$$S = \frac{a_1(1 - r^n)}{1 - r} \quad \text{approaches} \quad \frac{a_1(1 - 0)}{1 - r} = \frac{a_1}{1 - r}$$

Sum of an Infinite Geometric Series

Given an infinite geometric series $a_1 + a_1r + a_1r^2 + \cdots$, with $|r| < 1$, the sum S of all terms in the series is given by

$$S = \frac{a_1}{1 - r}$$

Note: If $|r| \geq 1$, then the sum does not exist.

Example 6 Finding the Sum of an Infinite Geometric Series

Find the sum of the series. $\displaystyle\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1}$

Solution:

In this example, the upper limit of summation is ∞. This indicates that we have an infinite series where the pattern for the sum of terms continues indefinitely.

$$\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$$

Answer

6. 1533

The series is geometric with $a_1 = 1$ and $r = \frac{1}{3}$. Because $|r| = |\frac{1}{3}| < 1$, we have

$$S = \frac{a_1}{1 - r} = \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

The sum is $\frac{3}{2}$.

Skill Practice Find the sum of the series.

7. $\displaystyle\sum_{i=1}^{\infty} 4\left(\frac{3}{4}\right)^{i-1}$

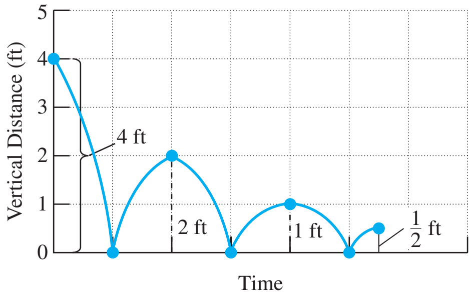
Example 7 Using a Geometric Series in a Physics Application

A child drops a ball from a height of 4 ft. With each bounce, the ball rebounds to 50% of its original height. Determine the total distance traveled by the ball.

Solution:

The heights (in ft) from which the ball drops are given by the sequence

$$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$$



After the ball falls from its initial height of 4 ft, the distance traveled for every bounce thereafter is doubled (the ball travels up and down). Therefore, the total distance traveled is given by the series

$$4 + 2 \cdot 2 + 2 \cdot 1 + 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + \cdots$$

or equivalently $$4 + 4 + 2 + 1 + \frac{1}{2} + \cdots$$

The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ is an infinite geometric series with $a_1 = 4$ and $r = \frac{1}{2}$.

$$\underset{\substack{\text{Initial height}\\ \text{from which the}\\ \text{ball was dropped}}}{4} + \overbrace{4 + 2 + 1 + \frac{1}{2} + \cdots}^{\text{Infinite geometric series}}$$

$$= 4 + \frac{4}{1 - \frac{1}{2}}$$

$$= 4 + \frac{4}{\frac{1}{2}}$$

$$= 4 + 8 = 12$$

The ball traveled a total of 12 ft.

Skill Practice

8. A child drops a ball from a height of 3 ft. With each bounce, the ball rebounds to $\frac{1}{3}$ of its original height. Determine the total distance traveled by the ball.

Answers

7. 16 **8.** 6 ft

Section 10.4 Activity

A.1. Kevin earns a salary of \$50,000 during his first year of work. He receives a raise of 4% per year thereafter.

a. Fill in the table representing the sequence for Kevin's salary for year n.

Year	Salary	Simplified
Year 1	$\$50{,}000(1.04)^{0}$	\$50,000
Year 2	$\$50{,}000(1.04)^{1}$ (Kevin has 1 raise of 4% of \$50,000 or equivalently 104% of his first year's salary.)	\$52,000
Year 3	$\$50{,}000(1.04)^{2}$ (Kevin has 2 raises of 4%. This is 104% of 104% of his first year's salary.)	\$54,080
Year 4	$\$50{,}000(1.04)^{3}$	
Year 5	$\$50{,}000(1.04)^{4}$	
Year n	$\$50{,}000(1.04)^{\boxed{n-1}}$	

b. Use the formula from the last row in the table to determine Kevin's salary in year 25.

c. Refer to Exercise A.1 on page 891. Joyce received a salary of \$55,000 in her first year and then a raise of \$2000 each year thereafter. Kevin makes less money initially (\$50,000), and his first raise is for \$2000. But Kevin's yearly salary eventually overtakes Joyce's salary. Explain why.

d. Write a series using summation notation that represents Kevin's total earnings over 25 years.

e. The series in part (d) is a finite geometric series of the form $\sum_{i=1}^{n} a_1 r^{i-1}$. The sum can be evaluated by using the formula $\sum_{i=1}^{n} a_1 r^{i-1} = \dfrac{a_1(1-r^n)}{1-r}$. Find Kevin's total earnings over 25 years.

f. From Exercise A.4(b) on page 891, you computed Joyce's total earnings over 25 years to be $\sum_{i=1}^{25}(53{,}000 + 2000i) = 1{,}975{,}000$. How much more money does Kevin make in 25 years?

For Exercises A.2–A.4, refer to the sequences a_n, b_n, and c_n.

$$a_n = 16,\ 8,\ 4,\ 2,\ \ldots,\ \tfrac{1}{16} \qquad b_n = 16,\ 8,\ 0,\ -8,\ \ldots,\ -360 \qquad c_n = 16,\ 8,\ 1,\ -5,\ \ldots,\ -20$$

A.2. **a.** Is sequence a_n arithmetic, geometric, or neither? Why?

b. Write a formula for the nth term of a_n.

c. Find the eighth term of a_n.

d. How many terms does the sequence a_n have?

A.3. **a.** Is sequence b_n arithmetic, geometric, or neither?

b. Write a formula for the nth term of b_n.

c. Find the 12th term of b_n.

d. How many terms does the sequence b_n have?

A.4. Is sequence c_n arithmetic, geometric, or neither?

For Exercises A.5–A.8, find the sum if possible. If the sum does not exist, explain why.

A.5. $\sum_{i=1}^{9} 16\left(\dfrac{1}{2}\right)^{i-1}$

A.6. $\sum_{i=1}^{\infty} 16\left(\dfrac{1}{2}\right)^{i-1}$

A.7. $\sum_{i=1}^{9} 16(2)^{i-1}$

A.8. $\sum_{i=1}^{\infty} 16(2)^{i-1}$

Section 10.4 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, find the function values.

R.1. Find the function value for the function defined by $f(x) = 50\left(\frac{2}{5}\right)^x$.

a. $f(1)$ **b.** $f(2)$ **c.** $f(3)$ **d.** $f(4)$

R.2. Find the function value for the function defined by $g(x) = 8\left(\frac{1}{2}\right)^x$.

a. $g(1)$ **b.** $g(2)$ **c.** $g(3)$ **d.** $g(4)$

For Exercises R.3–R.8, simplify the expression.

R.3. $\dfrac{16}{1-\frac{1}{2}}$

R.4. $\dfrac{12}{1-\frac{2}{3}}$

R.5. $\dfrac{10(1-2^4)}{1-2}$

R.6. $\dfrac{24(1-3^3)}{1-3}$

R.7. $\dfrac{16\left[1-\left(\frac{1}{2}\right)^3\right]}{1-\frac{1}{2}}$

R.8. $\dfrac{27\left[1-\left(\frac{2}{3}\right)^3\right]}{1-\frac{2}{3}}$

Vocabulary and Key Concepts

1. a. A __________ sequence is a sequence in which the ratio between each term and its predecessor is constant.

b. The common __________ between a term and its predecessor in a geometric sequence is often denoted by r.

c. The nth term of a geometric sequence is given by $a_n =$ __________, where a_n is the nth term in the sequence, a_1 is the first term, and r is the common ratio.

d. The sum S_n of the first n terms of a geometric series $\sum_{i=1}^{n} a_1 r^{i-1}$ is given by $S_n =$ __________, where a_1 is the first term and r is the common ratio.

e. Given an infinite geometric series $a_1 + a_2 r + a_1 r^2 + \cdots$ with $|r| < 1$, the sum of all terms in the sequence is given by $S =$ __________. If $|r| \geq$ __________, then the sum does not exist.

Concept 1: Geometric Sequences

2. Explain how to determine if a sequence is geometric.

For Exercises 3–6, write the first four terms of the sequence.

3. $a_n = 4 \cdot 2^{n-1}$

4. $b_n = \frac{1}{3} \cdot 3^{n-1}$

5. $c_n = (-5)^{n-1}$

6. $d_n = (-1)^{n+1} 4^{n-1}$

For Exercises 7–12, the first term of a geometric sequence is given along with the common ratio. Write the first four terms of the sequence.

7. $a_1 = 1;\ r = 10$

8. $a_1 = 2;\ r = 5$

9. $a_1 = 64;\ r = \frac{1}{2}$

10. $a_1 = 3;\ r = \frac{2}{3}$

11. $a_1 = 8;\ r = -\frac{1}{4}$

12. $a_1 = 9;\ r = -\frac{1}{3}$

For Exercises 13–18, determine the common ratio, r, for the geometric sequence.

13. $5, 10, 20, 40, \ldots$

14. $-2, -1, -\frac{1}{2}, -\frac{1}{4}, \ldots$

15. $8, -2, \frac{1}{2}, -\frac{1}{8}, \ldots$

16. $4, -12, 36, -108, \ldots$

17. $3, -6, 12, -24, \ldots$

18. $1, 4, 16, 64, \ldots$

For Exercises 19–24, write the first five terms of the geometric sequence.

19. $a_1 = -3, r = -2$

20. $a_1 = -4, r = -1$

21. $a_1 = 6, r = \frac{1}{2}$

22. $a_1 = 8, r = \frac{1}{4}$

23. $a_1 = -1, r = 6$

24. $a_1 = 2, r = -3$

For Exercises 25–30, find the nth term of each geometric sequence. **(See Example 1.)**

25. $3, 12, 48, 192, \ldots$

26. $2, 6, 18, 54, \ldots$

27. $-5, 15, -45, 135, \ldots$

28. $-6, 12, -24, 48, \ldots$

29. $\frac{1}{2}, 2, 8, 32, \ldots$

30. $\frac{16}{3}, 4, 3, \frac{9}{4}, \ldots$

For Exercises 31–40, find the indicated term of each geometric sequence. **(See Examples 2–3.)**

31. Given $a_n = 2(\frac{1}{2})^{n-1}$, find a_8.

32. Given $a_n = -3(\frac{1}{2})^{n-1}$, find a_8.

33. Given $a_n = 4(-\frac{3}{2})^{n-1}$, find a_6.

34. Given $a_n = 6(-\frac{1}{3})^{n-1}$, find a_6.

35. Given $a_n = -3(2)^{n-1}$, find a_5.

36. Given $a_n = 5(3)^{n-1}$, find a_4.

37. Given $a_5 = -\frac{16}{9}$ and $r = -\frac{2}{3}$, find a_1.

38. Given $a_6 = \frac{5}{16}$ and $r = -\frac{1}{2}$, find a_1.

39. Given $a_7 = 8$ and $r = 2$, find a_1.

40. Given $a_6 = 27$ and $r = 3$, find a_1.

41. If the second and third terms of a geometric sequence are 16 and 64, what is the first term?

42. If the second and third terms of a geometric sequence are $\frac{1}{3}$ and $\frac{1}{9}$, what is the first term?

Concept 2: Geometric Series

43. Explain the difference between a geometric sequence and a geometric series.

44. **a.** Write the series in expanded form. $\sum_{i=1}^{5} 4(2)^{i-1}$

b. Find the sum. $\sum_{i=1}^{5} 4(2)^{i-1}$

45. **a.** Write the series in expanded form. $\sum_{i=1}^{4} 3(4)^{i-1}$

b. Find the sum. $\sum_{i=1}^{4} 3(4)^{i-1}$

46. **a.** Write the series in expanded form. $\sum_{i=1}^{4} 6\left(\frac{1}{2}\right)^{i-1}$

b. Find the sum. $\sum_{i=1}^{4} 6\left(\frac{1}{2}\right)^{i-1}$

For Exercises 47–56, find the sum of the geometric series. **(See Examples 4–5.)**

47. $10 + 2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125}$

48. $1 + 3 + 9 + 27 + 81 + 243$

49. $-2 + 1 + \left(-\frac{1}{2}\right) + \frac{1}{4} + \left(-\frac{1}{8}\right)$

50. $\frac{1}{4} + (-1) + 4 + (-16) + 64$

51. $12 + 16 + \frac{64}{3} + \frac{256}{9} + \frac{1024}{27}$

52. $9 + 6 + 4 + \frac{8}{3} + \frac{16}{9}$

53. $1 + \frac{2}{3} + \frac{4}{9} + \cdots + \frac{32}{243}$

54. $\frac{8}{3} + 2 + \frac{3}{2} + \cdots + \frac{243}{512}$

55. $-4 + 8 + (-16) + \cdots + (-256)$

56. $-\frac{7}{3} + 7 + (-21) + \cdots + 5103$

57. A deposit of \$1000 is made in an account that earns 5% interest compounded annually. The balance in the account after n years is given by

$$a_n = 1000(1.05)^n \quad \text{for } n \geq 1$$

a. List the first four terms of the sequence. Round to the nearest cent.

b. Find the balance after 10 years, 20 years, and 40 years by computing a_{10}, a_{20}, and a_{40}. Round to the nearest cent.

58. A home purchased for \$125,000 increases by 4% of its value each year. The value of the home after n years is given by

$$a_n = 125{,}000(1.04)^n \quad \text{for } n \geq 1$$

a. List the first four terms of the sequence. Round to the nearest dollar.

b. Find the value of the home after 5 years, 10 years, and 20 years by computing a_5, a_{10}, and a_{20}. Round to the nearest dollar.

Steve Mason/Photodisc/Getty Images

For Exercises 59–64, first find the common ratio r. Then determine the sum of the infinite series, if it exists. **(See Example 6.)**

59. $1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \cdots$

60. $-2 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{8}\right) + \left(-\frac{1}{32}\right) + \cdots$

61. $\sum_{i=1}^{\infty} \left(-\frac{1}{4}\right)^{i-1}$

62. $\sum_{i=1}^{\infty} \left(-\frac{1}{5}\right)^{i-1}$

63. $\frac{2}{3} + (-1) + \frac{3}{2} + \left(-\frac{9}{4}\right) + \cdots$

64. $3 + 5 + \frac{25}{3} + \frac{125}{9} + \cdots$

65. Suppose \$200 million is spent by tourists at a certain resort town. Further suppose that 75% of the revenue is respent in the community and then respent over and over, each time at a rate of 75%. The series

$$200 + 200(0.75) + 200(0.75)^2 + 200(0.75)^3 + \cdots$$

gives the total amount spent (and respent) in the community. Find the sum of the infinite series. **(See Example 7.)**

66. A bungee jumper jumps off a platform and stretches the cord 80 ft before rebounding upward. Each successive bounce stretches the cord 60% of its previous length. The total vertical distance traveled is given by

$$80 + 2(0.60)(80) + 2(0.60)^2(80) + 2(0.60)^3(80) + \cdots$$

After the first term, the series is an infinite geometric series. Compute the total vertical distance traveled.

67. A ball drops from a height of 4 ft. With each bounce, the ball rebounds to $\frac{3}{4}$ of its height. The total vertical distance traveled is given by

$$4 + 2\left(\frac{3}{4}\right)(4) + 2\left(\frac{3}{4}\right)^2(4) + 2\left(\frac{3}{4}\right)^3(4) + \cdots$$

After the first term, the series is an infinite geometric series. Compute the total vertical distance traveled.

68. The repeating decimal number $0.\overline{2}$ can be written as an infinite geometric series by

$$\frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots$$

a. What is a_1? **b.** What is r? **c.** Find the sum of the series.

69. The repeating decimal number $0.\overline{7}$ can be written as an infinite geometric series by

$$\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots$$

a. What is a_1? **b.** What is r? **c.** Find the sum of the series.

Expanding Your Skills

70. The yearly salary for Job A is \$40,000 initially with an annual raise of \$3000 per year. The yearly salary for Job B is \$38,000 initially with an annual raise of 6% per year.

a. Find the total earnings for Job A over 20 years. Is this an arithmetic or geometric series?

b. For Job B, what is the amount of the raise after 1 year?

c. Find the total earnings for Job B over 20 years. Round to the nearest dollar. Is this an arithmetic or geometric series?

d. What is the difference in total salary earned over 20 years between Job A and Job B?

71. **a.** Brook has a job that pays \$48,000 the first year. She receives a 4% raise each year. Find the sum of her yearly salaries over a 20-year period. Round to the nearest dollar.

b. Chamille has a job that pays \$48,000 the first year. She receives a 4.5% raise each year. Find the sum of her yearly salaries over a 20-year period. Round to the nearest dollar.

c. Chamille's raise each year was 0.5% higher than Brook's raise. How much more total income did Chamille receive than Brook over 20 years?

Problem Recognition Exercises

Identifying Arithmetic and Geometric Sequences

For Exercises 1–18, determine if the sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find *d*. If the sequence is geometric, find *r*.

1. $5, -\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \ldots$

2. $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

3. $-2, -4, -8, -16, -32, \ldots$

4. $-2, -4, -7, -11, -16, \ldots$

5. $-\frac{1}{3}, \frac{1}{3}, 1, \frac{5}{3}, \ldots$

6. $1, -\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \ldots$

7. $2, 6, 11, 17, 24, \ldots$

8. $2, -2, 2, -2, 2, \ldots$

9. $-2, -4, -6, -8, -10, \ldots$

10. $1, 3, 1, 3, 1, \ldots$

11. $0, 1, 0, 1, 0, 1, \ldots$

12. $2, 6, 10, 14, 18, 22, \ldots$

13. $\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi, \cdots$

14. $-1, 1, -1, 1, -1, \ldots$

15. $1, 4, 9, 16, 25, \ldots$

16. $\frac{1}{3}e, \frac{2}{3}e, e, \frac{4}{3}e, \frac{5}{3}e, \cdots$

17. $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}, \cdots$

18. $2, 6, 18, 54, 162, \ldots$

Chapter 10 Summary

Section 10.1 Binomial Expansions

Key Concepts

For a whole number n, the expression $\boldsymbol{n!}$ (read as "n factorial") is defined as the product of integers from 1 to n, for $n \geq 1$.

$n! = n(n-1)(n-2)\ldots(2)(1)$

$0! = 1$ by definition

For a positive integer, n, the expression $(a+b)^n$ can be expanded using the **binomial theorem**:

$$(a+b)^n = \frac{n!}{n!\cdot 0!}a^n + \frac{n!}{(n-1)!\cdot 1!}a^{(n-1)}b + \frac{n!}{(n-2)!\cdot 2!}a^{(n-2)}b^2 + \cdots + \frac{n!}{0!\cdot n!}b^n$$

The coefficients of the expansion may also be found by using **Pascal's triangle**.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Let n and k be positive integers such that $k \leq n$. Then the kth term in the expansion of $(a+b)^n$ is

$$\frac{n!}{[n-(k-1)]!\cdot(k-1)!}\cdot a^{n-(k-1)}\cdot b^{k-1}$$

Examples

Example 1

$6! = (6)(5)(4)(3)(2)(1) = 720$

Example 2

$$\frac{7!}{2!\cdot 5!} = \frac{7\cdot \overset{3}{\cancel{6}}\cdot \cancel{5\cdot 4\cdot 3\cdot 2\cdot 1}}{(\cancel{2}\cdot 1)(\cancel{5\cdot 4\cdot 3\cdot 2\cdot 1})} = 21$$

Example 3

$(2x-4y)^4 = [2x+(-4y)]^4$

$$= \frac{4!}{4!\cdot 0!}(2x)^4 + \frac{4!}{3!\cdot 1!}(2x)^3(-4y) + \frac{4!}{2!\cdot 2!}(2x)^2(-4y)^2 + \frac{4!}{1!\cdot 3!}(2x)(-4y)^3 + \frac{4!}{0!\cdot 4!}(-4y)^4$$

$$= 1(2x)^4 + 4(2x)^3(-4y) + 6(2x)^2(-4y)^2 + 4(2x)(-4y)^3 + 1(-4y)^4$$

$$= 16x^4 - 128x^3y + 384x^2y^2 - 512xy^3 + 256y^4$$

Example 4

The third term of $(3x+y^2)^7$ is

$$\frac{7!}{[7-(3-1)]!\cdot(3-1)!}\cdot(3x)^{7-(3-1)}\cdot(y^2)^{3-1}$$

$$= \frac{7!}{5!\cdot 2!}(3x)^5(y^2)^2$$

$$= 5103x^5y^4$$

Section 10.2 Sequences and Series

Key Concepts

An **infinite sequence** is a function whose domain is the set of positive integers.

A **finite sequence** is a function whose domain is the set of the first n positive integers.

The expression $\boldsymbol{a_n}$ defines the nth term (or general term) of a sequence.

A **series** is the sum of a sequence.

Examples

Example 1

$$a_n = 8 - 3n$$
$$= 5, 2, -1, -4, -7, -10, \ldots$$

Example 2

$$a_n = (-1)^n \frac{n}{n+1} \qquad \text{for } 1 \le n \le 5$$
$$= -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}$$

Example 3

$$\sum_{i=1}^{4} 5i = 5 + 10 + 15 + 20 = 50$$

Section 10.3 Arithmetic Sequences and Series

Key Concepts

An **arithmetic sequence** is a sequence in which the **common difference,** d, between two consecutive terms is a constant.

The nth term of an arithmetic sequence is given by

$$a_n = a_1 + (n-1)\,d$$

This formula has four variables, a_n, a_1, n, and d. Sometimes it is necessary to solve for one or more of the variables before the nth term can be identified.

Examples

Example 1

Write the nth term of the sequence:
10, 8, 6, 4, . . .

$a_1 = 10$ (the first term)

$d = -2$ (the common difference)

$a_n = 10 + (n-1)\cdot(-2)$

$a_n = -2n + 12$

Example 2

Find the 12th term of the arithmetic sequence in which $a_1 = -6$ and $a_{20} = 51$.

$a_n = a_1 + (n-1)d$ Find d first.

$51 = -6 + (20-1)d$ Substitute known values for a_1 and a_{20}.

$57 = 19d$

$3 = d$

$a_n = -6 + (n-1)\cdot 3 \longrightarrow a_{12} = -6 + (12-1)\cdot 3$

$a_{12} = 27$

The sum of an arithmetic sequence is an **arithmetic series**. The sum of the first n terms of an arithmetic series is given by

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Example 3

Find the sum. $\sum_{i=1}^{50}(3i - 1)$

$n = 50, a_1 = 2, a_{50} = 149$

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{50}{2}(2 + 149)$$
$$= 3775$$

Section 10.4 Geometric Sequences and Series

Key Concepts

A **geometric sequence** is a sequence in which the ratio between each term and its predecessor is constant. This constant is called the **common ratio**, ***r***.

The nth term of a geometric sequence is given by

$$a_n = a_1 r^{n-1}$$

The sum of a geometric sequence is called a **geometric series**. The sum of the first n terms of a geometric series is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

The sum of an infinite geometric series is given by

$$S_n = \frac{a_1}{1 - r} \quad \text{provided } |r| < 1$$

Examples

Example 1

Find the common ratio: $\frac{8}{5}, \frac{6}{5}, \frac{9}{10}, \frac{27}{40}, \ldots$

$$r = \frac{6}{5} \div \frac{8}{5} = \frac{6}{5} \cdot \frac{5}{8} = \frac{30}{40} = \frac{3}{4}$$

Example 2

Find the fourth term.

$$a_n = 3\left(\frac{2}{3}\right)^{n-1} \longrightarrow a_4 = 3\left(\frac{2}{3}\right)^{4-1} = 3\left(\frac{2}{3}\right)^3 = \frac{8}{9}$$

Example 3

Find the sum. $48 + 24 + 12 + 6 + 3 + \frac{3}{2}$

$a_1 = 48, r = \frac{1}{2}, n = 6$

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{48\left[1 - \left(\frac{1}{2}\right)^6\right]}{1 - \frac{1}{2}} = \frac{189}{2}$$

Example 4

Find the sum of the series: $4 - 2 + 1 - \frac{1}{2} + \cdots$

$a_1 = 4$ and the common ratio is $-2 \div 4 = -\frac{1}{2}$.

Since $\left|-\frac{1}{2}\right| < 1$,

$$S_n = \frac{a_1}{1 - r} = \frac{4}{1 - \left(-\frac{1}{2}\right)} = \frac{4}{\frac{3}{2}} = \frac{8}{3}$$

Chapter 10 Review Exercises

Section 10.1

For Exercises 1–4, simplify the expression.

1. $8!$ **2.** $5!$ **3.** $\dfrac{12!}{10! \cdot 2!}$ **4.** $\dfrac{9!}{6! \cdot 3!}$

For Exercises 5–6, expand the binomial.

5. $(x^2 + 4)^5$

6. $(c - 3d)^4$

7. Find the first three terms of the binomial expansion. $(a - 2b)^{11}$

8. Find the fourth term of the binomial expansion. $(x^2 + 3y)^8$

9. Find the eighth term of the binomial expansion. $(5x - y^3)^{10}$

10. Find the middle term of the binomial expansion. $(a + 2b)^6$

Section 10.2

For Exercises 11–14, write the terms of the sequence.

11. $a_n = -3n + 4;\ 1 \le n \le 5$

12. $a_n = -2n^3;\ 1 \le n \le 3$

13. $a_n = (-1)^{n+1} \cdot \dfrac{n}{n+2};\ 1 \le n \le 4$

14. $a_n = \left(-\dfrac{2}{3}\right)^n;\ 1 \le n \le 4$

For Exercises 15–16, write a formula for the nth term of the sequence.

15. $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$

16. $-\dfrac{3}{2}, \dfrac{3}{4}, -\dfrac{3}{8}, \dfrac{3}{16}, \ldots$

17. What is the index of summation for the series?

$$\sum_{k=1}^{7} (2n + 3)$$

18. How many terms are in the series? $\displaystyle\sum_{j=1}^{5} j^3$

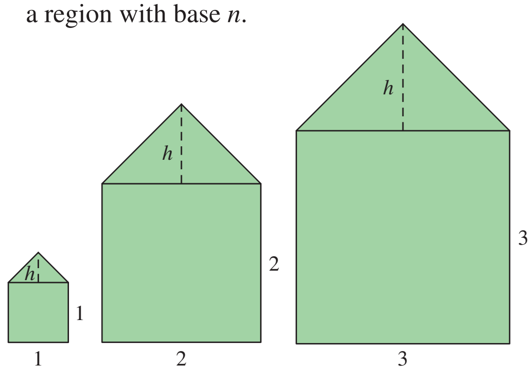
For Exercises 19–20, find the sum of the series.

19. $\displaystyle\sum_{i=1}^{7} 5(-1)^{i-1}$ **20.** $\displaystyle\sum_{n=1}^{5} (-2)^n$

21. Write the sum in summation notation.

$$\frac{4}{1} + \frac{5}{2} + \frac{6}{3} + \frac{7}{4} + \frac{8}{5} + \frac{9}{6} + \frac{10}{7}$$

22. Study the figures and the corresponding expression for the area of each region. Write the nth term of a sequence representing the area of a region with base n.



$n = 1$ $\quad$ $n = 2$ $\quad$ $n = 3$

$A = 1 + \frac{1}{2}h$ $\quad$ $A = 4 + h$ $\quad$ $A = 9 + \frac{3}{2}h$

Section 10.3

For Exercises 23–24, write the first five terms of the arithmetic sequence.

23. $a_1 = -12,\ d = 1.5$ **24.** $a_1 = \dfrac{2}{3},\ d = \dfrac{1}{3}$

For Exercises 25–26, write an expression for the nth term of the sequence.

25. $1, 11, 21, 31, \ldots$ **26.** $6, -8, -22, -36, \ldots$

For Exercises 27–28, find the indicated term of the arithmetic sequence.

27. $a_n = \dfrac{1}{2} + (n - 1) \cdot \dfrac{1}{4};$ 17th term

28. $a_1 = -11,\ d = -6;$ 25th term

For Exercises 29–30, find the number of terms.

29. $3, 8, 13, 18, \ldots, 118$

30. $4, 1, -2, -5, \ldots, -50$

31. Find the common difference for an arithmetic sequence, given $a_1 = -40$ and $a_{19} = 140$.

32. Write an expression for the nth term of the arithmetic sequence, 14, 9, 4, −1, −6,….

For Exercises 33–36, find the sum of the arithmetic series.

33. $\sum_{k=1}^{40}(5-k)$

34. $\sum_{i=1}^{20}(0.25i+2)$

35. $-6+(-4)+(-2)+\cdots+34$

36. $4+5+6+\cdots+109$

Section 10.4

For Exercises 37–38, find the common ratio.

37. 5, 15, 45, 135, . . .

38. 2, 2.4, 2.88, 3.456

For Exercises 39–40, write the first four terms of the geometric series.

39. $a_1 = -1$ and $r = -\frac{1}{4}$

40. $a_n = 10(2)^{n-1}$

For Exercises 41–42, write an expression for the nth term of the geometric series.

41. −4, −8, −16, −32, …

42. $6, -2, \frac{2}{3}, -\frac{2}{9}, \ldots$

43. Given $a_n = 4\left(\frac{2}{3}\right)^{n-1}$, find a_6.

44. Given $a_n = 10(3)^{n-1}$, find a_4.

45. Given $a_7 = \frac{1}{16}$ and $r = \frac{1}{2}$, find a_1.

46. Given $a_5 = 81$ and $r = 3$, find a_1.

47. Find the sum of the series. $\sum_{n=1}^{8} 5(2)^{n-1}$

48. Find the sum of the series.

$$\frac{1}{3}+(-1)+3+(-9)+\cdots+(-81)$$

49. Find the sum of the series.

$$6+4+\frac{8}{3}+\frac{16}{9}+\cdots$$

50. Find the sum of the series.

$$10+2+\frac{2}{5}+\frac{2}{25}+\cdots$$

51. An investment of \$10,000 is originally invested in a mutual fund. After 10 years, the yearly return averaged approximately 7%. The balance on the account is given by $b_n = 10{,}000(1.07)^n$.

a. Find the first three terms of the sequence.

b. Interpret the value of b_3.

c. Find the amount in the account after 10 years.

Chapter 10 Test

For Exercises 1–2, simplify.

1. 0!

2. $\frac{10!}{6!\cdot 4!}$

3. Write the expansion of $(a+b)^4$. Use Pascal's triangle to determine the coefficients of the expansion.

4. Expand. $(3y-2x^2)^4$

5. Find the sixth term. $(a-c^3)^8$

6. Write the terms of the sequence.

$$a_n = -\frac{3}{n+2};\ 1 \le n \le 4$$

7. Find the sum. $\sum_{i=1}^{5}(i^2+2)$

8. a. An 8-in. tomato seedling is planted on Sunday. Each day, it grows an average of 1.5 in. Write five terms representing the height of the plant each day, beginning with Sunday.

b. Write the nth term of the sequence.

9. Write the series in summation notation.
$x^3 + x^6 + x^9 + x^{12}$

10. Find the common difference. $3, \frac{13}{4}, \frac{7}{2}, \frac{15}{4}, \ldots$

11. Find the common ratio. $9, 3, 1, \frac{1}{3}, \ldots$

12. Given an arithmetic sequence with $a_1 = -15$ and $d = -3$,

a. Write the first four terms of the sequence.

b. Write an expression for the nth term of the sequence.

c. Find the 40th term of the sequence.

13. Given a geometric sequence with $a_1 = 4$ and $r = -2$,

a. Write the first four terms of the sequence.

b. Write an expression for the nth term of the sequence.

c. Find the 10th term of the sequence.

14. Write an expression for the nth term of the sequence. $\frac{3}{5}, \frac{3}{10}, \frac{3}{20}, \frac{3}{40}, \ldots$

15. Write an expression for the nth term of the sequence. $20, 18, 16, 14, \ldots$

16. Find the number of terms in the sequence.
$3, 10, 17, 24, \ldots, 213$

17. Find the number of terms in the sequence.
$1, \frac{2}{3}, \frac{4}{9}, \ldots, \frac{64}{729}$

18. Find the sum of the arithmetic series.
$$\sum_{i=1}^{100}(8 + 2i)$$

19. Find the sum of the geometric series.
$$\sum_{n=1}^{10} 8\left(\frac{1}{2}\right)^{n-1}$$

20. Find the sum of the infinite geometric series.
$4 + 2 + 1 + \frac{1}{2} + \cdots$

21. Given a geometric series with $a_6 = 9$ and $r = 3$, find a_1.

22. Find the 18th term of the arithmetic sequence in which $a_1 = -66$ and $a_{40} = -378$.

23. Suppose a student smokes one pack of cigarettes per day at a cost of \$5.50 per pack.

a. If the student quits smoking and puts the \$5.50 saved per day in a jar, how much money will the student have saved at the end of 1 year?

b. If the student takes the first year's savings from part (a) and invests the money in a bond fund paying 4% annual interest, how much will this money be worth 1 year later?

c. Suppose that the student reinvests the principal and interest each year so that the money is compounded annually. The student will have:

End of year 1: \$2007.50

End of year 2: $(2007.50)(1.04) = \$2087.80$

End of year 3: $(2007.50)(1.04)^2 = \$2171.31$

The sequence $a_n = \$2007.50(1.04)^{n-1}$ represents the value of the investment at the end of n years. Find the value of the investment after 30 years. (Round to two decimal places.)

d. Now suppose that the student saves \$2007.50 per year each year for 30 years. Further suppose that the student invests each year's saving at 4% interest compounded annually. The following series represents the total savings plus interest over 30 years. Find the sum.

$$\sum_{n=1}^{30} 2007.5(1.04)^{n-1}$$

Additional Topics Appendix

Determinants and Cramer's Rule

Section A.1

Concepts

1. Introduction to Determinants
2. Determinant of a 3×3 Matrix
3. Cramer's Rule

1. Introduction to Determinants

Associated with every square matrix is a real number called the **determinant** of the matrix. The determinant of a square matrix **A**, denoted det **A**, is written by enclosing the elements of the matrix within two vertical bars. For example,

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 2 & -1 \\ 6 & 0 \end{bmatrix} \qquad \text{then} \quad \det \mathbf{A} = \begin{vmatrix} 2 & -1 \\ 6 & 0 \end{vmatrix}$$

$$\text{If} \quad \mathbf{B} = \begin{bmatrix} 0 & -5 & 1 \\ 4 & 0 & \frac{1}{2} \\ -2 & 10 & 1 \end{bmatrix} \qquad \text{then} \quad \det \mathbf{B} = \begin{vmatrix} 0 & -5 & 1 \\ 4 & 0 & \frac{1}{2} \\ -2 & 10 & 1 \end{vmatrix}$$

Determinants have many applications in mathematics, including solving systems of linear equations, finding the area of a triangle, determining whether three points are collinear, and finding an equation of a line between two points.

The determinant of a 2×2 matrix is defined as follows:

Definition of a Determinant of a 2×2 Matrix

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the real number $ad - bc$. It is written as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1 Evaluating a 2×2 Determinant

Evaluate the determinants.

a. $\begin{vmatrix} 6 & -2 \\ 5 & \frac{1}{3} \end{vmatrix}$ **b.** $\begin{vmatrix} 2 & -11 \\ 0 & 0 \end{vmatrix}$

Solution:

a. $\begin{vmatrix} 6 & -2 \\ 5 & \frac{1}{3} \end{vmatrix}$ For this determinant, $a = 6$, $b = -2$, $c = 5$, and $d = \frac{1}{3}$.

$$\begin{aligned} ad - bc &= (6)\left(\frac{1}{3}\right) - (-2)(5) \\ &= 2 + 10 \\ &= 12 \end{aligned}$$

TIP: Example 1(b) illustrates that the value of a determinant having a row of all zeros is 0. The same is true for a determinant having a column of all zeros.

b. $\begin{vmatrix} 2 & -11 \\ 0 & 0 \end{vmatrix}$ For this determinant, $a = 2$, $b = -11$, $c = 0$, $d = 0$.

$$\begin{aligned} ad - bc &= (2)(0) - (-11)(0) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

Skill Practice Evaluate the determinants.

1. $\begin{vmatrix} 2 & 8 \\ -1 & 5 \end{vmatrix}$ **2.** $\begin{vmatrix} -6 & 0 \\ 4 & 0 \end{vmatrix}$

2. Determinant of a 3 × 3 Matrix

To find the determinant of a 3×3 matrix, we first need to define the **minor** of an element of the matrix. For any element of a 3×3 matrix, the minor of that element is the determinant of the 2×2 matrix obtained by deleting the row and column in which the element resides. For example, consider the matrix

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix}$$

The minor of the element 5 is found by deleting the first row and first column and then evaluating the determinant of the remaining 2×2 matrix:

$\begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix}$ Now evaluate the determinant: $\begin{vmatrix} -7 & 1 \\ 2 & 6 \end{vmatrix} = (-7)(6) - (1)(2)$

$= -44$

For this matrix, the minor for the element 5 is -44.

To find the minor of the element -7, delete the second row and second column, and then evaluate the determinant of the remaining 2×2 matrix.

$\begin{bmatrix} 5 & -1 & 6 \\ 0 & -7 & 1 \\ 4 & 2 & 6 \end{bmatrix}$ Now evaluate the determinant: $\begin{vmatrix} 5 & 6 \\ 4 & 6 \end{vmatrix} = (5)(6) - (6)(4) = 6$

For this matrix, the minor for the element -7 is 6.

Example 2 Determining the Minor for Elements in a 3 × 3 Matrix

Find the minor for each element in the first column of the matrix.

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$$

Solution:

For 3: $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$ The minor is: $\begin{vmatrix} -4 & 5 \\ 1 & -6 \end{vmatrix} = (-4)(-6) - (5)(1) = 19$

Answers

1. 18 **2.** 0

For 2: $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$ The minor is: $\begin{vmatrix} 4 & -1 \\ 1 & -6 \end{vmatrix} = (4)(-6) - (-1)(1) = -23$

For 0: $\begin{bmatrix} 3 & 4 & -1 \\ 2 & -4 & 5 \\ 0 & 1 & -6 \end{bmatrix}$ The minor is: $\begin{vmatrix} 4 & -1 \\ -4 & 5 \end{vmatrix} = (4)(5) - (-1)(-4) = 16$

Skill Practice Find the minor for the element 3.

3. $\begin{bmatrix} -1 & 8 & -6 \\ \frac{1}{2} & 3 & 2 \\ 5 & 7 & 4 \end{bmatrix}$

The determinant of a 3×3 matrix is defined as follows:

Definition of a Determinant of a 3×3 Matrix

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

From this definition, we see that the determinant of a 3×3 matrix can be written as

$$a_1 \cdot (\text{minor of } a_1) - a_2 \cdot (\text{minor of } a_2) + a_3 \cdot (\text{minor of } a_3)$$

Evaluating determinants in this way is called *expanding minors*.

Example 3 Evaluating a 3×3 Determinant

Evaluate the determinant. $\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix} = 2 \cdot \begin{vmatrix} -3 & 0 \\ 5 & -1 \end{vmatrix} - (1) \cdot \begin{vmatrix} 4 & 2 \\ 5 & -1 \end{vmatrix} + (-5) \cdot \begin{vmatrix} 4 & 2 \\ -3 & 0 \end{vmatrix}$$

$= 2[(-3)(-1) - (0)(5)] - 1[(4)(-1) - (2)(5)] - 5[(4)(0) - (2)(-3)]$

$= 2(3) - 1(-14) - 5(6)$

$= 6 + 14 - 30$

$= -10$

Skill Practice Evaluate the determinant.

4. $\begin{vmatrix} -2 & 4 & 9 \\ 5 & -1 & 2 \\ 1 & 1 & 6 \end{vmatrix}$

Answers

3. $\begin{vmatrix} -1 & -6 \\ 5 & 4 \end{vmatrix} = 26$ **4.** -42

TIP: There is another method to determine the signs for each term of the expansion. For the a_{ij} element, multiply the term by $(-1)^{i+j}$.

Although we defined the determinant of a matrix by expanding the minors of the elements in the first column, *any row or column can be used*. However, we must choose the correct sign to apply to each term in the expansion. The following array of signs is helpful.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

The signs alternate for each row and column, beginning with + in the first row, first column.

Example 4 Evaluating a 3 × 3 Determinant

Evaluate the determinant by expanding minors about the elements in the second row.

$$\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix}$$

Solution:

Signs obtained from the array of signs

$$\begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ -5 & 5 & -1 \end{vmatrix} = -(1)\cdot\begin{vmatrix} 4 & 2 \\ 5 & -1 \end{vmatrix} + (-3)\cdot\begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} - (0)\cdot\begin{vmatrix} 2 & 4 \\ -5 & 5 \end{vmatrix}$$

$$= -1[(4)(-1) - (2)(5)] - 3[(2)(-1) - (2)(-5)] - 0$$

$$= -1(-14) - 3(8)$$

$$= 14 - 24$$

$$= -10$$

Notice that the value of the determinant is the same as the result obtained in Example 3.

Skill Practice Evaluate the determinant.

5. $\begin{vmatrix} 4 & -1 & 2 \\ 3 & 6 & -8 \\ 0 & \frac{1}{2} & 5 \end{vmatrix}$

In Example 4, the third term in the expansion of minors was zero because the element 0 when multiplied by its minor is zero. To simplify the arithmetic in evaluating a determinant of a 3 × 3 matrix, expand about the row or column that has the most 0 elements.

3. Cramer's Rule

In this section, we will learn another method to solve a system of linear equations. This method is called **Cramer's rule**.

Cramer's Rule for a 2 × 2 System of Linear Equations

The solution to the system $\quad a_1x + b_1y = c_1$

$\quad a_2x + b_2y = c_2$

is given by $\quad x = \dfrac{\mathbf{D}_x}{\mathbf{D}} \quad$ and $\quad y = \dfrac{\mathbf{D}_y}{\mathbf{D}}$

where $\mathbf{D} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ (and $\mathbf{D} \neq 0$) $\quad \mathbf{D}_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \mathbf{D}_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Answer

5. 154

Example 5 Using Cramer's Rule to Solve a 2 × 2 System of Linear Equations

Solve the system by using Cramer's rule.

$$3x - 5y = 11$$
$$-x + 3y = -5$$

Solution:

For this system: $a_1 = 3 \quad b_1 = -5 \quad c_1 = 11$

$a_2 = -1 \quad b_2 = 3 \quad c_2 = -5$

$$\mathbf{D} = \begin{vmatrix} 3 & -5 \\ -1 & 3 \end{vmatrix} = (3)(3) - (-5)(-1) = 9 - 5 = 4$$

$$\mathbf{D}_x = \begin{vmatrix} 11 & -5 \\ -5 & 3 \end{vmatrix} = (11)(3) - (-5)(-5) = 33 - 25 = 8$$

$$\mathbf{D}_y = \begin{vmatrix} 3 & 11 \\ -1 & -5 \end{vmatrix} = (3)(-5) - (11)(-1) = -15 + 11 = -4$$

Therefore, $x = \dfrac{\mathbf{D}_x}{\mathbf{D}} = \dfrac{8}{4} = 2 \qquad y = \dfrac{\mathbf{D}_y}{\mathbf{D}} = \dfrac{-4}{4} = -1$

Check the ordered pair $(2, -1)$ in both original equations.

<u>Check</u>: $3x - 5y = 11 \longrightarrow 3(2) - 5(-1) \stackrel{?}{=} 11$ ✓

$-x + 3y = -5 \longrightarrow -(2) + 3(-1) \stackrel{?}{=} -5$ ✓

The solution set is $\{(2, -1)\}$.

Skill Practice Solve using Cramer's rule.

6. $2x + y = 5$
$-x - 3y = 5$

TIP: Here are some memory tips to help you remember Cramer's rule to solve:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Coefficients of
x terms y terms

$$\mathbf{D} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

1. The determinant $\mathbf{D}$ is the determinant of the coefficients of x and y.

x coefficients
replaced by c_1 and c_2

$$\mathbf{D}_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

2. The determinant $\mathbf{D}_x$ has the column of x-term coefficients replaced by c_1 and c_2.

y coefficients
replaced by c_1 and c_2

$$\mathbf{D}_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

3. The determinant $\mathbf{D}_y$ has the column of y-term coefficients replaced by c_1 and c_2.

Answer

6. $\{(4, -3)\}$

It is important to note that the linear equations must be written in standard form to apply Cramer's rule.

Example 6 **Using Cramer's Rule to Solve a 2 × 2 System of Linear Equations**

Solve the system by using Cramer's rule.

$$-16y = -40x - 7$$
$$40y = 24x + 27$$

Solution:

$$\begin{aligned} -16y &= -40x - 7 &\longrightarrow\quad 40x - 16y &= -7 \\ 40y &= 24x + 27 &\longrightarrow\quad -24x + 40y &= 27 \end{aligned}$$

Rewrite each equation in standard form.

For this system: $a_1 = 40$ $\quad b_1 = -16$ $\quad c_1 = -7$

$a_2 = -24$ $\quad b_2 = 40$ $\quad c_2 = 27$

$$\mathbf{D} = \begin{vmatrix} 40 & -16 \\ -24 & 40 \end{vmatrix} = (40)(40) - (-16)(-24) = 1216$$

$$\mathbf{D}_x = \begin{vmatrix} -7 & -16 \\ 27 & 40 \end{vmatrix} = (-7)(40) - (-16)(27) = 152$$

$$\mathbf{D}_y = \begin{vmatrix} 40 & -7 \\ -24 & 27 \end{vmatrix} = (40)(27) - (-7)(-24) = 912$$

Therefore, $x = \dfrac{\mathbf{D}_x}{\mathbf{D}} = \dfrac{152}{1216} = \dfrac{1}{8}$ $\qquad y = \dfrac{\mathbf{D}_y}{\mathbf{D}} = \dfrac{912}{1216} = \dfrac{3}{4}$

The ordered pair $\left(\frac{1}{8}, \frac{3}{4}\right)$ checks in both original equations.

The solution set is $\left\{\left(\frac{1}{8}, \frac{3}{4}\right)\right\}$.

Skill Practice Solve by using Cramer's rule.

7. $9x = 12y - 8$
$30y = -18x - 7$

Cramer's rule can be used to solve a 3 × 3 system of linear equations by using a similar pattern of determinants.

Cramer's Rule for a 3 × 3 System of Linear Equations

The solution to the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

is given by $\quad x = \dfrac{\mathbf{D}_x}{\mathbf{D}} \quad y = \dfrac{\mathbf{D}_y}{\mathbf{D}} \quad$ and $\quad z = \dfrac{\mathbf{D}_z}{\mathbf{D}}$

Answer

7. $\left\{\left(-\frac{2}{3}, \frac{1}{6}\right)\right\}$

Where $\mathbf{D} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (and $\mathbf{D} \neq 0$) $\quad \mathbf{D}_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$$\mathbf{D}_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \mathbf{D}_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Example 7 Using Cramer's Rule to Solve a 3 × 3 System of Linear Equations

Solve the system by using Cramer's rule.

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ -5x + 7y - 2z &= -6 \\ 9x - 2y + 3z &= 4 \end{aligned}$$

Solution:

$$\mathbf{D} = \begin{vmatrix} 2 & -3 & 5 \\ -5 & 7 & -2 \\ 9 & -2 & 3 \end{vmatrix} = 2 \cdot \begin{vmatrix} 7 & -2 \\ -2 & 3 \end{vmatrix} - (-5) \cdot \begin{vmatrix} -3 & 5 \\ -2 & 3 \end{vmatrix} + 9 \cdot \begin{vmatrix} -3 & 5 \\ 7 & -2 \end{vmatrix}$$

$$= 2(17) + 5(1) + 9(-29)$$

$$= -222$$

TIP: In Example 7, we expanded the determinants about the first column.

$$\mathbf{D}_x = \begin{vmatrix} 11 & -3 & 5 \\ -6 & 7 & -2 \\ 4 & -2 & 3 \end{vmatrix} = 11 \cdot \begin{vmatrix} 7 & -2 \\ -2 & 3 \end{vmatrix} - (-6) \cdot \begin{vmatrix} -3 & 5 \\ -2 & 3 \end{vmatrix} + 4 \cdot \begin{vmatrix} -3 & 5 \\ 7 & -2 \end{vmatrix}$$

$$= 11(17) + 6(1) + 4(-29)$$

$$= 77$$

$$\mathbf{D}_y = \begin{vmatrix} 2 & 11 & 5 \\ -5 & -6 & -2 \\ 9 & 4 & 3 \end{vmatrix} = 2 \cdot \begin{vmatrix} -6 & -2 \\ 4 & 3 \end{vmatrix} - (-5) \cdot \begin{vmatrix} 11 & 5 \\ 4 & 3 \end{vmatrix} + 9 \cdot \begin{vmatrix} 11 & 5 \\ -6 & -2 \end{vmatrix}$$

$$= 2(-10) + 5(13) + 9(8)$$

$$= 117$$

$$\mathbf{D}_z = \begin{vmatrix} 2 & -3 & 11 \\ -5 & 7 & -6 \\ 9 & -2 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 7 & -6 \\ -2 & 4 \end{vmatrix} - (-5) \cdot \begin{vmatrix} -3 & 11 \\ -2 & 4 \end{vmatrix} + 9 \cdot \begin{vmatrix} -3 & 11 \\ 7 & -6 \end{vmatrix}$$

$$= 2(16) + 5(10) + 9(-59)$$

$$= -449$$

$$x = \frac{\mathbf{D}_x}{\mathbf{D}} = \frac{77}{-222} = -\frac{77}{222}$$

$$y = \frac{\mathbf{D}_y}{\mathbf{D}} = \frac{117}{-222} = -\frac{39}{74}$$

$$z = \frac{\mathbf{D}_z}{\mathbf{D}} = \frac{-449}{-222} = \frac{449}{222}$$

The solution $\left(-\frac{77}{222}, -\frac{39}{74}, \frac{449}{222}\right)$ checks in each of the original equations.

The solution set is $\left\{\left(-\frac{77}{222}, -\frac{39}{74}, \frac{449}{222}\right)\right\}$.

Skill Practice Solve by using Cramer's rule.

8. $x + 3y - 3z = -14$
$x - 4y + z = 2$
$x + y + 2z = 6$

Cramer's rule may seem cumbersome for solving a 3×3 system of linear equations. However, it provides convenient formulas that can be programmed into a computer or calculator to solve for x, y, and z. Cramer's rule can also be extended to solve a 4×4 system of linear equations, a 5×5 system of linear equations, and in general an $n \times n$ system of linear equations.

It is important to remember that Cramer's rule does not apply if $\mathbf{D} = 0$. In such a case, either the equations are dependent or the system is inconsistent, and another method may be needed to analyze the system.

Example 8 Analyzing a Dependent System of Equations

Solve the system. Use Cramer's rule if possible.

$$2x - 3y = 6$$
$$-6x + 9y = -18$$

Solution:

$$\mathbf{D} = \begin{vmatrix} 2 & -3 \\ -6 & 9 \end{vmatrix} = (2)(9) - (-3)(-6) = 18 - 18 = 0$$

Because $\mathbf{D} = 0$, Cramer's rule does not apply. Using the addition method to solve the system, we have

$$2x - 3y = 6 \xrightarrow{\text{Multiply by 3.}} 6x - 9y = 18$$
$$-6x + 9y = -18 \longrightarrow \underline{-6x + 9y = -18}$$
$$0 = 0 \quad \text{The equations are dependent.}$$

The solution set is $\{(x, y) \mid 2x - 3y = 6\}$.

Skill Practice Solve. Use Cramer's rule if possible.

9. $x - 6y = 2$
$2x - 12y = -2$

TIP: When Cramer's rule does not apply, that is, when $\mathbf{D} = 0$, you may also use the substitution method or the Gauss-Jordan method to get a solution.

Answers

8. $\{(-2, 0, 4)\}$
9. { }; inconsistent system

TIP: In a 2×2 system of equations, if $\mathbf{D} = 0$, then the equations are dependent or the system is inconsistent.

- If $\mathbf{D} = 0$ and both $\mathbf{D}_x = 0$ and $\mathbf{D}_y = 0$, then the equations are dependent and the system has infinitely many solutions.
- If $\mathbf{D} = 0$ and either $\mathbf{D}_x \neq 0$ or $\mathbf{D}_y \neq 0$, then the system is inconsistent and has no solution.

Practice Exercises

Section A.1

Vocabulary and Key Concepts

1. **a.** Given the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the __________ of $\mathbf{A}$ is denoted det $\mathbf{A}$ and is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. The value of det $\mathbf{A}$ is the real number equal to __________.

b. Given a 3×3 matrix, the __________ of an element in the matrix is the determinant of the 2×2 matrix formed by deleting the row and column in which the element resides.

c. Complete the expression on the right to represent the value of the 3×3 determinant shown here.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - \square \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix}$$

Concept 1: Introduction to Determinants

For Exercises 2–7, evaluate the determinant of the 2×2 matrix. **(See Example 1.)**

2. $\begin{vmatrix} -3 & 1 \\ 5 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix}$

4. $\begin{vmatrix} -2 & 2 \\ -3 & -5 \end{vmatrix}$

5. $\begin{vmatrix} 5 & -1 \\ 1 & 0 \end{vmatrix}$

6. $\begin{vmatrix} \frac{1}{2} & 3 \\ -2 & 4 \end{vmatrix}$

7. $\begin{vmatrix} -3 & \frac{1}{4} \\ 8 & -2 \end{vmatrix}$

Concept 2: Determinant of a 3 × 3 Matrix

For Exercises 8–11, evaluate the minor corresponding to the given element from matrix **A.** **(See Example 2.)**

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 8 \\ 2 & 6 & 0 \\ -7 & 5 & 3 \end{bmatrix}$$

8. 4

9. -1

10. 2

11. 3

For Exercises 12–15, evaluate the minor corresponding to the given element from matrix **B.**

$$\mathbf{B} = \begin{bmatrix} -2 & 6 & 0 \\ 4 & -2 & 1 \\ 5 & 9 & -1 \end{bmatrix}$$

12. 6

13. 5

14. 1

15. 0

16. Construct the sign array for a 3×3 matrix.

17. Evaluate the determinant of matrix **B**, using expansion by minors. **(See Exercises 3–4.)**

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & -1 & 2 \\ 3 & 2 & -2 \end{bmatrix}$$

a. About the first column

b. About the second row

18. Evaluate the determinant of matrix **C**, using expansion by minors.

$$\mathbf{C} = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

a. About the first row

b. About the second column

19. When evaluating the determinant of a 3×3 matrix, explain the advantage of being able to choose any row or column about which to expand minors.

For Exercises 20–25, evaluate the determinant. **(See Examples 3–4.)**

20. $\begin{vmatrix} 8 & 2 & -4 \\ 4 & 0 & 2 \\ 3 & 0 & -1 \end{vmatrix}$

21. $\begin{vmatrix} 5 & 2 & 1 \\ 3 & -6 & 0 \\ -2 & 8 & 0 \end{vmatrix}$

22. $\begin{vmatrix} -2 & 1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{vmatrix}$

23. $\begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 4 \end{vmatrix}$

24. $\begin{vmatrix} -5 & 4 & 2 \\ 0 & 0 & 0 \\ 3 & -1 & 5 \end{vmatrix}$

25. $\begin{vmatrix} 0 & 5 & -8 \\ 0 & -4 & 1 \\ 0 & 3 & 6 \end{vmatrix}$

For Exercises 26–31, evaluate the determinant.

26. $\begin{vmatrix} x & 3 \\ y & -2 \end{vmatrix}$

27. $\begin{vmatrix} a & 2 \\ b & 8 \end{vmatrix}$

28. $\begin{vmatrix} a & 5 & -1 \\ b & -3 & 0 \\ c & 3 & 4 \end{vmatrix}$

29. $\begin{vmatrix} x & 0 & 3 \\ y & -2 & 6 \\ z & -1 & 1 \end{vmatrix}$

30. $\begin{vmatrix} p & 0 & q \\ r & 0 & s \\ t & 0 & u \end{vmatrix}$

31. $\begin{vmatrix} f & e & 0 \\ d & c & 0 \\ b & a & 0 \end{vmatrix}$

Concept 3: Cramer's Rule

For Exercises 32–34, evaluate the determinants represented by **D**, $\mathbf{D}_x$, and $\mathbf{D}_y$.

32. $x - 4y = 2$
$3x + 2y = 1$

33. $4x + 6y = 9$
$-2x + y = 12$

34. $-3x + 8y = -10$
$5x + 5y = -13$

For Exercises 35–40, solve the system by using Cramer's rule. **(See Examples 5–6.)**

35. $2x + y = 3$
$x - 4y = 6$

36. $2x - y = -1$
$3x + y = 6$

37. $4y = x - 8$
$3x = -7y + 5$

38. $7x - 4 = -3y$
$5x = 4y + 9$

39. $4x - 3y = 5$
$2x + 5y = 7$

40. $2x + 3y = 4$
$6x - 12y = -5$

For Exercises 41–46, solve for the indicated variable by using Cramer's rule. **(See Example 7.)**

41. $2x - y + 3z = 9$
$x + 4y + 4z = 5$ for x
$3x + 2y + 2z = 5$

42. $x + 2y + 3z = 8$
$2x - 3y + z = 5$ for y
$3x - 4y + 2z = 9$

43. $3x - 2y + 2z = 5$
$6x + 3y - 4z = -1$ for z
$3x - y + 2z = 4$

44. $\begin{aligned} 4x + 4y - 3z &= 3 \\ 8x + 2y + 3y &= 0 \\ 4x - 4y + 6z &= -3 \end{aligned}$ for x

45. $\begin{aligned} 5x \qquad + 6z &= 5 \\ -2x + y \qquad &= -6 \\ 3y - z &= 3 \end{aligned}$ for y

46. $\begin{aligned} 8x + y \qquad &= 1 \\ 7y + z &= 0 \\ x \qquad - 3z &= -2 \end{aligned}$ for y

47. When does Cramer's rule not apply in solving a system of equations?

48. How can a system be solved if Cramer's rule does not apply?

For Exercises 49–58, solve the system by using Cramer's rule, if possible. Otherwise, use another method. If a system does not have a unique solution, determine the number of solutions and whether the system is inconsistent or the equations are dependent. **(See Example 8.)**

49. $\begin{aligned} 4x - 2y &= 3 \\ -2x + y &= 1 \end{aligned}$

50. $\begin{aligned} 6x - 6y &= 5 \\ x - y &= 8 \end{aligned}$

51. $\begin{aligned} 4x + y &= 0 \\ x - 7y &= 0 \end{aligned}$

52. $\begin{aligned} -3x - 2y &= 0 \\ -x + 5y &= 0 \end{aligned}$

53. $\begin{aligned} x + 5y &= 3 \\ 2x + 10y &= 6 \end{aligned}$

54. $\begin{aligned} -2x - 10y &= -4 \\ x + 5y &= 2 \end{aligned}$

55. $\begin{aligned} x \qquad\qquad &= 3 \\ -x + 3y \qquad &= 3 \\ y + 2z &= 4 \end{aligned}$

56. $\begin{aligned} 4x \qquad + z &= 7 \\ y \qquad &= 2 \\ x \qquad + z &= 4 \end{aligned}$

57. $\begin{aligned} x + y + 8z &= 3 \\ 2x + y + 11z &= 4 \\ x \qquad + 3z &= 0 \end{aligned}$

58. $\begin{aligned} -8x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x \qquad - z &= 0 \end{aligned}$

Expanding Your Skills

For Exercises 59–62, solve the equation.

59. $\begin{vmatrix} 6 & x \\ 2 & -4 \end{vmatrix} = 14$

60. $\begin{vmatrix} y & -2 \\ 8 & 7 \end{vmatrix} = 30$

61. $\begin{vmatrix} 3 & 1 & 0 \\ 0 & 4 & -2 \\ 1 & 0 & w \end{vmatrix} = 10$

62. $\begin{vmatrix} -1 & 0 & 2 \\ 4 & t & 0 \\ 0 & -5 & 3 \end{vmatrix} = -4$

For Exercises 63–64, evaluate the determinant by using expansion by minors about the first column.

63. $\begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 4 \\ -2 & 0 & 0 & 1 \\ 4 & -1 & -2 & 0 \end{vmatrix}$

64. $\begin{vmatrix} 5 & 2 & 0 & 0 \\ 0 & 4 & -1 & 1 \\ -1 & 0 & 3 & 0 \\ 0 & -2 & 1 & 0 \end{vmatrix}$

For Exercises 65–66, refer to the following system of four variables.

$$\begin{aligned} x + y + z + w &= 0 \\ 2x \qquad - z + w &= 5 \\ 2x + y \qquad - w &= 0 \\ y + z \qquad &= -1 \end{aligned}$$

65. **a.** Evaluate the determinant $\mathbf{D}$.

b. Evaluate the determinant $\mathbf{D}_x$.

c. Solve for x by computing $\dfrac{\mathbf{D}_x}{\mathbf{D}}$.

66. **a.** Evaluate the determinant $\mathbf{D}_y$.

b. Solve for y by computing $\dfrac{\mathbf{D}_y}{\mathbf{D}}$.

67. Two angles are complementary. The measure of one angle is $\frac{5}{7}$ the measure of the other. Find the measures of the two angles.

68. Two angles are supplementary. The measure of the larger angle is 61° more than $\frac{3}{4}$ the measure of the smaller angle. Find the measures of the angles.

69. A theater charges \$80 per ticket for seats in section A, \$50 per ticket for seats in section B, and \$30 per ticket for seats in section C. For one performance, 3000 tickets were sold for a total of \$165,000 in revenue. If the total number of tickets in sections A and C is equal to the number of tickets sold in section B, how many tickets in each section were sold?

70. The measure of the largest angle in a triangle is 80° larger than the sum of the measures of the other two angles. The measure of the smallest angle is 22° less than the measure of the middle angle. Find the measure of each angle.

71. Suppose 1000 people were surveyed in southern California, and 445 said that they worked out at least three times a week. If $\frac{1}{2}$ of the women and $\frac{3}{8}$ of the men said that they worked out at least three times a week, how many men and how many women were in the survey?

72. During a 1-hr television program, there were 22 commercials. Some commercials were 15 sec long and some were 30 sec long. Find the number of 15-sec commercials and the number of 30-sec commercials if the total playing time for commercials was 9.5 min.

Student Answer Appendix

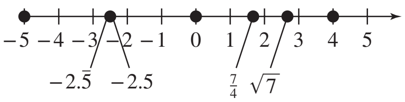
Chapter R

Section R.1 Activity, p. 9

A.1. 4 **A.2.** 0, 4 **A.3.** $-5, 0, 4$

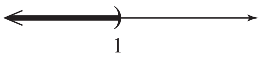
A.4. $-5, -2.\overline{5}, 0, \frac{7}{4}, 4, -2.5$ **A.5.** $\sqrt{7}$

A.6.

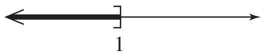


A.7. $x \ge 44$ in. **A.8.** $x \le 35$ mph **A.9.** $16 \le x < 18$ years

A.10. a.



b.



c. Use parentheses when an "endpoint" is not included in the set. Use brackets when an "endpoint" is included in the set. **d.** $(-\infty, 1)$ **e.** $(-\infty, 1]$ **f.** Use parentheses when an "endpoint" is not included in the set. Use brackets when an "endpoint" is included in the set. Use parentheses for ∞ and $-\infty$. The ∞ and $-\infty$ symbols do not actually represent numerical values, but rather indicate that the set continues indefinitely.

A.11. $(-4, -1]$ **A.12.** $\left[\frac{3}{2}, \infty\right)$

Section R.1 Practice Exercises, pp. 9–13

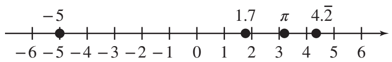
R.1. $<$ **R.3.** $<$

R.5. a. 0.835353 **b.** 0.835 **c.** 0.84

R.7. a. $2.44999, 2.45, 2.4\overline{5}, 2.\overline{45}$ **b.** $-2.\overline{45}, -2.4\overline{5}, -2.45, -2.44999$

1. a. set **b.** inequalities **c.** a is less than b **d.** c is greater than or equal to d **e.** 5 is not equal to 6 **f.** infinity; negative infinity **g.** $\{x \mid x > 5\}$; interval **h.** excludes; includes **i.** parenthesis

3.



5. $\frac{-10}{1}$ **7.** $\frac{-3}{5}$ **9.** $\frac{0}{1}$

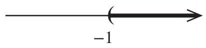
11.

	Real Numbers	Irrational Numbers	Rational Numbers	Integers	Whole Numbers	Natural Numbers
5	✓		✓	✓	✓	✓
$-\sqrt{9}$	✓		✓	✓		
-1.7	✓		✓			
$\frac{1}{2}$	✓		✓			
$\sqrt{7}$	✓	✓				
$\frac{0}{4}$	✓		✓	✓	✓	
$0.\overline{2}$	✓		✓			

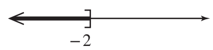
13. $<$ **15.** $>$ **17.** $>$ **19.** $<$ **21.** $(2, \infty)$

23. $(-\infty, 0]$ **25.** $(-5, 0]$ **27.** $[-4.7, \infty)$

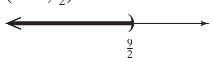
29. $(-1, \infty)$



31. $(-\infty, -2]$

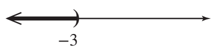


33. $(-\infty, \frac{9}{2})$

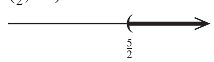


35. $(-2.5, 4.5]$

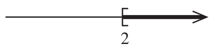
37. $(-\infty, -3)$



39. $(\frac{5}{2}, \infty)$

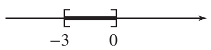


41. $[2, \infty)$



43. $(-4, 4)$

45. $[-3, 0]$



47. All real numbers less than -4 **49.** All real numbers greater than -2 and less than or equal to 7 **51.** All real numbers between -180 and 90, inclusive **53.** All real numbers

55. $a \ge 18$ **57.** $c \le 25$ **59.** $s \ge 261$

61. $r \le 4.5$ **63.** $18 \le a \le 25$ **65.** $p < 130$

67. $130 \le p \le 139$ **69.** $2.2 \le \text{pH} \le 2.4$ acidic

71. $3.0 \le \text{pH} \le 3.5$ acidic

Section R.2 Activity, pp. 24–25

A.1. a. $|-6|$ is the absolute value of -6. **b.** 6 **c.** 10 **d.** -16 **e.** Add the absolute values of the numbers and keep the common sign.

A.2. a. 23 **b.** 4 **c.** -19 **d.** Find the absolute value of each addend. Then subtract the smaller absolute value from the larger absolute value. Apply the sign of the number with the larger absolute value to the sum.

A.3. a. 8 **b.** $2 + 8$ **c.** 10 **d.** $a + (-b)$

	Original Expression	Equivalent Addition Expression	Result
A.4.	$-4.2 - 12.6$	$-4.2 + (-12.6)$	-16.8
A.5.	$\frac{7}{8} - \frac{11}{6}$	$\frac{7}{8} + \left(-\frac{11}{6}\right)$	$-\frac{23}{24}$
A.6.	$-1\frac{3}{4} - \left(-5\frac{1}{2}\right)$	$-1\frac{3}{4} + 5\frac{1}{2}$	$3\frac{3}{4}$

A.7. a. negative **b.** $4 \cdot (-3)$ **c.** negative; positive

A.8. a. $-\frac{1}{2}$ **b.** $12 \cdot \left(-\frac{1}{2}\right)$ **c.** -6 **d.** $a \cdot \frac{1}{b}$ **e.** positive; negative

A.9. a. 5; 5 **b.** 0; 0 **c.** There is no real number that when multiplied by 0 will equal 4. **A.10. a.** -16 **b.** -24 **c.** -80 **d.** -5

A.11. $-\frac{25}{6}$ **b.** $\frac{5}{2}$ **c.** $\frac{25}{9}$ **d.** $\frac{1}{4}$ **A.12. a.** Undefined **b.** 0

A.13. a. 1 **b.** 0 **A.14. a.** 9 **b.** 27 **c.** 81

A.15. a. 9 **b.** -27 **c.** 81

A.16. a. -25 **b.** 25 **c.** -125 **d.** -125

A.17. a. 4 **b.** -4 **c.** Not a real number

A.18. -73 **A.19.** 1

Section R.2 Practice Exercises, pp. 25–29

R.1. a. $\frac{2}{15}$ **b.** $\frac{22}{15}$ **R.3. a.** $\frac{1}{4}$ **b.** $\frac{25}{9}$

R.5. a. $3\frac{5}{6}$ **b.** $1\frac{1}{2}$ **R.7. a.** 9 **b.** $\frac{36}{25}$ or $1\frac{11}{25}$

1. a. opposites **b.** $|a|$; 0 **c.** base; n **d.** radical; square **e.** $\frac{1}{a}$; 1 **f.** 0 **g.** 0; undefined

3. positive; negative **5.** $>$

7. Distance can never be negative. **9.** Negative

11.

Number	Opposite	Reciprocal	Absolute Value
6	-6	$\frac{1}{6}$	6
$\frac{1}{11}$	$-\frac{1}{11}$	11	$\frac{1}{11}$
-8	8	$-\frac{1}{8}$	8
$-\frac{13}{10}$	$\frac{13}{10}$	$-\frac{10}{13}$	$\frac{13}{10}$
0	0	Undefined	0
-3	3	$-0.\overline{3}$	3

13. $<$ **15.** $=$ **17.** $<$ **19.** $<$ **21.** -4 **23.** -19

25. -7 **27.** 14 **29.** -22.1 **31.** -8.1 **33.** $-\frac{5}{3}$ or $-1\frac{2}{3}$

35. $-\frac{67}{45}$ **37.** -32 **39.** $\frac{8}{21}$ **41.** $\frac{3}{5}$ **43.** $-\frac{18}{5}$

45. Undefined **47.** 0 **49.** 3.72 **51.** $\frac{5}{11}$ **53.** 64 **55.** -49

57. 49 **59.** $\frac{125}{27}$ **61.** 3 **63.** Not a real number **65.** $\frac{1}{2}$

67. -7 **69.** 32 **71.** 40 **73.** 25 **75.** 13 **77.** 17

79. -11 **81.** -603 **83.** $\frac{109}{150}$ **85.** 5.4375 **87.** $\frac{2}{3}$

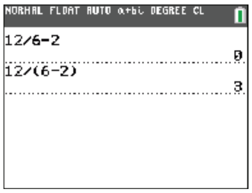
89. -1 **91.** 21 **93.** Undefined **95.** $\frac{9}{10}$ **97.** -10.1°C

99. a. 25°C **b.** 100°C **c.** 0°C **d.** -40°C **101.** $4\frac{1}{6}$ gal

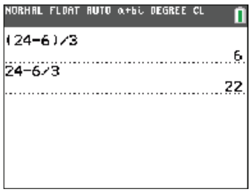
103. 9 in.^{2} **105.** 8.06 cm^2 **107.** 14.1 ft^3 **109.** 26.8 ft^3

111. 141.4 in.^{3}

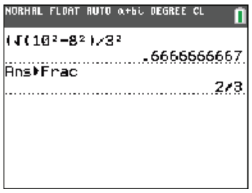
113. 12/(6 – 2)



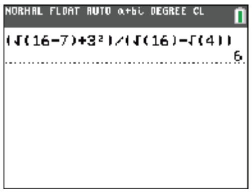
114. (24 – 6)/3



115. $\frac{2}{3}$ or 0.6666666667



116. 6



Section R.3 Activity, p. 35

A.1. a. 6 **b.** 6 **c.** Commutative property of addition

A.2. a. 30 **b.** 30 **c.** Commutative property of multiplication

A.3. The order in which two real numbers are added or multiplied does not affect the result.

A.4. a. 9 **b.** 9 **c.** Associative property of addition

A.5. a. -60 **b.** -60 **c.** Associative property of multiplication

A.6. The manner in which real numbers are grouped when added or multiplied does not affect the result.

A.7. a. The sum is the number itself. **b.** 0

A.8. a. The product is the number itself. **b.** 1

A.9. a. Opposite: 5; reciprocal: $-\frac{1}{5}$. **b.** The sum is 0. **c.** The product is 1. **d.** additive **e.** multiplicative

A.10. a. -30 **b.** -30 **c.** Distributive property of multiplication over addition. **A.11. a.** $4a^2, -ab, 6b^2, -7$ **b.** 4, -1, 6, -7 **c.** -7

A.12. $-13x$ **A.13.** $\frac{1}{2}a^2 + \frac{3}{10}a$ **A.14.** $-8b - 5c + 3d$

A.15. $-2x^2 - 3x + 5$ **A.16.** $9m + 5n$ **A.17.** $-5c^2 + 30c - 65$

Section R.3 Practice Exercises, pp. 36–38

R.1. a. 2 **b.** 2 **R.3. a.** 40 **b.** 40

R.5. a. -6 **b.** 6 **R.7. a.** -2 **b.** $-\frac{1}{2}$

R.9. a. 20 **b.** 20 **R.11. a.** 0 **b.** 1

R.13. a. 3 **b.** 3 **R.15. a.** -48 **b.** -48

1. a. constant **b.** coefficient **c.** 1; 1 **d.** *like*

3. a. $-7 + x$ **b.** $-4y$ **5. a.** 1 **b.** $\frac{1}{5}$

7. a. 3 **b.** $-\frac{1}{6}$ **9. a.** 3 terms **b.** 6 **c.** 2, -5, 6

11. a. 5 terms **b.** -7 **c.** 1, -7, 1, -4, 1 **13.** a **15.** f

17. e **19.** i **21.** b **23.** g **25.** d **27.** h **29.** a

31. $2x - 6y + 16$ **33.** $-40s + 90t + 30$ **35.** $7w - 5z$

37. $\frac{1}{2}a - 2b + \frac{8}{5}$ **39.** $7.8x - 12.3$ **41.** $14c - 16 - 30d + 5f$

43. $14y - 2x$ **45.** $6p^2 + p - 6$ **47.** $-p^2 - 3p$

49. $n^3 + m - 6$ **51.** $7ab + 8a$ **53.** $16xy^2 - 5y^2$ **55.** $8x - 23$

57. $-4c - 6$ **59.** $-9w + 10$ **61.** $4z - 16$ **63.** $7s - 26$

65. $-12w + 13$ **67.** 0 **69.** $4c + 2$ **71.** $1.4x + 10.2$

73. $-2a^2 + 3a + 38$ **75.** $2y^2 - 3y - 5$ **77.** $-62.7x + 220$

79. $-4m + 15n + 2$ **81.** 0, for example: $3 + 0 = 3$

83. Reciprocal **85.** No, for example: $6 - 5 \neq 5 - 6$; $1 \neq -1$

87. a. $x(y + z)$ **b.** xy **c.** xz **d.** $xy + xz$ **e.** $x(y + z) = xy + xz$; The distributive property of multiplication over addition

Chapter R Review Exercises, pp. 41–42

1. 0 **2.** For example: $-\frac{1}{2}, -\frac{3}{4}, \frac{5}{2}$

3. For example: $-2, -1, 0, 1, 2$

4. All real numbers between 7 and 16

5. All real numbers greater than 0 but less than or equal to 2.6

6. All real numbers between -6 and -3, inclusive

7. All real numbers greater than 8

8. All real numbers less than or equal to 13 **9.** All real numbers

10. $(-\infty, 2)$

11. $[0, \infty)$

12. $(-1, 5)$

13. True **14.** 8, $-\frac{1}{8}$, 8 **15.** $-\frac{4}{9}, \frac{9}{4}, \frac{4}{9}$ **16.** 16, 2

17. 625, 5 **18.** -2 **19.** 3 **20.** -21.6 **21.** -8.151

22. $-\frac{25}{13}$ or $-1\frac{12}{13}$ **23.** $\frac{4}{11}$ **24.** $\frac{119}{100}$ or $1\frac{19}{100}$ **25.** $\frac{7}{6}$ or $1\frac{1}{6}$

26. 3 **27.** -8 **28.** 6 **29.** 48 **30.** 11 **31.** 37

32. 75 **33.** 4 **34.** 256 **35.** 756 in.^{2} **36.** $3x + 15y$

37. $\frac{1}{2}x + 4y - \frac{5}{2}$ **38.** $4x - 10y + z$ **39.** $-13a + b + 5c$

40. $7q - 14$ **41.** $9p + 3$ **42.** $-6y - 5$ **43.** $9x - 1$

44. For example: $3 + x = x + 3$ **45.** For example: $5(2y) = (5 \cdot 2)y$

Chapter R Test, p. 42

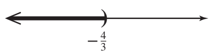
1. a. $-5, -4, -3, -2, -1, 0, 1, 2$ **b.** For example: $\frac{3}{2}, \frac{5}{4}, \frac{8}{5}$

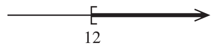
2. a. $\frac{1}{2}, -2, \frac{1}{2}$ **b.** $-4, \frac{1}{4}, 4$ **c.** 0, no reciprocal exists, 0

3. The interval $[4, \infty)$ includes all real numbers 4 and greater, whereas the interval $(4, \infty)$ does not include the endpoint, 4.

4. True

5. $\left(-\infty, -\frac{4}{3}\right)$ **6.** $[12, \infty)$





7. $x \le 5$ **8.** $p \ge 7$ **9.** 6 **10.** -17 **11.** $\frac{1}{4}$ **12.** $-\frac{16}{3}$
13. $z = 1.1$ **14.** $-2b - 6$ **15.** $2x - 1$ **16.** $-2x + 1$
17. False **18.** True **19.** True **20.** True

Chapter 1

Section 1.1 Activity, pp. 53–54

A.1. **a.** $\{16\}$. Add 4 to both sides.
b. $\{8\}$. Subtract 4 from both sides or add –4 to both sides.
c. $\{3\}$. Divide both sides by 4 or multiply both sides by $\frac{1}{4}$.
d. $\{48\}$. Multiply both sides by 4.
A.2. Add 3 to both sides. Then divide both sides by 2 or multiply both sides by $\frac{1}{2}$.
A.3. $\{-7\}$ (Explanations vary.) **A.4.** $\{6\}$ (Explanations vary.)
A.5. $\{-13\}$ (Explanations vary.) **A.6.** **a.** 12, 24, 36, 48, 60 **b.** $4x$ **c.** -30 **d.** 12 **e.** $3x$ **f.** $\{42\}$
A.7. **a.** 100, 1000, 10,000, and 100,000 **b.** -390 **c.** $-120x$ **d.** 784 **e.** $-100x$ **f.** $\{-58.7\}$ **A.8.** $\{5\}$ (Explanations vary.)
A.9. $\{1\}$ (Explanations vary.) **A.10.** $\{12\}$ (Explanations vary.)
A.11. After subtracting x from both sides, the equation reduces to the contradiction $3 = 5$. This is false for all real numbers. Thus, there is no solution to the equation, and the solution set is the empty set, $\{\ \}$.
A.12. After simplifying both sides, the equation reduces to $3x + 7 = 3x + 7$. The left and right sides of the equation are identical for all values of x. Therefore, the solution set is the set of real numbers. **A.13.** 5; $\{5\}$

Section 1.1 Practice Exercises, pp. 55–58

R.1. t **R.3.** n **R.5.** $11y - 4$ **R.7.** $26x - 43$
R.9. 30 **R.11.** 10,000 **R.13.** -19 **R.15.** -2

1. **a.** equation **b.** solution **c.** linear **d.** first **e.** solution; set **f.** solution **3.** contradiction **5.** identity **7.** Linear
9. Nonlinear **11.** Linear **13.** b **15.** $\{12\}$ **17.** $\{-2\}$
19. $\left\{\frac{21}{20}\right\}$ **21.** $\{-40\}$ **23.** $\{-1.1\}$ **25.** $\{6.7\}$ **27.** $\{11\}$
29. $\{13\}$ **31.** $\{-7\}$ **33.** $\{13\}$ **35.** $\left\{\frac{3}{2}\right\}$ **37.** $\{0\}$
39. $\{3\}$ **41.** $\left\{\frac{7}{4}\right\}$ **43.** $\{0\}$ **45.** $\left\{\frac{4}{3}\right\}$ **47.** $\{-4\}$
49. $\{3\}$ **51.** $\{-6\}$ **53.** $\{1\}$ **55.** $\{2\}$
57. It is an equation that is true for some values of the variable but false for other values. **59.** Identity; $\{x \mid x$ is a real number$\}$
61. Conditional equation; $\{0\}$ **63.** Contradiction; $\{\ \}$ **65.** $\{16\}$
67. $\{2\}$ **69.** $\{3\}$ **71.** $\{b \mid b$ is a real number$\}$ **73.** $\{\ \}$
75. $\left\{\frac{8}{5}\right\}$ **77.** $\left\{-\frac{3}{2}\right\}$ **79.** $\{p \mid p$ is a real number$\}$
81. $\left\{\frac{33}{5}\right\}$ **83.** $\{\ \}$ **85.** $\left\{\frac{3}{7}\right\}$ **87.** $\left\{\frac{3}{4}\right\}$ **89.** $\{60\}$
91. $\{6\}$ **93.** $\{\ \}$ **95.** $\left\{-\frac{9}{2}\right\}$ **97.** The family used 1024 kWh.
99. **a.** $y + 8$ **b.** $\{-8\}$ **c.** To simplify an expression clear parentheses and combine *like* terms. To solve an equation, isolate the variable to find a solution.

Chapter 1 Problem Recognition Exercises, p. 58

1. Expression; $-4x + 4$ **2.** Expression; $-7y + 5$
3. Equation; $\{1\}$ **4.** Equation; $\{0\}$
5. Expression; $-10a - 39$ **6.** Expression; $28x - 10$
7. Equation; $\left\{\frac{1}{2}\right\}$ **8.** Equation; $\left\{\frac{3}{4}\right\}$
9. Equation; $\left\{-\frac{19}{6}\right\}$ **10.** Equation; $\left\{\frac{25}{22}\right\}$
11. Expression; $-\frac{1}{6}v - \frac{1}{10}$ **12.** Expression; $-\frac{17}{8}t + \frac{1}{2}u$
13. Equation; $\{\ \}$ **14.** Equation; $\{\ \}$ **15.** Equation; $\left\{\frac{39}{8}\right\}$
16. Equation; $\left\{\frac{2}{25}\right\}$ **17.** Expression; $0.17c + 4.495$
18. Expression; $-1.006k - 0.78$ **19.** Equation; $\{p \mid p$ is a real number$\}$ **20.** Equation; $\{u \mid u$ is a real number$\}$

Section 1.2 Activity, pp. 67–68

A.1. **a.** $x + 1, x + 2$ **b.** $x - 1, x - 2$
A.2. **b.** $x + 2, x + 4$ **c.** (First integer) + 2 times (the second integer) = (the third integer) + 54
d. $x + 2(x + 2) = (x + 4) + 54$ **e.** $x = 27$
f. The integers are 27, 29, and 31.
A.3. **a.** \$300; \$900 **b.** Discount: $0.15x$; New price: $0.85x$
A.4. **b.** One possibility: Let x represent the original price. **c.** Original price − discount = \$216
d. $x - 0.4x = 216$ **e.** $x = 360$ **f.** The original price was \$360.
A.5. **a.** \$420 **b.** $0.07x$
A.6. **b.** One possibility: Let x represent the amount of money borrowed at 7%. (Alternatively, we can let x represent the amount borrowed at 2%, and the table would be adjusted accordingly.)
c.

	7% Loan	2% Loan	Total
Amount borrowed (\$)	x	$12{,}000 - x$	12,000
Interest owed (\$)	$0.07x$	$0.02(12{,}000 - x)$	415

d. $0.07x + 0.02(12{,}000 - x) = 415$ **e.** $x = 3500$
f. The amount borrowed at 7% is \$3500 and the amount borrowed at 2% is \$8500.
A.7. **a.** Acid: 5 mL; Non-acid: 45 mL **b.** $0.20x$
A.8. **b.** One possibility: Let x represent the amount of 10% acid solution.
c.

	10% Acid Solution	25% Acid Solution	20% Acid Solution
Amount of solution (L)	x	6	$x + 6$
Amount of pure acid (L)	$0.1x$	$0.25(6)$	$0.2(x + 6)$

d. $0.1x + 0.25(6) = 0.2(x + 6)$ **e.** $x = 3$
f. Mix 3 L of the 10% acid solution with 6 L of the 25% solution. This will make 9 L of 20% acid solution.
A.9. **a.** 10 mi **b.** $2x$ **A.10.** **b.** One possibility: Let x represent the rate at which Kesha walks to the campsite.
c.

	Distance (mi)	Rate (mph)	Time (hr)
Walking to the campsite	$1.5x$	x	1.5
Walking back to the car	$3(x - 2)$	$x - 2$	3

d. $1.5x = 3(x - 2)$ **e.** $x = 4$ **f.** Kesha walks 4 mph to the campsite and 2 mph back to the car. The campsite is 6 mi from the car.

Section 1.2 Practice Exercises, pp. 69–73

R.1. 30 **R.3.** $\{1000\}$ **R.5.** \$2.52 **R.7.** **a.** \$120 **b.** $15x$
R.9. **a.** 4 mL **b.** $0.2(40 - x)$ **R.11.** **a.** 900 mi **b.** $450(t + 2)$

1. **a.** consecutive **b.** even; odd **c.** 1; 2; 2 **d.** $x + 1$ **e.** $x + 2$ **f.** $x + 2; x + 4$ **g.** Prt; interest **h.** \$1300

3. $x+5$ **5.** $2t-7$ **7.** The numbers are 5 and 13. **9.** The number is -3. **11.** The page numbers are 111 and 112. **13.** The integers are -75 and -73. **15.** The integers are -22 and -20. **17.** The integers are 31, 33, and 35. **19.** She would pay \$5100 for 4 yr at 8.5% and \$5812.50 for 5 yr at 7.75%; the 8.5% option for 4 yr requires less interest. **21.** She must sell \$60,000. **23.** The total for merchandise was \$1197.02 and the sales tax was \$96.36. **25.** The price before markup was \$35.90. **27.** He invested \$8500 in the 2% account and \$4000 in the 5% account. **29.** \$12,000 was borrowed at 6% and \$6000 was borrowed at 11%. **31.** She invested \$12,000 in the 4% account and \$8000 in the 3% account. **33.** 8 oz should be used. **35.** 2 L should be used. **37.** 12.5 L of 18% solution should be added to 7.5 L of 10% solution. **39.** 2 oz should be added. **41.** The plane flies 300 mph from Atlanta to Fort Lauderdale and 240 mph on the return trip. **43.** The speeds are 46 mph and 50 mph. **45.** The integers are 10 and 20. **47.** The original price was \$199. **49.** The speeds are 20 mph and 40 mph. **51.** She deposited \$4500 at 5% and \$9000 at 6%. **53.** Mix 2.5 lb of black tea and 1.5 lb of green tea. **55.** The median price the previous year was \$215,000.

Section 1.3 Activity, p. 78

A.1. The sum of their measures is 90°.

A.2. The sum of their measures is 180°. **A.3.** 180°

A.4. $P=2l+2w$ **A.5.** The width is 80 ft and the length is 120 ft.

A.6. The angles are 35°, 40°, and 105°.

A.7. The angles are 23° and 157°.

A.8. **a.** $r=\dfrac{d}{t}$ **b.** 75 mph **c.** Yes, the vehicle was traveling on average 75 mph. This is 15 mph over the posted speed limit.
d. Sometimes it is necessary to find the value of a different variable within a formula. In this case, a digital camera, image recognition software, and a little mathematics can monitor roadways to keep vehicles traveling at safe speeds.

A.9. **a.** $\{-1\}$ **b.** $y=\dfrac{m-d}{c}$ **c.** In each equation, combine the non-y terms on the right and divide by the coefficient of the y term.

A.10. **a.** $y=\dfrac{5x+8}{-4}$ or $y=-\dfrac{5}{4}x-2$ **b.** Yes. The single fraction $y=\dfrac{5x+8}{-4}$ implies that the entire numerator is divided by -4. Therefore, the expression can be written as the sum of two fractions as $y=\dfrac{5x}{-4}+\dfrac{8}{-4}$ or equivalently, $y=-\dfrac{5}{4}x-2$.

Section 1.3 Practice Exercises, pp. 79–82

R.1. **a.** 19° **b.** 109° **R.3.** 77° **R.5.** **a.** $P=2l+2w$ **b.** 40 ft

1. The court is 29.5 ft wide and 59 ft long. **3.** The sides are 6 m, 8 m, and 10 m. **5.** **a.** The dimensions are $12\frac{1}{2}$ yd by 8 yd. **b.** Raoul needs 41 yd. **7.** The width is 9 ft and the length is 11 ft. **9.** The angles are 30°, 30°, and 120°. **11.** The angles are 15° and 75°. **13.** $x=20$; 139°, 41° **15.** $x=27.5$; 60°, 30° **17.** $x=18$; 36°, 91°, 53° **19.** $x=23$; 42°, 48° **21.** **a.** $r=\dfrac{d}{t}$ **b.** The average speed was 161.3 mph.

23. **a.** $t=\dfrac{I}{Pr}$ **b.** 7 yr **25.** $l=\dfrac{A}{w}$ **27.** $P=\dfrac{I}{rt}$

29. $K_1=K_2-W$ **31.** $C=\frac{5}{9}(F-32)$ or $C=\dfrac{5F-160}{9}$

33. $v^2=\dfrac{2K}{m}$ **35.** $a=\dfrac{v-v_0}{t}$

37. $v_2=\dfrac{w}{p}+v_1$ or $v_2=\dfrac{w+pv_1}{p}$ **39.** $y=\dfrac{c-ax}{b}$

41. $B=\dfrac{3V}{h}$ **43.** $y=-3x+6$ **45.** $y=\dfrac{5}{4}x-5$

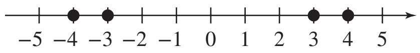
47. $y=-3x-\dfrac{13}{2}$ **49.** $y=x-2$

51. $y=-\dfrac{27}{4}x+\dfrac{15}{4}$ **53.** $y=\dfrac{3}{2}x$ **55.** **a.** $x=z\sigma+\mu$ **b.** $x=130$ **57.** a, b, c **59.** a, b **61.** $t=\dfrac{12}{6-r}$

63. $x=\dfrac{-2}{a-6}$ or $x=\dfrac{2}{6-a}$ **65.** $P=\dfrac{A}{1+rt}$

67. $m=\dfrac{T}{g-f}$ **69.** $x=\dfrac{z-by}{a-c}$ or $x=\dfrac{by-z}{c-a}$

Section 1.4 Activity, p. 88

A.1. 

a. < **b.** >

A.2. The direction of the inequality sign must be reversed. That is, if $3<4$, then $-1\cdot 3>-1\cdot 4$, or equivalently, $-3>-4$.

A.3. **a.** 

$(-\infty,-4]$

b. 

$[15,\infty)$

c. 

$(-\infty,-36)$

d. 

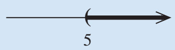
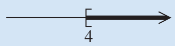
$(9,\infty)$

A.4. Parts (a) and (c). The inequality in part (a) required that we divide both sides by a negative number. In part (c) we multiplied both sides by a negative number.

A.5. $\left[\dfrac{8}{5},\infty\right)$ **A.6.** $(-\infty,3)$ **A.7.** $\left(-\infty,-\dfrac{8}{13}\right)$

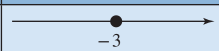
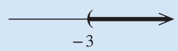
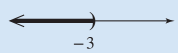
A.8. Greg would need a score of 96 or better to earn an "A" in the class.

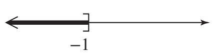
Section 1.4 Practice Exercises, pp. 89–92

	Set-Builder Notation	Interval Notation	Graph
R.1.	$\{x\mid x>5\}$	$(5,\infty)$	
R.3.	$\{x\mid -3<x\le 6\}$	$(-3,6]$	
R.5.	$\{x\mid x\ge 4\}$	$[4,\infty)$	

R.7. **a.** False **b.** True **c.** True

1. **a.** linear; inequality **b.** negative

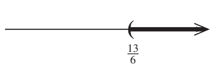
	Set Notation	Interval Notation	Graph
3.a.	$\{-3\}$	n/a	
b.	$\{x\mid x>-3\}$	$(-3,\infty)$	
c.	$\{x\mid x<-3\}$	$(-\infty,-3)$	

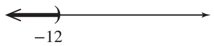
5. 

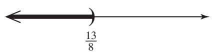
a. $\{y\mid y\le -1\}$ **b.** $(-\infty,-1]$

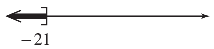
7. 

a. $\{x\mid x\ge 10\}$ **b.** $[10,\infty)$

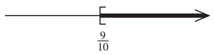
9.  **a.** $\{z \mid z > \frac{13}{6}\}$ **b.** $\left(\frac{13}{6}, \infty\right)$

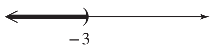
11.  **a.** $\{t \mid t < -12\}$ **b.** $(-\infty, -12)$

13.  **a.** $\{y \mid y < \frac{13}{8}\}$ **b.** $\left(-\infty, \frac{13}{8}\right)$

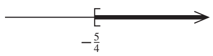
15.  **a.** $\{a \mid a \le -21\}$ **b.** $(-\infty, -21]$

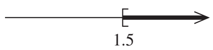
17.  **a.** $\{x \mid x > -3\}$ **b.** $(-3, \infty)$

19.  **a.** $\{x \mid x \ge \frac{9}{10}\}$ **b.** $\left[\frac{9}{10}, \infty\right)$

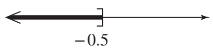
21.  **a.** $\{p \mid p < -3\}$ **b.** $(-\infty, -3)$

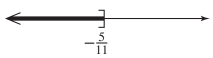
23.  **a.** $\{t \mid t < 5\}$ **b.** $(-\infty, 5)$

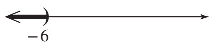
25.  **a.** $\{y \mid y \ge -\frac{5}{4}\}$ **b.** $\left[-\frac{5}{4}, \infty\right)$

27.  **a.** $\{k \mid k \ge 1.5\}$ **b.** $[1.5, \infty)$

29.  **a.** $\{x \mid x \le 12\}$ **b.** $(-\infty, 12]$

31.  **a.** $\{b \mid b \le -0.5\}$ **b.** $(-\infty, -0.5]$

33.  **a.** $\{c \mid c \le -\frac{5}{11}\}$ **b.** $\left(-\infty, -\frac{5}{11}\right]$

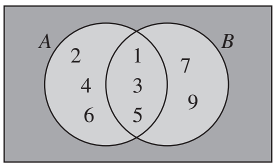
35.  **a.** $\{y \mid y < -6\}$ **b.** $(-\infty, -6)$

37.  **a.** $\{x \mid x > -3\}$ **b.** $(-3, \infty)$

39.  **a.** $\{a \mid a \le 0\}$ **b.** $(-\infty, 0]$

41. a. Nadia needs to score at least a 70 on the fifth quiz. **b.** It would be impossible for Nadia to get an "A" because she would have to earn 120 on her last quiz and it is impossible to earn more than 100. **43.** Boys 8 yr old or older will be on average at least 51 in. tall. **45.** Boys 6 yr old or younger will be on average no more than 46 in. tall. **47. a.** She needs to sell in excess of \$375,000. **b.** She needs to sell in excess of \$1,375,000. **c.** The base salary is still the same. The increase comes solely from commission. **49.** More than 73 jackets must be sold. **51.** $>$ **53.** $<$

Section 1.5 Activity, pp. 101–102

A.1. 

A.2. a. union; $A \cup B$ **b.** $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$

A.3. a. intersection; $A \cap B$ **b.** $\{1, 3, 5\}$

A.4. a.  **b.** 

c.  **d.** $(-\infty, \infty)$

e.  **f.** $[-3, 2)$

A.5. a. $(-5, \infty)$ **b.** $[-1, \infty)$ **c.** $[-1, \infty)$ **d.** $(-5, \infty)$ **e.** intersection **f.** union

A.6. a. $-3 \le \frac{1}{2}x - 5$ and $\frac{1}{2}x - 5 < 1$ **b.** $[4, 12)$

A.7. a. $4{,}200{,}000 \le x \le 5{,}900{,}000$ **b.** $x < 4{,}200{,}000$ or $x > 5{,}900{,}000$

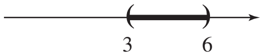
Section 1.5 Practice Exercises, pp. 102–107

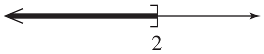
R.1. $(-\infty, 8)$ **R.3.** $(-9, -6]$ **R.5.** $[-6, \infty)$
R.7. $[-2, 6)$ **R.9.** $[18, \infty)$

1. a. union; $A \cup B$ **b.** intersection; $A \cap B$ **c.** intersection **d.** $a < x < b$ **e.** union **3.** $\{2, 4\}$ **5.** $\{2, 4, 5, 6, 7, 8, 10, 11\}$ **7. a.** $\{-3, -1\}$ **b.** $\{-4, -3, -2, -1, 0, 1, 3, 5\}$ **9.** $[-7, -4)$ **11.** $(-\infty, -4) \cup (2, \infty)$ **13.** $\{\ \}$ **15.** $[-7, \infty)$ **17.** $[0, 5)$ **19.** $[-7, \infty)$ **21. a.** $[-1, 5)$ **b.** $(-2, \infty)$

23. a. $(-1, 3)$ **b.** $\left(-\frac{5}{2}, \frac{9}{2}\right)$ **25. a.** $(0, 2]$ **b.** $(-4, 5]$

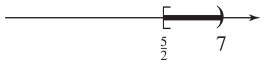
27. $[-2, 3]$ 

29. $(3, 6)$ 

31. $(-\infty, 2]$ 

33. $[8, \infty)$ 

35. $\{\ \}$ **37.** $-4 \le t$ and $t < \frac{3}{4}$ **39.** The statement $6 < x < 2$ is equivalent to $6 < x$ and $x < 2$. However, no real number is greater than 6 and also less than 2. **41.** The statement $-5 > y > -2$ is equivalent to $-5 > y$ and $y > -2$. However, no real number is less than -5 and also greater than -2.

43. $\left[\frac{5}{2}, 7\right)$ 

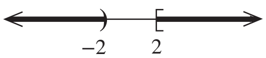
45. $(-6, 6]$ 

47. $(2, 8)$ 

49. $\left[-\frac{10}{3}, -\frac{7}{3}\right]$ 

51. $\left[-\frac{1}{2}, \frac{3}{2}\right)$ 

53. $(-0.4, 0.2)$ 

55. $(-\infty, -2) \cup [2, \infty)$ 

57. $(-\infty, 2]$ 

59. $[-6, \infty)$ 

61. $(-\infty, \infty)$ 

63. $(-\infty, -4] \cup (4.5, \infty)$ 

65. a. $(-10, 8)$ **b.** $(-\infty, \infty)$
67. a. $(-\infty, -8] \cup [1.3, \infty)$ **b.** $\{\ \}$
69. $\left(-\frac{11}{2}, \frac{7}{2}\right]$ **71.** $\left(-\frac{13}{2}, \infty\right)$ **73.** $(-\infty, 1) \cup (5, \infty)$
75. a. $4800 \le x \le 10{,}800$ **b.** $x < 4800$ or $x > 10{,}800$
77. a. $44\% < x < 48\%$ **b.** $x \le 44\%$ or $x \ge 48\%$
79. All real numbers between $-\frac{3}{2}$ and 6
81. All real numbers greater than 2 or less than -1

83. **a.** Amy would need 82% or better on her final exam.
b. If Amy scores at least 32% and less than 82% on her final exam she will receive a “B” in the class.
85. $32° \le F \le 42.08°$

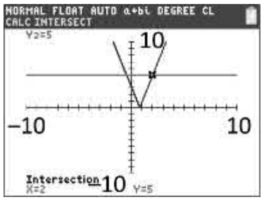
Section 1.6 Activity, p. 113

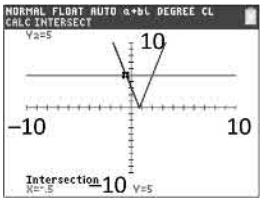
A.1. **a.** zero **b.** $\{-3, 3\}$ **c.** $\{-11, 11\}$ **d.** $\{0\}$
e. There are no real numbers for which the absolute value is negative. The absolute value of any real number is greater than or equal to zero.
A.2. **a.** $x - 4 = 5$ or $x - 4 = -5$ **b.** $\{-1, 9\}$
A.3. **a.** Isolate the absolute value by adding 6 to both sides.
b. $\left\{3, -\frac{11}{3}\right\}$
A.4. **a.** $\{\ \}$ **b.** $\left\{\frac{6}{5}\right\}$
A.5. $x = y$ or $x = -y$
A.6. **a.** $x - 5 = 2x - 1$ or $x - 5 = -(2x - 1)$ **b.** $\{-4, 2\}$
A.7. $\left\{-\frac{1}{4}\right\}$

Section 1.6 Practice Exercises, pp. 113–115

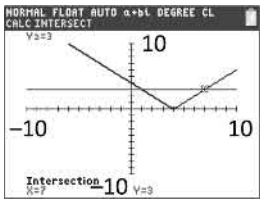
R.1. **a.** $\{5\}$ **b.** $\left\{-\frac{39}{5}\right\}$ **R.3.** **a.** $\{-13\}$ **b.** $\left\{-\frac{1}{5}\right\}$
R.5. $\{\ \}$ **R.7.** $\{x \mid x$ is a real number$\}$ **R.9.** $\{5\}$

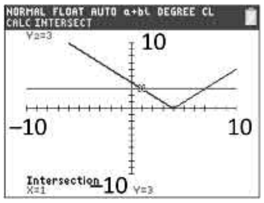
1. **a.** absolute; $\{a, -a\}$ **b.** Subtract 5 from both sides.
c. y; $-y$ **d.** $\{\ \}$; $\{-4\}$
3. **a.** $\{-5, 5\}$ **b.** $\{\ \}$ **c.** $\{0\}$ **5.** **a.** $\{1, 7\}$ **b.** $\{\ \}$ **c.** $\{4\}$
7. $\{7, -7\}$ **9.** $\{6, -6\}$ **11.** $\{\ \}$ **13.** $\{-2, 2\}$
15. $\{0\}$ **17.** $\left\{4, -\frac{4}{3}\right\}$ **19.** $\left\{\frac{9}{2}, -\frac{1}{2}\right\}$ **21.** $\left\{\frac{10}{7}, -\frac{8}{7}\right\}$
23. $\{\ \}$ **25.** $\{-12, 28\}$ **27.** $\left\{\frac{5}{2}, -\frac{7}{2}\right\}$ **29.** $\left\{\frac{7}{3}\right\}$
31. $\{\ \}$ **33.** $\left\{\frac{3}{2}\right\}$ **35.** $\left\{-4, \frac{16}{5}\right\}$ **37.** $\left\{-\frac{1}{2}, \frac{7}{6}\right\}$
39. $\left\{\frac{5}{2}, -\frac{3}{2}\right\}$ **41.** $\left\{-4, \frac{1}{3}\right\}$ **43.** $\left\{\frac{1}{2}\right\}$ **45.** $\left\{-\frac{1}{16}\right\}$
47. $\left\{\frac{1}{4}\right\}$ **49.** $\{-1.44, -0.4\}$ **51.** $\{\ \}$
53. $\{w \mid w$ is a real number$\}$ **55.** $\{\ \}$
57. $\left\{2, -\frac{1}{2}\right\}$



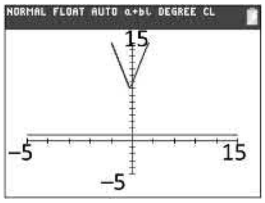


58. $\{7, 1\}$

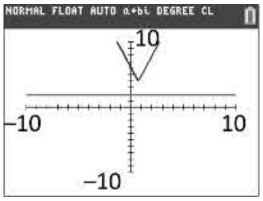




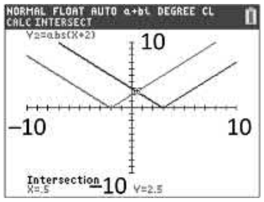
59. $\{\ \}$



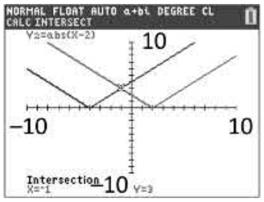
60. $\{\ \}$



61. $\left\{\frac{1}{2}\right\}$

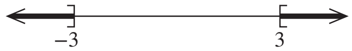


62. $\{-1\}$



63. $|x| = 6$ **65.** $|x| = \frac{4}{3}$

Section 1.7 Activity, pp. 122–123

A.1. **a.** 

b. $(-\infty, -3] \cup [3, \infty)$
c. 

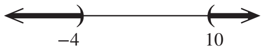
d. $[-3, 3]$
A.2. **a.** Isolate the absolute value by adding 1 to both sides.
b. $-8 < 2x + 6 < 8$ **c.** $(-7, 1)$
A.3. **a.** Isolate the absolute value by subtracting 3 from both sides and then dividing both sides by –1. (Remember to reverse the direction of the inequality sign when dividing by –1.)
b. $x - 9 < -10$ or $x - 9 > 10$ **c.** $(-\infty, -1) \cup (19, \infty)$
A.4. **a.** $\{\ \}$. By definition, the absolute value of a real number is greater than or equal to zero. Therefore, it cannot be less than a negative number.
b. $\{\ \}$. As with part (a), the absolute value of a real number cannot be less than or equal to a negative number.
c. $(-\infty, \infty)$. Because the absolute value of any real number is greater than or equal to zero, it is automatically greater than a negative number.
d. $(-\infty, \infty)$. Because the absolute value of any real number is greater than or equal to zero, it is automatically greater than or equal to a negative number.
A.5. **a.** $\{\ \}$. By definition, the absolute value of a real number is greater than or equal to zero. Therefore, it cannot be less than a negative number.
b. $\{3\}$. Although the absolute value of a real-valued expression cannot be negative, it *can* be equal to zero. Therefore, the only solution is the value of x for which $2x - 6 = 0$.
c. $(-\infty, \infty)$. By definition, the absolute value of any real number is greater than or equal to zero.
d. $(-\infty, 3) \cup (3, \infty)$. The absolute value of any real number is greater than or equal to zero. Therefore, the solution set to the inequality $|2x - 6| > 0$ includes all real numbers, *except* those that make $2x - 6 = 0$. The value $x = 3$ must be excluded from the solution set.

Section 1.7 Practice Exercises, pp. 123–126

R.1. $(-\infty, -4) \cup (1, \infty)$ **R.3.** $[-5, -1]$ **R.5.** $\left\{-\frac{4}{3}, 4\right\}$
R.7. $\{\ \}$ **R.9.** $\left\{\frac{3}{5}\right\}$

1. **a.** $-a$; a **b.** $-a$; $>$ **c.** $\{\ \}$; $(-\infty, \infty)$
3. **a.** $\{-5, 5\}$ **b.** $(-\infty, -5) \cup (5, \infty)$ 

c. $(-5, 5)$ 

5. **a.** $\{10, -4\}$ **b.** $(-\infty, -4) \cup (10, \infty)$ 

c. $(-4, 10)$ 

7. **a.** $\{\ \}$ **b.** $(-\infty, \infty)$  **c.** $\{\ \}$

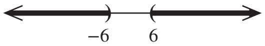
9. **a.** $\{\ \}$ **b.** $(-\infty, \infty)$  **c.** $\{\ \}$

11. **a.** $\{0\}$ **b.** $(-\infty, 0) \cup (0, \infty)$ 

c. $\{\ \}$

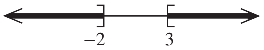
13. **a.** $\{7\}$ **b.** $(-\infty, 7) \cup (7, \infty)$ 

c. $\{\ \}$

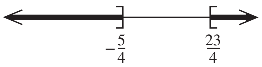
15. $(-\infty, -6) \cup (6, \infty)$ 

17. $[-3, 3]$ 

19. $(-\infty, \infty)$ 

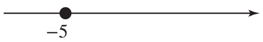
21. $(-\infty, -2] \cup [3, \infty)$ 

23. $\{\ \}$ **25.** $[-10, 14]$ 

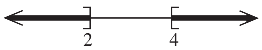
27. $\left(-\infty, -\frac{5}{4}\right] \cup \left[\frac{23}{4}, \infty\right)$ 

29. $\left(-\frac{21}{2}, \frac{19}{2}\right)$ 

31. $(-\infty, \infty)$ 

33. $\{\ \}$ **35.** $\{-5\}$ 

37. $(-\infty, 6) \cup (6, \infty)$ 

39. $(-\infty, 2] \cup [4, \infty)$ 

41. $(4, 8)$ 

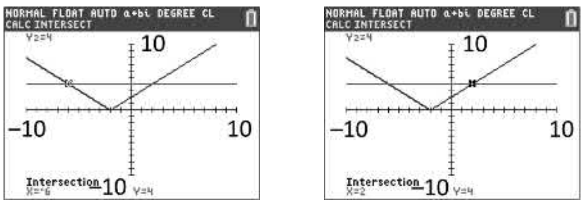
43. $(-10.5, 4.5)$ 

45. $|x| > 7$ **47.** $|x - 2| \le 13$ **49.** $|x - 32| \le 0.05$

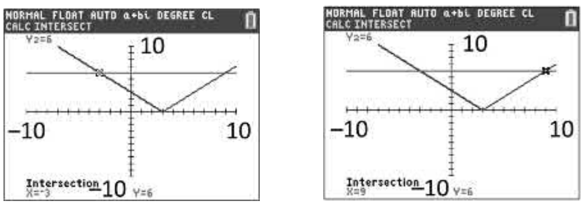
51. $\left|x - 6\frac{3}{4}\right| \le \frac{1}{8}$

53. The solution set is $\{w \mid 1.99 \le w \le 2.01\}$, or equivalently in interval notation, $[1.99, 2.01]$. This means that the actual width of the bolt could be between 1.99 cm and 2.01 cm inclusive.

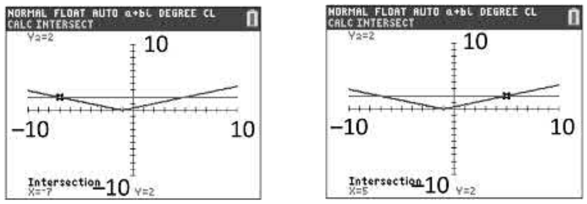
55. $(-\infty, -6) \cup (2, \infty)$



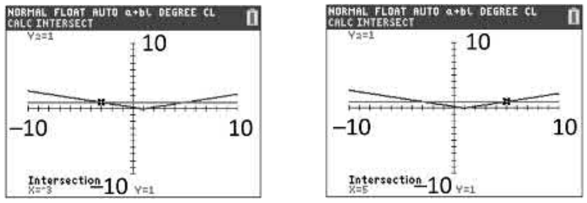
56. $(-\infty, -3) \cup (9, \infty)$



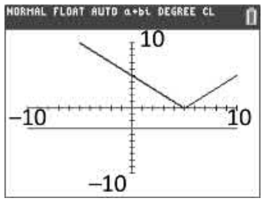
57. $(-7, 5)$



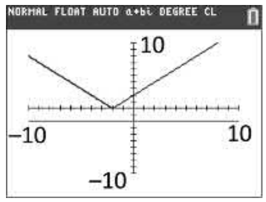
58. $(-3, 5)$



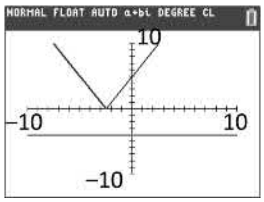
59. $\{\ \}$



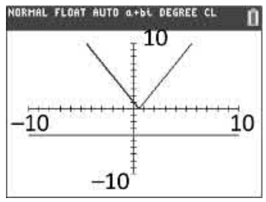
60. $\{\ \}$



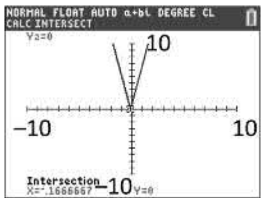
61. $(-\infty, \infty)$



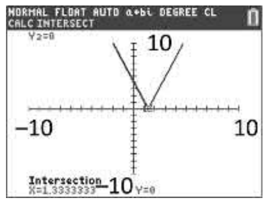
62. $(-\infty, \infty)$



63. $\left\{-\frac{1}{6}\right\}$



64. $\left\{\frac{4}{3}\right\}$



Chapter 1 Problem Recognition Exercises, pp. 126–127

1. **a.** $\{9\}$ **b.** $\{9, -3\}$ **c.** $(-3, 9)$ **d.** $(-\infty, -3] \cup [9, \infty)$

2. **a.** $\left\{-\frac{22}{5}\right\}$ **b.** $\{\ \}$ **c.** $\{\ \}$ **d.** $(-\infty, \infty)$

3. **a.** $\{-7\}$ **b.** $(-\infty, -7)$ **c.** $[-7, \infty)$

4. **a.** $\{29\}$ **b.** $[29, \infty)$ **c.** $(-\infty, 29)$

5. **a.** $\left\{\frac{1}{2}, -\frac{1}{6}\right\}$ **b.** $\left\{\frac{1}{2}\right\}$ **6.** **a.** $(-7, 6]$ **b.** $(-7, 6]$

7. **a.** $(-5, \infty)$ **b.** $[1, \infty)$ **8.** **a.** $(-\infty, -7) \cup (2, \infty)$ **b.** $\{\ \}$

9. **a.** Linear equation **b.** $\{-6\}$ **10.** **a.** Linear equation **b.** $\{9\}$ **11.** **a.** Absolute value inequality **b.** $[-6, -2]$

12. **a.** Absolute value inequality **b.** $\{\ \}$

13. **a.** Compound inequality **b.** $(-6, 9)$

14. **a.** Compound inequality **b.** $(-\infty, 2) \cup [7, \infty)$

15. **a.** Absolute value equation **b.** $\{4, -16\}$

16. **a.** Absolute value equation **b.** $\{1, -6\}$

17. **a.** Linear inequality **b.** $(-\infty, 12]$ **18.** **a.** Linear inequality **b.** $[-1, \infty)$ **19.** **a.** Absolute value inequality

b. $(-\infty, -3] \cup [12, \infty)$ **20. a.** Absolute value inequality **b.** $(-5, 25)$ **21. a.** Absolute value equation **b.** { }

22. a. Absolute value equation **b.** $\left\{-\frac{1}{5}\right\}$

23. a. Linear equation **b.** $\{-16\}$ **24. a.** Linear equation **b.** $\{-11\}$ **25. a.** Compound inequality **b.** $(8, 12]$ **26. a.** Compound inequality **b.** $[-6, -3)$ **27. a.** Linear equation **b.** $(-\infty, \infty)$ **28. a.** Linear equation **b.** { }

Chapter 1 Review Exercises, pp. 135–138

1. The empty set; no solution **2.** All real numbers

3. $\{-5\}$; conditional equation **4.** $\left\{\frac{1}{8}\right\}$; conditional equation **5.** $\{-18.075\}$; conditional equation

6. $\{0.22\}$; conditional equation **7.** $\left\{\frac{31}{6}\right\}$; conditional equation **8.** $\{3\}$; conditional equation

9. $\left\{-\frac{21}{8}\right\}$; conditional equation **10.** { }; contradiction

11. { }; contradiction **12.** $\{m \mid m \text{ is a real number}\}$; identity

13. $x, x+1, x+2$ **14.** $x, x+2$ **15.** Distance equals rate times time. **16.** Simple interest equals principal times the annual interest rate times the time in years. **17. a.** \$23,856 **b.** \$61,344

18. There were 6.7 million men. **19.** There were 16,600 deaths due to alcohol-related accidents. **20.** The integers are −4, −2, and 0.

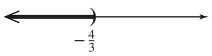
21. The pieces are 2 ft and $\frac{2}{3}$ ft. **22.** She invested \$1500 in the 6% account and \$3500 in the 9% account. **23.** Mix 2 L of 10% solution.

24. Lynn drives 45 mph and Linda drives 30 mph. **25.** The width is 9 ft, and the length is 11 ft. **26.** $x = 52$; 27°

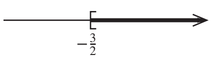
27. $x = 30$; 29°, 61° **28.** $y = \dfrac{3x-4}{2}$ or $y = \dfrac{3}{2}x - 2$

29. $y = 6x + 12$ **30.** $h = \dfrac{S - 2\pi r}{\pi r^2}$ or $h = \dfrac{s}{\pi r^2} - \dfrac{2}{r}$

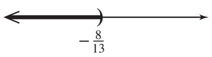
31. $b = \dfrac{2A}{h}$ **32. a.** $\pi = \dfrac{C}{2r}$ **b.** $\pi \approx 3.14$

33. 

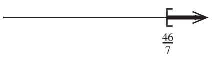
a. $\{x \mid x < -\frac{4}{3}\}$ **b.** $(-\infty, -\frac{4}{3})$

34. 

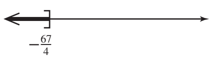
a. $\{x \mid x \geq -\frac{3}{2}\}$ **b.** $[-\frac{3}{2}, \infty)$

35. 

a. $\{x \mid x < -\frac{8}{13}\}$ **b.** $(-\infty, -\frac{8}{13})$

36. 

a. $\{x \mid x \geq \frac{46}{7}\}$ **b.** $[\frac{46}{7}, \infty)$

37. 

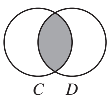
a. $\{x \mid x \leq -\frac{67}{4}\}$ **b.** $(-\infty, -\frac{67}{4}]$

38. 

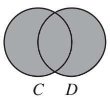
a. $\{x \mid x \leq \frac{29}{2}\}$ **b.** $(-\infty, \frac{29}{2}]$

39. Dave must earn at least 95 on his fifth test.

40. $C \cap D$ is the set of elements common to both sets C and D.



$C \cup D$ is the set of elements in either set C, set D, or both sets.



41. $[-10, 1)$ **42.** $(-\infty, \infty)$ **43.** $(-\infty, \infty)$ **44.** $(-1, 1)$

45. $(-\infty, -3] \cup (-1, \infty)$ **46.** { } **47.** $\left(-\frac{11}{4}, 4\right]$ **48.** $[-5, 2)$

49. { } **50.** { } **51.** $(-\infty, 6] \cup (12, \infty)$

52. $(-\infty, -2) \cup [2, \infty)$ **53.** $\left(-\infty, \frac{1}{2}\right)$ **54.** $(-5, \infty)$

55. $\left[0, \frac{4}{3}\right]$ **56.** $[-7, -2)$

57. All real numbers between −6 and 12

58. a. $140 \leq x \leq 225$ **b.** $x < 140$ or $x > 225$

59. a. $125 \leq x \leq 200$ **b.** $x < 125$ or $x > 200$

60. a. For example, for a 47-year-old person, the maximum recommended heart rate is 173. **b.** Given a maximum recommended heart rate of 173, the interval is (104, 130).

61. $\{10, -10\}$ **62.** $\{17, -17\}$ **63.** $\{1.3, 7.4\}$

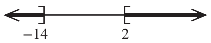
64. $\{-0.44, 2.54\}$ **65.** $\{5, -9\}$ **66.** $\{3, 1\}$

67. { } **68.** { } **69.** $\left\{\frac{3}{7}\right\}$ **70.** $\left\{-\frac{5}{4}\right\}$

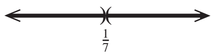
71. $\left\{6, \frac{4}{5}\right\}$ **72.** $\left\{-\frac{1}{2}\right\}$ **73.** $\{d \mid d \text{ is a real number}\}$

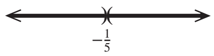
74. { } **75.** Both expressions give the distance between 3 and −2.

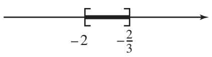
76. $|x| > 5$ **77.** $|x| < 4$ **78.** $|x| < 6$ **79.** $|x| > \frac{2}{3}$

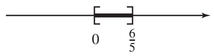
80. $(-\infty, -14] \cup [2, \infty)$ 

81. $[-11, -5]$ 

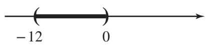
82. $\left(-\infty, \frac{1}{7}\right) \cup \left(\frac{1}{7}, \infty\right)$ 

83. $\left(-\infty, -\frac{1}{5}\right) \cup \left(-\frac{1}{5}, \infty\right)$ 

84. $\left[-2, -\frac{2}{3}\right]$ 

85. $\left[0, \frac{6}{5}\right]$ 

86. $(2, 22)$ 

87. $(-12, 0)$ 

88. $(-\infty, \infty)$ 

89. $(-\infty, \infty)$ 

90. $(-\infty, \infty)$ 

91. $(-\infty, \infty)$ 

92. { } **93.** { } **94.** If an absolute value is less than a negative number, there will be no solution. **95.** If an absolute value is greater than a negative number, then all real numbers are solutions. **96.** $0.17 \leq p \leq 0.23$ or, equivalently in interval notation, [0.17, 0.23]. This means that the actual percentage of viewers is estimated to be between 17% and 23%, inclusive.

97. $3\frac{1}{8} \leq L \leq 3\frac{5}{8}$ or, equivalently in interval notation, $\left[3\frac{1}{8}, 3\frac{5}{8}\right]$. This means that the actual length of the screw may be between $3\frac{1}{8}$ in. and $3\frac{5}{8}$ in., inclusive.

Chapter 1 Test, pp. 138–139

1. $\{133\}$ **2.** $\left\{\frac{4}{5}\right\}$ **3.** $\{142{,}500\}$ **4.** $\left\{\frac{14}{9}\right\}$

5. $\{10, -22\}$ **6.** $\{-8, 2\}$ **7.** $\{3, 0\}$ **8.** { } **9.** $\{-1\}$

10. a. Identity **b.** Contradiction **c.** Conditional equation

11. The numbers are 18 and 90.

12. a. It took 0.4 hr (24 min). **b.** The distance is 1.8 mi.

13. She invested \$1000 in the 5% account.

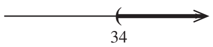
14. Each side is 27 in.

15. The numbers are 31, 33, and 35.

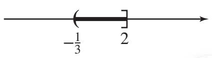
16. 24 gal of 20% solution must be used.

17. $y = -2x + 3$

18. $z = \dfrac{x - \mu}{\sigma}$

19. $(34, \infty)$ 

20. $(-\infty, \frac{18}{5}]$ 

21. $(-\frac{1}{3}, 2]$ 

22. $\left[-\frac{1}{3}, 2\right]$ **23.** $(-2, 13]$ **24.** $(-\infty, -24] \cup [-15, \infty)$
25. $[-3, 0)$ **26.** $(-\infty, \infty)$ **27.** { } **28.** { }
29. $\left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{17}{3}, \infty\right)$
30. $(-18.75, 17.25)$ **31.** $(-\infty, \infty)$ **32.** $[-3, 8]$
33. It can carry at most seven more passengers.
34. **a.** $9 \le x \le 33$ **b.** $x < 9$ or $x > 33$
35. $|x - 15.41| \le 0.01$

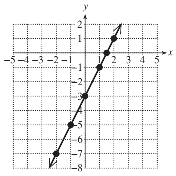
Chapter 2

Section 2.1 Activity, pp. 152–153

A.1. **a.**

x	y
0	−3
$\frac{3}{2}$	0
1	−1
2	1
−2	−7
−1	−5

b.–c.

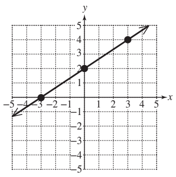


d. The equation $2x - y = 3$ is called a linear equation because the graph representing all solutions to the equation is a line. **e.** A linear equation in two variables is an equation that can be written in the form $Ax + By = C$ where A and B are not both zero. The equation $2x - y = 3$ is linear because it can be written as $(2)x + (-1)y = 3$.

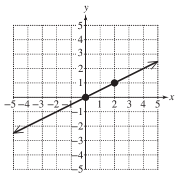
A.2. An x-intercept is a point where the graph of an equation intersects the x-axis. A y-intercept is a point where the graph of an equation intersects the y-axis.

A.3. To find the x-intercept, substitute 0 for y and solve for x. To find the y-intercept, substitute 0 for x and solve for y.

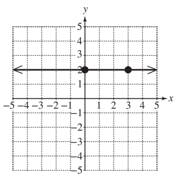
A.4. **a.** $(-3, 0)$ **b.** $(0, 2)$
c. For example: $(3, 4)$
d.



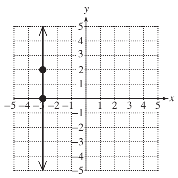
A.5. **a.** $(0, 0)$ **b.** $(0, 0)$
c. For example, $(2, 1)$
d.



A.6. **a.** No x-intercept **b.** $(0, 2)$
c. For example, $(3, 2)$
d.



A.7. **a.** $(-3, 0)$ **b.** No y-intercept
c. For example, $(-3, 2)$
d.



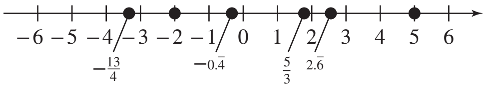
A.8. An equation represents a horizontal line if the equation has a y variable, but no x variable. In such a case, the equation can be written in the form $y = k$, where k is a constant.

A.9. An equation represents a vertical line if the equation has an x variable, but no y variable. In such a case, the equation can be written in the form $x = k$, where k is a constant.

A.10. An equation represents a slanted line passing through the origin if the equation has both x and y variables and the constant term is zero. In such a case, the equation can be written in the form $Ax + By = 0$, where neither A nor B is zero.

Section 2.1 Practice Exercises, pp. 153–160

R.1.–R.6.

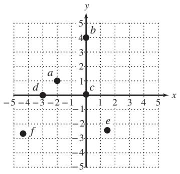


R.7. {5} **R.9.** {11} **R.11.** 27 **R.13.** 7

1. **a.** x; y-axis **b.** ordered **c.** origin; $(0, 0)$ **d.** quadrants **e.** negative **f.** III **g.** $Ax + By = C$ **h.** x-intercept **i.** y-intercept **j.** vertical **k.** horizontal

3. For (x, y), if $x > 0$, $y > 0$ the point is in quadrant I. If $x < 0$, $y > 0$ the point is in quadrant II. If $x < 0$, $y < 0$ the point is in quadrant III. If $x > 0$, $y < 0$ the point is in quadrant IV.

5.

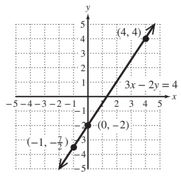


7. 0 **9.** $A(-4, 5)$, II; $B(-2, 0)$, x-axis; $C(1, 1)$, I; $D(4, -2)$, IV; $E(-5, -3)$, III **11.** **a.** Yes **b.** No **c.** Yes

13. **a.** No **b.** Yes **c.** No

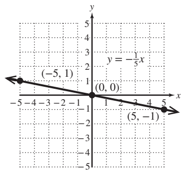
15.

x	y
0	−2
4	4
−1	$-\frac{7}{2}$

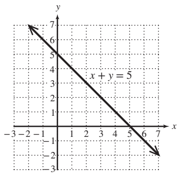


17.

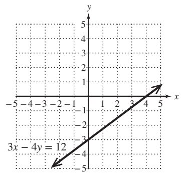
x	y
0	0
5	−1
−5	1



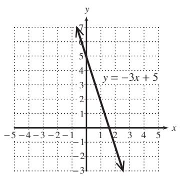
19.



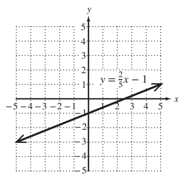
21.



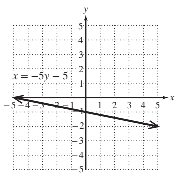
23.



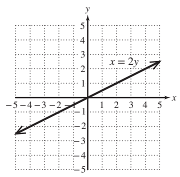
25.



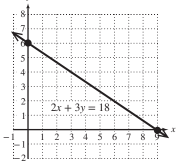
27.

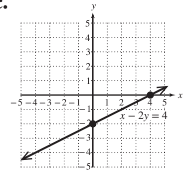


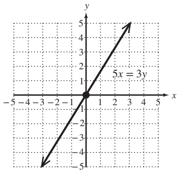
29.

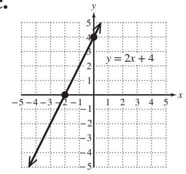


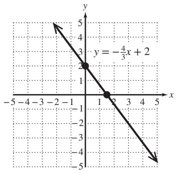
31. To find an x-intercept, substitute $y = 0$ and solve for x. To find a y-intercept, substitute $x = 0$ and solve for y.

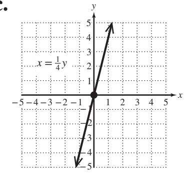
33. **a.** (9, 0) **b.** (0, 6)
c.


35. **a.** (4, 0) **b.** (0, −2)
c.


37. **a.** (0, 0) **b.** (0, 0)
c.


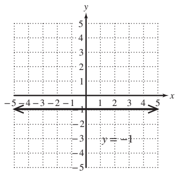
39. **a.** (−2, 0) **b.** (0, 4)
c.


41. **a.** $(\frac{3}{2}, 0)$ **b.** (0, 2)
c.


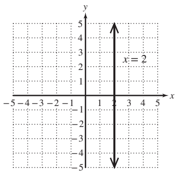
43. **a.** (0, 0) **b.** (0, 0)
c.


45. **a.** \$55,000 **b.** \$39,000 **c.** The y-intercept is (0, 15,000). For \$0 in sales, the salary will be \$15,000.
d. Total sales cannot be negative. **47.** **a.** \$1200 **b.** 4 yr
c. (0, 1500); The y-intercept represents the initial value.
d. (5, 0); The x-intercept indicates that once the computer is 5 yr old, its value is \$0.

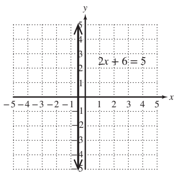
49. Horizontal; no x-intercept; y-intercept (0, −1)



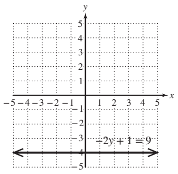
51. Vertical; x-intercept (2, 0); no y-intercept



53. Vertical; x-intercept $(-\frac{1}{2}, 0)$; no y-intercept

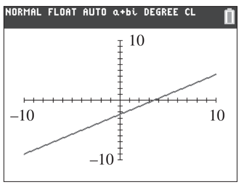


55. Horizontal; no x-intercept; y-intercept (0, −4)

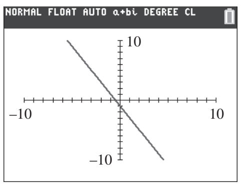


57. A horizontal line parallel to the x-axis will not have an x-intercept. A vertical line parallel to the y-axis will not have a y-intercept. **59.** b, c, d

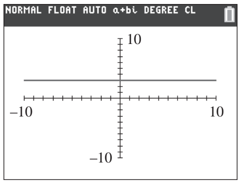
61. $y = \frac{2}{3}x - \frac{7}{3}$



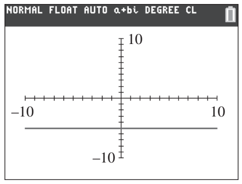
62. $y = -2x - 1$



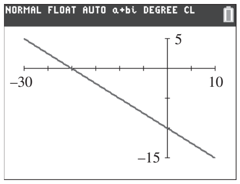
63. $y = 3$



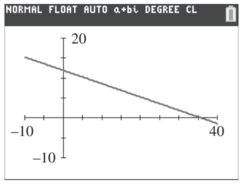
64. $y = -5$



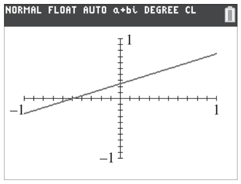
65.



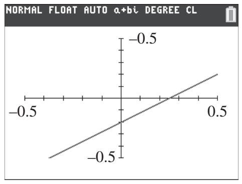
66.



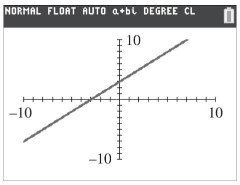
67.



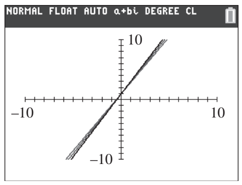
68.



69. The lines look nearly indistinguishable. However, the y-intercepts are different so the lines are different.



70. The lines look nearly indistinguishable. However, the slopes are different so the lines are different.



71. x-intercept (2, 0); y-intercept (0, 3) **73.** x-intercept $(a, 0)$; y-intercept $(0, b)$

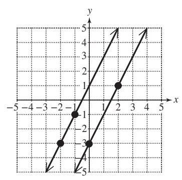
Section 2.2 Activity, p. 167

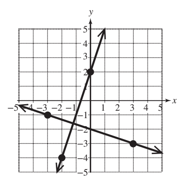
A.1. $m = \frac{1}{24} = 0.041\overline{6}$. Yes, the slope of the shower is approximately 4.2%, which is slightly greater than 4%.

A.2. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ **A.3.** $m = 4$; graph ii

A.4. Slope is undefined; graph iii **A.5.** $m = 0$; graph iv

A.6. $m = -\dfrac{1}{3}$; graph i

A.7. **a.** Line l_1: $m = 2$, Line l_2: $m = 2$ **b.** parallel
c.


A.8. **a.** Line l_1: $m = -\dfrac{1}{3}$, Line l_2: $m = 3$ **b.** perpendicular
c.


A.9. **a.** $m = \frac{9}{5} = 1.8$. The slope is 1.8 and means that the book club grew on average by 1.8 members per month between months 1 and 6.
b. $m = \frac{23}{6} \approx 3.8$. The slope is 3.8 and means that the book club grew on average by 3.8 members per month between months 6 and 12.
c. The growth of the book club increased at a faster rate during the last 6 months.

Section 2.2 Practice Exercises, pp. 168–173

R.1. $-\dfrac{5}{3}$ **R.3.** 0 **R.5.** $\dfrac{5}{9}$

R.7. **a.** $-\dfrac{7}{3}$ **b.** $\dfrac{7}{3}$ **R.9.** \$2.99/week

1. a. slope; $\frac{y_2 - y_1}{x_2 - x_1}$ **b.** parallel; same **c.** right **d.** -1

3. Zero slope **5.** Nonlinear shape

7. $m = \frac{24}{7}$ **9.** $m = \frac{1}{9}$ **11.** $m = \frac{4}{100}$ or $\frac{1}{25}$

13. $m = \frac{1}{2}$ **15.** $m = -\frac{5}{3}$ **17.** $m = -2$

19. $m = -\frac{3}{4}$ **21.** m is undefined. **23.** $m = 0$

25. $m = \frac{1}{2}$ **27.** $m = -\frac{1}{6}$ **29.** $m = 0$

31. Line rising to the right: positive slope Line falling to the right: negative slope Horizontal line: zero slope Vertical line: undefined slope **33.** $m = 0$

35. $m = \frac{1}{10}$ **37.** $m = -1$ **39. a.** $m = 5$ **b.** $m = -\frac{1}{5}$

41. a. $m = -\frac{4}{7}$ **b.** $m = \frac{7}{4}$ **43. a.** $m = 0$ **b.** m is undefined.

45. No, because the product of slopes of perpendicular lines must be -1. The product of two positive numbers is not negative.

47. Undefined **49.** $m = 0$

51. Undefined **53.** $m_1 = 2$; $m_2 = -\frac{1}{2}$; perpendicular

55. $m_1 = -1$; $m_2 = 3$; neither **57.** m_1 is undefined; $m_2 = 0$; perpendicular. One line is horizontal and one is vertical.

59. $m_1 = 1$; $m_2 = 1$; parallel

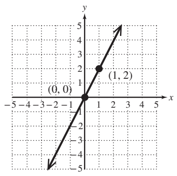
61. a. $m = 20.25$ **b.** The number of cell phone subscriptions increased at a rate of 20.25 million per year during this period.

63. a. $m = 6$ **b.** The weight of boys tends to increase at a rate of 6 lb/yr during this period of growth.

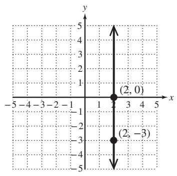
65. a. $m = 2$ **b.** $m = 2$ **c.** $m = 2$

67. a. $m = -2$ **b.** $m = 2$ **c.** $m = 5$

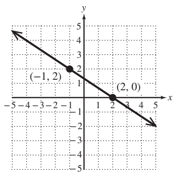
69. For example: (1, 2)



71. For example: (2, 0)



73. For example: (2, 0)



75. $y = 15$

77. a. Pitch: $\frac{1}{6}$ **b.** Slope: $\frac{1}{3}$

Section 2.3 Activity, pp. 180–181

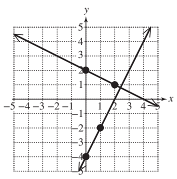
A.1. $y = mx + b$

A.2. a. $y = 2x - 4$; slope: 2; y-intercept $(0, -4)$

b. $y = -\frac{1}{2}x + 2$; slope: $-\frac{1}{2}$; y-intercept $(0, 2)$

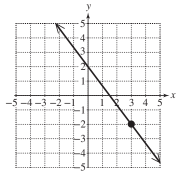
c. Perpendicular. The slope of one line is the opposite of the reciprocal of the slope of the other line.

d.–e. Yes. The two lines appear perpendicular.



A.3. a. $b = 2$ **b.** $y = -\frac{4}{3}x + 2$

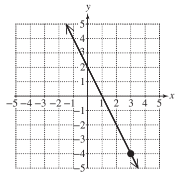
c.



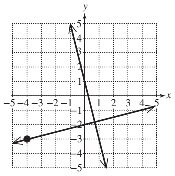
A.4. $y - y_1 = m(x - x_1)$

A.5. a. $y = -2x + 2$

b.



A.6. a. $y = -4x + 1$ **b.** $\frac{1}{4}$ **c.** $y = \frac{1}{4}x - 2$ **d.** The line $y = \frac{1}{4}x - 2$ does indeed pass through $(-4, -3)$ and is perpendicular to the line $4x + y = 1$.

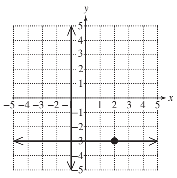


A.7. The slopes of the lines would each be –4 because the slopes of parallel lines are equal.

A.8. a. $m = \frac{5}{2}$ **b.** $y = \frac{5}{2}x + 1$ **c.** $y = \frac{5}{2}x + 1$ **d.** The equations are the same. (This shows that either point on the line can be used with the slope of the line to find an equation of the line.)

A.9. a. Vertical line, and the slope is undefined. **b.** Horizontal line, and the slope is 0.

A.10. a.–b.



c. $y = -3$

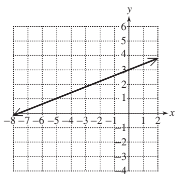
Section 2.3 Practice Exercises, pp. 182–187

R.1. $-\frac{5}{2}$ **R.3.** Parallel: $-\frac{5}{4}$; perpendicular: $\frac{4}{5}$

R.5. Parallel: 0; perpendicular: undefined

R.7. $\{-2\}$ **R.9.** $y = \frac{3}{7}x - 2$

R.11. a. x-intercept: $\left(-\frac{15}{2}, 0\right)$; y-intercept: $(0, 3)$; **b.** Slope: $\frac{2}{5}$

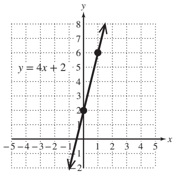


1. a. $y = mx + b$ **b.** standard **c.** horizontal **d.** vertical **e.** slope; y-intercept **f.** $y - y_1 = m(x - x_1)$

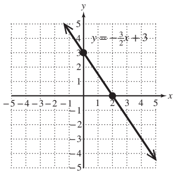
3. Check whether the product of the slopes is -1. If so, this means that the slope of one line is the opposite of the reciprocal of the slope of the other line.

5. Slope: 3; y-intercept: $(0, 1)$ **7.** Slope: $-\frac{2}{3}$; y-intercept: $(0, -4)$ **9.** Slope: 3; y-intercept: $(0, 2)$ **11.** Slope: -17; y-intercept: $(0, 0)$ **13.** Slope: 0; y-intercept: $(0, 9)$ **15.** Slope: $-\frac{2}{3}$; y-intercept: $\left(0, \frac{3}{4}\right)$ **17.** Slope: 0.625; y-intercept: $(0, -1.2)$ **19.** d **21.** f **23.** b

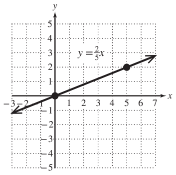
25.



27.

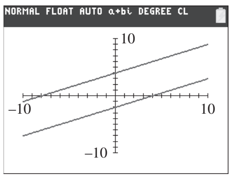


29.

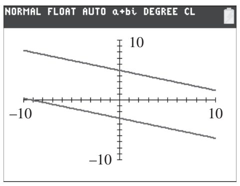


31. $y = -\frac{A}{B}x + \frac{C}{B}$. The slope is given by $m = -\frac{A}{B}$. The y-intercept is $\left(0, \frac{C}{B}\right)$. **33.** Perpendicular **35.** Parallel **37.** Neither **39.** $y = 3x + 5$ **41.** $y = 2x - 11$ **43.** $y = -\frac{4}{5}x + 8$ **45.** $y = 3x - 2$ or $3x - y = 2$ **47.** $y = 2x + 3$ or $2x - y = -3$ **49.** $y = -3x - 11$ or $3x + y = -11$ **51.** $y = -\frac{4}{5}x + \frac{9}{5}$ or $4x + 5y = 9$ **53.** $y = -\frac{4}{3}x + 4$ or $4x + 3y = 12$ **55.** $y = x + 6$ or $x - y = -6$ **57.** $y = 2$ **59.** $y = -\frac{3}{4}x + \frac{17}{4}$ or $3x + 4y = 17$ **61.** $y = \frac{4}{3}x - 2$ or $4x - 3y = 6$ **63.** $y = \frac{3}{4}x - \frac{13}{2}$ or $3x - 4y = 26$ **65.** $y = -\frac{1}{5}x - \frac{13}{5}$ or $x + 5y = -13$ **67.** $y = \frac{3}{2}x - 6$ or $3x - 2y = 12$ **69.** $y = \frac{3}{2}x + 1$ or $3x - 2y = -2$ **71.** $y = -\frac{1}{2}x + \frac{7}{2}$ or $x + 2y = 7$ **73.** $y = -3x$ or $3x + y = 0$ **75.** $y = -3$ **77.** $x = 2$ **79.** $y = 5$ **81.** $x = 5$

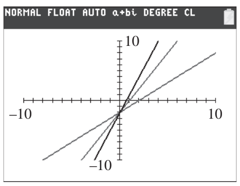
83. The lines have the same slope but different y-intercepts; they are parallel lines.



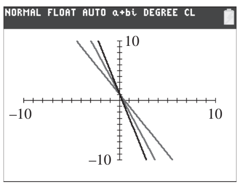
84. The lines have the same slope but different y-intercepts; they are parallel lines.



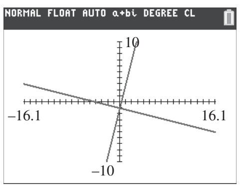
85. The lines have different slopes but the same y-intercept.



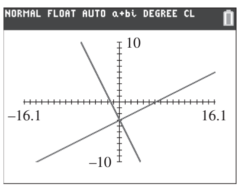
86. The lines have different slopes but the same y-intercept.



87. The lines are perpendicular.



88. The lines are perpendicular.



89. $x = -2$ is not in slope-intercept form. No y-intercept, undefined slope. **91.** $y = 3$ is in slope-intercept form, $y = 0x + 3$. Slope is 0 and y-intercept is $(0, 3)$. **93.** $y = -2x + 3$ **95.** $y = 2$

Chapter 2 Problem Recognition Exercises, p. 187

1. b, f **2.** a, c, d, h **3.** a **4.** b, g **5.** c, e **6.** a **7.** c, h **8.** b **9.** e **10.** g **11.** c, h **12.** a **13.** g **14.** e **15.** h **16.** f **17.** e **18.** g **19.** d, h **20.** b, f

Section 2.4 Activity, p. 192

A.1. a. $y = 2x + 5000$ **b.** 5208 points **c.** \$1500

A.2. a. 24 **b.** 25 **c.** A school with 420 students would have 1 more teacher than a school with 400 students. **d.** The slope is $\frac{1}{20}$ and means that 1 additional teacher is added for every 20 additional students.

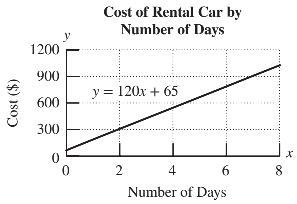
A.3. a. (1, 720) means that in year 1, the crime rate was 720 crimes per 100,000 people in the city. **b.** The slope is -20.5, which means that the crime rate went down by approximately 20.5 crimes per 100,000 people in the city per year. **c.** $y = -20.5x + 740.5$ **d.** The y-intercept is $(0, 740.5)$ and means that in year 0 when the study began, the crime rate was approximately 740.5 crimes per 100,000 people. **e.** 597 crimes per 100,000 people

Section 2.4 Practice Exercises, pp. 192–198

R.1. Slope: $\frac{2}{9}$; y-intercept: $(0, -2)$ **R.3.** 8 **R.5.** $y = -\frac{1}{2}x - \frac{7}{2}$ **R.7.** $y = -7x - 27$

1. a. $y = 120x + 65$

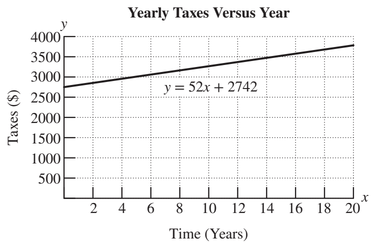
b.



c. (0, 65); The base cost to rent the car is \$65. **d.** A 2-day rental costs \$305. A 5-day rental costs \$665, and a 7-day rental costs \$905. **e.** Yes; \$799 is less expensive than \$905. **f.** \$577.70 **g.** No, the car cannot be driven for a negative number of days.

3. a. $y = 52x + 2742$

b.

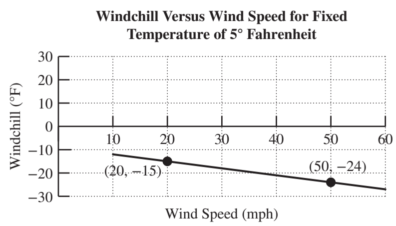


c. $m = 52$. Taxes increase at a rate of \$52 per year. **d.** (0, 2742). Initial year ($x = 0$) taxes were \$2742. **e.** 10 years: \$3262; 15 years: \$3522

5. a. 0.8 mi, 2.4 mi, and 3.2 mi **b.** 21 sec

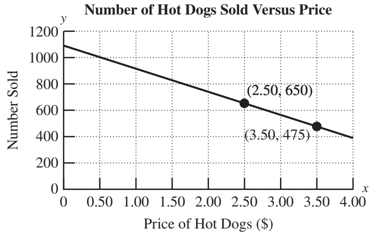
7. a. \$73.30 **b.** \$54.50; The approximate value differs from the actual value by \$1.30. **c.** $m = 9.4$; The amount spent per person on video games increased by an average rate of \$9.40 per year. **d.** (0, 35.7); The y-intercept means that the average amount spent on video games per person was \$35.70 at the start of the study (year 0).

9. a.



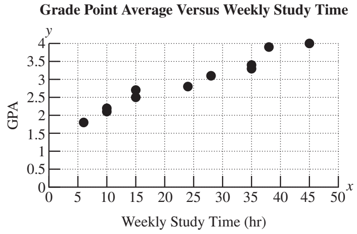
b. $y = -0.3x - 9$ **c.** −21°F **d.** −22.8°F **e.** $m = -0.3$; This means that windchill decreases at a rate of 0.3°F for every 1 mph increase in wind speed. **11. a.** $y = 15x + 155$ **b.** The slope is 15 and means that the number of associate degrees awarded in the United States increased by 15 thousand per year. **c.** 830 thousand (equivalently 830,000)

13. a.



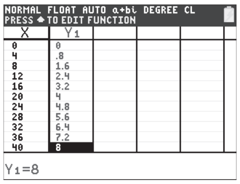
b. $y = -175x + 1087.5$ **c.** Approximately 388 hot dogs would be sold.

15. a.



b. yes **c.** $y = 0.05x + 1.7$ **d.** 3.2 **e.** This model is not reasonable for study times greater than 46 hr per week, because the GPA would exceed 4.0.

17.



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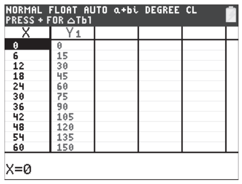

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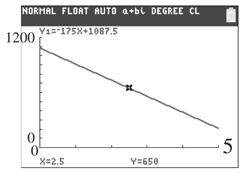
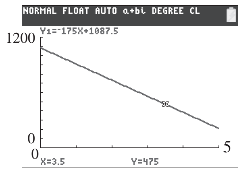
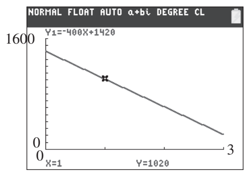
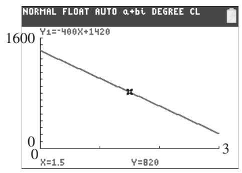




20.





21. Collinear **23.** Not collinear

Section 2.5 Activity, p. 204

A.1. a. x values **b.** $\{-3, 0, 2, -1\}$ **c.** range **d.** $\{1, 4, -5, -4\}$

A.2. a. $\{A, E, I, O, U\}$ **b.** $\{1, 9, 15, 21, 5\}$

A.3. a. $\{-4, -2, 0, 2, 3\}$ **b.** $\{-3, -2, 1, 2, 4\}$

A.4. a. −4; 4 **b.** $[-4, 4]$ **c.** −2; 4 **d.** $[-2, 4]$

A.5. a. The point $(1, -1)$ is *not* included in the graph.

b. Infinitely far to the left; up to but not including $x = 1$ to the right.

c. $(-\infty, 1)$ **d.** −2; no **e.** $[-2, \infty)$

Section 2.5 Practice Exercises, pp. 205–209

R.1. $(-6, \infty)$ **R.3.** $\left(-\infty, \frac{3}{4}\right]$ **R.5.** $(0.5, 1.2]$

R.7. $(-4, 1)$ **R.9.** $(1, -4)$ **R.11.** $(4, 0)$

1. a. relation **b.** domain **c.** range

3. a. {(Northeast, 54.1), (Midwest, 65.6), (South, 110.7), (West, 70.7)} **b.** Domain: {Northeast, Midwest, South, West}; Range: {54.1, 65.6, 110.7, 70.7} **5. a.** {(USSR, 1961), (USA, 1962), (Poland, 1978), (Vietnam, 1980), (Cuba, 1980)} **b.** Domain: {USSR, USA, Poland, Vietnam, Cuba}; range: {1961, 1962, 1978, 1980}

7. a. {(A, 1), (A, 2), (B, 2), (C, 3), (D, 5), (E, 4)}

b. Domain: {A, B, C, D, E}; range: {1, 2, 3, 4, 5}

9. a. {(6, A), (6, B), (6, C)} **b.** Domain: {6}; range: {A, B, C}

11. a. $\{(-4, 1), (0, 3), (1, -1), (3, 0)\}$

b. Domain: $\{-4, 0, 1, 3\}$; range: $\{-1, 0, 1, 3\}$

13. a. $\{(-4, 4), (1, 1), (2, 1), (3, 1), (4, -2)\}$

b. Domain: $\{-4, 1, 2, 3, 4\}$; range: $\{-2, 1, 4\}$

15. Domain: $[0, 4]$; range: $[-2, 2]$ **17.** Domain: $[-5, 3]$; range: $[-2.1, 2.8]$ **19.** Domain: $(-\infty, 2]$; range: $(-\infty, \infty)$

21. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$ **23.** Domain: $\{-3\}$; range: $(-\infty, \infty)$ **25.** Domain: $(-\infty, 2)$; range: $[-1.3, \infty)$

27. Domain: $\{-3, -1, 1, 3\}$; range: $\{0, 1, 2, 3\}$

29. Domain: $[-4, 5)$; range: $\{-2, 1, 3\}$ **31. a.** 2.85

b. 9.33 **c.** Dec. **d.** Nov. **e.** 7.63 **f.** {Jan., Feb., Mar., Apr., May, June, July, Aug., Sept., Oct., Nov., Dec.}

33. a. 34.2 million or 34,200,000 **b.** The year 2019

35. a. For example: {(Julie, New York), (Peggy, Florida), (Stephen, Kansas), (Pat, New York)} **b.** Domain: {Julie, Peggy, Stephen, Pat}; range: {New York, Florida, Kansas}

37. $y = 2x - 1$ **39.** $y = x^2$

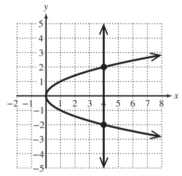
Section 2.6 Activity, p. 216

A.1. The relation defines y as a function of x if for each element x in the domain, there is exactly one value of y in the range.

A.2. a. Yes. The ordered pairs $(-5, 1)$ and $(-5, 2)$ have the same x value, but different y values. **b.** No

A.3. a. No **b.** Yes

A.4. a. Yes. The vertical line intersects the graph at $(4, 2)$ and $(4, -2)$.



b. No **c.** Two points on a graph that are aligned vertically must have the same x-coordinate, but different y-coordinates. In such a case the graph fails to define y as a function of x.

A.5. a. 7 **b.** (3, 7) **A.6. a.** $f(3) = 7$ **b.** (3, 7)

c. The results are the same because evaluating a function for a given value of x indicates that we substitute the value for x into the function and solve for y.

A.7. a. $g(2) = 4$ **b.** $g(-3) = -1$ **c.** $g(0) = -4$

d. Not possible. Substituting 1 for x would make the denominator zero.

A.8. a. $x = 1$ must be excluded because substituting 1 into the function would make the denominator equal to zero. Division by zero is undefined.

b. $(-\infty, 1) \cup (1, \infty)$

A.9. a. $h(-1) = 1$ **b.** $h(2) = 2$ **c.** $h(7) = 3$

d. Not possible. Substituting −6 for x would result in the square root of a negative number.

A.10. a. The radicand $x + 2$ must be greater than or equal to zero. That is, $x + 2 \geq 0$. This means that $x \geq -2$. **b.** $[-2, \infty)$

A.11. a. $f(-2) = 14$ **b.** $f(t) = t^2 - 3t + 4$ **c.** $f(a + 2) = a^2 + a + 2$

Section 2.6 Practice Exercises, pp. 217–222

R.1. -16 **R.3.** 17 **R.5.** $x = 5$ **R.7.** $x = -3, x = 3$

R.9. $[4, \infty)$ **R.11.** $(-6, -5]$ **R.13.** $\left[\frac{3}{2}, \infty\right)$ **R.15.** $(-\infty, 4]$

1. a. function **b.** vertical **c.** $2x + 1$ **d.** x values **e.** y values **f.** denominator **g.** negative **3.** h **5.** Function **7.** Not a function **9.** Function **11.** Not a function **13.** Function **15.** Not a function **17.** When x is 2, the function value y is 5. **19.** $(0, -2)$ **21.** -11 **23.** 1 **25.** 2 **27.** $6t - 2$ **29.** 7 **31.** 4 **33.** 4 **35.** $6x + 4$ **37.** $-4x^2 - 8x + 1$ **39.** $-\pi^2 + 4\pi + 1$ **41.** 7 **43.** $-6a - 2$ **45.** $|-c - 2|$ **47.** 1 **49.** 7 **51.** -18.8 **53.** -7 **55.** 2π **57.** -5 **59.** 4 **61. a.** 3 **b.** 1 **c.** 1 **d.** $x = -3$ **e.** $x = 0, x = 2$ **f.** $(-\infty, 3]$ **g.** $(-\infty, 5]$

63. a. 3 **b.** $H(4)$ is not defined because 4 is not in the domain of H. **c.** 4 **d.** $x = -3$ and $x = 2$ **e.** All x on the interval $[-2, 1]$ **f.** $[-4, 4)$ **g.** $[2, 5)$

65. a. -4 **b.** 0 **c.** -3 **d.** $x = -1$ **e.** There are no such values of x. **f.** $(-\infty, \infty)$ **g.** $(-\infty, -3] \cup (-2, \infty)$

67. $\{-3, -7, -\frac{3}{2}, 1.2\}$ **69.** $\{6, 0\}$ **71.** $x = -3$ and $x = 1.2$ **73.** $x = 6$ and $x = 1$ **75.** -3

77. The domain is the set of all real numbers for which the denominator is not zero. Set the denominator equal to zero, and solve the resulting equation. The solution(s) to the equation must be excluded from the domain. In this case, setting $x - 2 = 0$ indicates that $x = 2$ must be excluded from the domain. The domain is $(-\infty, 2) \cup (2, \infty)$.

79. $(-\infty, -6) \cup (-6, \infty)$ **81.** $(-\infty, 0) \cup (0, \infty)$ **83.** $(-\infty, \infty)$ **85.** $[-7, \infty)$ **87.** $[3, \infty)$ **89.** $(-\infty, \frac{1}{2}]$ **91.** $(-\infty, \infty)$ **93.** $(-\infty, \infty)$

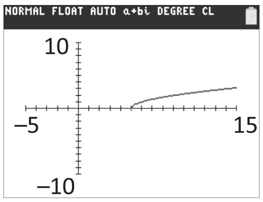
95. a. $h(1) = 64$ and $h(1.5) = 44$ **b.** $h(1) = 64$ means that after 1 sec, the height of the ball is 64 ft. $h(1.5) = 44$ means that after 1.5 sec, the height of the ball is 44 ft.

97. a. $d(1) = 11.5$ and $d(1.5) = 17.25$ **b.** $d(1) = 11.5$ means that after 1 hr, the distance traveled is 11.5 mi. $d(1.5) = 17.25$ means that after 1.5 hr, the distance traveled is 17.25 mi.

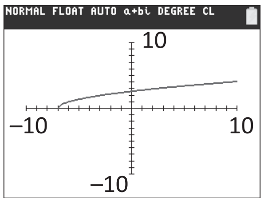
99. $f(x) = 2x + 3$

101. $f(x) = |x| - 10$

103.



104.



105. $(-2, \infty)$

Section 2.7 Activity, p. 228

A.1. a. Slanted line **b.** $f(x) = -3x - 2$ **c.** Linear function

A.2. a. Horizontal line **b.** $f(x) = 5$ **c.** Constant function

A.3. a. Parabola **b.** $f(x) = x^2 - 4x + 3$ **c.** Quadratic function

A.4. a. Substitute 0 for $f(x)$ and solve the resulting equation for x.

b. $\left(-\frac{2}{3}, 0\right)$

c. Substitute 0 for x. Using function notation, this is equivalent to evaluating $f(0)$.

d. $(0, -2)$

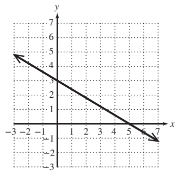
A.5. x-intercept: none; y-intercept: $(0, 5)$

A.6. x-intercepts: $(1, 0)$ and $(3, 0)$; y-intercept: $(0, 3)$

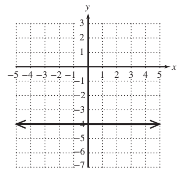
Section 2.7 Practice Exercises, 229–234

R.1. Vertical line **R.3.** Slanted line

R.5. x-intercept: $(5, 0)$; y-intercept: $(0, 3)$



R.7. x-intercept: none; y-intercept: $(0, -4)$

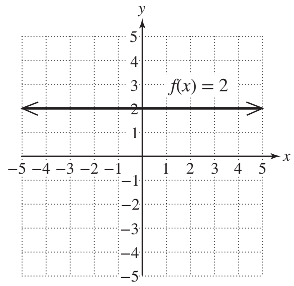


1. a. linear **b.** constant **c.** quadratic **d.** parabola **e.** 0 **f.** y

3. a. $g(1) = -3$ **b.** $g(2) = -3$ **c.** $g(3) = -3$

5. a. $k(1) = -3$ **b.** $k(2) = -6$ **c.** $k(3) = -9$ **7.** horizontal

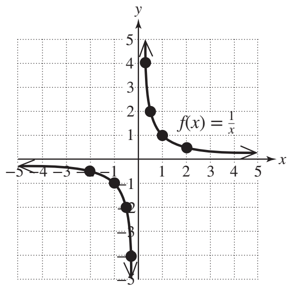
9. Domain $(-\infty, \infty)$; range $\{2\}$



11.

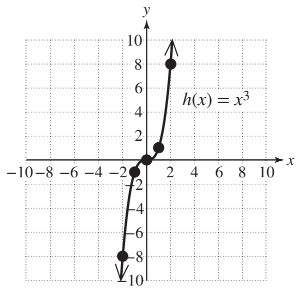
x	$f(x)$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{4}$	-4

x	$f(x)$
$\frac{1}{4}$	4
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$



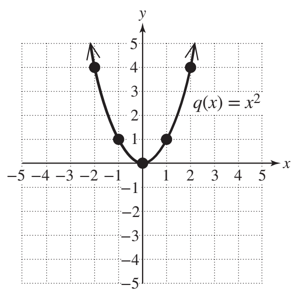
13.

x	$h(x)$
-2	-8
-1	-1
0	0
1	1
2	8



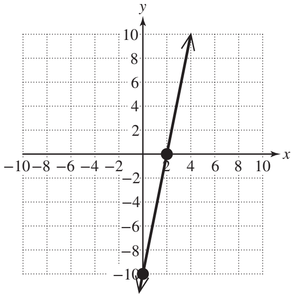
15.

x	$q(x)$
-2	4
-1	1
0	0
1	1
2	4

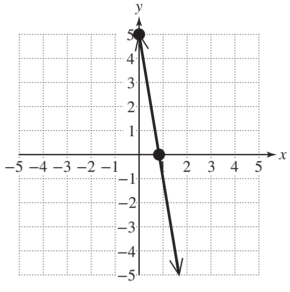


17. Quadratic **19.** Linear **21.** Constant **23.** None of these **25.** Linear **27.** None of these

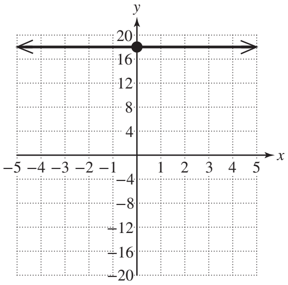
29. x-intercept: (2, 0); y-intercept: (0, −10)



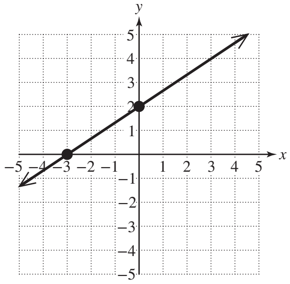
31. x-intercept: $\left(\frac{5}{6}, 0\right)$; y-intercept: (0, 5)



33. x-intercept: none; y-intercept: (0, 18)



35. x-intercept: (−3, 0); y-intercept: (0, 2)



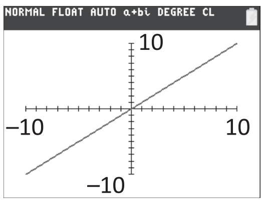
37. a. $x = -1$ **b.** $f(0) = 1$ **39. a.** $x = -2$ and $x = 2$ **b.** $f(0) = -2$ **41. a.** None **b.** $f(0) = 2$ **43. a.** $(-\infty, \infty)$ **b.** (0, 0) **c.** vi **45. a.** $(-\infty, \infty)$ **b.** (0, 1) **c.** viii **47. a.** $[-1, \infty)$ **b.** (0, 1) **c.** vii

49. a. $(-\infty, 3) \cup (3, \infty)$ **b.** $\left(0, -\frac{1}{3}\right)$ **c.** ii

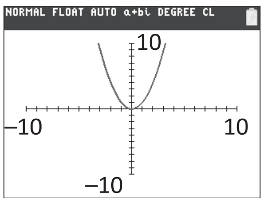
51. a. $(-\infty, \infty)$ **b.** (0, 2) **c.** iv **53. a.** Linear
b. $G(90) = 82.5$. This means that if the student gets a 90% score on her final exam, then her overall course average is 82.5%.
c. $G(50) = 72.5$. This means that if the student gets a 50% score on her final exam, then her overall course average is 72.5%.

55. $f(x) = x^2 + 2$ **57.** $f(x) = 3$ **59.** $f(x) = \frac{1}{2}x - 2$

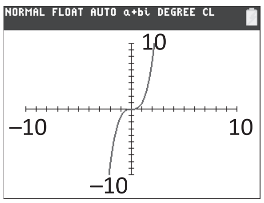
61.



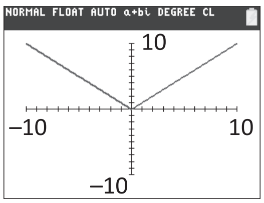
62.



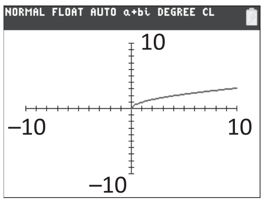
63.



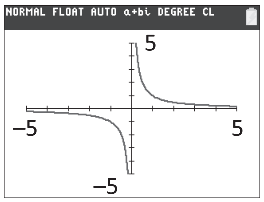
64.



65.



66.



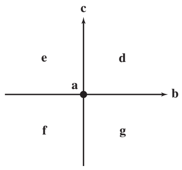
67. x-intercept: (8, 0); y-intercept: (0, 1) **68.** x-intercept: (−6, 0); y-intercept: (0, −3) **69.** x-intercept: (−5, 0); y-intercept: (0, 4) **70.** x-intercept: (2, 0); y-intercept: (0, −7)

Chapter 2 Problem Recognition Exercises, p. 234

1. a, c, d, f, g **2.** a, b, d, f, h **3.** 4 **4.** 2
5. $\{0, 1, \frac{1}{2}, -3, 2\}$ **6.** {4, 3, 6, 1, 10} **7.** [−2, 4]
8. $[0, \infty)$ **9.** 0 **10.** $\left(\frac{9}{5}, 0\right)$ **11.** (0, −1), (0, 1)
12. 1 **13.** c **14.** d **15.** 3

Chapter 2 Review Exercises, pp. 241–246

1.

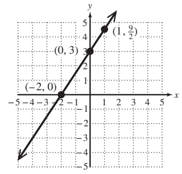


2. No **3.** Yes

4. $A(0, 0)$; $B(2, 1)$; $C(0, -4)$; $D(-2, -4)$; $E(-2, 0)$; $F(-5, 1)$; $G(4, -3)$

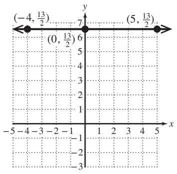
5.

x	y
0	3
−2	0
1	$\frac{9}{2}$



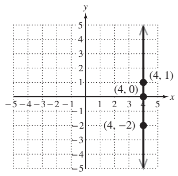
6.

x	y
0	$\frac{13}{2}$
5	$\frac{13}{2}$
−4	$\frac{13}{2}$

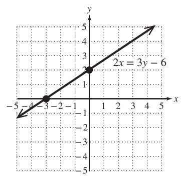


7.

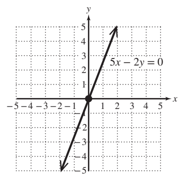
x	y
4	0
4	1
4	−2



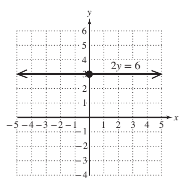
8. x-intercept (−3, 0); y-intercept (0, 2)



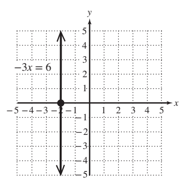
9. x- and y-intercept (0, 0)



10. No x-intercept; y-intercept (0, 3)

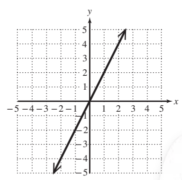


11. x-intercept (−2, 0); no y-intercept

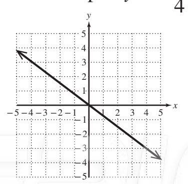


12. a. $m = \frac{1}{2}$ **b.** $m = -3$ **c.** $m = 0$

13. For example: $y = 2x$



14. For example: $y = -\frac{3}{4}x$



15. $m = 2$ **16.** $m = \frac{7}{10}$ **17.** $m = 0$

18. Undefined slope **19.** Perpendicular

20. Perpendicular **21.** Neither **22.** Parallel

23. a. $m = 53$ **b.** The enrollment increased at a rate of 53 students per year.

24. $m = \frac{3}{4}$ **25. a.** $y = k$ **b.** $y - y_1 = m(x - x_1)$ **c.** $Ax + By = C$ **d.** $x = k$ **e.** $y = mx + b$

26. $y = \frac{1}{9}x + 6$ or $x - 9y = -54$

27. $y = -\frac{2}{3}x + 2$ or $2x + 3y = 6$

28. $y = \frac{10}{3}x + \frac{77}{3}$ or $10x - 3y = -77$

29. $y = 3x - 20$ or $3x - y = 20$

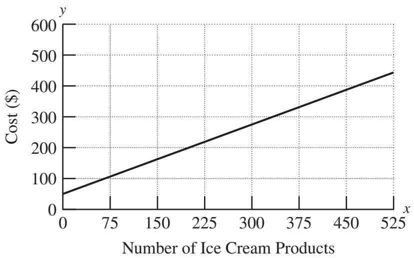
30. $y = -\frac{4}{3}x - 3$ or $4x + 3y = -9$

31. a. $y = -2$ **b.** $x = -3$ **c.** $x = -3$ **d.** $y = -2$

32. Yes, (a) and (d) are the same; (b) and (c) are the same.

33. a. $y = 0.75x + 50$

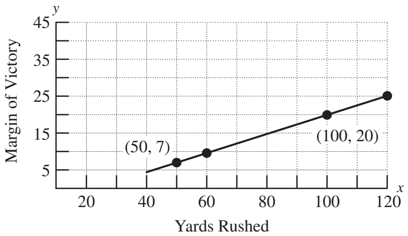
b.



c. The daily fixed cost is $50 if no ice cream is sold.

d. $387.50 **e.** 0.75 **f.** The cost increases at a rate of $0.75 per ice cream product.

34. a.



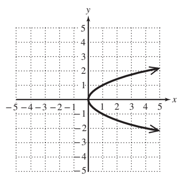
b. $y = 0.26x - 6$ **c.** The margin of victory would be −6 points, which means they would lose by 6 points.

35. Domain $\left\{\frac{1}{3}, 6, \frac{1}{4}, 7\right\}$; range $\left\{10, -\frac{1}{2}, 4, \frac{2}{5}\right\}$

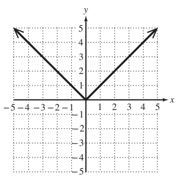
36. Domain $\{-3, -1, 0, 2, 3\}$; range $\left\{-2, 0, 1, \frac{5}{2}\right\}$

37. Domain $[-3, 9]$; range $[0, 60]$ **38.** Domain $[-4, 4]$; range $[1, 3]$

39.



40.



41. a. Not a function **b.** $[1, 3]$ **c.** $[-4, 4]$

42. a. Function **b.** $(-\infty, \infty)$ **c.** $(-\infty, 0.35]$

43. a. Function **b.** $\{1, 2, 3, 4\}$ **c.** $\{3\}$

44. a. Not a function **b.** $\{0, 4\}$ **c.** $\{2, 3, 4, 5\}$

45. a. Not a function **b.** $\{4, 9\}$ **c.** $\{2, -2, 3, -3\}$

46. a. Function **b.** $\{6, 7, 8, 9\}$ **c.** $\{9, 10, 11, 12\}$

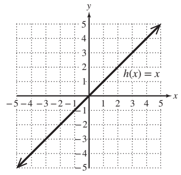
47. −4 **48.** 2 **49.** 2 **50.** $6t^2 - 4$ **51.** $6b^2 - 4$

52. $6\pi^2 - 4$ **53.** $6a^2 - 4$ **54.** 20 **55.** $(-\infty, \infty)$

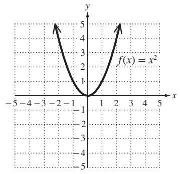
56. $(-\infty, 11) \cup (11, \infty)$ **57.** $[8, \infty)$ **58.** $[-2, \infty)$

59. a. $98 **b.** $123 **c.** $148

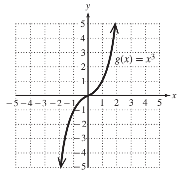
60.



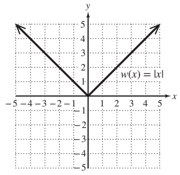
61.



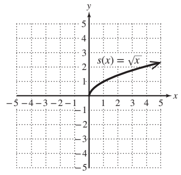
62.



63.



64.



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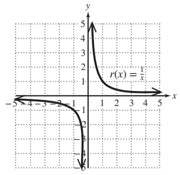
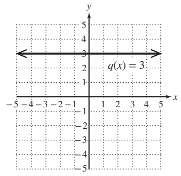
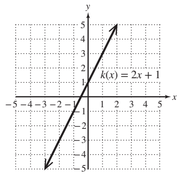


66.



67.



68. a. $s(4) = 4$, $s(-3) = 25$, $s(2) = 0$, $s(1) = 1$, $s(0) = 4$ **b.** $(-\infty, \infty)$ **69. a.** $r(2)$ is not a real number, $r(4) = 0$, $r(5) = 2$, $r(8) = 4$ **b.** $[4, \infty)$

70. a. $h(-3) = -\frac{1}{2}$, $h(0) = -1$, $h(2) = -3$, $h(5) = \frac{3}{2}$ **b.** $(-\infty, 3) \cup (3, \infty)$

71. a. $k(-5) = -2$, $k(-4) = -1$, $k(-3) = 0$, $k(2) = -5$ **b.** $(-\infty, \infty)$ **72.** x-intercept: $\left(\frac{7}{4}, 0\right)$; y-intercept: $(0, -7)$

73. x-intercept: $\left(\frac{9}{2}, 0\right)$; y-intercept: $(0, 9)$

74. a. $b(0) = 28.3$. In 2010 consumption was 28.3 gal of bottled water per capita. $b(5) = 36.5$. In 2010 consumption was 36.5 gal of bottled water per capita. **b.** $m = 1.64$. Consumption increased at a rate of 1.64 gal/year.

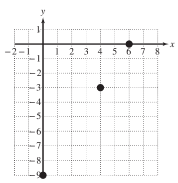
75. −1 **76.** 0 **77.** $x = 0$ and $x = 4$

78. There are no values of x for which $g(x) = 4$

79. $(-4, \infty)$ **80.** $(-\infty, 1]$

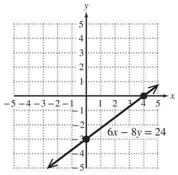
Chapter 2 Test, pp. 246–249

1. (0, −9) (6, 0) (4, −3)

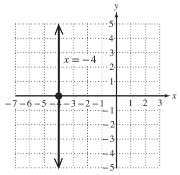


2. a. False, the product is positive in quadrant III also. **b.** False, the quotient is negative in quadrant II also. **c.** True **d.** True **3.** Yes **4.** To find the x-intercept, let $y = 0$ and solve for x. To find the y-intercept, let $x = 0$ and solve for y.

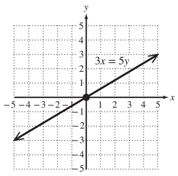
5. x-intercept: $(4, 0)$; y-intercept: $(0, -3)$



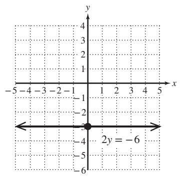
6. x-intercept: $(-4, 0)$; no y-intercept



7. x- and y-intercept: $(0, 0)$



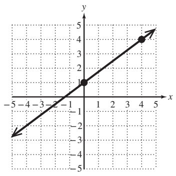
8. no x-intercept; y-intercept: $(0, -3)$



9. a. $m = \dfrac{5}{8}$ **b.** $m = \dfrac{6}{5}$

10. a. $y = \frac{3}{4}x + 1$ **b.** Slope: $\frac{3}{4}$; y-intercept: $(0, 1)$

c.



11. a. -7 **b.** $\dfrac{1}{7}$ **12. a.** The slopes are the same. **b.** The slope of one line is the opposite of the reciprocal of the slope of the other. **13.** Neither

14. a. Perpendicular **b.** Parallel **c.** Perpendicular **d.** Neither

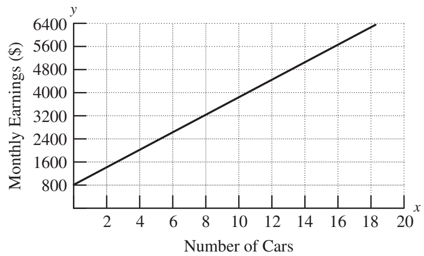
15. a. For example: $y = 3x + 2$ **b.** For example: $x = 2$ **c.** For example: $y = 3$; $m = 0$ **d.** For example: $y = -2x$

16. $y = \frac{3}{2}x - 6$ or $3x - 2y = 12$ **17.** $y = 2x - 11$ or $2x - y = 11$

18. $y = -2x + \dfrac{31}{2}$ **19.** $y = \dfrac{1}{3}x + \dfrac{1}{3}$

20. a. $y = 300x + 800$

b.

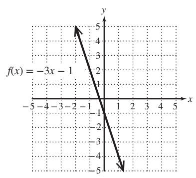


c. Jack earns a base monthly salary of $800. **d.** $5900

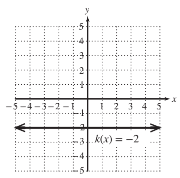
21. a. $(0, 66)$. For a woman born in 1940, life expectancy was about 66 yr. **b.** $m = \dfrac{3}{10}$ or $m = 0.3$; Life expectancy rises at a rate of 3 yr for every 10 yr. **c.** $y = \dfrac{3}{10}x + 66$

d. 82.2 yr; It is greater than the reported value.

22.



23.



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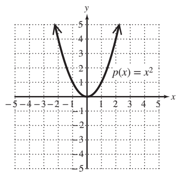
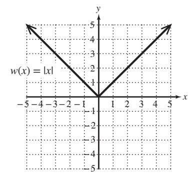


25.



26. a. Not a function **b.** $\{-3, -1, 1, 3\}$ **c.** $\{-2, -1, 1, 3\}$ **27. a.** Function **b.** $(-\infty, \infty)$ **c.** $(-\infty, 0]$ **28.** $(-\infty, -7) \cup (-7, \infty)$ **29.** $[-7, \infty)$ **30.** $(-\infty, \infty)$ **31. a.** $r(-2) = 9$, $r(0) = 1$, $r(3) = 4$ **b.** $(-\infty, \infty)$ **32. a.** $s(0) = 0.96$. At the beginning of the study, (year 0), the per capita consumption was 0.96 c per day. $s(20) = 0.8$. In year 20, the per capita consumption was 0.8 c per day. **b.** $m = -0.008$. Consumption decreased at a rate of 0.008 c per day. **33.** Quadratic **34.** Linear **35.** Constant **36.** None of these **37.** To find the x-intercept(s), solve for the real solutions of the equation $f(x) = 0$. To find the y-intercept, find $f(0)$. **38.** x-intercept: $(-12, 0)$; y-intercept: $(0, 9)$ **39.** 1 **40.** 2 **41.** $(-1, 7]$ **42.** $[-1, 4)$ **43.** False **44.** $(6, 0)$ **45.** $x = 6$ **46.** All x in the interval $[1, 3]$ and $x = 5$

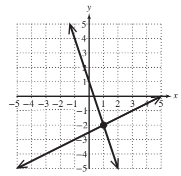
Chapter 3

Section 3.1 Activity, pp. 256–257

A.1. a. $y = -3x + 1$ and $y = \dfrac{1}{2}x - \dfrac{5}{2}$ **b.** -3 and $\dfrac{1}{2}$

c. The lines intersect at a single point

d.

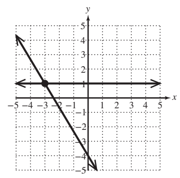


e. $(1, -2)$ **f.** The point $(1, -2)$ checks in both equations.

e. $(1, -2)$ **f.** The point $(1, -2)$ checks in both equations.

A.2. a. horizontal; slanted **b.** one

c.

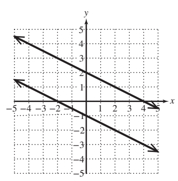


d. $\{(-3, 1)\}$

A.3. a. $y = -\dfrac{1}{2}x + 2$ and $y = -\dfrac{1}{2}x - 1$ **b.** $-\dfrac{1}{2}$ and $-\dfrac{1}{2}$

c. The lines are parallel.

d.

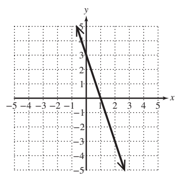


e. No solution; { } **f.** The system is inconsistent.

A.4. a. $y = -3x + 3$ and $y = -3x + 3$ **b.** -3 and -3

c. The equations represent the same line.

d.



e. There are infinitely many solutions.

f. $\{(x, y) \mid y = -3x + 3\}$

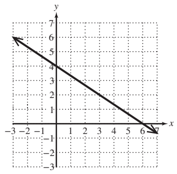
g. The equations are dependent.

Section 3.1 Practice Exercises, pp. 257–262

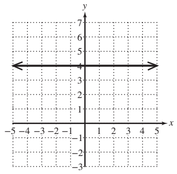
R.1. Slope: $-\frac{2}{3}$; x-intercept: (6, 0); y-intercept: (0, 4)

R.3. Slope: 0; x-intercept: none; y-intercept: (0, 4)

R.5.

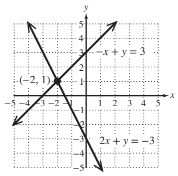


R.7.

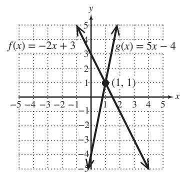


1. a. system **b.** solution **c.** intersect **d.** consistent **e.** the empty set, { } **f.** dependent **g.** independent **3.** (2, 11) is a solution. **5.** $(-1, 4)$ is a solution. **7.** None **9. a.** Consistent **b.** Independent **c.** One solution **11. a.** Inconsistent **b.** Independent **c.** Zero solutions **13. a.** Consistent **b.** Dependent **c.** Infinitely many solutions

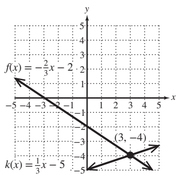
15. $\{(-2, 1)\}$



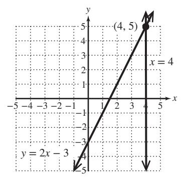
17. $\{(1, 1)\}$



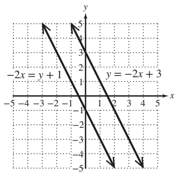
19. $\{(3, -4)\}$



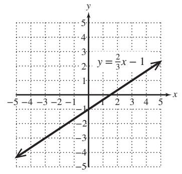
21. $\{(4, 5)\}$



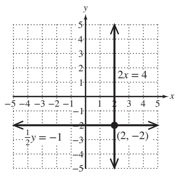
23. No solution; { }; inconsistent system



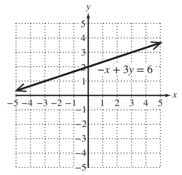
25. Infinitely many solutions; $\left\{(x, y) \middle| y=\frac{2}{3}x-1\right\}$; dependent equations



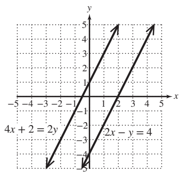
27. $\{(2, -2)\}$



29. Infinitely many solutions; $\{(x, y) \mid -x+3y=6\}$; dependent equations

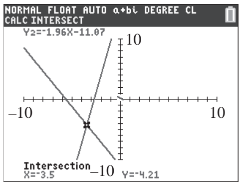


31. No solution; { }; inconsistent system

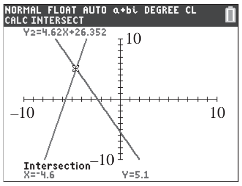


33. False **35.** True

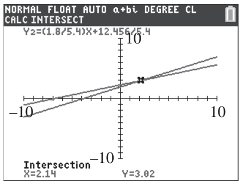
37. $\{(-3.5, -4.21)\}$



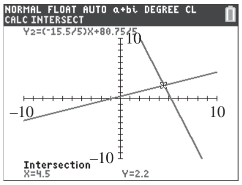
38. $\{(-4.6, 5.1)\}$



39. $\{(2.14, 3.02)\}$



40. $\{(4.5, 2.2)\}$



41. For example: $x+y=9$
$2x+y=13$ **43.** $C=5, D=4$

Section 3.2 Activity, p. 267

A.1. a. $5x+3(2x-6)=-7$ **b.** $x=1$ **c.** $y=-4$; $\{(1, -4)\}$ **d.** The ordered pair $(1, -4)$ checks in both equations.

A.2. a. The x variable in the second equation is easiest to isolate. **b.** $\{(-3, 5)\}$ **c.** The ordered pair $(-3, 5)$ checks in both equations.

A.3. a. The x variable in the first equation is easiest to isolate. **b.** { } **c.** No solution **d.** Inconsistent

A.4. a. The y variable in the first equation is easiest to isolate. **b.** $\left\{(x, y) \middle| y=\frac{4}{5}x-1\right\}$ **c.** Infinitely many **d.** Dependent

Section 3.2 Practice Exercises, pp. 267–270

R.1. $y=3x-4$; $y=3x-4$; coinciding lines

R.3. $y=\frac{2}{3}x-\frac{5}{3}$; $y=-\frac{5}{6}x+\frac{13}{3}$; intersecting lines

R.5. $y=\frac{3}{5}x$; $y=\frac{3}{5}x-\frac{4}{5}$; parallel lines

R.7. $x=5$ **R.9.** $y=9$ **R.11. a.** Contradiction **b.** { }

R.13. a. Identity **b.** The set of real numbers

R.15. a. Conditional equation **b.** {11}

1. $5-2y$ **3.** $x=3$; $\{(3, 1)\}$ **5.** No solution **7.** No; the ordered pair is not a solution to the second equation. **9.** $\{(-1, 2)\}$ **11.** $\{(3, 1)\}$ **13.** $\{(12, 0)\}$ **15.** $\{(5, -3)\}$ **17.** $\left\{\left(-\frac{1}{4}, \frac{1}{2}\right)\right\}$ **19.** $\{(2, 0)\}$ **21.** $\{(1, -2)\}$ **23.** Infinitely many solutions; $\{(x, y) \mid x=3y-1\}$; dependent equations **25.** No solution; { }; inconsistent system **27.** No solution; { }; inconsistent system **29.** Infinitely many solutions; $\{(x, y) \mid 3x-y=7\}$; dependent equations **31.** When solving a system, if you get an identity, such as $0=0$ or $5=5$, then the equations are dependent. **33.** $\{(8, 5)\}$ **35.** $\{(5, 3)\}$ **37.** No solution; { }; inconsistent system **39.** $\{(0, 3)\}$ **41.** $\{(-7, 6)\}$ **43.** Infinitely many solutions; $\{(x, y) \mid 2x-y=6\}$; dependent equations **45.** $\{(10, -32.1)\}$ **47.** $\left\{\left(\frac{1}{2}, \frac{3}{4}\right)\right\}$ **49.** $\{(-2, 4)\}$ **51.** $\{(1, 1)\}$ **53.** $\{(6, 1)\}$ **55.** $\{(5, -1)\}$ **57. a.** $y=\frac{4}{9}x+\frac{25}{9}$ **b.** $y=-\frac{4}{7}x+\frac{23}{7}$ **c.** $\left(\frac{1}{2}, 3\right)$ **59. a.** At Glendale Lakes: $y=800x+250$
At the Breakers: $y=750x+500$ **b.** 5 months

Section 3.3 Activity, pp. 275–276

A.1. a. $7x + 0y = -28$ **b.** $-3x + 9y = 15$ **c.** In part (a), the variable y was eliminated because the coefficients on the terms $-3y$ and $3y$ are opposites. **A.2. a.** $\{(-4, -2)\}$ **b.** The ordered pair $(-4, -2)$ checks in both equations. **A.3. a.** The x variable **b.** $\{(-2, 1)\}$ **c.** The ordered pair $(-2, 1)$ checks in both equations.

A.4. a. 10 **b.** 100

c. $5x - 4y = -7$
$2x + 3y = 11$

d. $\{(1, 3)\}$ **e.** The ordered pair (1, 3) checks in both equations.

A.5. a. 15 **b.** $3x - 30y = 5$
$6x - 60y = 10$

c. $\{(x, y) \mid 3x - 30y = 5\}$ **d.** Infinitely many

A.6. a. { } **b.** No solution

Section 3.3 Practice Exercises, pp. 276–279

R.1. a. No **b.** Yes **R.3.** { }

R.5. $\left\{\frac{2}{5}\right\}$ **R.7.** The set of real numbers

1. a. -3 **b.** 5 **3.** Multiply the first equation by -5 and the second equation by 4. Then add the equations.
5. $\{(-2, -5)\}$ **7.** $\{(3, -1)\}$ **9.** $\{(12, -8)\}$
11. $\{(1, -1)\}$ **13.** $\{(2, -1)\}$ **15.** $\{(0, 0)\}$
17. Infinitely many solutions; $\{(x, y) \mid 3x - 2y = 1\}$; dependent equations **19.** No solution; { }; inconsistent system
21. Infinitely many solutions; $\{(x, y) \mid 12x - 4y = 2\}$; dependent equations **23.** No solution; { }; inconsistent system
25. Use the substitution method if one equation has x or y already isolated. **27.** False **29.** True **31.** True
33. $\{(-2, -3)\}$ **35.** $\left\{\left(-\frac{1}{2}, 2\right)\right\}$ **37.** $\{(4, 0)\}$
39. No solution; { }; inconsistent system **41.** $\{(12, 30)\}$
43. Infinitely many solutions; $\left\{(x, y) \middle| x = \frac{3}{2}y\right\}$; dependent equations
45. $\{(0, 4)\}$ **47.** $\left\{\left(-\frac{1}{2}, -\frac{5}{2}\right)\right\}$ **49.** $\{(30, 10)\}$ **51.** $\left\{\left(\frac{4}{3}, -1\right)\right\}$
53. $\{(1, 5)\}$ **55.** No solution; { }; inconsistent system
57. 112 mi in the city and 120 mi on the highway.
59. $\left\{\left(-\frac{56}{41}, \frac{221}{41}\right)\right\}$ **61.** $\left\{\left(-\frac{13}{7}, -\frac{37}{14}\right)\right\}$

Chapter 3 Problem Recognition Exercises, p. 279

1. $\{(2, 4)\}$ **2.** Infinitely many solutions; $\{(x, y) \mid 3x - 2y = 4\}$; dependent equations **3.** No solution; { }; inconsistent system
4. $\{(1, 2)\}$ **5.** $\left\{\left(\frac{1}{2}, -11\right)\right\}$ **6.** $\{(-3, 1)\}$
7. $\{(-2, -4)\}$ **8.** $\{(2, 2)\}$

Section 3.4 Activity, pp. 285–286

A.1. a. $x + y = 790$
$4x + 1.5y = 1785$

b. $\{(240, 550)\}$ **c.** There were 240 burritos sold and 550 tacos sold.

A.2. a. 12 g **b.** $0.24x$ **c.** $0.54y$

A.3. a. Let y represent the amount of the 54% alloy.

b.

	24% Alloy	54% Alloy	30% Alloy
Amount of alloy (g)	x	y	40
Amount of pure silver (g)	$0.24x$	$0.54y$	0.30(40)

c. $x + y = 40$
$0.24x + 0.54y = 0.30(40)$

d. $\{(32, 8)\}$ **e.** 32 g of the 24% alloy should be mixed with 8 g of the 54% alloy to get 40 g of an alloy that is 30% silver.

A.4. a. \$125 **b.** $0.025x$ **c.** $0.04y$

A.5. a. Let y represent the amount invested at 4%.

b.

	4% Account	2.5% Account	Total
Amount invested (\$)	x	y	
Interest earned (\$)	$0.025x$	$0.04y$	652.50

c. $y = 3x$
$0.025x + 0.04y = 652.50$

d. $\{(4500, 13{,}500)\}$ **e.** \$4500 was invested at 2.5% and \$13,500 was invested at 4%.

A.6. a. 16 mph; 32 mi **b.** 10 mph; 20 mi **c.** $(b + c)$; $(b + c) \cdot 1.5$ **d.** $(b - c)$; $(b - c) \cdot 2.4$

A.7. a. Let c represent the speed of the current.

b.

	Distance (mi)	Rate (mph)	Time (hr)
With the current	24	$b + c$	1.5
Against the current	24	$b - c$	2.4

c. $24 = (b + c) \cdot 1.5$; $24 = (b - c) \cdot 2.4$ **d.** $\{(13, 3)\}$ **e.** The speed of the boat in still water is 13 mph, and the speed of the current is 3 mph.

Section 3.4 Practice Exercises, pp. 286–290

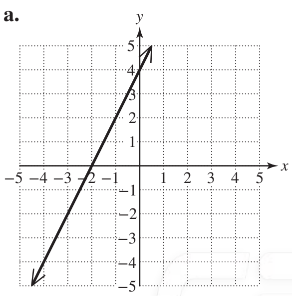
R.1. $\{(25, 15)\}$ **R.3. a.** \$275 **b.** $\$55x$
R.5. a. \$840 **b.** $0.035y$
R.7. a. Bleach: 0.5 L; water: 1.5 L **b.** $0.25x$ liters
R.9. a. 3 mph **b.** 5 mph **c.** 7.5 mi **d.** 4.5 mi **e.** $b - c$

1. 180° **3.** 90° **5.** 552 tickets were sold at \$20 each, and 638 were sold at \$30 each.
7. Hamburgers cost \$4.60 and fish sandwiches cost \$5.20.
9. Vanilla has 12 g of fat per scoop and mud pie has 16 g of fat per scoop. **11.** Mix 4 oz of 18% moisturizer and 8 oz of 24% moisturizer. **13.** Mix 2 L of 8% nitrogen fertilizer and 6 L of 12% nitrogen fertilizer. **15.** Mix 1 oz of pure bleach and 11 oz of 4% solution. **17.** He invested \$4500 in the stock fund and \$1500 in the bond fund. **19.** He borrowed \$2800 at 5.5% and \$2600 at 3.5%. **21.** Alina borrowed \$10,000 from the bank charging 6% interest and \$5000 from the bank charging 7% interest.
23. The boat travels 6 mph in still water, and the current is 2 mph.
25. The plane's still air speed is 720 km/hr, and the speed of the wind is 80 km/hr. **27.** Stephen's walking speed on nonmoving ground is 3.5 ft/sec. The sidewalk moves at 1.5 ft/sec. **29.** The angles are 69° and 21°. **31.** The angles are 134.5° and 45.5°.
33. The angles are 28° and 62°. **35.** 7.5 g of pure gold is needed. **37.** The boat's speed in still water is 5.2 mph and the current speed is 1.2 mph. **39.** Grandstand tickets cost \$330 each and general admission tickets cost \$175 each. **41.** Svetlana invested \$1750 at 2% and \$1250 at 1.3%. **43.** The length is 11 m and the width is 10 m. **45.** There are eleven 50¢ pieces and ten \$1 coins.
47. a. $f(x) = 60x$ **b.** $g(x) = 50x + 100$ **c.** 10 months

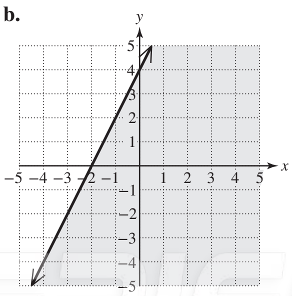
Section 3.5 Activity, p. 298

A.1. a. Yes **b.** No **c.** No

A.2. a.



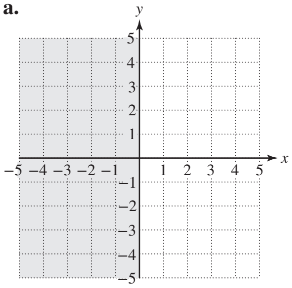
b.



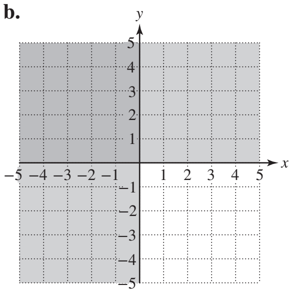
c. The solution set would be the region on and *above* the line.

d. The solution set would be the set of points *strictly below* the line. To show that the points on the line are *not* included in the solution set, the line would be drawn as a dashed line.

A.3. a.

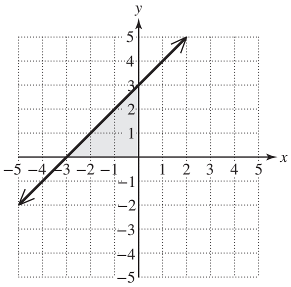


b.



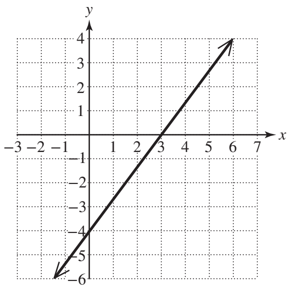
c. The area of "overlap" is the set of points in the second quadrant. That is, the solution set to the system of inequalities $x < 0$ and $y > 0$ is the set of points in Quadrant II.

d.

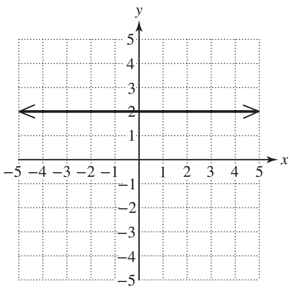


Section 3.5 Practice Exercises, pp. 299–305

R.1. x-intercept: (3, 0); y-intercept: (0, −4)



R.3. x-intercept: None: y-intercept: (0, 2)



R.5. a. No **b.** Yes **c.** No

1. a. linear **b.** is not; is

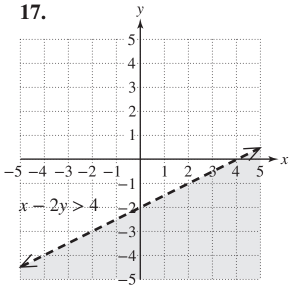
3. The graph of $y > x + 2$ is the set of points *above* the line $y = x + 2$.

5. No. **7. a.** No **b.** No **c.** Yes **d.** Yes

9. a. No **b.** Yes **c.** Yes **d.** No

11. $>$ **13.** $\geq$ **15.** $\geq, \leq$

17.



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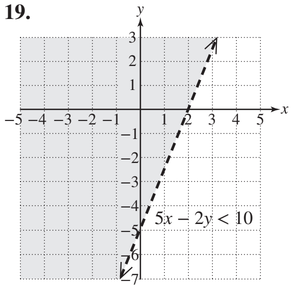

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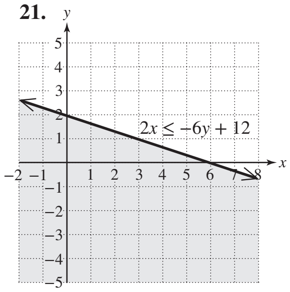

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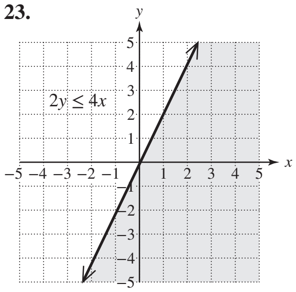

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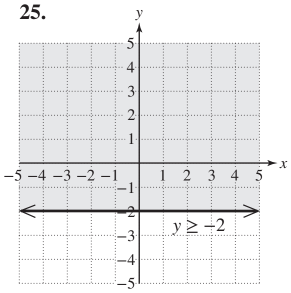
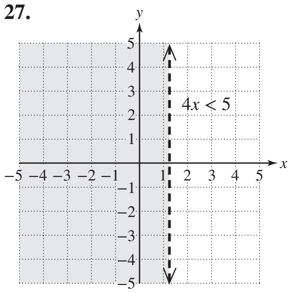


27.



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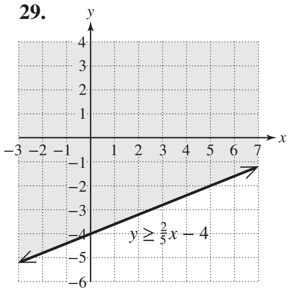

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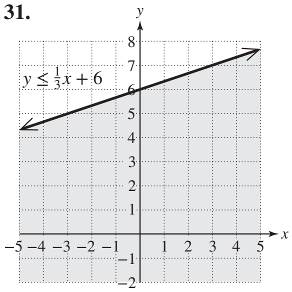

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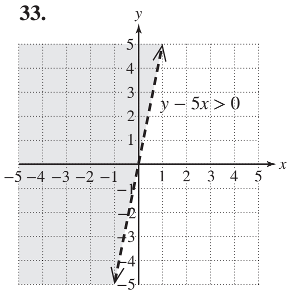

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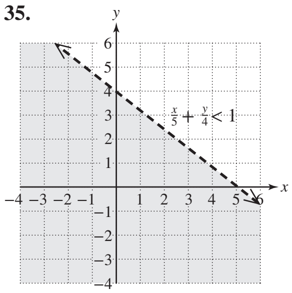

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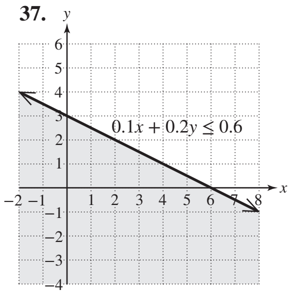

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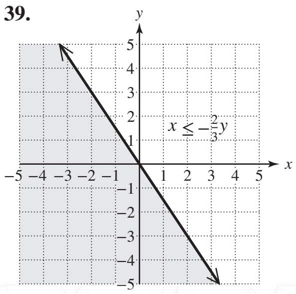

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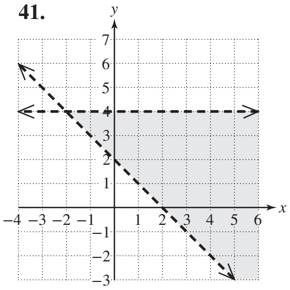

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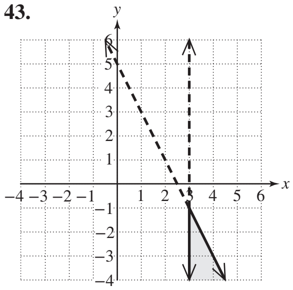

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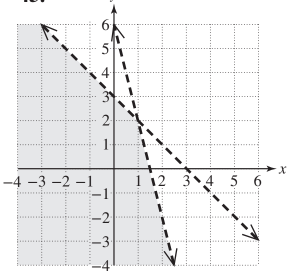

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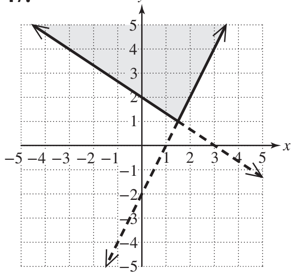

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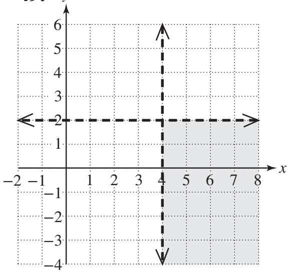

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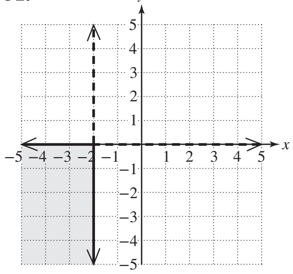

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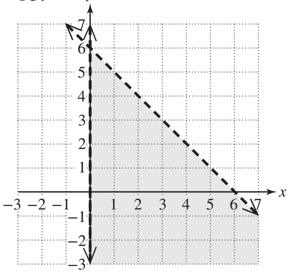

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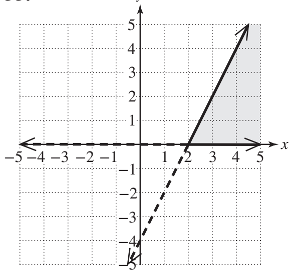

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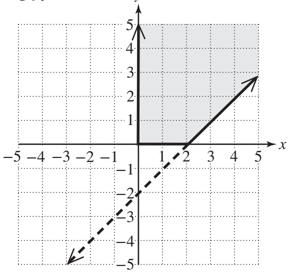

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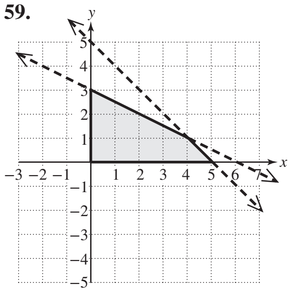
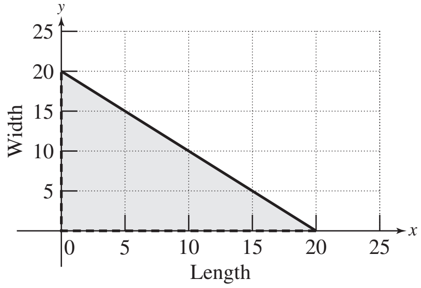


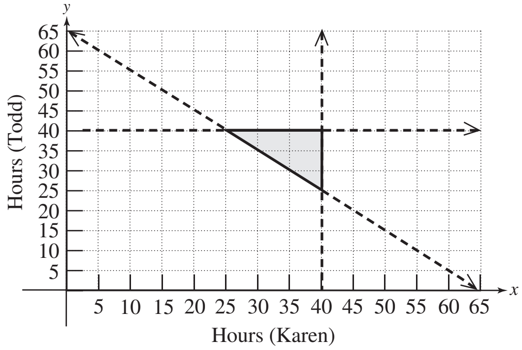
61. a. $2x + 2y \le 40$

b.



63. a. $x \ge 0, y \ge 0$ **b.** $x \le 40, y \le 40$ **c.** $x + y \ge 65$

d.



e. Yes. The point (35, 40) means that Karen works 35 hr and Todd works 40 hr. **f.** No. The point (20, 40) means that Karen works 20 hr and Todd works 40 hr. This does not satisfy the constraint that there must be at least 65 hr total.

Section 3.6 Activity, pp. 312–314

A.1. a. plane **b.** pair; triple **A.2. a.** No **b.** Yes

A.3. a. $-8x + 5y = 4$ $\boxed{\text{D}}$ **b.** $14x - 7y = 0$ $\boxed{\text{E}}$ **c.** $x = 2, y = 4$ **d.** $z = -3$ **e.** $\{(2, 4, -3)\}$. The ordered triple checks in each original equation. **A.4. a.** $3x + 7z = -29$ $\boxed{\text{D}}$ **b.** $x = 2, z = -5$ **c.** $y = -1$ **d.** $\{(2, -1, -5)\}$. The ordered triple checks in each original equation.

A.5. a. $-4x - 6z = -6$ $\boxed{\text{D}}$ **b.** Identity; infinitely many solutions

A.6. { } **A.7. a.** One possibility: Let y represent Randall's second test score. Let z represent Randall's third test score.

b. $\dfrac{x + y + z}{3} = 92$ **c.** $x = y + 6$ **d.** $z = x - 3$ **e.** $\{(95, 89, 92)\}$

f. Randall's first test score is 95, his second is 89, and his third is 92.

Section 3.6 Practice Exercises, pp. 314–317

R.1. a. No **b.** Yes **R.3.** $\{(5, -1)\}$ **R.5.** $\{(x, y) \mid 4x + y = 6\}$; dependent equations **R.7.** { }; inconsistent system

1. linear **3.** Infinitely many solutions

5. $x = 1, y = -3$, $\{(1, -3, 2)\}$

7. (4, 0, 2) is a solution. **9.** (1, 1, 1) is a solution.

11. $\{(1, -2, 4)\}$ **13.** $\{(-1, 2, 0)\}$ **15.** $\{(1, -4, -2)\}$

17. $\{(1, 2, 3)\}$ **19.** $\{(-6, 1, 7)\}$ **21.** $\left\{\left(\frac{1}{2}, \frac{2}{3}, -\frac{5}{6}\right)\right\}$

23. The angles are 67°, 82°, and 31°. **25.** The sides are 10 cm, 18 cm, and 27 cm. **27.** The fiber supplement has 3 g; the oatmeal has 4 g; and the cereal has 1.5 g. **29.** There are three par 3s, nine par 4s, and six par 5s. **31.** Hats cost \$15, T-shirts cost \$25, and jackets cost \$45. **33.** Walter invested \$7000 in small caps, \$6000 in the balanced fund, and \$12,000 in global markets.

35. Dependent equations **37.** Inconsistent system

39. $\{(1, 3, 1)\}$ **41.** $\{(-2, -1, -3)\}$

43. Dependent equations **45.** Inconsistent system

47. (0, 0, 0) is the only solution. **49.** Dependent equations

Section 3.7 Activity, pp. 323–324

A.1. a. A **b.** B **c.** D **d.** E **e.** E **f.** D **g.** A, C **h.** A

A.2. a.
$$\begin{aligned} x - 3y + z &= 1 \\ y + 2z &= 10 \\ z &= 4 \end{aligned}$$

b. $\{(3, 2, 4)\}$

A.3. a.
$$\begin{aligned} x &= 4 \\ y &= -5 \end{aligned}$$

b. $\{(4, -5)\}$

A.4. a. $\left[\begin{array}{cc|c} 2 & -2 & -4 \\ 4 & 2 & 10 \end{array}\right]$ **b.** $\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 4 & 2 & 10 \end{array}\right]$ **c.** $\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 6 & 18 \end{array}\right]$

d. $\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & 3 \end{array}\right]$ **e.** $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right]$ **f.** $\{(1, 3)\}$

A.5. a. $\left[\begin{array}{rrr|r} 2 & -4 & 2 & 10 \\ 3 & -1 & -2 & 5 \\ 2 & 6 & 3 & -10 \end{array}\right]$

b. Multiply row 1 by $\frac{1}{2}$. $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 5 \\ 3 & -1 & -2 & 5 \\ 2 & 6 & 3 & -10 \end{array}\right]$ **c.** $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 5 \\ 0 & 5 & -5 & -10 \\ 0 & 10 & 1 & -20 \end{array}\right]$

d. Multiply row 2 by $\frac{1}{5}$. $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 5 \\ 0 & 1 & -1 & -2 \\ 0 & 10 & 1 & -20 \end{array}\right]$

e. $\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 11 & 0 \end{array}\right]$ **f.** $\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

g. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$ **h.** $\{(1, -2, 0)\}$

A.6. The last row of the matrix corresponds to the contradiction $0 = 10$. This indicates that the system has no solution.

A.7. The last row of the matrix corresponds to the identity $0 = 0$. This indicates that the equations represented by this system are dependent and that the system has infinitely many solutions.

Section 3.7 Practice Exercises, pp. 324–327

R.1. $\{(6, -2)\}$ **R.3.** $\{(6, 1, -1)\}$ **R.5.** The solution set is { }; inconsistent

1. a. matrix; rows; columns **b.** column; one; square **c.** coefficient; augmented **3.** Row 1, column 4 **5.** Row 2, column 3

7. a. 3×1 **b.** column matrix

9. a. 2×2 **b.** square matrix **11. a.** 1×4 **b.** row matrix

13. a. 2×3 **b.** none of these

15. $\left[\begin{array}{rr|r} 1 & -2 & -1 \\ 2 & 1 & -7 \end{array}\right]$ **17.** $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 5 \\ 2 & 6 & 3 & -2 \\ 3 & -1 & -2 & 1 \end{array}\right]$

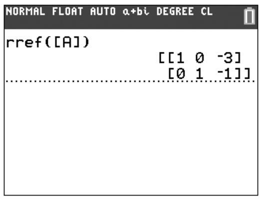
19. $4x + 3y = 6$
$12x + 5y = -6$
21. $x = 4, y = -1, z = 7$

23. a. 7 **b.** -2 **25.** $\left[\begin{array}{rr|r} 1 & \frac{1}{2} & \frac{11}{2} \\ 2 & -1 & 1 \end{array}\right]$

27. $\left[\begin{array}{rr|r} 1 & -4 & 3 \\ 5 & 2 & 1 \end{array}\right]$ **29.** $\left[\begin{array}{rr|r} 1 & 5 & 2 \\ 0 & 11 & 5 \end{array}\right]$

31. a. $\left[\begin{array}{rrr|r} 1 & 3 & 0 & -1 \\ 0 & -11 & -5 & 10 \\ -2 & 0 & -3 & 10 \end{array}\right]$ **b.** $\left[\begin{array}{rrr|r} 1 & 3 & 0 & -1 \\ 0 & -11 & -5 & 10 \\ 0 & 6 & -3 & 8 \end{array}\right]$

33. True **35.** True **37.** Interchange rows 1 and 2.
39. Multiply row 1 by -3 and add to row 2. Replace row 2 with the result. **41.** $\{(-3, -1)\}$ **43.** $\{(-21, 9)\}$ **45.** Infinitely many solutions; $\{(x, y) \mid x + 3y = 3\}$; dependent equations
47. $\{(3, -1)\}$ **49.** $\{(-10, 3)\}$ **51.** No solution; { }; inconsistent system **53.** $\{(1, 2, 3)\}$ **55.** $\{(1, -2, 0)\}$

57.


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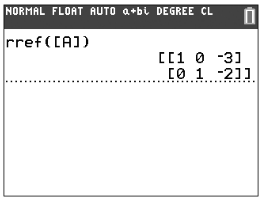

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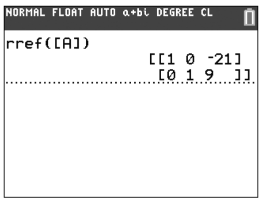

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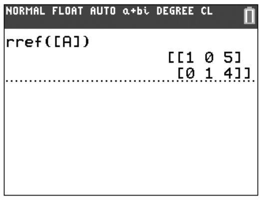

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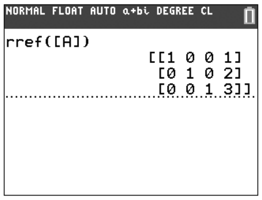

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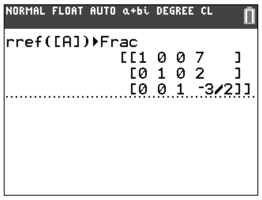
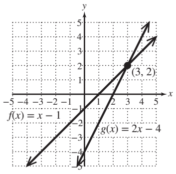


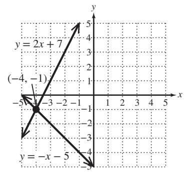
Chapter 3 Review Exercises, pp. 335–338

1. a. No **b.** Yes **2.** False **3.** True **4.** True

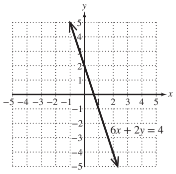
5. $\{(3, 2)\}$



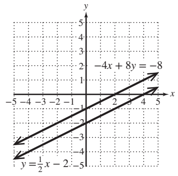
6. $\{(-4, -1)\}$



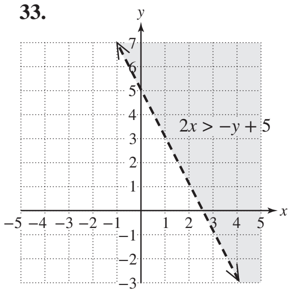
7. Infinitely many solutions; $\{(x, y) \mid 6x + 2y = 4\}$; dependent equations



8. No solution; { }; inconsistent system



9. $\{(4, -1)\}$ **10.** $\{(2, -7)\}$ **11.** No solution; { }; inconsistent system **12.** No solution; { }; inconsistent system
13. Infinitely many solutions; $\{(x, y) \mid 10x - 2y = 3\}$; dependent equations **14.** Infinitely many solutions; $\{(x, y) \mid 4x + y = 7\}$; dependent equations **15.** 30 months **16.** 8 days
17. $\{(1, 1)\}$ **18.** $\{(2, -1)\}$ **19.** $\{(2, -1)\}$
20. Infinitely many solutions; $\{(x, y) \mid 3x + y = 1\}$; dependent equations **21.** No solution; { }; inconsistent system
22. $\{(6, -2)\}$ **23.** $\{(2, -3)\}$ **24.** $\{(-10, 25)\}$
25. $\{(0, 6)\}$ **26.** $\{(-2, 7)\}$ **27.** She invested \$4500 at 5%.
28. There were 21 student tickets and 33 adult tickets.
29. 10 L of 20% solution must be mixed with 6 L of 50% solution.
30. The plane travels 150 mph in still air, and the wind speed is 10 mph. **31. a.** $f(x) = 915x + 275$ **b.** $g(x) = 965x$ **c.** 5.5 months **32.** The angles are 76° and 14°.

33.


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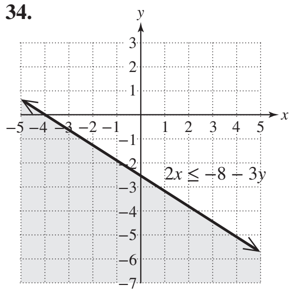

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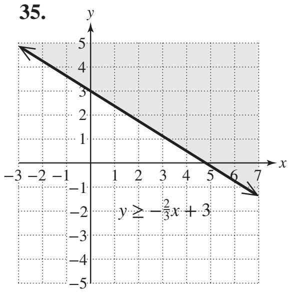

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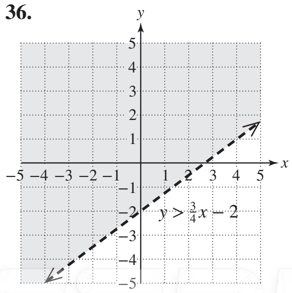

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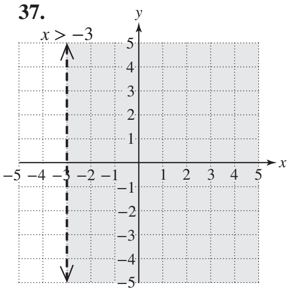

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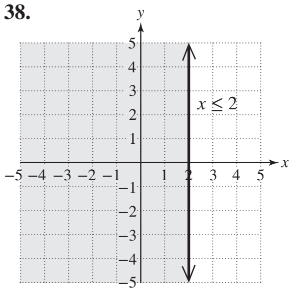

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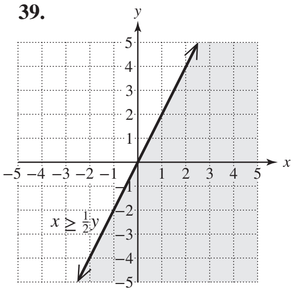

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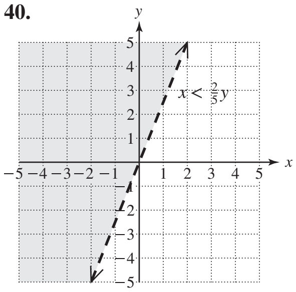

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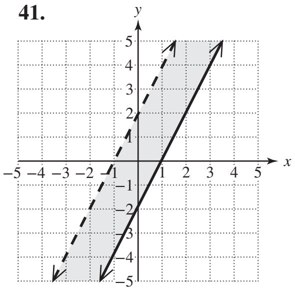

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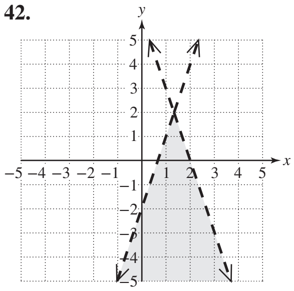

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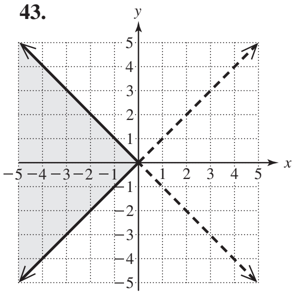

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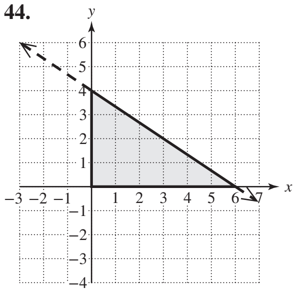
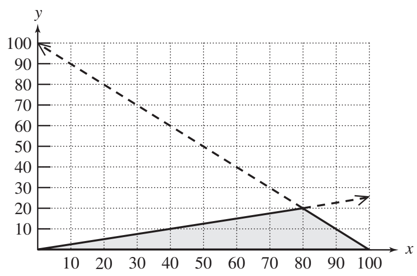


45. a. $x \ge 0, y \ge 0$ **b.** $x + y \le 100$ **c.** $x \ge 4y$

d.



46. $\{(2, -1, 5)\}$ **47.** Inconsistent system
48. Dependent equations **49.** $\{(-1, -2, 2)\}$
50. The sides are 5 ft, 12 ft, and 13 ft.
51. The pumps can drain 250, 300, and 400 gal/hr.
52. The angles measure 113°, 38°, and 29°.
53. 3×3 **54.** 3×2 **55.** 1×4 **56.** 3×1

57. $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right]$ **58.** $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ -2 & 2 & -1 & -9 \end{array}\right]$

59. $x = 9, y = -3$ **60.** $x = -5, y = 2, z = -8$ **61. a.** 4

b. $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -13 & 2 \end{array}\right]$ **62. a.** $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & -9 & 1 & 12 \\ -3 & 2 & 2 & 5 \end{array}\right]$

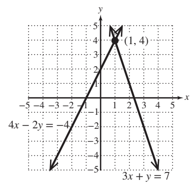
b. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & -9 & 1 & 12 \\ 0 & 8 & 2 & -4 \end{array}\right]$ **63.** $\{(1, 2)\}$ **64.** $\{(-3, 6)\}$

65. $\{(1, 3, -2)\}$ **66.** $\{(6, 1, -1)\}$

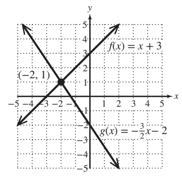
Chapter 3 Test, pp. 339–340

1. Yes **2.** b **3.** c **4.** a

5. $\{(1, 4)\}$



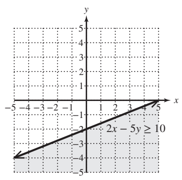
6. $\{(-2, 1)\}$



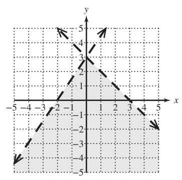
7. $\{(-4, 5)\}$ **8.** $\left\{\left(\frac{1}{2}, \frac{1}{4}\right)\right\}$ **9.** $\{(0, -3)\}$

10. Infinitely many solutions; $\{(x, y) \mid 3x - 5y = -7\}$; dependent equations **11.** $\{(2, -6)\}$ **12.** No solution; { }; inconsistent system **13.** $\{(5, 0)\}$

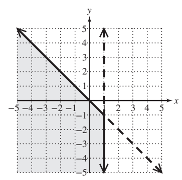
14.



15.

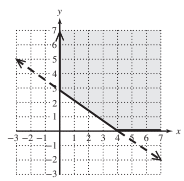


16.



17. a. $x \ge 0, y \ge 0$ **b.** $300x + 400y \ge 1200$

c.



18. $\{(16, -37, 9)\}$ **19.** $\{(2, -2, 5)\}$ **20.** She borrowed \$1200 at 6.5% and \$3800 at 5%. **21.** Mix 80 L of 20% solution with 120 L of 60% solution. **22.** The angles are 80° and 10°.
23. Joanne can process 142 orders, Kent can process 162 orders, and Geoff can process 200 orders.

24. For example: $\begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2.6 & 7 \end{bmatrix}$

25. a. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ 0 & -8 & -3 & 10 \\ -5 & -6 & 3 & 0 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ 0 & -8 & -3 & 10 \\ 0 & 4 & 8 & -15 \end{array}\right]$

26. $\{(6, -1)\}$ **27.** $\{(2, -4, 3)\}$

Chapter 4

Section 4.1 Activity, pp. 348–349

A.1. a. $(x \cdot x \cdot x \cdot x \cdot x)(x \cdot x) = x^7$ **b.** $x^m \cdot x^n = x^{m+n}$

Example	Expanded Form	Simplified Form	Observation	General Rule
A.2. $x^4 \cdot x^5$	$(x \cdot x \cdot x \cdot x)(x \cdot x \cdot x \cdot x \cdot x)$	x^9	$x^4 \cdot x^5 = x^{4+5} = x^9$	$b^m \cdot b^n = b^{m+n}$
A.3. $\dfrac{w^6}{w^3}$	$\dfrac{w \cdot w \cdot w \cdot w \cdot w \cdot w}{w \cdot w \cdot w}$	w^3	$\dfrac{w^6}{w^3} = w^{6-3} = w^3$	$\dfrac{b^m}{b^n} = b^{m-n}$
A.4. $(t^4)^3$	$(t \cdot t \cdot t \cdot t)(t \cdot t \cdot t \cdot t)(t \cdot t \cdot t \cdot t)$	t^{12}	$(t^4)^3 = t^{4 \cdot 3} = t^{12}$	$(b^m)^n = b^{m \cdot n}$
A.5. $(2c)^3$	$(2c)(2c)(2c)$	2^3c^3 or $8c^3$	$(2c)^3 = 2^3c^3$	$(ab)^m = a^mb^m$
A.6. $\left(\dfrac{x^4}{3}\right)^2$	$\left(\dfrac{x^4}{3}\right)\left(\dfrac{x^4}{3}\right)$	$\dfrac{x^8}{3^2}$ or $\dfrac{x^8}{9}$	$\left(\dfrac{x^4}{3}\right)^2 = \dfrac{(x^4)^2}{(3)^2} = \dfrac{x^8}{9}$	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$
A.7. $\dfrac{b^4}{b^4}$	$\dfrac{b \cdot b \cdot b \cdot b}{b \cdot b \cdot b \cdot b}$	1	$\dfrac{b^4}{b^4} = b^{4-4} = b^0 = 1$	$b^0 = 1$
A.8. $\dfrac{b^2}{b^5}$	$\dfrac{b \cdot b}{b \cdot b \cdot b \cdot b \cdot b}$	$\dfrac{1}{b^3}$	$\dfrac{b^2}{b^5} = b^{2-5} = b^{-3} = \dfrac{1}{b^3}$	$\dfrac{b^m}{b^n} = b^{m-n}$

A.9. 34 **A.10.** $\dfrac{b^8}{16a^6}$ **A.11.** $\dfrac{2x^{23}y^7}{9}$

A.12. a. 16 **b.** -16 **c.** -64 **d.** -64 **e.** $\dfrac{1}{16}$ **f.** $-\dfrac{1}{16}$ **g.** $-\dfrac{1}{64}$ **h.** $-\dfrac{1}{64}$ **A.13. a.** 1 **b.** -1 **c.** 1 **d.** -6

A.14.

Standard Form	Expanded Form	Scientific Notation
63,000	$6.3 \times 10{,}000$	6.3×10^4
0.008	$8 \times \dfrac{1}{1000}$ or $8 \times \dfrac{1}{10^3}$	8×10^{-3}
8,120,000	$8.12 \times 1{,}000{,}000$	8.12×10^6
2000	2×1000	2×10^3
0.034	$3.4 \times \dfrac{1}{100}$ or $3.4 \times \dfrac{1}{10^2}$	3.4×10^{-2}
0.56	$5.6 \times \dfrac{1}{10}$ or $5.6 \times \dfrac{1}{10^1}$	5.6×10^{-1}

A.15. a. 5×10^6 **b.** 5×10^{12} cells/L **c.** 2.35×10^{12} cells/pint

A.16. 8.125×10^1 or equivalently 81.25. Earth is 81.25 times as massive as the Moon.

Section 4.1 Practice Exercises, pp. 349–352

R.1. 64 **R.3.** 64 **R.5.** 125 **R.7.** -125

R.9. a. 100 **b.** 10^4 **R.11. a.** $\dfrac{1}{100}$ **b.** 10^{-4} **c.** 10^{-2}

1. a. exponent **b.** 1 **c.** $\left(\dfrac{1}{b}\right)^n$ or $\dfrac{1}{b^n}$ **d.** scientific notation

3. $ab^3 = a \cdot b \cdot b \cdot b$
$(ab)^3 = (ab) \cdot (ab) \cdot (ab) = a \cdot a \cdot a \cdot b \cdot b \cdot b = a^3b^3$

5. For example: $(xy)^3 = x^3y^3$ **7.** For example: $\dfrac{x^5}{x^2} = x^3$

9. For example: $x^0 = 1 \quad (x \neq 0)$ **11.** 3 **13.** $\dfrac{1}{25}$ **15.** $-\dfrac{1}{25}$

17. $\dfrac{1}{25}$ **19.** -64 **21.** $\dfrac{16}{81}$ **23.** $-\dfrac{125}{8}$ **25.** 1 **27.** $10a$

29. y^8 **31.** 13^2 or 169 **33.** y^8 **35.** 3^4x^8 or $81x^8$ **37.** $\dfrac{1}{p^3}$

39. $\dfrac{1}{7^3}$ or $\dfrac{1}{343}$ **41.** $\dfrac{1}{w^2}$ **43.** $\dfrac{1}{a^7}$ **45.** r^2 **47.** $\dfrac{1}{z^4}$ **49.** a^3b^2

51. 1 **53.** $\dfrac{65}{4}$ **55.** $\dfrac{26}{25}$ **57.** 3 **59.** $\dfrac{5}{2}$ **61.** $\dfrac{q^2}{p^3}$

63. $-\dfrac{3b^7}{2a^3}$ **65.** $\dfrac{x^{16}}{81y^{20}z^8}$ **67.** $-\dfrac{4m^4}{n^2}$ **69.** $\dfrac{4q^{11}}{p^4}$ **71.** $\dfrac{5x^8}{y^2}$

73. $\dfrac{16a^2}{b^6}$ **75.** $-\dfrac{27y^{27}}{8x^{24}}$ **77.** $\dfrac{4x^{18}}{y^{10}}$ **79.** $27x^3y^9$

81. a. $\$8 \times 10^9$ **b.** 3×10^6 DVDs **c.** 1.4×10^{13} eV **d.** 1.602×10^{-19} J

83. a. 200,000,000,000 **b.** 0.000004 m **c.** 108,200,000,000 m

85. 3.5×10^5 **87.** Proper

89. Proper **91.** 3.38×10^{-4}

93. 1.608×10^4 **95.** 3.4×10^{13}

97. 2.5×10^3 **99.** 1.204×10^{24} hydrogen atoms and 6.02×10^{23} oxygen atoms

101. There are 2×10^4 or 20,000 people per square mile. **103.** $\$5.25 \times 10^{10}$

105. a. 540 months **b.** \$10,800 **c.** \$55,395.45

107. y^{2a+2} **109.** x^{2a-4}

111. $x^{a-6}y^6$

Section 4.2 Activity, pp. 358–359

A.1. a. $6t^3$ **b.** $-4x^2 + x - 5$

A.2. a. For example: 4 **b.** For example: $\dfrac{1}{2}x^4 + 3x$

A.3. a. $6x^2 - 9x - 7$ **b.** Add *like* terms by applying the distributive property. **A.4. a.** $7y^2 - 3y + 4$ **b.** Multiply each term by -1, which changes the sign of each term.

A.5. a. $\dfrac{1}{2}w^2 + w - \dfrac{1}{2}$ **b.** $5x^2y^2 + 3xy - 7$ **c.** The difference of a and b is translated as $a - b$. On the other hand, a subtracted from b is translated as $b - a$.

A.6. a. $x + 4$ **b.** $P(x) = 2(x + 4) + 2(x)$ or equivalently $P(x) = 4x + 8$ **c.** $A(x) = (x + 4) \cdot x$ or equivalently $A(x) = x^2 + 4x$ **d.** $P(12) = 56$ means that if the width of the rectangle is 12 cm, then the perimeter is 56 cm. **e.** $A(7) = 77$ means that if the width of the rectangle is 7 cm, then the area is 77 cm^2.

Section 4.2 Practice Exercises, pp. 359–363

R.1. $-3t$ **R.3.** $-\dfrac{3}{4}x$ **R.5.** $-8y - 16z + 22$ **R.7.** $6m - 13$

R.9. a. 11 **b.** -11 **R.11. a.** $f(-4) = 18$ **b.** $f(0) = 6$ **c.** $f\left(\dfrac{1}{3}\right) = 5$

1. a. polynomial **b.** coefficient; n **c.** 1; 1 **d.** leading; leading; coefficient **e.** greatest **f.** zero **g.** exponents **h.** polynomial

3. binomial **5.** 6

7. $-6a^3 + a^2 - a$; leading coefficient -6; degree 3

9. $3x^4 + 6x^2 - x - 1$; leading coefficient 3; degree 4

11. $-t^2 + 100$; leading coefficient -1; degree 2

13. For example: $3x^5$ **15.** For example: $x^2 + 2x + 1$

17. For example: $6x^4 - x^2$ **19.** $m^2 + 10m$

21. $3x^4 + 2x^3 - 8x^2 + 2x$ **23.** $2w^3 + \dfrac{1}{9}w^2 + 0.9w$

25. $17x^2y - 4xy - 14$ **27.** $4a^2 - 11a - 7$

29. $9x^3 - 5x^2 + 2x - 8$ **31.** $30y^3$ **33.** $-4p^3 - 2p + 12$

35. $11ab^2 - a^2b$ **37.** $6z^5 - 6z^2$ **39.** $-2x^3 + 4x^2 + 5$

41. $-2xy^3 + 3x^2y + xy + 5$ **43.** $t^3 - 13t^2 - 9t - 13$

45. $\dfrac{1}{2}a^2 - \dfrac{9}{10}ab + \dfrac{3}{5}b^2 + 8$ **47.** $-x^2 + 6x - 16$

49. $3x^5 - x^4 - 4x^3 + 11$ **51.** $4y^3 + 5y^2$ **53.** $-7r^4 - 11r$

55. $9x^2 + 5xy + 11y$ **57.** $18ab - 42b^2$ **59.** $3p - 9$

61. $2m^2 + 6$ **63.** $7x^3 + 4x - 5$ **65.** $12a^2b + ab - 5ab^2$

67. $12x^3 - 6x^2 + 1$ **69.** $8a^2b - 5ab^2 + 4ab$

71. $-3x^3 - 16x^2 + 10x + 1$

73. $-2.2p^5 - 15.5p^4 - 8.5p^3 - 5p^2 - 7.9p$ **75.** $12x^3 + 2x$

77. Yes; degree 2 **79.** No; the term $-\frac{3}{x} = 3x^{-1}$ and -1 is not a whole number. **81.** Yes; degree 0 **83.** No; the term $|x|$ is not of the form ax^n. **85. a.** -17 **b.** -8 **c.** -5 **d.** -4

87. a. $\frac{1}{4}$ **b.** $\frac{9}{4}$ **c.** $-\frac{7}{4}$ **d.** $\frac{3}{4}$ **89.** $P(x) = 4x + 6$

91. a. $P(x) = 6.6x - 99$ **b.** \$231

93. a. $D(0) = 1636$ means that at the beginning of the study, (year 0) the annual dormitory charge was \$1636. $D(18) = 4048$ means that in year 18, the annual dormitory charge was \$4048. **b.** \$5896 **95.** $G(20) = 24$ means that when the car travels 20 mph, the gas mileage is 24 mpg. $G(40) = 36$ means that when the car travels 40 mph, the gas mileage is 36 mpg. $G(50) = 33$ means that when the car travels 50 mph, the gas mileage is 33 mpg. **97. a.** (0, 0); at $t = 0$ sec, the position of the rocket is at the origin. **b.** (25, 27.3); after 1 sec, the position of the rocket is (25, 27.3). **c.** (50, 22.6)

Section 4.3 Activity, p. 370

A.1. $4a^4b^5$ **A.2. a.** 2, 3, 6 **b.** $2w^3 - 13w^2 + 31w - 24$

A.3. a. conjugates **b.** $25c^2 + 10c - 10c - 4 = 25c^2 - 4$ **c.** $(5c)^2 - (2)^2 = 25c^2 - 4$ **A.4. a.** $9x^2 - 21x - 21x + 49 = 9x^2 - 42x + 49$ **b.** $(3x)^2 - 2(3x)(7) + (7)^2 = 9x^2 - 42x + 49$

A.5. $x^3 + 6x^2y + 12xy^2 + 8y^3$

Section 4.3 Practice Exercises, pp. 371–374

R.1. $9y$ **R.3.** $18y^2$ **R.5.** $-4a^4b$ **R.7.** $-21a^8b^2$
R.9. $5k^2 - 2k + 8$ **R.11.** $-15a + 30b + 35c$ **R.13.** $9x + 12y$

1. a. distributive **b.** $4x - 7$ **3.** $14x^2$ **5.** $-4xy^2$

7. $-42x^5y^6$ **9.** $11a^7b^8c^{10}$ **11.** $\frac{2}{5}a - \frac{3}{5}$

13. $2m^5n^5 - 6m^4n^4 + 8m^3n^3$ **15.** $3x^2y^2 - 4x^2y^3$

17. $x^2 - xy - 2y^2$ **19.** $12x^2 + 28x - 5$ **21.** $2y^4 - 21y^2 - 36$

23. $25s^2 + 5st - 6t^2$ **25.** $5n^3 + 3n^2 + 50n + 30$

27. $3.25a^2 - 0.9ab - 28b^2$ **29.** $6x^3 + 7x^2y + 4xy^2 + y^3$

31. $x^3 - 343$ **33.** $4a^4 - 17a^3b + 8a^2b^2 - 5ab^3 + b^4$

35. $\frac{1}{2}a^2 + ab + \frac{1}{2}ac - 12b^2 + 8bc - c^2$

37. $-3x^3 + 11x^2 - 7x - 5$ **39.** $\frac{1}{10}y^2 - 8y + 150$ **41.** $a^2 - 64$

43. $9p^2 - 1$ **45.** $x^2 - \frac{1}{9}$ **47.** $9h^2 - k^2$ **49.** $9h^2 - 6hk + k^2$

51. $t^2 - 14t + 49$ **53.** $u^2 + 6uv + 9v^2$ **55.** $h^2 + \frac{1}{3}hk + \frac{1}{36}k^2$

57. $4z^4 - w^6$ **59.** $25x^4 - 30x^2y + 9y^2$ **61. a.** Yes **b.** No

63. a. $A^2 - B^2$ **b.** $x^2 + 2xy + y^2 - B^2$ Both are examples of multiplying conjugates to get a difference of squares.

65. $w^2 + 2wv + v^2 - 4$ **67.** $4 - x^2 - 2xy - y^2$

69. $9a^2 - 24a + 16 - b^2$ **71.** Write $(x + y)^3$ as $(x + y)^2(x + y)$. Square the binomial and then use the distributive property to multiply the resulting trinomial by the remaining factor of $(x + y)$.

73. $8x^3 + 12x^2y + 6xy^2 + y^3$ **75.** $64a^3 - 48a^2b + 12ab^2 - b^3$

77. Multiply and simplify the first two binomials. Then multiply the resulting trinomial by the third binomial, using the distributive property. **79.** $6a^4 + 32a^3 + 10a^2$ **81.** $x^3 + 5x^2 - 9x - 45$

83. $-16x^2 + 2x - 22$ **85.** $-3y^2 - 10y - 8$ **87.** $(r + t)^2$

89. $x^2 - y^3$ **91.** The sum of p cubed and q squared **93.** The product of x and the square of y **95.** $A(x) = 4x^2 + 70x + 300$

97. a. $V(x) = 4x^3 - 32x^2 + 64x$ **b.** 36 in.^{3} **99.** $x^2 - 4x + 4$

101. $x^2 - 4$ **103.** $x^2 - 9$ **105.** $9x^3 + 30x^2$ **107.** $2x + h - 3$

109. Multiply $(x + 2)^2(x + 2)^2$ by squaring the binomials. Then multiply the resulting trinomials using the distributive property.

111. $(5x - 6)$ **113.** $(2y - 1)$

Section 4.4 Activity, pp. 382–383

A.1. a. $\frac{12a^3}{-6a} - + \frac{-6a^2}{-6a} + \frac{18a}{-6a}$ **b.** $-2a^2 + a - 3$

A.2. $6k^3 - k + 4 - \frac{3}{2k^2}$ **A.3.** Use long division when the divisor has two or more terms. **A.4.** $1626\frac{11}{31}$

A.5. a. $2x^3 - 3x^2 + x - 4 + \frac{-2}{x - 3}$ **b.** $2x^3 - 3x^2 + x - 4$
c. $x - 3$ **d.** $2x^4 - 9x^3 + 10x^2 - 7x + 10$ **e.** -2
f. $2x^4 - 9x^3 + 10x^2 - 7x + 10 = (x - 3)(2x^3 - 3x^2 + x - 4) + (-2)$ ✓

A.6. a. $(27x^3 + 0x^2 + 0x + 8) \div (3x + 2)$ **b.** $9x^2 - 6x + 4$
c. $9x^2 - 6x + 4$ **d.** $3x + 2$ **e.** $27x^3 + 8$ **f.** 0
g. $27x^3 + 8 = (3x + 2)(9x^2 - 6x + 4)$ ✓

A.7. a. $\frac{3y^4 + 2y^3 + 0y^2 - 6y + 1}{y^2 + 0y - 2}$ **b.** $3y^2 + 2y + 6 + \frac{-2y + 13}{y^2 - 2}$
c. $3y^4 + 2y^3 - 6y + 1 = (y^2 - 2)(3y^2 + 2y + 6) + (-2y + 13)$ ✓

A.8. a.
$$\begin{array}{r|rrrrr} 3 & 2 & -9 & 10 & -7 & 10 \\ & & 6 & -9 & 3 & -12 \\ \hline & 2 & -3 & 1 & -4 & \boxed{-2} \end{array}$$
b. $2x^3 - 3x^2 + x - 4$

c. -2 **d.** $2x^3 - 3x^2 + x - 4 + \frac{-2}{x - 3}$; Yes

e. Answers will vary.

A.9. a. $t^2 + 5t + 25$ **b.** $t^3 - 125 = (t - 5)(t^2 + 5t + 25)$ ✓

A.10. Synthetic division can be used if the divisor is of the form $x - r$ where r is a constant.

A.11. Long division **A.12.** Monomial division

A.13. Long division or synthetic division **A.14.** Long division

Section 4.4 Practice Exercises, pp. 384–386

R.1. Dividend: 8945; divisor: 9; quotient: 993; remainder: 8

R.3. $8x^3 - 24x^2 + 16x - 8$ **R.5.** $16b^4 - 40b^3 + 96b^2$

R.7. $11x^2$ **R.9.** $2x^2 - 2x$

1. a. division; quotient; remainder **b.** Synthetic

3. p^5 **5.** $7a^4$ **7.** $\frac{2x^2}{3y}$ **9.** $-4t^3 + t - 5$

11. $12 + 8y + 2y^2$ **13.** $x^2 + 3xy - y^2$ **15.** $2y^2 + 3y - 8$

17. $-\frac{1}{2}p^3 + p^2 - \frac{1}{3}p + \frac{1}{6}$ **19.** $a^2 + 5a + 1 - \frac{5}{a}$

21. $-3s^2t + 4s - \frac{5}{t^2}$ **23.** $8p^2q^6 - 9p^3q^5 - 11p - \frac{4}{p^2q}$

25. a. Divisor: $(x - 2)$; quotient: $(2x^2 - 3x - 1)$; remainder: (-3) **b.** Multiply the quotient and the divisor, then add the remainder. The result should equal the dividend.

27. $x + 7 + \frac{-9}{x + 4}$ **29.** $3y^2 + 2y + 2 + \frac{9}{y - 3}$

31. $-4a + 11$ **33.** $6y - 5$ **35.** $6x^2 + 4x + 5 + \frac{22}{3x - 2}$

37. $4a^2 - 2a + 1$ **39.** $x^2 - 2x + 2$

41. $x^2 + 2x + 5$ **43.** $x^2 - 1 + \frac{8}{x^2 - 2}$

45. $n^3 + 2n^2 + 4n + 8$ **47.** $3y^2 - 3 + \frac{2y + 6}{y^2 + 1}$

49. The divisor must be of the form $x - r$.

51. No. The divisor is not of the form $x - r$.

53. a. $x - 5$ **b.** $x^2 + 3x + 11$ **c.** 58

55. $x + 6$ **57.** $t - 4$ **59.** $5y + 10 + \frac{11}{y - 1}$

61. $3y^2 - 2y + 2 + \frac{-3}{y + 3}$ **63.** $x^2 - x - 2$

65. $a^4 + 2a^3 + 4a^2 + 8a + 16$ **67.** $x^2 + 6x + 36$

69. $4w^3 + 2w^2 + 6$ **71.** $-x^2 - 4x + 13 + \dfrac{-54}{x + 4}$

73. $2x - 1 + \dfrac{3}{x}$ **75.** $4y - 3 + \dfrac{4y + 5}{3y^2 - 2y + 5}$

77. $2x^2 + 3x - 1$ **79.** $2k^3 - 4k^2 + 1 - \dfrac{5}{k^4}$

81. $x + \dfrac{9}{5} + \dfrac{2}{x}$

Chapter 4 Problem Recognition Exercises, p. 387

1. a. $9x^2 + 6x + 1$ **b.** $9x^2 - 1$ **c.** 2 **2. a.** -10 **b.** $81m^2 - 25$ **c.** $81m^2 - 90m + 25$ **3. a.** $2x + 4 - \dfrac{5}{x}$ **b.** $2x + 5 + \dfrac{-5}{2x - 1}$ **c.** $4x + 12 + \dfrac{2}{x - 1}$ **4. a.** $y - 5 + \dfrac{4}{3y}$ **b.** $y - 7 + \dfrac{46}{3y + 6}$ **c.** $3y - 33 + \dfrac{202}{y + 6}$ **5. a.** -30 **b.** $-10p - 50$ **c.** 0 **6. a.** $-8x - 32$ **b.** -20 **c.** 0 **7.** $2t^2 + t - 1$ **8.** $-15x^4 - 5x^3 + 10x^2$ **9.** $36z^2 - 25$ **10.** $9y^3 + 2y^2 - 3y + 1$ **11.** $6b^2 - 11b + 4$ **12.** $10a^3 + 19a^2 + 11a + 2$ **13.** $t^3 - 6t^2 + 8t + 3$ **14.** $2b^2 + 4b + 5$ **15.** $k^2 + 4k + 25$ **16.** $3x^3 + x^2 - 5$ **17.** $7t^3 - 12t^2 + 2t$ **18.** $-xy^2 + 2y + \dfrac{x}{7y}$ **19.** $\dfrac{11}{12}p^3 - \dfrac{1}{2}p^2 + \dfrac{1}{5}p + 5$ **20.** $-30.6w^6 + 15.6w^5 - 7.2w^4$ **21.** $36a^4 - 48a^2b + 16b^2$ **22.** $\dfrac{1}{4}z^4 - \dfrac{1}{9}$ **23.** $m^2 - 8m - 7$ **24.** $x^2 + 3x - 14$ **25.** $2m^4 - 8m^3 - 13m^2 + 46m - 21$ **26.** $x^2 + 4x + 16$ **27.** $25 - 10a - 10b + a^2 + 2ab + b^2$ **28.** $a^2 - x^2 + 2xy - y^2$ **29.** $4xy$ **30.** $a^3 - 12a^2 + 48a - 64$ **31.** $-\dfrac{1}{8}x^2 + \dfrac{1}{3}x - \dfrac{1}{6}$ **32.** $-\dfrac{1}{2}x^6y^4z^5w^3$

Section 4.5 Activity, p. 393

A.1. a.
$$18x^3y^2 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot y \cdot y$$
$$12x^2y^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y \cdot y \cdot y$$
$$9xy^4 = 3 \cdot 3 \cdot x \cdot y \cdot y \cdot y \cdot y$$
b. $3xy^2$ **c.** $3xy^2(6x^2 + 4xy + 3y^2)$

A.2. a. $2(a + 2b)$ **b.** $2(a + 2b)(2x + 3)$

A.3. a. $8x(-x + 3)$ **b.** $-8x(x - 3)$ **c.** The sign of each term is changed.

A.4. $10x^2 + 45x + 6xy + 27y$ **A.5. a.** $5x(2x + 9) + 3y(2x + 9)$ **b.** $(2x + 9)(5x + 3y)$ **c.** The product is equal to the polynomial $10x^2 + 45x + 6xy + 27y$. **A.6.** $(c + 5)(c - 2d)$

A.7. a. $1(8x^2 + 15y) + 2x(5 + 6y)$ **b.** The two binomial expressions do not match. **c.** $(2x + 3y)(4x + 5)$ **d.** $(2x + 3y)(4x + 5) = 8x^2 + 10x + 12xy + 15y$ ✓

Section 4.5 Practice Exercises, pp. 393–396

R.1. $2^4 \cdot 3 \cdot 5$ **R.3.** $-3x^2 + 7x - 8$ **R.5.** $-10d^3 + 40d^2$ **R.7.** $15x^4y - 10x^3y + 5x^2y$ **R.9.** $18ax - 66x + 15a - 55$

1. a. product **b.** greatest; common; factor **3.** $3(x + 4)$ **5.** $2z(3z + 2)$ **7.** $4p(p^5 - 1)$ **9.** $12x^2(x^2 - 3)$ **11.** $9t(st + 3)$ **13.** $9a^2b^3(a^2 + 3ab - 2b^2)$ **15.** $5xy(2x + 3y - 1)$ **17.** $b(13b - 11a^2 - 12a)$ **19.** $-1(x^2 + 10x - 7)$ **21.** $-3xy(4x^2 + 2x + 1)$ **23.** $-t(2t^2 - 11t + 3)$ **25.** $(3z - 2b)(2a - 5)$ **27.** $(2x - 3)(2x^2 + 1)$ **29.** $(2x + 1)^2(y - 3)$ **31.** $3(x - 2)^2(y + 2)$ **33.** For example: $3x^3 + 6x^2 + 12x^4$ **35.** For example: $6(c + d) + y(c + d)$ **37. a.** $(2x - y)(a + 3b)$ **b.** $(2w - 1)(5w - 3b)$ **c.** In part (b), $-3b$ was factored out so that the signs in the last two terms were changed. The resulting binomial factor matches the binomial factor from the first two terms. **39.** $(y + 4)(y^2 + 3)$ **41.** $(p - 7)(6 + q)$ **43.** $(m + n)(2x + 3y)$ **45.** $(2x - 3y)(5a - 4b)$ **47.** $(x^2 - 3)(x - 1)$ **49.** $6p(p + 3)(q - 5)$ **51.** $100(x - 3)(x^2 + 2)$ **53.** $(3a + b)(2x - y)$ **55.** $(4 - b)(a + 3)$ **57.** Cannot be factored **59.** It is not possible to get a common binomial factor regardless of the order of terms. **61.** $A = \dfrac{U}{v + cw}$ **63.** $y = \dfrac{bx}{c - a}$ or $y = \dfrac{-bx}{a - c}$ **65.** Length $= 2w + 1$ **67.** $(a + 3)^4(6a + 19)$ **69.** $18(3x + 5)^2(4x + 5)$ **71.** $(t + 4)(t + 3)$ **73.** $5w^2(2w - 1)^2(7w - 3)$

Section 4.6 Activity, pp. 407–408

A.1. a. 1 **b.** Yes **c.** $a = 6, b = 1, c = -12$ **d.** -72 **e.** -8 and 9 **f.** $6x^2 - 8x + 9x - 12$ **g.** $(3x - 4)(2x + 3)$ **h.** $(2x + 3)(3x - 4)$ is the same as $(3x - 4)(2x + 3)$ by the commutative property of multiplication.

A.2. a. 1 **b.** No; $2x^2 + 13x + 6$ **c.** $(2x + 1)(x + 6)$
d. $(2x + 1)(x + 6)$
$= 2x^2 + 12x + x + 6$
$= 2x^2 + 13x + 6$ ✓

A.3. a. 1 **b.** Yes **c.** 5 **d.** $5x^2$ **e.** $(5x + 4)(x + 6)$
f. $(5x + 4)(x + 6)$
$= 5x^2 + 30x + 4x + 24$
$= 5x^2 + 34x + 24$ ✓

A.4. a. 1 **b.** All of these **c.** Both negative **d.** $(4c - 3d)(2c - 3d)$
e. $(4c - 3d)(2c - 3d)$
$= 8c^2 - 12cd - 6cd + 9d^2$
$= 8c^2 - 18cd + 9d^2$ ✓

A.5. a. $3y$ **b.** $-3y(y^2 - 4y - 32)$ **c.** Different **d.** $-3y(y - 8)(y + 4)$
e. $-3y(y - 8)(y + 4)$
$= -3y(y^2 + 4y - 8y - 32)$
$= -3y(y^2 - 4y - 32)$
$= -3y^3 + 12y^2 + 96y$ ✓

A.6. a. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121 **b.** $x^2, x^4, x^6, x^8, x^{10}, x^{12}$ **c.** Even powers of n **A.7. a.** $a^2 + 2ab + b^2$ **b.** $(a + b)^2$ **c.** $9p^2 + 42p + 49$ **d.** $(3p + 7)^2$ **e.** The first and third terms are perfect squares. The middle term is twice the product of the square roots of the first and third terms. **A.8.** $2(2m - 5n)^2$

Section 4.6 Practice Exercises, pp. 408–411

R.1. -4 and 10 **R.3.** $5t^2 + 14t + 13$ **R.5.** $4x(x^2 + 5x + 3)$ **R.7.** $x^2 - 14x + 33$ **R.9.** $16p^2 - 34p - 15$ **R.11.** $6c^3d^2 - 44c^2d^2 - 32cd^2$ **R.13.** $9c^2 + 48c + 64$ **R.15.** $(3a - b)(2x - 1)$ **R.17.** $(2x - 5)(4x + 3)$

1. a. positive **b.** opposite **c.** Both are correct. **d.** $2(3x - 5)(x + 1)$ **e.** $(a + b)^2$; $(a - b)^2$ **3.** Both negative **5.** $(2x + 3)(x - 4)$ **7.** $(5x + 3)(2x + 5)$ **9.** $(b - 8)(b - 4)$ **11.** $(y + 12)(y - 2)$ **13.** $(x + 10)(x + 3)$ **15.** $(c - 8)(c + 2)$ **17.** $(2x + 3)(x - 5)$ **19.** $(6a - 5)(a + 1)$ **21.** $(s + 3t)(s - 2t)$ **23.** $3(x - 18)(x - 2)$ **25.** $2(c - 4)(c + 3)$ **27.** $2(x - y)(x + 5y)$ **29.** Prime **31.** $(3x + 5y)(x + 3y)$ **33.** $5uv(u - 3v)^2$ **35.** $x(x - 7)(x + 2)$ **37.** $(2z - 5)(5z + 1)$ **39.** Prime **41.** $-2(t + 4)(t - 10)$ **43.** $(7a - 4)(2a + 3)$ **45.** $2b(3a + 2)(a + 3)$ **47. a.** $x^2 + 10x + 25$ **b.** $(x + 5)^2$ **49. a.** $9x^2 - 12xy + 4y^2$ **b.** $(3x - 2y)^2$ **51.** $30x$ **53.** $16z^2t$ **55.** $(y - 4)^2$ **57.** $(8m + 5)^2$ **59.** Not a perfect square trinomial **61.** $(3a - 5b)^2$ **63.** $4(4t^2 - 20tv + 5v^2)$; Not a perfect square trinomial **65.** $5(b^2 - 2)^2$ **67. a.** $(u - 5)^2$ **b.** $(x^2 - 5)^2$ **c.** $(a - 4)^2$ **69. a.** $(u + 13)(u - 2)$ **b.** $(w^3 + 13)(w^3 - 2)$ **c.** $(y + 9)(y - 6)$ **71.** $(3x - 4)(3x + 1)$ **73.** $(2x - 9)(x - 1)$ **75.** $(3y + 11)(y + 6)$ **77.** $(3y^3 + 2)(y^3 + 3)$ **79.** $(4p^2 + 1)(p^2 + 1)$

81. $(x^2 + 12)(x^2 + 3)$ **83.** The factorization $(2y - 1)(2y - 4)$ is not factored completely because the factor $2y - 4$ has a GCF of 2.
85. $(w^2 + 6)^2$ **87.** $(9w + 5)^2$ **89.** $3(a + b)(x - 2)$
91. $2abc^2(6a + 2b - 3c)$ **93.** $-2x(5x - 6)(2x - 5)$
95. Prime **97.** $(2w^2 - 15)(w^2 - 2)$
99. $(1 - d)(1 - 3d)$ or $(3d - 1)(d - 1)$
101. $(a + 2b)(x - 5a)$ **103.** $8(z - 4w)(z + 7w)$
105. $(y + x)(a - 5c)$ **107.** $g(x) = (3x + 2)(x + 4)$
109. $n(t) = (t + 10)^2$ **111.** $Q(x) = x^2(x + 4)(x + 2)$
113. $k(a) = (a - 4)(a^2 + 2)$

Section 4.7 Activity, pp. 418–419

A.1. a. $a^2 - b^2$ **b.** $(a - b)(a + b)$ **c.** $25y^2 - 16$ **d.** $(5y - 4)(5y + 4)$ **e.** The two terms have opposite signs and their absolute values are perfect squares.
A.2. $(8p - 7)(8p + 7)$ **A.3.** $5(3m^2 - 2n)(3m^2 + 2n)$
A.4. $(9t^2 + 4)(3t - 2)(3t + 2)$ **A.5. a.** Factoring by grouping **b.** $(2x + 3)(x^2 - 25)$ **c.** Yes. $x^2 - 25$ can be factored further as a difference of squares. **d.** $(2x + 3)(x - 5)(x + 5)$
A.6. a. The first pair of terms and the second pair of terms do not share a common binomial factor. **b.** $y^2 - (z^2 + 8z + 16)$ **c.** $y^2 - (z + 4)^2$ **d.** $[y - (z + 4)][y + (z + 4)]$, which simplifies to $(y - z - 4)(y + z + 4)$
A.7. a. 1, 8, 27, 64, 125 **b.** x^3, x^6, x^9, x^{12} **c.** n must be a multiple of 3.
A.8. a. $a^3 - b^3$ **b.** $(a - b)(a^2 + ab + b^2)$
A.9. a. $(a + b)(a^2 - ab + b^2)$ **b.** $(5x)^3$ **c.** $(4)^3$ **d.** $a = 5x, b = 4$
e. $(5x + 4)(25x^2 - 20x + 16)$
f. $(5x + 4)(25x^2 - 20x + 16)$
$= 125x^3 - 100x^2 + 80x + 100x^2 - 80x + 64$
$= 125x^3 + 64$ ✓
A.10. a. $(a - b)(a^2 + ab + b^2)$ **b.** $(c^2)^3$ **c.** $(2d)^3$
d. $a = c^2, b = 2d$ **e.** $(c^2 - 2d)(c^4 + 2c^2d + 4d^2)$
f. $(c^2 - 2d)(c^4 + 2c^2d + 4d^2)$
$= c^6 + 2c^4d + 4c^2d^2 - 2c^4d - 4c^2d^2 - 8d^3$
$= c^6 - 8d^3$ ✓

Section 4.7 Practice Exercises, pp. 419–422

R.1. $5y^2$ **R.3.** $4p$ **R.5.** $-3t^2$ **R.7.** $9x^2 - 25$
R.9. $t^4 - 256$ **R.11.** $x^2 - 16x + 64$ **R.13.** $y^3 - 8$

1. a. difference; $(a + b)(a - b)$ **b.** sum **c.** is not **d.** square
3. difference; cubes **5.** $(a + b)(a^2 - ab + b^2)$ **7.** 8, 64, x^3, x^9
9. No, the sum of squares is prime. **11.** $(x - 3)(x + 3)$
13. $(4 - 7w)(4 + 7w)$ **15.** $2(2a - 9b)(2a + 9b)$ **17.** Prime
19. $2(a^2 + 4)(a - 2)(a + 2)$ **21.** $(7 - k^3)(7 + k^3)$
23. $(x - 4)(x + 4)(x - 1)$ **25.** $(2x + 1)(2x - 1)(x + 3)$
27. $(9y + 7)(y - 2)(y + 2)$ **29.** $(7x + 2 - y)(7x + 2 + y)$
31. $(w - 3n + 1)(w + 3n - 1)$ **33.** $(p^2 - 5 - t^2)(p^2 - 5 + t^2)$
35. $(3u^2 - 2v^2 + 5)(3u^2 + 2v^2 - 5)$ **37.** Look for a binomial of the form $a^3 + b^3$. This factors as $(a + b)(a^2 - ab + b^2)$.
39. $(2x - 1)(4x^2 + 2x + 1)$ **41.** $(5c + 3)(25c^2 - 15c + 9)$
43. $(x - 10)(x^2 + 10x + 100)$ **45.** $(4t^2 + 1)(16t^4 - 4t^2 + 1)$
47. $2(10y^2 + x)(100y^4 - 10y^2x + x^2)$
49. $2z(2z - 3)(4z^2 + 6z + 9)$ **51.** $(p^4 - 5)(p^8 + 5p^4 + 25)$
53. $\left(6y - \frac{1}{5}\right)\left(6y + \frac{1}{5}\right)$ **55.** $2(3d^6 - 4)(3d^6 + 4)$
57. $2(121v^2 + 16)$ **59.** $4(x - 2)(x + 2)$ **61.** $(5 - 7q)(5 + 7q)$
63. $(t + 2s - 6)(t + 2s + 6)$ **65.** $(3 - t)(9 + 3t + t^2)$
67. $\left(3a + \frac{1}{2}\right)\left(9a^2 - \frac{3}{2}a + \frac{1}{4}\right)$ **69.** $2(m + 2)(m^2 - 2m + 4)$
71. $(x - y)(x + y)(x^2 + y^2)$
73. $(a + b)(a^2 - ab + b^2)(a^6 - a^3b^3 + b^6)$
75. $\left(\frac{1}{2}p - \frac{1}{5}\right)\left(\frac{1}{4}p^2 + \frac{1}{10}p + \frac{1}{25}\right)$ **77.** Prime
79. $\left(\frac{1}{5}x - \frac{1}{2}y\right)\left(\frac{1}{5}x + \frac{1}{2}y\right)$
81. $(a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$
83. $(2 + y)(4 - 2y + y^2)(2 - y)(4 + 2y + y^2)$
85. $(h^2 + k^2)(h^4 - h^2k^2 + k^4)$ **87.** $(2x^2 + 5)(4x^4 - 10x^2 + 25)$
89. $4x^2 - 9$ **91.** $8a^3 - 27$ **93.** $64x^6 + y^3$ **95. a.** $x^2 - y^2$ **b.** $(x + y)(x - y)$ **c.** 20 in.^{2} **97.** $(x + y)(x - y + 1)$
99. $(x + y)(x^2 - xy + y^2 + 1)$
101. $(3a - c)(3a + c)(4a - 1)(16a^2 + 4a + 1)$

Chapter 4 Problem Recognition Exercises, pp. 422–424

1. A polynomial whose only factors are 1 and itself
2. Factor out the GCF. **3.** Difference of squares $a^2 - b^2$, difference of cubes $a^3 - b^3$, or sum of cubes $a^3 + b^3$
4. Look for a perfect square trinomial, $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$.
5. Try factoring by grouping 2 terms by 2 terms or by grouping 3 terms by 1 term. **6.** Let $u = (4x^2 + 1)$. The polynomial becomes $3u^2 + 20u + 12$. Factor this simpler expression and then back substitute.
7. a. Trinomial **b.** $3(2x + 3)(x - 5)$
8. a. Trinomial **b.** $m(4m + 1)(2m - 3)$
9. a. Difference of squares **b.** $2(2a - 5)(2a + 5)$
10. a. Grouping **b.** $(b + y)(a - b)$
11. a. Trinomial **b.** $(2u - v)(7u - 2v)$
12. a. Perfect square trinomial **b.** $(3p - 2q)^2$
13. a. Difference of cubes **b.** $2(2x - 1)(4x^2 + 2x + 1)$
14. a. Sum of squares **b.** Prime
15. a. Sum of cubes **b.** $(3y + 5)(9y^2 - 15y + 25)$
16. a. None of these **b.** Prime
17. a. Sum of cubes **b.** $2(4p^2 + 3q)(16p^4 - 12p^2q + 9q^2)$
18. a. Perfect square trinomial **b.** $5(b - 3)^2$
19. a. Difference of squares **b.** $(2a - 1)(2a + 1)(4a^2 + 1)$
20. a. Perfect square trinomial **b.** $(9u - 5v)^2$
21. a. Grouping **b.** $(p - 6 - c)(p - 6 + c)$
22. a. Sum of squares **b.** $4(x^2 + 4)$
23. a. Grouping **b.** $2(2x - y)(3a + b)$
24. a. Difference of cubes **b.** $(5y - 2)(25y^2 + 10y + 4)$
25. a. Trinomial **b.** $(5y - 1)(y + 3)$
26. a. Difference of squares **b.** $2(m^2 - 8)(m^2 + 8)$
27. a. Difference of squares **b.** $(t - 10)(t + 10)$
28. a. Difference of squares **b.** $(2m - 7n)(2m + 7n)$
29. a. Sum of cubes **b.** $(y + 3)(y^2 - 3y + 9)$
30. a. Sum of cubes **b.** $(x + 1)(x^2 - x + 1)$
31. a. Trinomial **b.** $(d - 4)(d + 7)$
32. a. Trinomial **b.** $(c + 8)(c - 3)$
33. a. Perfect square trinomial **b.** $(x - 6)^2$
34. a. Perfect square trinomial **b.** $(p + 8)^2$
35. a. Grouping **b.** $(ax + b)(2x - 5)$
36. a. Grouping **b.** $(4x + a)(2x - b)$
37. a. Trinomial **b.** $(2y - 1)(5y + 4)$
38. a. Trinomial **b.** $(3z + 2)(4z + 1)$
39. a. Difference of squares **b.** $10(p - 8)(p + 8)$
40. a. Difference of squares **b.** $2(5a - 6)(5a + 6)$
41. a. Difference of cubes **b.** $z(z - 4)(z^2 + 4z + 16)$
42. a. Difference of cubes **b.** $t(t - 2)(t^2 + 2t + 4)$
43. a. Trinomial **b.** $b(b + 5)(b - 9)$
44. a. Trinomial **b.** $y(y - 4)(y - 10)$

45. **a.** Perfect square trinomial **b.** $(3w + 4x)^2$
46. **a.** Perfect square trinomial **b.** $(2k - 5p)^2$
47. **a.** Grouping **b.** $10(2x + a)(3x - 1)$
48. **a.** Grouping **b.** $10(5x + c)(x - 4)$
49. **a.** Difference of squares **b.** $(w^2 + 4)(w - 2)(w + 2)$
50. **a.** Difference of squares **b.** $(k^2 + 9)(k - 3)(k + 3)$
51. **a.** Difference of cubes **b.** $(t^2 - 2)(t^4 + 2t^2 + 4)$
52. **a.** Sum of cubes **b.** $(p^2 + 3)(p^4 - 3p^2 + 9)$
53. **a.** Trinomial **b.** $(4p - 1)(2p - 5)$
54. **a.** Trinomial **b.** $(3m + 4)(3m - 5)$
55. **a.** Perfect square trinomial **b.** $(6y - 1)^2$
56. **a.** Perfect square trinomial **b.** $(3a + 7)^2$
57. **a.** Sum of squares **b.** $2(x^2 + 25)$
58. **a.** Sum of squares **b.** $4(y^2 + 16)$
59. **a.** Trinomial **b.** $s^2(3r - 2)(4r + 5)$
60. **a.** Trinomial **b.** $w^2(7z + 4)(z - 2)$
61. **a.** Trinomial **b.** $(x - 3y)(x + 11y)$
62. **a.** Trinomial **b.** $(s + 3t)(s - 12t)$
63. **a.** Sum of cubes **b.** $(m^2 + n)(m^4 - m^2n + n^2)$
64. **a.** Difference of cubes **b.** $(a - b^2)(a^2 + ab^2 + b^4)$
65. **a.** None of these **b.** $x(x - 4)$
66. **a.** None of these **b.** $y(y - 9)$ **67.** $(x - y)(x + y)^2$
68. $(u - v)^2(u + v)$ **69.** $(a + 3)^4(6a + 19)$
70. $(4 - b)^3(2 - b)$ **71.** $18(3x + 5)^2(4x + 5)$
72. $5(2y + 3)^2(6y + 11)$ **73.** $\left(\frac{1}{10}x + \frac{1}{7}\right)^2$ **74.** $\left(\frac{1}{5}a + \frac{1}{6}\right)^2$
75. $5x^2(5x^2 - 6)$ **76.** $x^3(x^3 - 2)$ **77.** $(4p^2 + q^2)(2p - q)(2p + q)$
78. $(s^2t^2 + 9)(st - 3)(st + 3)$ **79.** $\left(y + \frac{1}{4}\right)\left(y^2 - \frac{1}{4}y + \frac{1}{16}\right)$
80. $\left(z + \frac{1}{5}\right)\left(z^2 - \frac{1}{5}z + \frac{1}{25}\right)$ **81.** $(a + b)(a - b)(6a + b)$
82. $(2p - q)(2p + q)(p + 3q)$ **83.** $\left(\frac{1}{3}t + \frac{1}{4}\right)^2$ **84.** $\left(\frac{1}{5}y + \frac{1}{2}\right)^2$
85. $(x + 6 - a)(x + 6 + a)$ **86.** $(a + 5 + b)(a + 5 - b)$
87. $(p + q - 9)(p + q + 9)$ **88.** $(m - n - 3)(m - n + 3)$
89. $(b - x - 2)(b + x + 2)$ **90.** $(p - y + 3)(p + y - 3)$
91. $(2 + u - v)(2 - u + v)$ **92.** $(5 - a - b)(5 + a + b)$
93. $(3a + b)(2x - y)$ **94.** $(q + 3)(5p - 4)$
95. $(u - 2)(u + 2)(u^2 + 2u + 4)(u^2 - 2u + 4)$
96. $(1 - v)(1 + v)(1 + v + v^2)(1 - v + v^2)$
97. $(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$
98. $(y^4 + 16)(y^2 + 4)(y + 2)(y - 2)$
99. $(a + b)(a - b + 1)$ **100.** $(5c - 3d)(5c + 3d + 1)$
101. $(x + y)(x^2 - xy + y^2)(5w - 2z)$

Section 4.8 Activity, pp. 434–435

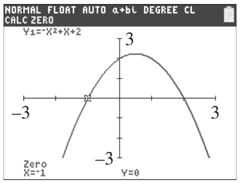
A.1. **a.** linear; 1 **b.** quadratic; 2
A.2. **a.** $a = 0$ or $b = 0$ **b.** $2x + 1 = 0$ or $x - 6 = 0$ **c.** $\left\{-\frac{1}{2}, 6\right\}$
A.3. **a.** $(3x - 1)(x + 4) = 0$ **b.** $3x - 1 = 0, x + 4 = 0$
c. $\left\{\frac{1}{3}, -4\right\}$ **d.** Yes
A.4. **a.** $4(x - 2)(x - 8) = 0$ **b.** $4 = 0, x - 2 = 0, x - 8 = 0$
c. $\{2, 8\}$ **d.** Yes **A.5.** Quadratic; $\{-9, -1\}$
A.6. Linear; $\{21\}$ **A.7.** Quadratic; $\{5\}$
A.8. Higher-order polynomial equation; $\left\{3, -\frac{1}{2}, 5\right\}$
A.9. Higher-order polynomial equation; $\left\{0, -\frac{1}{5}, \frac{1}{5}\right\}$
A.10. Higher-order polynomial equation; $\{-4, 3, 4\}$
A.11. **a.** $x + 2$ **b.** $x(x + 2)$ **c.** $x + (x + 2) + 14$ **d.** $x(x + 2) = x + (x + 2) + 14$ **e.** $x = -4, x = 4$ **f.** The numbers are –4 and –2 or 4 and 6.
A.12. **a.** $x + 11$ **b.** $x + 22$ **c.** $x^2 + (x + 11)^2 = (x + 22)^2$ **d.** $x = -11, x = 33$. The solution $x = -11$ does not make sense in this scenario because x represents the distance between two cities. **e.** The distance between Ann Arbor and Detroit is 33 mi. The distance between Ann Arbor and Toledo is 44 mi, and the distance between Toledo and Detroit is 55 mi.
A.13. **a.** Quadratic
b. $P(100) = 3200$ means that if Cassandra produces 100 boxes, she makes a profit of \$3200.
c. $P(200) = 4400$, $P(300) = 3600$, and $P(400) = 800$
d. This may be due to limited demand for boxes. If Cassandra makes too many boxes, she may not be able to sell them all.
e. (0, 0) and (420, 0) mean that if Cassandra makes 0 boxes or 420 boxes, she will have no profit. If she makes 0 boxes, she won't spend any money on materials, but she also won't earn any revenue. Thus, her profit is zero (she neither earns nor loses money). If she makes 420 boxes, she's made more than she can sell, and the cost of extra materials will counter any revenue made.

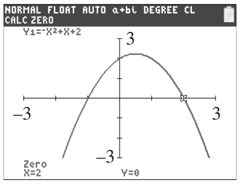
Section 4.8 Practice Exercises, pp. 435–440

R.1. $\{-11\}$ **R.3.** $\left\{\frac{9}{5}\right\}$ **R.5.** $(x - 8)(x + 9)$
R.7. $(4t - 3)(t + 2)$ **R.9.** $4h(h - 3)(h + 3)$
R.11. **a.** $f(0) = -20$ **b.** $f(5) = 0$ **c.** $f(-4) = 0$
R.13. x-intercept: $(-4, 0)$; y-intercept: $(0, -8)$

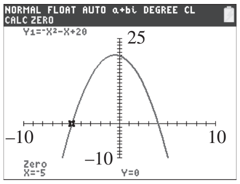
1. quadratic **3.** Pythagorean; c^2 **5.** $f(x) = 0$; y **7.** lw
9. The equation must be set equal to 0, and the polynomial must be factored. **11.** Correct form **13.** Incorrect form. Polynomial is not factored. **15.** Incorrect form. Equation is not set equal to 0.
17. **a.** $(w + 9)(w - 9)$ **b.** $\{-9, 9\}$ **19.** **a.** $(3x - 1)(x + 5)$
b. $\left\{\frac{1}{3}, -5\right\}$ **21.** $\{-3, -5\}$ **23.** $\left\{-\frac{9}{2}, \frac{1}{5}\right\}$ **25.** $\left\{0, -4, \frac{3}{10}\right\}$
27. $\{0.4, -2.1\}$ **29.** $\{-9, 3\}$ **31.** $\left\{-3, \frac{1}{2}\right\}$ **33.** $\left\{0, \frac{3}{2}\right\}$
35. $\left\{\frac{23}{3}\right\}$ **37.** $\{-3\}$ **39.** $\left\{-\frac{1}{3}, 2\right\}$ **41.** $\left\{-1, \frac{1}{2}, 3\right\}$
43. $\{-5, 4\}$ **45.** $\left\{\frac{5}{2}, -1\right\}$ **47.** $\{-12, 5\}$ **49.** $\left\{-\frac{1}{11}\right\}$
51. $\{0, 6, -2\}$ **53.** $\{0, 4, -4\}$ **55.** $\left\{3, -3, -\frac{5}{2}\right\}$ **57.** 5, –5
59. 4, –3 **61.** –7, –6 or 6, 7 **63.** –9, –7 or 7, 9
65. The length is 7 ft, and the width is 5 ft.
67. The length is 20 yd, and the width is 15 yd.
69. **a.** The base is 5 in., and the height is 6 in. **b.** The area is 15 in.^{2}
71. The base is 10 ft, and the height is 5 ft.
73. The integers are 4 and 5.
75. **a.** 14 mi **b.** The alternative route using superhighways
77. The lengths are 6 m, 8 m, and 10 m. **79.** The radius is 2 units.
81. **a.** 0, 3 **b.** $f(0) = 0$ **83.** **a.** 7, –1 **b.** $f(0) = -7$
85. x-intercepts: $(2, 0), (-1, 0), (0, 0)$; y-intercept: $(0, 0)$
87. x-intercept: $(1, 0)$; y-intercept: $(0, 1)$ **89.** $(3, 0), (-3, 0)$; d
91. $(-1, 0)$; a **93.** **a.** The function is in the form $s(t) = at^2 + bt + c$.
b. (0, 0) and (100, 0) **c.** At 0 sec and 100 sec, the rocket is at ground level (height = 0). **d.** At 1 sec and 99 sec
95. $f(x) = (x - 5)(x - 2)$; $x = 5$ and $x = 2$ represent x-intercepts.
97. $f(x) = (x + 1)^2$; $x = -1$ represents the x-intercept.
99. $f(x) = -(x + 1)(x + 5)$; $x = -1$ and $x = -5$ represent the x-intercepts.

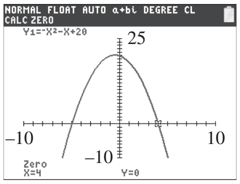
101. (2, 0), (−1, 0)



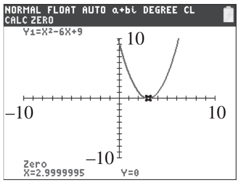


102. (−5, 0), (4, 0)

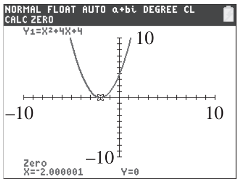




103. (3, 0)



104. (−2, 0)



105. The radius is 6 ft.
107. The length is 8 ft and the width is 6 ft.
109. $(x-2)(x+2)=0$ or $x^2-4=0$
111. $(x-0)(x+3)=0$ or $x^2+3x=0$

Chapter 4 Review Exercises, pp. 446–449

1. 3^5x^5 or $243x^5$ **2.** $-\dfrac{18}{x^{12}}$ **3.** $-3xy^2$ **4.** $\dfrac{3x^3}{2y^2}$ **5.** $-\dfrac{b^{15}}{8a^6}$
6. $\dfrac{a^4}{16b^6}$ **7.** $\dfrac{5^4y^{24}}{4^4x^{20}}$ **8.** $\dfrac{x^{10}y^5}{5^5}$ **9. a.** 3.6866×10^9 **b.** 1×10^{-6} **10. a.** 1×10^{-3} **b.** 5.1557×10^9
11. a. 0.001 **b.** 0.000000001 **12. a.** 5,230,000,000 ft^2 **b.** \$1,091,000,000,000 **13.** 6.25×10^9 **14.** 2×10^{-8}
15. 3.24×10^7 **16.** 3.64×10^{-8} **17.** Trinomial; degree 4
18. Monomial; degree 0 **19. a.** −7 **b.** −23 **c.** 5
20. a. 12 **b.** 10 **c.** −2 **21. a.** $A(5)\approx 25$ means that in year 5, Americans on average consumed approximately 25 gal of bottled water each. **b.** $A(15)\approx 49$ means that in year 15, Americans consumed approximately 49 gal of bottled water each. **22.** $-2x^2-3x-xy-1$
23. $20xy-18xz-3yz$ **24.** $3a^3+5a^2+6a$ **25.** $-2a^2+6a$
26. x^4-x^2-1 **27.** $x^4+\dfrac{3}{4}x^2$ **28.** $6x-6y$ **29.** $-5x+9y$
30. $-11x+1$ **31.** $4x^2-11x$ **32.** $-3x-11$ **33.** $-3x$
34. $2x^3-14x^2-8x$ **35.** $-18x^3+15x^2-12x$
36. x^2-x-42 **37.** $x^2-11x+18$ **38.** $\dfrac{1}{4}x^2-2x-5$
39. $2y^2+\dfrac{1}{5}y-\dfrac{1}{25}$ **40.** $27x^3+125$ **41.** x^3-y^3
42. $4x^2-20x+25$ **43.** $\dfrac{1}{4}x^2+4x+16$ **44.** $9y^2-121$
45. $36w^2-1$ **46.** $\dfrac{4}{9}t^2-16$ **47.** $z^2-\dfrac{1}{16}$
48. $x^2+4x+4-b^2$ **49.** c^2-w^2-6w-9
50. $8x^3+12x^2+6x+1$ **51.** $y^6-9y^4+27y^2-27$
52. a. $4x^2+12x+9$ **b.** $16x^2+24x+9$ **c.** $12x^2+12x$
53. a. $P(x)=8x+4$ **b.** $A(x)=3x^2+2x$
54. a. False; cannot add *unlike* terms True **b.** True False; when subtracting *like* terms, keep the variable the same.
55. $2x^2+4xy-3y^2$ **56.** $-2x^2-3x+4$

57. a. $3y^3-2y^2+6y-4$ **b.** Quotient $3y^3-2y^2+6y-4$; no remainder **c.** Multiply the quotient and the divisor. Then add the remainder.
58. $x+2$ **59.** $x+4+\dfrac{-32}{x+4}$
60. $2x^3+2x^2+8x+24+\dfrac{72x-4}{x^2-3x}$
61. $2x^3-2x^2+5x-4+\dfrac{4x-4}{x^2+x}$
62. Synthetic division can be used with a divisor in the form $x-r$.
63. a. $x-3$ **b.** $2x^3+11x^2+31x+99$ **c.** 298
64. t^2-t+6 **65.** $x+2+\dfrac{4}{x+5}$ **66.** $x+4+\dfrac{4}{x+4}$
67. $w^2-3w-9+\dfrac{-19}{w-3}$ **68.** p^3+2p^2+4p+8
69. $x(-x^2-4x+11)$ or $-x(x^2+4x-11)$ **70.** $7(3w^3-w+2)$
71. $(x-7)(5x-2)$ **72.** $(t+4)(3t+5)$
73. $(m-8)(m^2+1)$ **74.** $12(2x-3)(x^2+3)$
75. $x(2a+b)(2x-3)$ **76.** $(y-6)(y^2+1)$ **77.** The trinomial must be of the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$.
78. $(3x+2y)(6x+5y)$ **79.** $(3m-5t)(m+2t)$
80. $5a^2(3+4a)(4-a)$ or $-5a^2(4a+3)(a-4)$
81. $k^2(3k+2)(2k+1)$ **82.** $(7x-6)^2$ **83.** $2(5z+4)^2$
84. $(9w+7)(9w+1)$ **85.** $(4x-3)^2$
86. $3(3a^2-1)(2a^2+5)$ **87.** $(w^2+1)(3w^2-5)$
88. $(5-y)(5+y)$ **89.** $\left(x-\dfrac{1}{3}\right)\left(x^2+\dfrac{1}{3}x+\dfrac{1}{9}\right)$
90. Prime **91.** $h(h^2+9)$ **92.** $(a+4)(a^2-4a+16)$
93. $(k^2+4)(k-2)(k+2)$ **94.** $y(3y-2)(3y+2)$
95. $(x-4y-3)(x-4y+3)$ **96.** $(a+6-b)(a+6+b)$
97. $(t+8-5c)(t+8+5c)$ **98.** $(y-3-4x)(y-3+4x)$
99. It can be written in the form $ax^2+bx+c=0$ $(a\neq 0)$.
100. It is a parabola. **101.** Quadratic **102.** Quadratic
103. Linear **104.** Quadratic **105. a.** $(5x-4)(x+2)$
b. $\left\{\dfrac{4}{5},-2\right\}$ **106. a.** $(3x-7)(x-4)$ **b.** $\left\{\dfrac{7}{3},4\right\}$ **107.** $\{-3,5\}$
108. $\left\{\dfrac{3}{8},7\right\}$ **109.** $\left\{-\dfrac{4}{3},-1\right\}$ **110.** $\left\{1,-5,\dfrac{9}{2}\right\}$
111. (1, 0), (−1, 0), (0, 4); b **112.** (1, 0), (−1, 0), (0, −2); d
113. (2, 0), (−2, 0), (0, 40); c **114.** (2, 0), (−2, 0), $\left(0,\dfrac{1}{2}\right)$; a
115. Length 15 ft; width 8 ft; height 10 ft
116. a.

Time t (sec)	Height $h(t)$ (ft)
0	−1280
1	−624
3	592
10	3840
20	5760
30	4480
42	−1280

b. The position of the missile is below sea level. **c.** The missile will be at sea level after 2 sec and again after 40 sec.

Chapter 4 Test, pp. 450–451

1. $5a^{13}$ **2.** x^{11} **3.** $\dfrac{9x^{12}}{25y^{14}}$ **4.** $\dfrac{8y^{12}}{x^7}$ **5.** 5.68
6. 2.3×10^9 **7.** $F(-1)=1$ $F(2)=40$ $F(0)=8$
8. $8x^2-8x+8$ **9.** $2a^3-13a^2+2a+45$
10. $2x^2-\dfrac{23}{3}x-6$ **11.** $25x^2-16y^4$ **12.** The expression $25x^2+49$ does not include the middle term $70x$.

13. $49x^2 - 56x + 16$ **14.** $x^2y^3 + \frac{5}{2}xy - 3y^2 - \frac{1}{2}$

15. $5p^2 - p + 1$ **16.** $y^3 + 2y^2 + 4y + 6 + \frac{17}{y-2}$

17. $(3y-4)(y+7)$ **18.** Prime

19. $3(a+6b)(a+3b)$ **20.** $(c-1)(c+1)(c^2+1)$

21. $(y-7)(x+3)$ **22.** Prime **23.** $-10(u-2)(u-1)$

24. $3(2t-5)(2t+5)$ **25.** $5(y-5)^2$ **26.** $7q(3q+2)$

27. $(2x+1)(x-2)(x+2)$ **28.** $(y-5)(y^2+5y+25)$

29. $(x+4-y)(x+4+y)$ **30.** $r^2(r^2+16)(r-4)(r+4)$

31. $(x^2+3)(x^2+2)$ **32.** $(2-c)(6a+b)$ **33.** $\left\{\frac{3}{2}, -5\right\}$

34. $\{0, 7\}$ **35.** $\{8, -2\}$ **36.** $\left\{\frac{1}{5}, -1\right\}$ **37.** $\left\{0, \frac{1}{4}, -\frac{1}{4}\right\}$

38. $\left\{-\frac{1}{4}\right\}$ **39.** 256 ft **40. a.** $P(4) = 127.292$. In year 4, the population of Japan was approximately 127.292 million. **b.** 126.968 million **c.** 124.52 million **41.** $(4, 0), (2, 0), (0, 8)$; c

42. $(-4, 0), (3, 0), (-3, 0), (0, -36)$; b

43. $(-3, 0), (-1, 0), (0, -6)$; d **44.** $(4, 0), (-3, 0), (0, 0)$; a

Chapter 5

Section 5.1 Activity, p. 460

A.1. a. 3 **b.** 2 **c.** 6 **d.** Undefined

A.2. a. $x = -2$ makes the denominator zero. **b.** $(-\infty, -2) \cup (-2, \infty)$

A.3. a. $g(x) = \frac{2(x-2)(x+5)}{(x-2)(3x+1)}$ **b.** $x \neq 2, x \neq -\frac{1}{3}$

c. $\left(-\infty, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, 2\right) \cup (2, \infty)$

d. $g(x) = \frac{2(x+5)}{3x+1}$, provided that $x \neq 2$ and $x \neq -\frac{1}{3}$

A.4. a. $-y + 5$ or $5 - y$ b. -1 **A.5. a.** $4a + 2b$ **b.** -1

A.6. $-\frac{2(t+6)}{t+1}$ provided $t \neq 0$, $t \neq 6$, and $t \neq -1$

Section 5.1 Practice Exercises, pp. 461–464

R.1. $2^2 \cdot 3 \cdot 7$ **R.3.** $-\frac{4}{7}$ **R.5.** -1 **R.7.** x^2 **R.9.** $\frac{1}{a}$

R.11. $12x^3(2x-5)$ **R.13.** $(6c-5d)(6c+5d)$ **R.15.** $(7w-4)(2w-3)$

R.17. $(4a-b)(2c+3)$ **R.19.** $(4x-9y)^2$

R.21. $(-\infty, -3) \cup (-3, \infty)$;

1. a. rational **b.** denominator **c.** $\frac{p}{q}$ **d.** 1; -1

3. $k(0) = -\frac{3}{4}$, $k(-1) = -1$, $k(2) = -\frac{1}{2}$, $k(-4)$ is undefined

5. $n(1) = 2, n(0) = 1, n\left(-\frac{1}{3}\right) = 0, n(-1) = -1$

7. a. $\{x \mid x \text{ is a real number}, x \neq 0\}$ **b.** $(-\infty, 0) \cup (0, \infty)$

9. a. $\{v \mid v \text{ is a real number}, v \neq 7\}$ **b.** $(-\infty, 7) \cup (7, \infty)$

11. a. $\{x \mid x \text{ is a real number}, x \neq \frac{5}{2}\}$ **b.** $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$

13. a. $\{q \mid q \text{ is a real number}, q \neq -9, q \neq 3\}$ **b.** $(-\infty, -9) \cup (-9, 3) \cup (3, \infty)$ **15. a.** $\{c \mid c \text{ is a real number}\}$ **b.** $(-\infty, \infty)$ **17. a.** $\{x \mid x \text{ is a real number}, x \neq -5, x \neq 5\}$ **b.** $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$

19. a. $\{x \mid x \text{ is a real number}\}$ **b.** $(-\infty, \infty)$

21. $(-\infty, -4) \cup (-4, \infty)$; b **23.** $(-\infty, 4) \cup (4, \infty)$; d

25. a. $\frac{2x}{y}$ **b.** Cannot be simplified **27. a.** $\frac{(x+4)(x+2)}{(x+4)(x-1)}$ **b.** $x \neq -4, x \neq 1$ **c.** $\frac{x+2}{x-1}$ **29. a.** $\frac{(x-9)^2}{(x-9)(x+9)}$ **b.** $x \neq 9, x \neq -9$ **c.** $\frac{x-9}{x+9}$ **31.** $\frac{25x^2}{9y^3}$ **33.** $\frac{w^8z^3}{2}$ **35.** $-\frac{1}{4m^2n^3}$

37. $\frac{2}{3}$ **39.** $\frac{1}{x+5}$ **41.** $-\frac{1}{3c-5}$ **43.** $-\frac{t+4}{t+3}$

45. $\frac{(2p-1)^2}{p+1}$ **47.** $\frac{3-z}{2z-5}$ **49.** $\frac{2(z^2-4z+16)}{z+4}$

51. $-\frac{2x-5}{2}$ **53.** 1 **55.** -1 **57.** -1 **59.** -2

61. $\frac{c+4}{c-4}$; cannot be simplified **63.** $-\frac{1}{12(x+y)}$

65. $\frac{t-1}{t+6}$ **67.** $\frac{4}{p-3}$ **69.** $-\frac{2a}{b}$ **71.** $-\frac{x+y}{8}$

73. $\frac{1}{2b-3}$ **75.** $\frac{x-17}{2x-3}$ **77.** $\frac{(a-5)^2}{a-2}$ **79.** $-\frac{2x}{5}$

81. $\frac{x-5}{x+2}$ **83.** $\frac{t^2-2t+4}{3t-5}$ **85.** For example: $\frac{1}{x-2}$

87. For example: $f(x) = \frac{1}{x+5}$

Section 5.2 Activity, pp. 467–468

A.1. $\frac{6}{5}$ **A.2.** $-\frac{x}{5(2x+5)}$

A.3. a. $\frac{2y(y-5)^2}{-y^3(2y-3)} \cdot \frac{(y+5)(2y-3)}{4(y-5)(y+5)}$ **b.** $-\frac{y-5}{2y^2}$

A.4. $\frac{4}{9}$ **A.5. a.** $\frac{4ac^2d}{5cd^2} \cdot \frac{25ab}{8a^2c}$ **b.** $\frac{5b}{2d}$

A.6. a. $\frac{1-2x}{x^3+8} \cdot \frac{2x^2+5x+2}{4x^2-1}$

b. $\frac{1-2x}{(x+2)(x^2-2x+4)} \cdot \frac{(2x+1)(x+2)}{(2x-1)(2x+1)}$

c. $\frac{-1}{x^2-2x+4}$

Section 5.2 Practice Exercises, pp. 468–470

R.1. a. -7 **b.** $\frac{4}{3}$ **c.** $\frac{1}{9}$ **d.** $\frac{1}{y}$ **R.3.** $\frac{x^2+15x+14}{x-4}$

R.5. 6 **R.7.** $-\frac{3}{4}$ **R.9.** $\frac{7}{2}$ **R.11.** $\frac{3}{10}$

1. a. $\frac{pr}{qs}$ **b.** $\frac{ps}{qr}$ **3.** $\frac{3}{10}$ **5.** $-\frac{22}{3}$ **7.** $\frac{2}{z^3}$ **9.** $12r^3$

11. $\frac{x(2x-3)}{y^2(x+1)}$ **13.** $\frac{2(3w-7)}{5w+4}$ **15.** 15 **17.** $y^2(y-2)$

19. $\frac{14}{5}$ **21.** $-\frac{2}{5}$ **23.** $\frac{11}{5a^4b^3}$ **25.** $\frac{r+3}{4r^2}$

27. $\frac{1}{(p+2)(6p-7)}$ **29.** $\frac{4}{(b+2)(b+3)}$ **31.** $\frac{2s+5t}{s+5t}$

33. $\frac{2a+b^2}{a^2+b^2}$ **35.** $-\frac{4x}{3(x+2)}$ **37.** $\frac{9}{4y^2}$ **39.** $\frac{24}{y^2}$

41. $\frac{2a}{5}$ **43.** $\frac{2(x+3)}{x+2}$ **45.** $m - n$ **47.** $-\frac{x-3y}{x+3y}$ or $\frac{3y-x}{x+3y}$

49. 1 **51.** $\frac{a-3}{a+4}$ **53.** $\frac{40(2x+1)}{3}$ **55.** $\frac{(x-6)(3x+2)}{3x-2}$

57. $\frac{2k}{h^3}$ cm^2 **59.** $\frac{5x}{4}$ ft^2

Section 5.3 Activity, pp. 478–479

A.1. $\frac{1}{3}$ **A.2.** $x + 5$ **A.3. a.** $\frac{5}{2^2xy^4} + \frac{1}{2^3x^2}$ **b.** $2^3, x^2, y^4$, and the product is $2^3x^2y^4$ or $8x^2y^4$

c. $\frac{5}{4xy^4} \cdot \frac{\boxed{2x}}{\boxed{2x}} + \frac{1}{8x^2} \cdot \frac{\boxed{y^4}}{\boxed{y^4}}$ **d.** $\frac{10x+y^4}{8x^2y^4}$

A.4. a. $\frac{3}{5x} + \frac{7}{3 \cdot 5}$ **b.** $15x$ **c.** $\frac{9}{15x} + \frac{7x}{15x}$ **d.** $\frac{9+7x}{15x}$ **e.** $\frac{7x+9}{15x}$

A.5. a. $\frac{2}{3(x-6)} - \frac{8}{(x-6)(x+6)}$ **b.** $3(x-6)(x+6)$

c. $\dfrac{2(x+6)}{3(x-6)(x+6)} - \dfrac{8\cdot 3}{3\cdot(x-6)(x+6)}$ **d.** $\dfrac{2(x+6)-24}{3(x-6)(x+6)}$ **e.** $\dfrac{2}{3(x+6)}$

A.6. a. $\dfrac{y}{y+4}+\dfrac{1}{y-7}-\dfrac{11}{(y+4)(y-7)}$ **b.** $(y+4)(y-7)$

c. $\dfrac{y(y-7)}{(y+4)(y-7)}+\dfrac{1(y+4)}{(y+4)(y-7)}-\dfrac{11}{(y+4)(y-7)}$

d. $\dfrac{y(y-7)+1(y+4)-11}{(y+4)(y-7)}$ **e.** $\dfrac{y+1}{y+4}$

A.7. a. The expressions $x-3$ and $3-x$ differ only by a factor of -1.
b. $\dfrac{-5}{x-3}$ **c.** $\dfrac{5}{3-x}$ **d.** $\dfrac{-5}{x-3}=\left(\dfrac{-5}{x-3}\right)\cdot\left(\dfrac{-1}{-1}\right)=\dfrac{5}{-x+3}$ or equivalently $\dfrac{5}{3-x}$

Section 5.3 Practice Exercises, pp. 479–482

R.1. 72 **R.3.** 36 **R.5.** $\dfrac{x}{5y}$; $x\neq 0, y\neq 0$ **R.7.** $\dfrac{2y+1}{2}$; $y\neq 4$
R.9. -10; $x\neq 3$ **R.11.** $-5t^2+30t$ **R.13.** $15x^2-16x-7$
R.15. a, b, c

1. a. $\dfrac{p+r}{q}$; $\dfrac{p-r}{q}$ **b.** least common denominator

3. -3 **5.** $-\dfrac{5}{y}$ **7.** $\dfrac{2}{x}$ **9.** $\dfrac{1}{x+1}$

11. $\dfrac{2x+5}{(2x+9)(x-6)}$ **13.** 2 **15.** $40x$ **17.** $30m^4n^7$

19. $(x-4)(x+2)(x-6)$ **21.** $x^2(x-1)(x+7)^2$

23. $(x-6)(x-2)$ **25.** $a-4$ or $4-a$ **27.** $15xy$

29. $2x^3+4x^2$ **31.** y^2-y **33.** $\dfrac{8p-15}{6p^2}$ **35.** $\dfrac{-t-s}{st}$

37. $\dfrac{1}{3}$ **39.** $\dfrac{2b+20}{b(b+5)}$ **41.** $\dfrac{2x}{x-6}$ or $\dfrac{-2x}{6-x}$

43. $\dfrac{6b^2+5b+4}{(b-4)(b+1)}$ **45.** $\dfrac{10x}{(2x+1)(x-2)}$ **47.** $\dfrac{3y-1}{y+4}$

49. $\dfrac{11x+6}{(x-6)(x+6)(x+3)}$ **51.** $-\dfrac{3}{w}$ **53.** 1

55. $\dfrac{10-x}{3(x-5)(x+5)}$ or $\dfrac{x-10}{3(5-x)(5+x)}$

57. $\dfrac{m-5}{(m+5)(m+3)}$ **59.** $\dfrac{x^2+7x+6}{2x^2}$ **61.** $\dfrac{w^2-3}{w-2}$

63. $\dfrac{-x^2+13x-3}{x(x-1)^2}$ **65.** $-\dfrac{3}{t+3}$ **67.** $\dfrac{x^2+x+6}{2(x+1)}$

69. $\dfrac{2z^2+15z}{(z-3)(z+4)}$ **71.** $\dfrac{-2y}{(x-y)(x+y)}$ or $\dfrac{2y}{(y-x)(y+x)}$

73. $\dfrac{2p^2+2p-3}{3(p+4)}$ **75.** $\dfrac{x^2+13x+38}{(x+5)^3}$ **77.** $\dfrac{5}{(z+1)(z-1)}$

79. $\dfrac{5x^2+12x+24}{(x-2)(x+2)(x^2+2x+4)}$ **81.** $\dfrac{3x^2+5x+18}{3x^2}$ cm

83. $\dfrac{4x^2-2x+50}{(x-3)(x+5)}$ m

Section 5.4 Activity, pp. 487–488

A.1. a. $\dfrac{19}{18}$ **b.** $\dfrac{1}{2}$ **c.** $\dfrac{19}{9}$

A.2. a. $\dfrac{y^2-4x^2}{x^2y^2}$ **b.** $\dfrac{y+2x}{xy}$ **c.** $\dfrac{y-2x}{xy}$

A.3. a. 18 **b.** 19 **c.** 9 **d.** $\dfrac{19}{9}$

A.4. a. x^2y^2 **b.** y^2-4x^2 **c.** xy^2+2x^2y **d.** $\dfrac{y-2x}{xy}$

A.5. a. $\dfrac{\frac{1}{x}-\frac{4}{x^2}}{1-\frac{3}{x}-\frac{4}{x^2}}$ **b.** x^2 **c.** $\dfrac{1}{x+1}$

Section 5.4 Practice Exercises, pp. 488–490

R.1. $9x-3$ **R.3.** $y^2+10y+9$ **R.5.** $d-c$
R.7. a. $\dfrac{1}{2}$ **b.** $\dfrac{5}{6}$ **c.** 4 **R.9.** $\dfrac{10+3x}{2x^2}$ **R.11.** $\dfrac{7}{x^3}$

1. complex **3.** $\dfrac{5x^2}{27}$ **5.** $\dfrac{1}{x}$ **7.** $\dfrac{10}{3}$ **9.** -4

11. $28y$ **13.** -8 **15.** $\dfrac{3-p}{p-1}$ **17.** $\dfrac{2a+3}{4-9a}$

19. $-\dfrac{t}{t+1}$ **21.** $-\dfrac{4(w-1)}{w+2}$ **23.** $\dfrac{1}{y+4}$ **25.** $\dfrac{x+2}{x-1}$

27. $\dfrac{t^2}{(t+1)^2}$ **29.** $\dfrac{-a+2}{-a-3}$ **31.** $-\dfrac{y+1}{y-5}$ **33.** $-\dfrac{2}{x(x+h)}$

35. $\dfrac{1}{x(x^2+3)}$ **37.** $\dfrac{2b^2+3a}{b(b-a)}$ **39.** $\dfrac{-1}{4(4+h)}$ **41.** $\dfrac{-6}{x(x+h)}$

43. $m=\dfrac{y_2-y_1}{x_2-x_1}$ **45.** $\dfrac{63}{10}$ **47.** $\dfrac{4}{3}$

49. $(x^{-1}+y^{-1})^{-1}=\dfrac{1}{x^{-1}+y^{-1}}=\dfrac{1}{(1/x)+(1/y)}$ simplifies to $\dfrac{xy}{y+x}$

51. $-x(x-1)$

Chapter 5 Problem Recognition Exercises, pp. 490–491

1. $\dfrac{4y^2-8y+9}{2y(2y-3)}$ **2.** $\dfrac{x^2+x-13}{x-4}$ **3.** $\dfrac{4x-1}{4x-3}$ **4.** $\dfrac{a-5}{3a}$

5. $\dfrac{2y-5}{(y-1)(y+1)}$ **6.** $\dfrac{4}{w+4}$ **7.** $\dfrac{a+4}{2}$ **8.** $\dfrac{(t-3)(t+2)}{t}$

9. $\dfrac{a}{2a-1}$ **10.** $\dfrac{2xy^2}{5}$ **11.** $\dfrac{-x^2+4xy-y^2}{(x-y)(x+y)}$ or $\dfrac{x^2-4xy+y^2}{(y-x)(y+x)}$

12. $3(x-4)$ **13.** $\dfrac{-x^2+4x+14}{6(x-2)}$ **14.** $\dfrac{x^2+14x+99}{10(x+7)}$

15. $-\dfrac{1}{3}$ **16.** $\dfrac{1}{9}$ **17.** $y-1$ **18.** $\dfrac{5t^2-6t-17}{(t+2)(t-3)}$

19. $\dfrac{3(x+4)}{2x-3}$ **20.** $\dfrac{3z^3}{2x^2y^2}$ **21.** $\dfrac{9x^2+6x+15}{4(3x-1)}$ **22.** $\dfrac{2x+3}{1-5x}$

23. $\dfrac{y^2+11y-1}{(y+3)(y-2)}$ **24.** $a-10$

Section 5.5 Activity, p. 497

A.1. a. 15 **b.** $25t-6$ **c.** $15t-7$ **d.** $\left\{-\dfrac{1}{10}\right\}$

A.2. a. $\dfrac{1}{x+4}+\dfrac{x}{x-4}=\dfrac{-8}{(x+4)(x-4)}$ **b.** $x\neq -4, x\neq 4$
c. $(x+4)(x-4)$ **d.** $\{-1\}$; The value -4 does not check.

A.3. a. x^2 **b.** $2x^2-17x+21=0$ **c.** $\left\{\dfrac{3}{2}, 7\right\}$

A.4. a. $bc=ac+ab$ **b.** $bc=a(c+b)$ **c.** $a=\dfrac{bc}{c+b}$

Section 5.5 Practice Exercises, pp. 497–500

R.1. $2x(x-4)$ **R.3.** $8(w+5)$ **R.5.** p^3 **R.7.** $17x-28$
R.9. $-4p-8$ **R.11.** $x\neq 0, x\neq 4$ **R.13.** $\{4, 8\}$
R.15. $\left\{-\dfrac{7}{2}, \dfrac{3}{2}\right\}$ **R.17.** $w=\dfrac{A}{l}$ **R.19.** $y=\dfrac{2}{7}x-2$

1. a. rational **b.** denominator **c.** No **3.** $y\neq 0$

5. $c\neq 5, c\neq -1$ **7.** $\left\{\dfrac{13}{6}\right\}$ **9.** $\{-14\}$

11. $\{-24\}$ **13.** $\left\{-\dfrac{15}{22}\right\}$ **15.** $\{6\}$ **17.** $\{5\}$ **19.** $\{6\}$

21. $\{-25\}$ **23.** $\{3, 1\}$ **25.** $\{4, -4\}$ **27.** $\{8, -2\}$
29. $\{-2\}$ (The value 2 does not check.) **31.** $\{60\}$
33. $\{\ \}$ (The value 5 does not check.)
35. $\left\{\dfrac{31}{5}\right\}$ **37.** $\{5\}$ (The value -2 does not check.)

39. { } (The value 4 does not check.)

41. $\left\{-\frac{11}{4}\right\}$ **43.** $m=\frac{FK}{a}$ **45.** $E=\frac{IR}{K}$

47. $R=\frac{E-Ir}{I}$ or $R=\frac{E}{I}-r$ **49.** $B=\frac{2A-hb}{h}$ or $B=\frac{2A}{h}-b$

51. $t=\frac{b}{x-a}$ or $t=\frac{-b}{a-x}$ **53.** $x=\frac{y}{1-yz}$ or $x=\frac{-y}{yz-1}$

55. $h=\frac{2A}{a+b}$ **57.** $R=\frac{R_1R_2}{R_2+R_1}$

59. $t_2=\frac{s_2-s_1+vt_1}{v}$ or $t_2=\frac{s_2-s_1}{v}+t_1$ **61.** $\left\{-\frac{8}{5}\right\}$

63. $\{-6, 2\}$ **65.** $\left\{\frac{11}{2}\right\}$ **67.** $\left\{\frac{11}{2}\right\}$ **69.** $\{-5, 5\}$

71. $\{d \mid d$ is a real number$\}$ **73.** $\left\{-\frac{5}{2}\right\}$ **75.** $y=5$ **77.** $x=5$

Chapter 5 Problem Recognition Exercises, pp. 500–501

1. a. $\frac{2w+30}{(w-5)(w+5)}$ **b.** $\{-15\}$ **c.** The problem in part (a) is an expression, and the problem in part (b) is an equation.

2. a. $\frac{-3x-8}{6(x+2)}$ **b.** $\left\{-\frac{8}{3}\right\}$ **c.** The problem in part (a) is an expression, and the problem in part (b) is an equation.

3. $\frac{1}{a+1}$ **4.** $\frac{1}{c+2}$ **5.** $\{1\}$ **6.** $\{5\}$ **7.** $\frac{x^2-12}{x(x-1)}$

8. $\frac{23}{5(t-4)}$ **9.** $\{-1\}$ **10.** $\left\{\frac{3}{2}, -1\right\}$ **11.** $\frac{8p-11}{4(2p-3)}$

12. $\frac{-(2x+1)(x-1)}{2(x+2)}$ **13.** $\frac{x+3}{6x^2}$ **14.** $\frac{15a+2}{12a^2}$ **15.** $\left\{-\frac{4}{15}\right\}$

16. $\{-10\}$ **17.** $\frac{c+7}{(c+3)^2(c+1)}$ **18.** $\frac{-2y+13}{(y-5)^2(y-2)}$

19. { } (The value 4 does not check.)

20. { } (The value -2 does not check.)

Section 5.6 Activity, pp. 508–509

A.1. a. 9 **b.** $\frac{3 \text{ cookies}}{150 \text{ calories}}=\frac{\boxed{9 \text{ cookies}}}{450 \text{ calories}}$

A.2. a. Let x be the sales tax on an item that costs \$1250.

b. $\frac{\$11.70}{\$180}=\frac{x}{\$1250}$

c. \$81.25; This is the sales tax on an item that costs \$1250.

A.3. a. $1890-x$ **b.** $\frac{5}{4}=\frac{x}{1890-x}$; $x=1050$, which means that there are 1050 men at the school. Thus, there are 840 women at the school.

A.4. $a=6$ cm and $y=9$ cm

A.5. a. One possibility: Let x represent the average speed on the old bicycle.

b.

	Distance (mi)	Rate (mph)	Time (hr)
Riding old bike	21	x	$\frac{21}{x}$
Riding new bike	28	$x+4$	$\frac{28}{x+4}$

c. $\frac{21}{x}=\frac{28}{x+4}$ **d.** $x=12$ **e.** Mike rides his old bike an average of 12 mph and his new bike 16 mph.

A.6. a. $x+6$ **b.** $\frac{1 \text{ pond}}{\boxed{x+6}\text{hr}}+\frac{1 \text{ pond}}{\boxed{x}\text{hr}}=\frac{1 \text{ pond}}{4}$; $\frac{1}{x+6}+\frac{1}{x}=\frac{1}{4}$

c. $x=6, x=-4$ **d.** The solution $x=-4$ does not make sense in this scenario because the time required to fill a pond cannot be negative. Thus, it takes the large hose 6 hr to fill the pond and the small hose 12 hr to fill the pond.

Section 5.6 Practice Exercises, pp. 509–512

R.1. $\{-2, -3\}$ **R.3.** $\frac{11}{12(x-2)}$ **R.5.** $\left\{-\frac{3}{5}, 3\right\}$ **R.7.** $\frac{2}{a}$

1. proportion **3.** $\{8\}$ **5.** $\{6\}$ **7.** $\{-15\}$

9. $\left\{\frac{1}{3}\right\}$ **11.** $\left\{\frac{20}{9}\right\}$ **13.** $\{5, -1\}$ **15.** $\left\{\frac{3}{7}, -\frac{3}{7}\right\}$

17. $\{8, -5\}$ **19.** There must be 6 adults.

21. A 14-oz box contains 84 g of fat. **23.** 1000 swordfish were caught. **25.** Pam needs 11.5 gal of gas. **27.** There are approximately 4000 bison in the park. **29.** There are 31 men. **31.** 595 are men and 500 are women. **33.** $a=8$ ft, $z=8.4$ ft

35. $x=12$ in., $y=13$ in., $a=4.2$ in. **37.** The number is 3.

39. The number is $\frac{2}{5}$. **41. a.** $x+7$ **b.** $\frac{48}{x}$ **c.** $\frac{83}{x+7}$

43. The motorist drives 40 mph in the rain and 60 mph in sunny weather. **45.** The Broadmoor truck travels 70.4 mph and the Wescott truck travels 76.8 mph. **47.** The cyclist rode 10 mph against the wind. **49.** Celeste walks 5 ft/sec on the ground and 7 ft/sec on the moving walkway. **51.** Joe runs 6 mph and Beatrice runs 8 mph.

53. It will take them $\frac{24}{7}$ hr. **55.** It will take them $\frac{20}{3}$ hr.

57. a. The new pump will take 20 hr. **b.** The technician should return at noon on Friday. **59.** Gus would take 6 hr; Sid would take 12 hr.

Section 5.7 Activity, pp. 518–519

A.1. The equation $z=kp$ represents a direct relationship between z and p, and $z=\frac{k}{p}$ represents an inverse relationship between z and p.

A.2. a. increase **b.** decrease

A.3. a. $A=kw$ **b.** $k=\frac{1}{50}$; $A=\frac{1}{50}w$ **c.** more **d.** 3.6 g

e. 4.5 g **f.** 2.4 g

A.4. a. $C=\frac{k}{n}$ **b.** $k=2400$; $C=\frac{2400}{n}$ **c.** down **d.** \$0.40

e. \$0.30 **f.** \$1.00

A.5. a. $y=kb\sqrt{c}$ **b.** $k=3$ **c.** $y=3b\sqrt{c}$ **d.** 126

Section 5.7 Practice Exercises, pp. 519–522

R.1. $\{5\}$ **R.3.** $\{0.64\}$ **R.5.** 48 **R.7.** 2

1. a. kx **b.** $\frac{k}{x}$ **c.** kxw **3.** Inversely **5.** $T=kq$ **7.** $b=\frac{k}{c}$

9. $Q=\frac{kx}{y}$ **11.** $c=kst$ **13.** $L=kw\sqrt{v}$ **15.** $x=\frac{ky^2}{z}$

17. $k=\frac{9}{2}$ **19.** $k=512$ **21.** $k=1.75$ **23.** $x=70$

25. $b=6$ **27.** $Z=56$ **29.** $Q=9$ **31.** $L=9$

33. $B=\frac{15}{2}$ **35. a.** The heart weighs 0.92 lb. **b.** Answers will vary.

37. a. 5 turkeys **b.** 10 turkeys **c.** 50 turkeys

d. 8 turkeys; Note that the answer was rounded up, because it would be better for the chef to have too much food than too little.

39. 355,000 tons **41.** 42.6 ft **43.** 300 W **45.** 18.5 A

47. 20 lb **49.** \$3500

Chapter 5 Review Exercises, pp. 528–531

1. a. $k(2)=\frac{2}{3}$, $k(0)=0$, $k(1)$ undefined, $k(-1)$ undefined, $k\left(\frac{1}{2}\right)=-\frac{2}{3}$ **b.** $(-\infty, -1)\cup(-1, 1)\cup(1, \infty)$

2. a. $h(1)=\frac{1}{2}$, $h(0)=0$, $h(-1)=-\frac{1}{2}$, $h(-3)=-\frac{3}{10}$, $h\left(\frac{1}{2}\right)=\frac{2}{5}$

b. $(-\infty, \infty)$ **3.** $2a$ **4.** $\frac{xz^2}{5}$ **5.** $x-1$

6. $\frac{k+5}{k-3}$ **7.** $-\frac{x^2+3x+9}{3+x}$ **8.** $-(a^2+9)(a+3)$

9. $-\dfrac{2t+5}{t+7}$ **10.** $\dfrac{y(y+2)}{y-3}$ **11.** c; $(-\infty, 3) \cup (3, \infty)$
12. a; $(-\infty, -2) \cup (-2, \infty)$ **13.** b; $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$
14. d; $(-\infty, \infty)$ **15.** $\dfrac{a}{2}$ **16.** $-\dfrac{3}{2}$ **17.** $-\dfrac{x-y}{5x}$ or $\dfrac{y-x}{5x}$
18. $(x+8)^2$ **19.** $\dfrac{7(k-4)}{2(k-2)}$ **20.** $\dfrac{a+1}{a-1}$ **21.** $\dfrac{x-5}{x-4}$
22. $-\dfrac{1}{b}$ **23.** $\dfrac{8}{9w^2}$ **24.** $\dfrac{5(y^2+1)}{7(y^2-2y+4)}$ **25.** $\dfrac{5}{2}$
26. $(3k+5)(k+5)$ **27.** $\dfrac{x^2+x-1}{x^3}$ **28.** $\dfrac{2(3x+4)}{(x+2)(x-2)}$
29. $\dfrac{y-3}{2y-1}$ or $\dfrac{3-y}{1-2y}$ **30.** $\dfrac{a-4}{2(a+3)}$ **31.** $\dfrac{4k^2-k+3}{(k+1)^2(k-1)}$
32. $\dfrac{4x^2+17x+11}{x+4}$ **33.** $\dfrac{2(a^2-5)}{(a-5)(a+3)}$
34. $\dfrac{13x+19}{(x+1)(x+3)(x+2)}$ **35.** $\dfrac{6(7-4x)}{3x-5}$ or $\dfrac{-6(4x-7)}{3x-5}$
36. $\dfrac{2(16k-9)}{(4k+3)(k-1)(4k-3)}$ **37.** $\dfrac{9a^2+a+4}{(3a-1)(a-2)}$
38. $\dfrac{5y^2-17y-9}{(y-3)(y+2)}$ **39.** $\dfrac{1}{x+1}$ **40.** $\dfrac{(k-2)(k+2)}{15}$
41. $\dfrac{x(2y+1)}{4y}$ **42.** $-y$ **43.** $\dfrac{1+a}{1-a}$ or $-\dfrac{a+1}{a-1}$
44. $2x$ **45.** $\dfrac{y}{x-y}$ **46.** $\dfrac{a(5b+1)}{3b}$ **47.** $m=\dfrac{1}{18}$
48. $m=\dfrac{64}{23}$ **49.** $\{3\}$ (The value 1 does not check.)
50. $\{\ \}$ (The values 3 and -3 do not check.) **51.** $\{0, 17\}$
52. $\left\{-\dfrac{1}{3}\right\}$ **53.** $\{5, 1\}$ **54.** $\{1\}$
55. $x=\dfrac{b}{c-a}$ or $x=\dfrac{-b}{a-c}$ **56.** $P=\dfrac{A}{rt+1}$
57. $\left\{\dfrac{15}{2}\right\}$ **58.** $\left\{\dfrac{216}{7}\right\}$ **59.** $\left\{-\dfrac{7}{11}\right\}$ **60.** $\left\{-\dfrac{3}{4}\right\}$
61. The quarterback would gain 231 yd. **62.** Erik can buy \$253.80 Canadian. **63.** Tony averaged 20 mph the first day and 15 mph the second day. **64.** His speed driving was 60 mph. **65.** It would take $\frac{40}{9}$ hr. **66.** The larger pipe would take 9 hr and the smaller would take 18 hr. **67. a.** $F = kd$ **b.** $k = 3$ **c.** 12.6 lb
68. $y = 16$ **69.** $y = 12$ **70.** 48 km

Chapter 5 Test, pp. pp. 531–532

1. a. $h(0)=\dfrac{2}{7}$, $h(5)=\dfrac{1}{6}$, $h(7)$ is undefined, $h(-7)$ is undefined
b. $(-\infty, -7) \cup (-7, 7) \cup (7, \infty)$
2. $(-\infty, \infty)$ **3. a.** $\{x \mid x \text{ is a real number and } x \neq 4, x \neq -3\}$
b. $f(x)=\dfrac{2}{x-4}$ **4.** $\dfrac{2m^2}{3n}$ **5.** $\dfrac{9(x+1)}{3x+5}$ **6.** $m=-\dfrac{23}{9}$
7. $-(x-3)$ or $-x+3$ **8.** $x-4$ **9.** $\dfrac{x^2+5x+2}{x+1}$
10. $\dfrac{3}{4}$ **11.** $\dfrac{u^2v^2}{2(v^2-uv+u^2)}$ **12.** $-a$ **13.** $\dfrac{1}{(x+5)(x+3)}$
14. $\{3\}$ **15.** $\{0, 4\}$ **16.** $\{\ \}$ (The value 4 does not check.)
17. $T=\dfrac{1}{p-v}$ or $T=\dfrac{-1}{v-p}$ **18.** $m_1=\dfrac{Fr^2}{Gm_2}$
19. $\dfrac{1}{6}$ or 2 **20.** $a = 14$ m, $y = 15$ m
21. The cities are 1960 mi apart. **22.** Lance rides 16 mph against the wind and 20 mph with the wind.
23. It would take $\frac{20}{7}$ hr working together.
24. $x=\dfrac{ky}{t^2}$ **25.** 3.3 sec

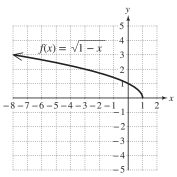
Chapter 6

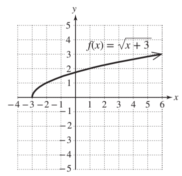
Section 6.1 Activity, p. 542

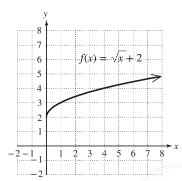
A.1. a. $b^2 = a$ **b.** $3^2 = 9$ **c.** $(-3)^2 = 9$ **d.** $\sqrt{a}$; nonnegative **e.** 3
A.2. a. $b^n = a$ **b.** $(-4)^3 = -64$ **c.** $2^5 = 32$ **d.** 3; 125 **e.** 5
A.3. a. No real number when squared equals a negative number.
b. No real number when raised to the fourth power (or any even power) equals a negative number.
c. $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$. In general, a negative real number when raised to an odd power is negative.
A.4. a. $\sqrt{a^2}$ is the principal square root of a^2. Therefore, the simplified form must not be negative. Because we do not know the value of a (which could be a positive or negative number), the absolute value bars ensure that the answer is nonnegative.
b. For a root with an odd index, the absolute value bars are not needed because the value can be positive, negative, or zero.
A.5. a. even **b.** $[0, \infty)$ **c.** $[5, \infty)$ **d.** $(-\infty, 5]$
A.6. a. odd **b.** $(-\infty, \infty)$ **c.** $(-\infty, \infty)$ **d.** $(-\infty, \infty)$

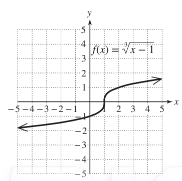
Section 6.1 Practice Exercises, pp. 542–546

R.1. 100 **R.3.** 0.16 **R.5.** $-\dfrac{27}{125}$
R.7. $\dfrac{1}{64}$ **R.9.** x^{14} **R.11.** $81t^{16}$

1. a. b; a **b.** principal **c.** b^n; a **d.** index; radicand **e.** cube **f.** is not; is **g.** even; odd **h.** Pythagorean; c^2 **i.** $[0, \infty)$; $(-\infty, \infty)$ **j.** -5 and -4
3. a. 8, -8 **b.** 8 **c.** There are two square roots for every positive number. $\sqrt{64}$ identifies the positive square root.
5. a. 9 **b.** -9 **7.** There is no real number b such that $b^2 = -36$. **9.** 7 **11.** -7 **13.** Not a real number
15. $\dfrac{8}{3}$ **17.** 0.9 **19.** -0.4 **21. a.** 8 **b.** 4 **c.** -8 **d.** -4 **e.** Not a real number **f.** -4
23. -3 **25.** $\dfrac{1}{2}$ **27.** 2 **29.** $-\dfrac{5}{4}$ **31.** Not a real number **33.** 10 **35.** -0.2 **37.** 0.5 **39.** $|a|$
41. a **43.** $|a|$ **45.** $|x+1|$ **47.** $|x|y^2$ **49.** $-\dfrac{x}{y}$
51. $\dfrac{2}{|x|}$ **53.** -92 **55.** 2 **57.** -923 **59.** y^4
61. $\dfrac{a^3}{b}$ **63.** $-\dfrac{5}{q}$ **65.** $3xy^2z$ **67.** $\dfrac{hk^2}{4}$ **69.** $-\dfrac{t}{3}$
71. $2y^2$ **73.** $2p^2q^3$ **75.** 9 cm **77.** 13 ft
79. They were 5 mi apart. **81.** They are 25 mi apart.
83. a. Not a real number **b.** Not a real number **c.** 0 **d.** 1 **e.** 2; Domain: $[2, \infty)$ **85. a.** -2 **b.** -1 **c.** 0 **d.** 1; Domain: $(-\infty, \infty)$ **87.** $\left(-\infty, \dfrac{5}{2}\right]$
89. $(-\infty, \infty)$ **91.** $[5, \infty)$ **93.** $(-\infty, \infty)$
95. a. $(-\infty, 1]$
b.


97. a. $[-3, \infty)$
b.


99. a. $[0, \infty)$
b.


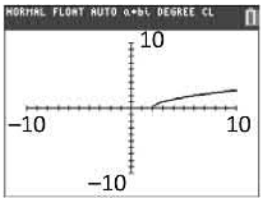
101. a. $(-\infty, \infty)$
b.


103. $q+p^2$ **105.** $\frac{6}{\sqrt[3]{x}}$ **107.** 8 in.

109. 8.3066 **110.** 76.1446 **111.** 3.7100

112. −0.5566 **113.** 15.6525 **114.** −6.2403

115. −0.1235 **116.** 1.0622

117.


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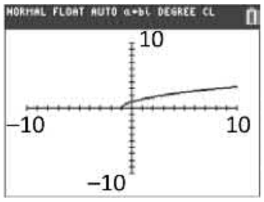

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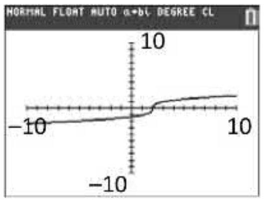

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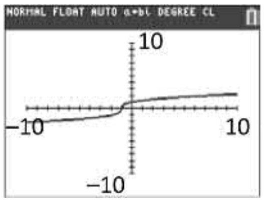


Section 6.2 Activity, p. 511

A.1. a. $\sqrt[n]{a}$ **b.** $\sqrt{9}=3$ **c.** $\sqrt[3]{-125}=-5$
d. The definition $a^{1/n}=\sqrt[n]{a}$ is true only if the radical is a real number. The expression $\sqrt{-16}$ is not a real number.

A.2. a. $\sqrt[n]{a^m}$ or $\left(\sqrt[n]{a}\right)^m$ **b.** $\sqrt[3]{125^2}$ or $\left(\sqrt[3]{125}\right)^2=25$
c. $\sqrt[4]{16^3}$ or $\left(\sqrt[4]{16}\right)^3=8$

A.3. a. t^7 **b.** t^3 **A.4. a.** y^4 **b.** $y^{13/15}$ **A.5. a.** z^{15} **b.** z^4

A.6. a. $\frac{1}{n^6}$ **b.** $\frac{1}{n^{1/2}}$ **A.7. a.** $2m^3$ **b.** $2m^{1/3}$

A.8. a. a^4b^8 **b.** ab^2 **A.9. a.** $\frac{c^9}{d^{12}}$ **b.** $\frac{c^4}{d^3}$

Section 6.2 Practice Exercises, pp. 552–555

R.1. a^5 **R.3.** $\frac{1}{y^3}$ **R.5.** $\frac{1}{n^8}$ **R.7.** $\frac{1}{125}a^6b^9$

R.9. 1 **R.11. a.** 4 **b.** 16 **c.** 2 **R.13.** 27 **R.15.** $t+2$

1. a. $\sqrt[n]{a}$ **b.** $(\sqrt[n]{a})^m$ or $\sqrt[n]{a^m}$

3. a. $\sqrt{49}$ **b.** $-\sqrt{49}$ **c.** $\frac{1}{\sqrt{49}}$

5. a. $\sqrt[3]{-64}$ **b.** $-\sqrt[3]{64}$ **c.** $\frac{1}{\sqrt[3]{64}}$ **7.** 12 **9.** −12

11. Not a real number **13.** −4 **15.** $\frac{1}{5}$ **17.** $-\frac{1}{7}$

19. $a^{m/n}=\sqrt[n]{a^m}$; The numerator of the exponent represents the power of the base. The denominator of the exponent represents the index of the radical. **21. a.** 8 **b.** −8 **c.** Not a real number **d.** $\frac{1}{8}$ **e.** $-\frac{1}{8}$ **f.** Not a real number

23. a. 125 **b.** −125 **c.** Not a real number **d.** $\frac{1}{125}$ **e.** $-\frac{1}{125}$ **f.** Not a real number **25.** $\frac{1}{512}$ **27.** 27

29. $-\frac{1}{81}$ **31.** $\frac{27}{1000}$ **33.** Not a real number **35.** −2

37. −2 **39.** 6 **41.** 10 **43.** $\frac{3}{4}$ **45.** 1 **47.** $\frac{9}{2}$

49. $\sqrt[3]{q^2}$ **51.** $6\sqrt[4]{y^3}$ **53.** $\sqrt[3]{x^2y}$ **55.** $\frac{6}{\sqrt[5]{r^2}}$

57. $x^{1/3}$ **59.** $10b^{1/2}$ **61.** $y^{2/3}$ **63.** $(a^2b^3)^{1/4}$

65. $\frac{1}{x}$ **67.** p **69.** y^2 **71.** $6^{2/5}$ **73.** $\frac{4}{t^{5/3}}$

75. a^7 **77.** $\frac{25a^4d}{c}$ **79.** $\frac{y^9}{x^8}$ **81.** $\frac{2z^3}{w}$ **83.** $5xy^2z^3$

85. $x^{13}z^{4/3}$ **87.** $\frac{x^3y^2}{z^5}$ **89. a.** 10.9% **b.** 8.8% **c.** The account in part (a). **91.** 2.7 in. **93.** $\sqrt{y}$

95. $\sqrt[4]{z}$ **97.** $\sqrt[3]{x^2}$ **99.** $y\sqrt{x}$ **101.** $4x^4y^3$

103. $2x\sqrt[3]{y^2z}$ **105.** $\sqrt[6]{x}$ **107.** $\sqrt[15]{w}$ **109.** 3

111. 0.3761 **113.** 2.9240 **115.** 31.6228

Section 6.3 Activity, p. 560

A.1. a. $\sqrt[n]{ab}$ **b.** $2\sqrt{5}$ **c.** $\sqrt{50}=\sqrt{25\cdot 2}=\sqrt{25}\cdot\sqrt{2}=5\sqrt{2}$
d. $\sqrt[3]{x^7}=\sqrt[3]{x^6\cdot x}=\sqrt[3]{x^6}\cdot\sqrt[3]{x}=x^2\sqrt[3]{x}$

A.2. a. $2^4\cdot 3$ **b.** $\sqrt{2^4\cdot 3x^5y^6}$
c. $\sqrt{(2^4x^4y^6)\cdot(3x)}$ or $\sqrt{(16x^4y^6)\cdot(3x)}$ **d.** $4x^2y^3\sqrt{3x}$

A.3. $5x^2$ **A.4.** $10\sqrt{2}$

Section 6.3 Practice Exercises, pp. 560–563

R.1. $2\cdot 3^2\cdot 7^2$ **R.3.** 4, 9, 25, 100, y^2, y^4, y^{10}

R.5. 1, 16, 81, b^4, b^8 **R.7.** $\frac{9}{8}$ **R.9.** 5

R.11. −6 **R.13.** −4 **R.15.** b^5 **R.17.** $3c^2d^3$

1. a. $\sqrt[n]{a}$; $\sqrt[n]{b}$ **b.** The exponent within the radicand is greater than the index. **c.** is not **d.** 3 **e.** t^{12}

3. $\sqrt{x^{11}}=\sqrt{x^{\boxed{10}}\cdot x}=\sqrt{x^{\boxed{10}}}\cdot x=x^5\sqrt{x}$

5. $\sqrt[3]{x^{11}}=\sqrt[3]{x^{\boxed{9}}\cdot x^2}=\sqrt[3]{x^{\boxed{9}}}\cdot\sqrt[3]{x^2}=x^3\sqrt[3]{x^2}$

7. $\sqrt[4]{x^{11}}=\sqrt[4]{x^{\boxed{8}}\cdot x^3}=\sqrt[4]{x^{\boxed{8}}}\cdot\sqrt[4]{x^3}=x^2\sqrt[4]{x^3}$

9. $x^2\sqrt{x}$ **11.** $q^2\sqrt[3]{q}$ **13.** $a^2b^2\sqrt{a}$ **15.** $-x^2y^3\sqrt[4]{y}$

17. $2\sqrt{7}$ **19.** $2\sqrt{5}$ **21.** $15\sqrt{2}$ **23.** $3\sqrt[3]{2}$ **25.** $5b\sqrt{ab}$

27. $2x^2\sqrt[3]{5x}$ **29.** $-2x^2z\sqrt[3]{2y}$ **31.** $2wz\sqrt[4]{5z^3}$ **33.** x **35.** p^2

37. 5 **39.** $\frac{1}{2}$ **41.** $\frac{5\sqrt[3]{2}}{3}$ **43.** $\frac{5\sqrt[3]{9}}{6}$ **45.** $4\sqrt{5}$

47. $-30\sqrt{3}$ **49.** $5x^2y\sqrt{y}$ **51.** $3yz\sqrt[3]{x^2z}$ **53.** $2w^2$

55. $\frac{1}{10y^6}$ **57.** $\frac{2}{b}$ **59.** $2a^7b^4c^{15}d^{11}\sqrt{2c}$

61. $3a^2b\sqrt[3]{2b}$ **63.** $-10a^2b^2\sqrt{3ac}$ **65.** $x\sqrt[4]{7xy}$

67. $3a^2b\sqrt{6}$ **69.** $2\sqrt{3}$ **71.** $\frac{3\sqrt{5}}{4}$

73. $\frac{1}{\sqrt[3]{w^6}}$ simplifies to $\frac{1}{w^2}$ **75.** $\sqrt{k^3}$ simplifies to $k\sqrt{k}$

77. $2\sqrt{41}$ ft **79.** $6\sqrt{5}$ m **81.** The distance is $90\sqrt{2}$ ft or approximately 127.3 ft. **83.** The path from A to B and B to C is faster.

Section 6.4 Activity, p. 566

A.1. index; radicand **A.2. a.** No; the radicands are different.
b. No; the indices are different. **c.** Yes **d.** Yes

A.3. a. $9x$ **b.** $9\sqrt{x}$ **c.** $9\sqrt{2}$
d. In each case, apply the distributive property to add *like* terms. For example, $4\sqrt{x}+7\sqrt{x}-2\sqrt{x}=\sqrt{x}(4+7-2)=\sqrt{x}\cdot(9)$ or $9\sqrt{x}$

A.4. a. 16, −12, 1 **b.** $5\sqrt{3t}$

A.5. a. $3x^2y\sqrt{5x}$ **b.** $-4x^2y\sqrt{5x}$ **c.** $6x^2y\sqrt{5x}$
d. $3x^2y\sqrt{5x}-4x^2y\sqrt{5x}+6x^2y\sqrt{5x}$ **e.** $5x^2y\sqrt{5x}$

A.6. a. $3x\sqrt{7}+5\sqrt{7}$ **b.** $\sqrt{7}(3x+5)$ or $(3x+5)\sqrt{7}$

Section 6.4 Practice Exercises, pp. 567–569

R.1. a. Yes **b.** No **c.** Yes **d.** No **R.3.** $-y$ **R.5.** c^2d

R.7. $3\sqrt{3}$ **R.9.** $5xy\sqrt[3]{2y}$

1. a. index; radicand **b.** $2\sqrt{3x}$ **c.** cannot; can

3. Not *like* radicals **5.** *Like* radicals **7.** Not *like* radicals

9. a. Both expressions can be simplified by using the distributive property. **b.** Neither expression can be simplified because the terms do not contain *like* radicals or *like* terms.

11. a. $8x$ **b.** $8\sqrt{x}$ **13. a.** $-7t$ **b.** $-7\sqrt[3]{t}$ **15.** $9\sqrt{5}$

17. $2\sqrt[3]{tw}$ **19.** $5\sqrt{10}$ **21.** $8\sqrt[4]{3}-\sqrt[4]{14}$ **23.** $2\sqrt{x}+2\sqrt{y}$

25. Cannot be simplified further

27. Cannot be simplified further **29.** $\frac{29}{18}z\sqrt[3]{6}$

31. $0.7x\sqrt{y}$ **33.** Simplify each radical: $3\sqrt{2}+35\sqrt{2}$. Then add *like* radicals: $38\sqrt{2}$ **35.** 15 **37.** $8\sqrt{3}$

39. $3\sqrt{7}$ **41.** $-5\sqrt{2}+3\sqrt{3}$ **43.** $\sqrt[3]{3}$ **45.** $-5\sqrt{2a}$

47. $8s^2t^2\sqrt[3]{s^2}$ **49.** $6x\sqrt[3]{x}$ **51.** $14p^2\sqrt{5}$

53. $-\sqrt[3]{a^2b}$ **55.** $(5x+6)\sqrt{x}$ **57.** $(5x-6)\sqrt{2}$

59. $33d\sqrt[3]{2c}$ **61.** $2a^2b\sqrt{6a}$ **63.** $5x\sqrt[3]{2}-6\sqrt[3]{x}$

65. False; $\sqrt{9}+\sqrt{16}\neq\sqrt{9+16}$; $7\neq 5$ **67.** True

69. False; $\sqrt{y}+\sqrt{y}=2\sqrt{y}\neq\sqrt{2y}$

71. False; $2w\sqrt{5}+4w\sqrt{5}=6w\sqrt{5}\neq 6w^2\sqrt{5}$

73. $\sqrt{48}+\sqrt{12}$ simplifies to $6\sqrt{3}$

75. $5\sqrt[3]{x^6}-x^2$ simplifies to $4x^2$ **77.** The difference of the principal square root of 18 and the square of 5

79. The sum of the principal fourth root of x and the cube of y

81. $9\sqrt{6}$ cm ≈ 22.0 cm **83.** $x=2\sqrt{2}$ ft

85. a. $10\sqrt{5}$ yd **b.** 22.36 yd **c.** \$105.95

Section 6.5 Activity, p. 575

A.1. $\sqrt[n]{ab}$ **A.2. a.** $10x^2$ **b.** $10x$ **c.** $10\sqrt[3]{x^2}$

d. Parts (b) and (c) illustrate that we multiply the coefficients of each term and multiply the radicals. In part (b), however, it was necessary to simplify the resulting radical.

A.3. a. $2x^3+4x^2y+xy^2$ **b.** $2x+4x\sqrt{y}+\sqrt{xy}$

A.4. a. $10x^2+3x-18$ **b.** $10x+3\sqrt{x}-18$

c. $10x-2\sqrt{6x}+5\sqrt{3x}-3\sqrt{2}$

A.5. a. $25x^2-30x+9$ **b.** $25x-30\sqrt{x}+9$ **c.** $25x-10\sqrt{3x}+3$

A.6. a. $4y^2-49$ **b.** $4y-49$ **c.** $4y-7$

A.7. a. $7^{1/3}$ **b.** $2^{1/5}$ **c.** $7^{1/3}\cdot 2^{1/5}$ **d.** $7^{5/15}\cdot 2^{3/15}$ **e.** $\sqrt[15]{7^5}\cdot\sqrt[15]{2^3}$

f. $\sqrt[15]{7^5\cdot 2^3}$

Section 6.5 Practice Exercises, pp. 576–578

R.1. $-40x^3y$ **R.3.** $-3a^3-5a^2b+2ab^2$ **R.5.** $3c^3-2c^2+7c+20$

R.7. $4a^2-20ab+25b^2$ **R.9.** $\frac{1}{4}m^2-16n^2$ **R.11.** $8y\sqrt{7}$

1. a. $\sqrt[n]{ab}$ **b.** x **c.** a **d.** conjugates **e.** $m-n$

3. a. x^2 **b.** $\sqrt{x}\cdot\sqrt{x}=\sqrt{x^{\boxed{2}}}=x$

5. a. t^6 **b.** $\sqrt[3]{t}\cdot\sqrt[3]{t^5}=\sqrt[3]{\boxed{t^6}}=t^2$

7. a. $50x^2$ **b.** $\sqrt{5x}\cdot\sqrt{10x}=\sqrt{50x^2}=5x\sqrt{2}$

9. $\sqrt[3]{21}$ **11.** $2\sqrt{5}$ **13.** $4\sqrt[4]{4}$ **15.** $8\sqrt[3]{20}$

17. $-24ab\sqrt{a}$ **19.** $6\sqrt{10}$ **21.** $6x\sqrt{2}$ **23.** $10a^3b^2\sqrt{b}$

25. $-24x^2y^2\sqrt{2y}$ **27.** $6ab\sqrt[3]{2a^2b^2}$ **29.** $12-6\sqrt{3}$

31. $2\sqrt{3}-\sqrt{6}$ **33.** $-2x-\frac{7}{3}\sqrt{x}$ **35.** $-8+7\sqrt{30}$

37. $x-5\sqrt{x}-36$ **39.** $\sqrt[3]{y^2}-\sqrt[3]{y}-6$

41. $9a-28\sqrt{ab}+3b$ **43.** $8\sqrt{p}+3p+5\sqrt{pq}+16\sqrt{q}-2q$

45. 15 **47.** $3y$ **49.** 6 **51.** 709 **53. a.** x^2-y^2 **b.** x^2-25 **55.** $29+8\sqrt{13}$ **57.** $p-2\sqrt{7p}+7$

59. $2a-6\sqrt{2ab}+9b$ **61.** $3-x^2$ **63.** 4

65. $\frac{4}{9}x-\frac{1}{4}y$ **67. a.** $3-x$ **b.** $3+2\sqrt{3x}+x$ **c.** $3-2\sqrt{3x}+x$ **69.** True **71.** False; $(x-\sqrt{5})^2=x^2-2x\sqrt{5}+5$ **73.** False; 5 is multiplied by 3 only.

75. True **77.** $6x$ **79.** $3x+1$

81. $x+19-8\sqrt{x+3}$ **83.** $2t+10\sqrt{2t-3}+22$

85. $12\sqrt{5}$ ft^2 **87.** $18\sqrt{15}$ in.^{2} **89.** $\sqrt[4]{x^3}$

91. $\sqrt[15]{(2z)^8}$ **93.** $p^2\sqrt[6]{p}$ **95.** $u\sqrt[6]{u}$

97. $\sqrt[6]{x^2y}$ **99.** $\sqrt[4]{2^3\cdot 3^2}$ or $\sqrt[4]{72}$ **101.** $\sqrt[12]{2^4 5^3 x^7 y^7}$

103. $2m\sqrt[6]{3^3 2mn^5}$ **105.** $a+b$

Section 6.6 Activity, pp. 585–586

A.1. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ **A.2.** $\frac{2x\sqrt[3]{5x^2}}{y}$ **A.3.** $5x$

A.4. a. $\frac{6}{\sqrt[4]{x}}\cdot\frac{\sqrt[4]{\boxed{x^3}}}{\sqrt[4]{\boxed{x^3}}}=\frac{6\sqrt[4]{x^3}}{\sqrt[4]{x^4}}$ **b.** $\frac{6\sqrt[4]{x^3}}{x}$

A.5. a. $\frac{14}{\sqrt[3]{7^2y}}$ **b.** $\frac{14}{\sqrt[3]{7^2y}}\cdot\frac{\sqrt[3]{\boxed{7y^2}}}{\sqrt[3]{\boxed{7y^2}}}=\frac{14\sqrt[3]{7y^2}}{\sqrt[3]{7^3y^3}}$ **c.** $\frac{2\sqrt[3]{7y^2}}{y}$

A.6. a. $49x^2-4$ **b.** $7x^2-4$ **c.** 3 **d.** No

e. $\frac{12}{(\sqrt{7}-2)}\cdot\frac{(\boxed{\sqrt{7}+2})}{(\boxed{\sqrt{7}+2})}$ **f.** $4(\sqrt{7}+2)$ or $4\sqrt{7}+8$

A.7. a. denominator **b.** conjugate **c.** $\frac{t-9\sqrt{t}+20}{t-16}$

Section 6.6 Practice Exercises, pp. 586–589

R.1. x **R.3.** ab^2 **R.5.** $p-16$ **R.7.** 3

1. a. radical **b.** $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ **c.** $\frac{4}{x^2}$ **d.** is; is not **3.** $\frac{7x^2}{y^3}$ **5.** $\frac{2a\sqrt{2}}{x^3}$

7. $\frac{-2j\sqrt[3]{2}}{k}$ **9.** $3b^2$ **11.** $\frac{1}{2b^2}$ **13.** $\frac{z\sqrt{3y}}{w^2}$ **15.** $\frac{\sqrt{5}}{\sqrt{5}}$

17. $\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$ **19.** $\frac{\sqrt{3z}}{\sqrt{3z}}$ **21.** $\frac{\sqrt[4]{2a^2}}{\sqrt[4]{2a^2}}$ **23.** $\frac{\sqrt{3}}{3}$

25. $\frac{\sqrt{x}}{x}$ **27.** $\frac{3\sqrt{2y}}{y}$ **29.** $\frac{a\sqrt{2a}}{2}$ **31.** $\frac{3\sqrt{2}}{2}$

33. $\frac{3\sqrt[3]{4}}{2}$ **35.** $\frac{-6\sqrt[4]{x^3}}{x}$ **37.** $\frac{7\sqrt[3]{2}}{2}$ **39.** $\frac{\sqrt[3]{4w}}{w}$

41. $\frac{2\sqrt[4]{27}}{3}$ **43.** $\frac{\sqrt[3]{2x}}{x}$ **45.** $\frac{2\sqrt{6}}{21}$ **47.** $\frac{\sqrt{x}}{x^4}$

49. $\frac{\sqrt{2x}}{2x^3}$ **51.** $\sqrt{2}+\sqrt{6}$ **53.** $\sqrt{x}-23$

55. $\frac{4\sqrt{2}-12}{-7}$ or $\frac{-4\sqrt{2}+12}{7}$ **57.** $4\sqrt{6}+8$

59. $-\sqrt{21}+2\sqrt{7}$ **61.** $\frac{-\sqrt{p}+\sqrt{q}}{p-q}$ **63.** $\sqrt{x}-\sqrt{5}$

65. $\frac{w+11\sqrt{w}+18}{81-w}$ **67.** $\frac{4\sqrt{xy}-3x-y}{y-x}$

69. $5-\sqrt{10}$ **71.** $\frac{5+\sqrt{21}}{4}$ **73.** $-6\sqrt{5}-13$

75. $\frac{16}{\sqrt[3]{4}}$ simplifies to $8\sqrt[3]{2}$ **77.** $\frac{4}{x-\sqrt{2}}$ simplifies to $\frac{4x+4\sqrt{2}}{x^2-2}$

79. $\frac{\pi\sqrt{2}}{4}$ sec ≈ 1.11 sec **81. a.** $\frac{\sqrt{2}}{2}$ **b.** $\frac{\sqrt[3]{4}}{2}$

83. a. $\frac{\sqrt{5a}}{5a}$ **b.** $\frac{\sqrt{5}-a}{5-a^2}$ **85.** $\frac{2\sqrt{6}}{3}$ **87.** $\frac{17\sqrt{15}}{15}$

89. $\frac{8\sqrt[3]{25}}{5}$ **91.** $\frac{-33}{2\sqrt{3}-12}$ **93.** $\frac{a-b}{a+2\sqrt{ab}+b}$

95. $\frac{3}{\sqrt{5+3h}+\sqrt{5}}$ **97.** $\frac{5}{\sqrt{4+5h}+2}$

Chapter 6 Problem Recognition Exercises, pp. 590–591

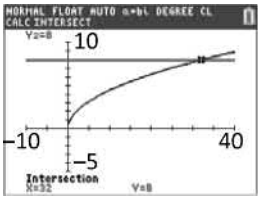
1. a. $2\sqrt{6}$ **b.** $2\sqrt[3]{3}$ **2. a.** $3\sqrt{6}$ **b.** $3\sqrt[3]{2}$
3. a. $10y^3\sqrt{2}$ **b.** $2y^2\sqrt[3]{25}$ **4. a.** $4z^7\sqrt{2z}$ **b.** $2z^5\sqrt[3]{4}$
5. a. $4\sqrt{5}$ **b.** $2\sqrt[3]{10}$ **c.** $2\sqrt[4]{5}$ **6. a.** $4\sqrt{3}$ **b.** $2\sqrt[3]{6}$ **c.** $2\sqrt[4]{3}$
7. a. $x^2y^3\sqrt{x}$ **b.** $xy^2\sqrt[3]{x^2}$ **c.** $xy\sqrt[4]{xy^2}$
8. a. $a^5b^4\sqrt{b}$ **b.** $a^3b^3\sqrt[3]{a}$ **c.** $a^2b^2\sqrt[4]{a^2b}$ **9. a.** $2st^2\sqrt[3]{4s^2}$ **b.** $2st\sqrt[4]{2st^2}$ **c.** $2st\sqrt[5]{t}$ **10. a.** $2v^2w^6\sqrt[3]{12vw^2}$ **b.** $2vw^5\sqrt[4]{6v^3}$ **c.** $2vw^4\sqrt[5]{3v^2}$ **11. a.** $2\sqrt{5}$ **b.** 5
12. a. $2\sqrt{10}$ **b.** 10 **13. a.** $-3\sqrt{6}$ **b.** 60
14. a. $-7\sqrt{7}$ **b.** 210 **15. a.** $3\sqrt{2}$ **b.** 4
16. a. $3\sqrt{3}$ **b.** 6 **17. a.** $7\sqrt{2}$ **b.** 240
18. a. $-\sqrt{2}$ **b.** 60 **19. a.** $14\sqrt[3]{3}$ **b.** $48\sqrt[3]{9}$
20. a. $\sqrt[3]{2}$ **b.** $30\sqrt[3]{4}$
21. a. $3\sqrt{2}$ **b.** Cannot be simplified further. **c.** $\sqrt{2}$
22. a. $\sqrt{7}$ **b.** $2\sqrt{7}$ **c.** Cannot be simplified further.
23. a. $9z$ **b.** $9+6\sqrt{z}+z$ **c.** $9-z$
24. a. $16-8\sqrt{x}+x$ **b.** $16-x$ **c.** $16x$
25. a. $\dfrac{6\sqrt{2x}}{x}$ **b.** $\dfrac{\sqrt{6x}}{x}$ **c.** $\dfrac{12\sqrt{2}-12x}{2-x^2}$
26. a. $\dfrac{45+15\sqrt{y}}{9-y}$ **b.** $\dfrac{5\sqrt{3y}}{y}$ **c.** $\dfrac{\sqrt{5y}}{y}$
27. a. $3\sqrt{5}-1$ **b.** $8-3\sqrt{5}$ **c.** $10-4\sqrt{5}$
28. a. $-8+11\sqrt{3}$ **b.** $12+16\sqrt{3}$ **c.** $3\sqrt{3}-9$
29. a. $4a^7\sqrt{a}$ **b.** $2a^5\sqrt[3]{2}$ **30. a.** $3y^3$ **b.** $3y^4\sqrt{3y}$

Section 6.7 Activity, pp. 598–599

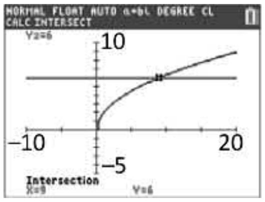
A.1. a. Divide both sides by 3 or multiply both sides by $\frac{1}{3}$.
b. Cube both sides of the equation.
A.2. a. { } **b.** $x = 25$. No, because $\sqrt{25} \neq -5$. **c.** even
A.3. a. Add 3 to both sides. Divide by 2 (or multiply by $\frac{1}{2}$) on both sides.
b. Add 3 to both sides. Divide by 2 (or multiply by $\frac{1}{2}$) on both sides.
c. {16} **d.** {5} **e.** {21} **f.** {85}
g. Parts (c), (d), and (f). In each part, it was necessary to check the potential solutions because we raised both sides of the equation to an *even* power.
A.4. a. $x^2 - 10x + 25$
b. {4}. (The value 7 does not check in the original equation.)
A.5. {1}
A.6. a. $\sqrt{x+2} = 1 - \sqrt{2x+5}$ **b.** $x+2 = 2x+6-2\sqrt{2x+5}$
c. $-x-4 = -2\sqrt{2x+5}$ **d.** $x+4 = 2\sqrt{2x+5}$
e. $x^2+8x+16 = 4(2x+5)$ **f.** $x = 2, x = -2$
g. {−2}. (The value 2 does not check.)

Section 6.7 Practice Exercises, pp. 599–602

R.1. $t^2+16t+64$ **R.3.** $3t+1$ **R.5.** $m-14\sqrt{m}+49$
R.7. {5} **R.9.** {−3, 7} **R.11.** $\left\{-\frac{1}{5}, \frac{7}{2}\right\}$
1. a. radical **b.** isolate; 7 **c.** extraneous
3. d **5.** e **7.** h **9.** f
11. {100} **13.** {4} **15.** {3} **17.** {42} **19.** {2}
21. {29} **23.** {140} **25.** { } (The value 25 does not check.)
27. {7} (The value −1 does not check.)
29. { } (The value 3 does not check.)
31. $V = \dfrac{4\pi r^3}{3}$ **33.** $h^2 = \dfrac{r^2-\pi^2r^2}{\pi^2}$ or $h^2 = \dfrac{r^2}{\pi^2} - r^2$
35. $a^2+10a+25$ **37.** $5a-6\sqrt{5a}+9$
39. $r+22+10\sqrt{r-3}$ **41.** {−3}
43. { } (The value $\frac{1}{2}$ does not check.)
45. {2} (The value 18 does not check.)
47. {9} **49.** {−4} **51.** $\left\{\frac{9}{5}\right\}$ **53.** {2}
55. { } (The value 9 does not check.)
57. $\left\{-\frac{11}{4}\right\}$ **59.** { } (The value $\frac{8}{3}$ does not check.)
61. {−1} (The value 3 does not check.) **63.** $\left\{\frac{1}{3}, -1\right\}$
65. { } (The values 3 and 23 do not check.)
67. {6} **69.** {2} (The value 18 does not check.)
71. a. 30.25 ft **b.** 34.5 m
73. a. \$2 million **b.** \$1.2 million **c.** 50,000 passengers
75. The x-coordinate of the point of intersection is the solution to the equation.



76. The x-coordinate of the point of intersection is the solution to the equation.



77. a. 12 lb **b.** $t(18) = 5.1$. An 18-lb turkey will take about 5.1 hr to cook. **79.** $b = \sqrt{25-h^2}$ **81.** $a = \sqrt{k^2-196}$

Section 6.8 Activity, p. 610

A.1. i; –1
A.2. a. $i\sqrt{b}$ **b.** $i\sqrt{13}$ **c.** $6i\sqrt{2}$ **d.** $i\sqrt{2}$ **e.** $\dfrac{3 \pm i\sqrt{7}}{2}$
A.3. a. Write the expression as $i\sqrt{4} \cdot i\sqrt{25}$. **b.** -10
A.4. a. Write the expression as $\dfrac{i\sqrt{50}}{i\sqrt{2}}$. **b.** 5
A.5. a. $i^3 = i^2 \cdot \boxed{i} = -1 \cdot \boxed{i} = -i$ **b.** $i^4 = i^2 \cdot \boxed{i^2} = -1 \cdot \boxed{-1} = 1$
A.6. a. $i^5 = i^4 \cdot \boxed{i} = 1 \cdot \boxed{i} = i$ **b.** $i^6 = i^4 \cdot \boxed{i^2} = 1 \cdot \boxed{-1} = -1$
c. $i^7 = i^4 \cdot i^2 \cdot \boxed{i} = -i$ **d.** $i^8 = i^4 \cdot \boxed{i^4} = 1$
A.7. a. $-i$ **b.** 1 **c.** –1 **d.** i
A.8. a. $3+(-7)i$. Real part: 3; imaginary part: –7
b. $4+2\sqrt{3}i$. Real part: 4; imaginary part: $2\sqrt{3}$
c. $9+0i$. Real part: 9; imaginary part: 0
d. $0+\frac{3}{4}i$. Real part: 0; imaginary part: $\frac{3}{4}$
A.9. a. $1+10x$ **b.** $1+10i$ **A.10. a.** $-12x^2-10x+12$ **b.** $24-10i$
A.11. a. $81-100x^2$ **b.** 181 **A.12. a.** $16x^2+40x+25$ **b.** $9+40i$
A.13. a. $\dfrac{5(3+4\sqrt{x})}{9-16x}$ or $\dfrac{15+20\sqrt{x}}{9-16x}$ **b.** $\dfrac{3}{5}+\dfrac{4}{5}i$ **A.14.** $\dfrac{3}{2}+\dfrac{\sqrt{7}}{2}i$

Section 6.8 Practice Exercises, pp. 611–613

R.1. $-2y-2$ **R.3.** $-9x+17$ **R.5.** $10n^2-19n-56$
R.7. $49d^2-25$ **R.9.** $100b^2-60b+9$ **R.11.** $3\sqrt{10}$
R.13. 3 **R.15.** $\dfrac{2}{3}+\dfrac{1}{3}x$

1. a. imaginary **b.** $\sqrt{-1}$; -1 **c.** $i\sqrt{b}$ **d.** $a+bi$; $\sqrt{-1}$ **e.** real; b **f.** $a+bi$ **3.** $\sqrt{-1}=i$ and $-\sqrt{1}=-1$ **5.** $7i$
7. -7 **9.** $\frac{1}{2}i$ **11.** $12i$ **13.** $i\sqrt{3}$ **15.** $-2i\sqrt{5}$
17. -60 **19.** $13i\sqrt{7}$ **21.** -7 **23.** -12
25. $-3\sqrt{10}$ **27.** $i\sqrt{2}$ **29.** 3 **31.** $-i$ **33.** 1
35. i **37.** 1 **39.** $-i$ **41.** -1 **43.** $a-bi$
45. Real: -5; imaginary: 12 **47.** Real: 0; imaginary: -6
49. Real: 35; imaginary: 0 **51.** Real: $\frac{3}{5}$; imaginary: 1
53. $7+6i$ **55.** $\frac{3}{10}+\frac{3}{2}i$ **57.** $5i\sqrt{2}$
59. $-1+10i$ **61.** $-24+0i$ **63.** $18+6i$
65. $26-26i$ **67.** $-29+0i$ **69.** $-9+40i$
71. $35+20i$ **73.** $-20+48i$ **75.** $\frac{13}{16}+0i$
77. $\frac{1}{5}-\frac{3}{5}i$ **79.** $\frac{3}{25}-\frac{4}{25}i$ **81.** $\frac{21}{29}+\frac{20}{29}i$
83. $-\frac{17}{10}-\frac{1}{10}i$ **85.** $-\frac{1}{2}-\frac{5}{2}i$ **87.** $\frac{1}{2}-\frac{1}{3}i$
89. $1+10i$ **91.** $\frac{1}{4}+\frac{1}{2}i$ **93.** $-1+i\sqrt{2}$
95. $-2-i\sqrt{3}$ **97.** $-\frac{1}{2}+\frac{\sqrt{2}}{2}i$ **99.** Yes **101.** Yes

Chapter 6 Review Exercises, pp. 620–622

1. a. False; $\sqrt{0}=0$ is not positive. **b.** False; $\sqrt[3]{-8}=-2$
2. $\sqrt{(-3)^2}=\sqrt{9}=3$ **3. a.** False **b.** True
4. $\frac{5}{4}$ **5.** 5 **6.** 6 **7. a.** 3 **b.** 0 **c.** $\sqrt{7}$ **d.** $[1,\infty)$ **8. a.** 0 **b.** 1 **c.** 3 **d.** $[-5,\infty)$
9. $\frac{\sqrt[3]{2x}}{\sqrt[4]{2x}}+4$ **10. a.** $|x|$ **b.** x **c.** $|x|$ **d.** $x+1$
11. a. $2|y|$ **b.** $3y$ **c.** $|y|$ **d.** y **12.** 8 cm
13. Yes, provided the expressions are defined. For example: $x^5\cdot x^3=x^8$ and $x^{1/5}\cdot x^{2/5}=x^{3/5}$ **14.** n represents the root.
15. Take the reciprocal of the base and change the exponent to positive. **16.** -5 **17.** $\frac{1}{2}$ **18.** 4
19. b^{10} **20.** $\frac{16y^{12}}{xz^9}$ **21.** $\frac{a^3c}{b^2}$ **22.** $x^{3/4}$
23. $(2y^2)^{1/3}$ **24.** 2.1544 **25.** 6.8173 **26.** 54.1819
27. a. The radicand has no factor raised to a power greater than or equal to the index. **b.** The radicand does not contain a fraction. **c.** There are no radicals in the denominator of a fraction.
28. $6\sqrt{3}$ **29.** $xz\sqrt[4]{xy}$ **30.** $-10ab^3\sqrt[3]{2b}$ **31.** $-\frac{2a}{b}$
32. a. The principal square root of the quotient of 2 and x **b.** The cube of the sum of x and 1 **33.** 31 ft
34. They are *like* radicals. **35.** Cannot be combined; the indices are different. **36.** Cannot be combined; one term has a radical, but the other does not. **37.** Can be combined: $3\sqrt[4]{3xy}$
38. Can be added after simplifying: $19\sqrt{2}$
39. $5\sqrt{7}$ **40.** $9\sqrt[3]{2}-8$ **41.** $10\sqrt{2}$ **42.** $13x\sqrt[3]{2x^2}$
43. False; 5 and $3\sqrt{x}$ are not *like* radicals.
44. False; $\sqrt{y}+\sqrt{y}=2\sqrt{y}$ (add the coefficients).
45. 6 **46.** $2\sqrt[4]{2}$ **47.** $-2\sqrt{21}+6\sqrt{33}$
48. $-6\sqrt{15}+15$ **49.** $4x-9$ **50.** $y-16$
51. $7y-2\sqrt{21xy}+3x$ **52.** $12w+20\sqrt{3w}+25$
53. $-2z-9\sqrt{6z}-42$ **54.** $3a+5\sqrt{5a}-10$
55. $u^2\sqrt[6]{u^5}$ **56.** $\sqrt[4]{4w^3}$ **57.** $\frac{y^2\sqrt{3y}}{5x^3}$
58. $\frac{-2x^2y^2\sqrt[3]{2x}}{z^3}$ **59.** $9w^2$ **60.** $\frac{t^4}{4}$ **61.** $\frac{\sqrt{14y}}{2y}$
62. $\frac{\sqrt{15w}}{3w}$ **63.** $\frac{4\sqrt[3]{3p}}{3p}$ **64.** $-\frac{\sqrt[3]{4x^2}}{x}$
65. $\sqrt{10}-\sqrt{15}$ **66.** $3\sqrt{5}-3\sqrt{7}$ **67.** $\sqrt{t}+\sqrt{3}$
68. $\sqrt{w}+\sqrt{7}$ **69.** The quotient of the principal square root of 2 and the square of x
70. $\left\{\frac{49}{2}\right\}$ **71.** $\{31\}$ **72.** $\{-12\}$ **73.** $\{7\}$
74. $\{9\}$ **75.** $\{\ \}$ (The value -2 does not check.)
76. $\left\{\frac{1}{2}, 4\right\}$ **77.** $\{-2\}$ (The value 2 does not check.)
78. $6\sqrt{5}$ m ≈ 13.4 m **79. a.** $v(20)\approx 25.3$ ft/sec. When the water depth is 20 ft, a wave travels about 25.3 ft/sec. **b.** 8 ft
80. $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$
81. $a+bi$, where $b\neq 0$
82. Multiply the numerator and denominator by $4-6i$, which is the complex conjugate of the denominator.
83. $4i$ **84.** $-i\sqrt{5}$ **85.** -15 **86.** $-2i$ **87.** -1
88. i **89.** $-i$ **90.** 0 **91.** $-5+5i$ **92.** $9+17i$
93. $25+0i$ **94.** $24-10i$
95. $-\frac{17}{4}+i$; real part: $-\frac{17}{4}$; imaginary part: 1
96. $-2-i$; real part: -2; imaginary part: -1
97. $\frac{4}{13}-\frac{7}{13}i$ **98.** $3+4i$ **99.** $-\frac{3}{2}+\frac{5}{2}i$
100. $-\frac{4}{17}+\frac{16}{17}i$ **101.** $-\frac{2}{3}+\frac{\sqrt{10}}{6}i$ **102.** $2-4i$

Chapter 6 Test, pp. 623–624

1. a. 6 **b.** -6 **2. a.** Real **b.** Not real **c.** Real **d.** Real **3. a.** y **b.** $|y|$ **4.** 3 **5.** $\frac{4}{3}$ **6.** $2\sqrt[3]{4}$
7. $a^2bc^2\sqrt{bc}$ **8.** $3x^2yz^2\sqrt{2xy}$ **9.** $4w^2\sqrt{w}$ **10.** $\frac{x^2}{5y}$
11. $\frac{3\sqrt{2}}{2}$ **12. a.** $f(-8)=2\sqrt{3}$; $f(-6)=2\sqrt{2}$; $f(-4)=2$; $f(-2)=0$ **b.** $(-\infty,-2]$ **13.** -0.3080
14. -3 **15.** $\frac{20x^2y^3}{z}$ **16.** $\sqrt[6]{7y^3}$ **17.** $\sqrt[12]{10}$
18. $3\sqrt{5}$ **19.** $3\sqrt{2x}-3\sqrt{5x}$ **20.** $40-10\sqrt{5x}-3x$
21. $\frac{-2\sqrt[3]{x^2}}{x}$ **22.** $\frac{x+6+5\sqrt{x}}{9-x}$ **23. a.** $2i\sqrt{2}$ **b.** $8i$ **c.** $\frac{1}{2}+\frac{\sqrt{2}}{2}i$ **24.** $1-11i$ **25.** $30+16i$ **26.** -28
27. $-33-56i$ **28.** 104 **29.** $\frac{17}{25}+\frac{6}{25}i$
30. $-\frac{15}{17}+\frac{9}{17}i$ **31.** $r(10)\approx 1.34$; the radius of a sphere of volume 10 cubic units is approximately 1.34 units.
32. 21 ft **33.** $\{-16\}$ **34.** $\left\{\frac{17}{5}\right\}$
35. $\{2\}$ (The value 42 does not check.)

Chapter 7

Section 7.1 Activity, pp. 632–633

A.1. a. $\{-7, 7\}$ **b.** $\{-4, 4\}$ **c.** $(5i)^2=-25$ and $(-5i)^2=-25$
d. $\sqrt{k}$ and $-\sqrt{k}$ or simply $\pm\sqrt{k}$
A.2. a. $\{\pm 8\}$ **b.** $\{-11, 5\}$ **c.** $\{-7, 1\}$
A.3. a. $\{\pm 2\sqrt{5}\}$ **b.** $\{4\pm 2\sqrt{5}\}$ **c.** $\{4\pm i\sqrt{6}\}$
A.4. a. 5 **b.** 25 **c.** $x^2+10x+25=(x+5)^2$ **d.** $\left(\frac{1}{2}b\right)^2$
A.5. a. $n=81$; $(y+9)^2$ **b.** $n=\frac{121}{4}$; $\left(t-\frac{11}{2}\right)^2$ **c.** $n=\frac{9}{100}$; $\left(x+\frac{3}{10}\right)^2$

A.6. **a.** $x^2 + 4x + 10 = 0$ **b.** $x^2 + 4x = -10$ **c.** $x^2 + 4x + 4 = -10 + 4$ **d.** $(x + 2)^2 = -6$ **e.** $\{-2 \pm i\sqrt{6}\}$

Section 7.1 Practice Exercises, pp. 633–637

R.1. $\{5\}$ **R.3.** $\left\{\frac{2}{3} + \frac{\sqrt{5}}{3}\right\}$ **R.5.** 11 and -11 **R.7.** $6\sqrt{2}$ **R.9.** $\frac{\sqrt{37}}{8}$ **R.11.** $y^2 - 14x + 49$ **R.13.** $9w^2 + 24w + 16$ **R.15.** $(p + 11)^2$ **R.17.** $(8x - 9)^2$

1. **a.** 0; 0 **b.** 0 **c.** $\sqrt{k}; -\sqrt{k}$ **d.** 2; $\{3, -3\}$ **e.** completing **f.** 100 **g.** 4; 1 **h.** 8 **3.** $\{\pm 2\}$ **5.** $\{\pm\sqrt{7}\}$ **7.** $\left\{\frac{11}{6}, -\frac{11}{6}\right\}$ **9.** $\{\pm 5i\}$ **11.** $\{-1, -5\}$ **13.** $\left\{\frac{-3 \pm \sqrt{7}}{2}\right\}$ **15.** $\{-5 \pm 3i\sqrt{2}\}$ **17.** $\left\{\pm\frac{\sqrt{33}}{3}\right\}$ **19.** $\left\{-\frac{4}{5} \pm \frac{\sqrt{3}}{5}i\right\}$ **21.** $\{4i, -4i\}$ **23.** 1. Factoring and applying the zero product rule. 2. Applying the square root property. $\{\pm 6\}$ **25.** **a.** $\{16\}$ **b.** $\{2, -2\}$ **27.** $n = 9$; $(x - 3)^2$ **29.** $n = 16$; $(t + 4)^2$ **31.** $n = \frac{1}{4}$; $\left(c - \frac{1}{2}\right)^2$ **33.** $n = \frac{25}{4}$; $\left(y + \frac{5}{2}\right)^2$ **35.** $n = \frac{1}{25}$; $\left(b + \frac{1}{5}\right)^2$ **37.** $n = \frac{1}{9}$; $\left(p - \frac{1}{3}\right)^2$ **39.** 1. Divide both sides by a to make the leading coefficient 1. 2. Isolate the variable terms on one side of the equation. 3. Complete the square. 4. Apply the square root property and solve for x. **41.** $\{-3, -5\}$ **43.** $\{-3 \pm i\sqrt{7}\}$ **45.** $\{-2 \pm i\sqrt{2}\}$ **47.** $\{5, -2\}$ **49.** $\left\{-1 \pm \frac{\sqrt{6}}{2}i\right\}$ **51.** $\left\{2 \pm \frac{2}{3}i\right\}$ **53.** $\left\{\frac{1}{5} \pm \frac{\sqrt{3}}{5}\right\}$ **55.** $\left\{-\frac{3}{4} \pm \frac{\sqrt{65}}{4}\right\}$ **57.** $\{2 \pm \sqrt{11}\}$ **59.** $\{1, -7\}$ **61.** **a.** $t = \frac{\sqrt{d}}{4}$ **b.** 8 sec **63.** $r = \sqrt{\frac{A}{\pi}}$ or $r = \frac{\sqrt{A\pi}}{\pi}$ **65.** $a = \sqrt{d^2 - b^2 - c^2}$ **67.** $r = \sqrt{\frac{3V}{\pi h}}$ or $r = \frac{\sqrt{3V\pi h}}{\pi h}$ **69.** The shelf extends 4.2 ft. **71.** The sides are 7.1 in. **73.** **a.** 4.5 thousand textbooks or 35.5 thousand textbooks **b.** Profit increases to a point as more books are produced. Beyond that point, the market is "flooded," and profit decreases. There are two points at which the profit is \$20,000. Producing 4.5 thousand books makes the same profit with fewer resources as producing 35.5 thousand books.

Section 7.2 Activity, pp. 648–649

A.1. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

A.2. **a.** $a = 2, b = -12, c = 16$ **b.** $\{2, 4\}$

A.3. **a.** $a = 1, b = -4, c = 5$ **b.** $\{2 \pm i\}$

A.4. **a.** $a = -1, b = -6, c = -9$ **b.** $\{-3\}$

A.5. **a.** Imaginary **b.** Real **c.** 1; 2 **d.** discriminant

A.6. **a.** 16 **b.** Two real solutions **c.** Yes

A.7. **a.** -4 **b.** Two imaginary solutions **c.** Yes

A.8. **a.** 0 **b.** One real solution **c.** Yes

A.9. **a.** The solutions to the equation give the x-intercepts of the graph.
b. The discriminant tells us the number of real solutions to the equation and thus the number of x-intercepts of the graph. In this case, the discriminant is 16, indicating that the equation $2x^2 - 12x + 16 = 0$ has two real solutions, and the graph of $f(x) = 2x^2 - 12x + 16$ has two x-intercepts.

A.10. **a.** 0 **b.** 1
c. Function g is shown in graph ii. Function h is shown in graph i.

Section 7.2 Practice Exercises, pp. 649–653

R.1. $2\sqrt{6}$ **R.3.** $2 + \sqrt{5}$ **R.5.** $2\sqrt{14}$ **R.7.** $\left\{-\frac{29}{3}\right\}$ **R.9.** $\{-59\}$ **R.11.** $\{-7 \pm 2\sqrt{11}\}$

1. **a.** quadratic; $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **b.** $ax^2 + bx + c = 0$ **c.** 8; -42; -27 **d.** 7; 97 **3.** Quadratic **5.** Neither **7.** Linear **9.** $\{-12, 1\}$ **11.** $\left\{\frac{1}{9} \pm \frac{2\sqrt{11}}{9}i\right\}$ **13.** $\left\{\frac{1}{6} \pm \frac{\sqrt{14}}{6}i\right\}$ **15.** $\{7, -5\}$ **17.** $\left\{\frac{-3 \pm \sqrt{41}}{2}\right\}$ **19.** $\left\{\frac{2}{5}\right\}$ **21.** $\{3 \pm i\sqrt{5}\}$ **23.** $\left\{\frac{1 \pm \sqrt{29}}{2}\right\}$ **25.** $\left\{\frac{-1 \pm \sqrt{13}}{4}\right\}$ **27.** $\left\{-\frac{2}{3} \pm \frac{2\sqrt{2}}{3}i\right\}$ **29.** $\left\{\frac{-5 \pm \sqrt{13}}{2}\right\}$ **31.** $\{-2, -4\}$ **33.** $\left\{\frac{-7 \pm \sqrt{109}}{6}\right\}$ **35.** **a.** $(x - 3)(x^2 + 3x + 9)$ **b.** $\left\{3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i\right\}$ **37.** **a.** $3x(x^2 - 2x + 2)$ **b.** $\{0, 1 \pm i\}$ **39.** The length of each side is 3 ft. **41.** The legs are approximately 1.6 in. and 3.6 in. **43.** The legs are approximately 8.2 m and 6.1 m. **45.** **a.** Approximately 4.5 fatalities per 100 million miles driven **b.** Approximately 1 fatality per 100 million miles driven **c.** Approximately 4.2 fatalities per 100 million miles driven **d.** For drivers 26 years old and 71 years old **47.** The times are $\frac{3 + \sqrt{5}}{2}$ sec $\approx$ 2.62 sec or $\frac{3 - \sqrt{5}}{2}$ sec $\approx$ 0.38 sec. **49.** **a.** $x^2 + 2x + 1 = 0$ **b.** 0 **c.** 1 rational solution **51.** **a.** $19m^2 - 8m + 0 = 0$ **b.** 64 **c.** 2 rational solutions **53.** **a.** $5p^2 + 0p - 21 = 0$ **b.** 420 **c.** 2 irrational solutions **55.** **a.** $n^2 + 3n + 4 = 0$ **b.** -7 **c.** 2 imaginary solutions **57.** Discriminant: 16; two x-intercepts **59.** Discriminant: 0; one x-intercept **61.** Discriminant: -39; no x-intercepts **63.** x-intercepts: $\left(\frac{5 + \sqrt{13}}{2}, 0\right), \left(\frac{5 - \sqrt{13}}{2}, 0\right)$; y-intercept: $(0, 3)$ **65.** x-intercepts: none; y-intercept: $(0, -1)$ **67.** x-intercepts: $\left(\frac{-5 - \sqrt{41}}{4}, 0\right), \left(\frac{-5 + \sqrt{41}}{4}, 0\right)$; y-intercept: $(0, -2)$ **69.** $\{-1 \pm 3i\}$ **71.** $\left\{-\frac{1}{3}, 6\right\}$ **73.** $\left\{\frac{5}{2} \pm \frac{3\sqrt{2}}{2}i\right\}$ **75.** $\left\{-\frac{1}{2} \pm 2i\right\}$ **77.** $\left\{\frac{3 \pm \sqrt{7}}{2}\right\}$ **79.** $\{1 \pm 3i\sqrt{2}\}$ **81.** $\{2 \pm \sqrt{2}\}$ **83.** $\left\{-\frac{1}{8}, \frac{3}{4}\right\}$ **85.** $\left\{\pm\frac{\sqrt{21}}{2}\right\}$ **87.** **a.** $\{-3 \pm \sqrt{14}\}$ **b.** $\{-3 \pm \sqrt{14}\}$ **c.** Answers will vary

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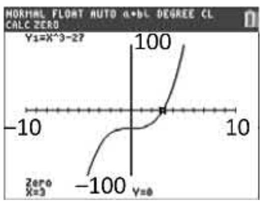

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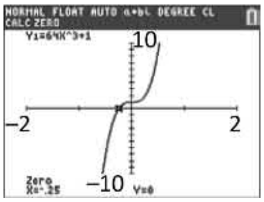

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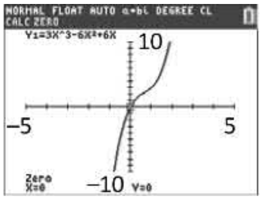

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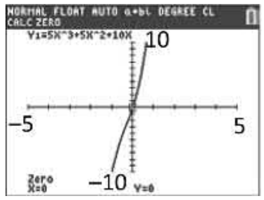


Section 7.3 Activity, pp. 657–658

A.1. a. quadratic **b.** $u^2 - 13u + 36 = 0$ **c.** $u = 4$ and $u = 9$ **d.** $\left\{\frac{5}{2}, 5\right\}$

A.2. $u = x^2 - 3$; $u^2 - 9u - 52 = 0$ **A.3.** $u = x^{1/3}$; $3u^2 - u - 4 = 0$

A.4. $u = x^2$; $u^2 - 29u + 100 = 0$ **A.5.** $u = 2 + \frac{3}{x}$; $u^2 - u - 12 = 0$

A.6. $\{\pm i, \pm 4\}$ **A.7.** $\left\{\frac{64}{27}, -1\right\}$ **A.8.** $\{\pm 5, \pm 2\}$ **A.9.** $\left\{\frac{3}{2}, -\frac{3}{5}\right\}$

Section 7.3 Practice Exercises, pp. 658–660

R.1. $\sqrt{x}$ **R.3.** $\sqrt[3]{z^2}$ or $\left(\sqrt[3]{z}\right)^2$ **R.5.** $\{-2, 9\}$ **R.7.** $\left\{\frac{-1 \pm \sqrt{37}}{6}\right\}$

R.9. $\{\pm 3\}$ **R.11.** $\{\pm\sqrt{5}i\}$ **R.13.** $\{125\}$ **R.15.** $\{-8\}$

1. a. quadratic **b.** $3x - 1$ **c.** $p^{1/3}$

3. Let $u = 3m + 4$; $u^2 - 4u - 5 = 0$

5. Let $u = y^2 + 1$; $u^2 - 7u + 10 = 0$

7. Let $u = \frac{1}{x}$; $3u^2 + 10u - 8 = 0$

9. a. $\{-4, -6\}$ **b.** $\{-4, -1, -2, -3\}$

11. $\{1 \pm i\sqrt{2}, 1 \pm \sqrt{2}\}$ **13.** $\{5, -1, 2\}$ **15.** $\{27, -8\}$

17. $\left\{-\frac{1}{32}, -243\right\}$ **19.** $\{4\}$ (The value 64 does not check.)

21. $\left\{\frac{1}{4}\right\}$ (The value 4 does not check.) **23.** $\{-7\}$

25. $\{16\}$; yes **27.** $\{\pm 3i, \pm\sqrt{5}\}$ **29.** $\left\{\pm\frac{\sqrt{3}}{2}, \pm i\sqrt{5}\right\}$

31. $\left\{\frac{-5 \pm \sqrt{13}}{2}\right\}$ **33.** $\left\{\frac{9 \pm \sqrt{73}}{4}\right\}$ **35.** $\{2 \pm \sqrt{3}\}$

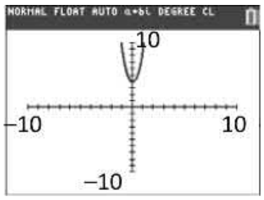
37. $\{\pm 2, \pm 2i\}$ **39.** $\left\{-\frac{7}{4}, -\frac{3}{2}\right\}$ **41.** $\left\{\frac{1}{2}, -\frac{1}{2}, \sqrt{2}, -\sqrt{2}\right\}$

43. $\left\{2, 1, -1 \pm i\sqrt{3}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right\}$ **45.** $\{5, 4\}$ **47.** $\{0, 2\}$

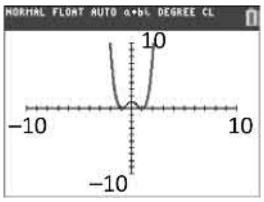
49. $\left\{1, \frac{17}{2}\right\}$ **51.** $\{64, -125\}$ **53.** $\{\pm\sqrt{2}, \pm 2i\}$

55. $\{\pm 4i, 1\}$ **57.** $\{4, \pm i\sqrt{5}\}$ **59.** $\left\{\frac{8}{3}, 2\right\}$

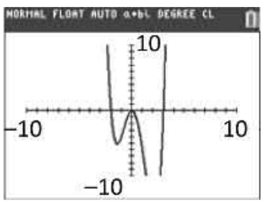
61. a. $\{\pm i\sqrt{2}\}$ **b.** Two imaginary solutions; no real solutions
c. No x-intercepts
d.



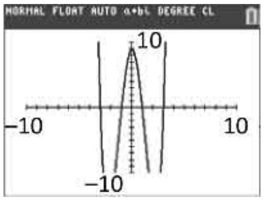
62. a. $\{1, -1\}$ **b.** Two real solutions; no imaginary solutions
c. Two x-intercepts
d.



63. a. $\{0, 3, -2\}$ **b.** Three real solutions; no imaginary solutions
c. Three x-intercepts
d.



64. a. $\{\pm 1, \pm 3\}$ **b.** Four real solutions; no imaginary solutions
c. Four x-intercepts
d.



Chapter 7 Problem Recognition Exercises, p. 661

1. $\{-5 \pm \sqrt{22}\}$ **2.** $\{8 \pm \sqrt{59}\}$ **3.** $\left\{-\frac{1}{6} \pm \frac{\sqrt{47}}{6}i\right\}$

4. $\left\{-\frac{3}{8} \pm \frac{\sqrt{71}}{8}i\right\}$ **5. a.** Quadratic **b.** $\{2, -7\}$

6. a. Quadratic **b.** $\{4, 5\}$ **7. a.** Quadratic form **b.** $\{-3, -1, 1, 3\}$ **8. a.** Quadratic form **b.** $\{2, -2, i, -i\}$

9. a. Quadratic form (or radical) **b.** $\{16\}$ (The value 1 does not check.) **10. a.** Quadratic **b.** $\left\{\frac{9 \pm \sqrt{89}}{2}\right\}$

11. a. Linear **b.** $\{4\}$ **12. a.** Linear **b.** $\{-1\}$

13. a. Quadratic **b.** $\{\pm i\sqrt{2}\}$ **14. a.** Quadratic **b.** $\{\pm i\sqrt{6}\}$ **15. a.** Rational **b.** $\{2, -1\}$

16. a. Rational **b.** $\{-2, 4\}$

17. a. Quadratic **b.** $\{10 \pm \sqrt{101}\}$

18. a. Quadratic **b.** $\{-9 \pm \sqrt{77}\}$

19. a. Quadratic **b.** $\left\{\frac{1 \pm \sqrt{17}}{2}\right\}$

20. a. Quadratic **b.** $\left\{\frac{1 \pm \sqrt{13}}{2}\right\}$

21. a. Radical **b.** $\{3\}$ (The value -1 does not check.)

22. a. Radical **b.** $\{6\}$ (The value -1 does not check.)

23. a. Quadratic form (or radical) **b.** $\{-125, 27\}$

24. a. Quadratic form (or radical) **b.** $\{-64, -1\}$

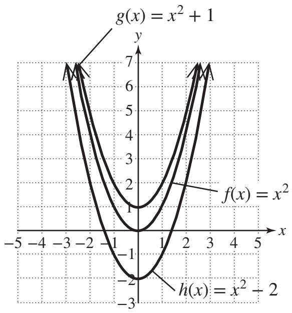
Section 7.4 Activity, pp. 669–670

Instructor Note

Students can complete the tables by hand and plotting points or, if time is a consideration, consider having them use a graphing utility.

A.1.

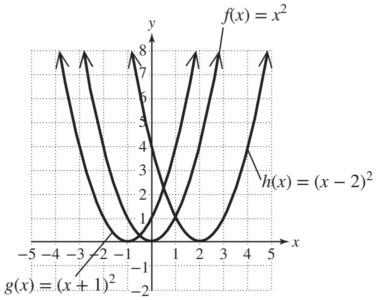
x	$f(x) = x^2$	$g(x) = x^2 + 1$	$h(x) = x^2 - 2$
0	0	1	−2
1	1	2	−1
2	4	5	2
3	9	10	7
−1	1	2	−1
−2	4	5	2
−3	9	10	7



A.2. The graph of $y = x^2 + k$ is the graph of $y = x^2$ shifted up k units. The graph of $y = x^2 - k$ is the graph of $y = x^2$ shifted down k units.

A.3.

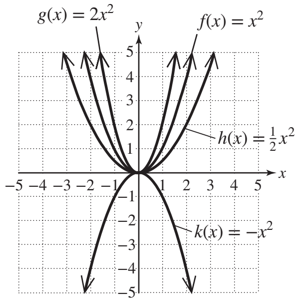
x	$f(x) = x^2$	$g(x) = (x + 1)^2$	$h(x) = (x - 2)^2$
0	0	1	4
1	1	4	1
2	4	9	0
3	9	16	1
−1	1	0	9
−2	4	1	16
−3	9	4	25



A.4. The graph of $y = (x + h)^2$ is the graph of $y = x^2$ shifted to the left h units. The graph of $y = (x - h)^2$ is the graph of $y = x^2$ shifted to the right h units.

A.5.

x	$f(x) = x^2$	$g(x) = 2x^2$	$h(x) = \frac{1}{2}x^2$	$k(x) = -x^2$
0	0	0	0	0
1	1	2	0.5	−1
2	4	8	2	−4
3	9	18	4.5	−9
−1	1	2	0.5	−1
−2	4	8	2	−4
−3	9	18	4.5	−9



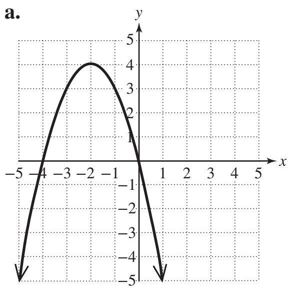
A.6. **a.** For $a > 1$, the graph of $y = ax^2$ is the graph of $y = x^2$ with a vertical stretch by a factor of a.

b. For $0 < a < 1$, the graph of $y = ax^2$ is the graph of $y = x^2$ with a vertical shrink by a factor of a.

c. For $a < 0$, the graph of $y = ax^2$ is the graph of $y = x^2$ reflected over the x-axis. There may also be a vertical shrink or stretch if $|a| \neq 1$.

A.7. **a.** i **b.** iii **c.** ii

A.8. **a.**



b. $(-2, 4)$

c. $x = -2$

d. Maximum value 4

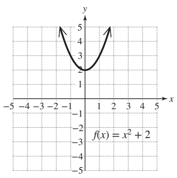
e. Domain: $(-\infty, \infty)$; range: $(-\infty, 4]$

Section 7.4 Practice Exercises, pp. 671–676

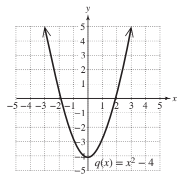
R.1. **a.** $(-\infty, \infty)$ **b.** $[-4, \infty)$ **c.** x-intercepts: $(-1, 0)$ and $(3, 0)$ **d.** y-intercept: $(0, -3)$ **R.3.** **a.** $(-\infty, \infty)$ **b.** $(-\infty, -1]$ **c.** x-intercepts: None **d.** y-intercept: $(0, -5)$

1. **a.** parabola **b.** $>$; $<$ **c.** lowest; highest **3.** Downward **5.** Maximum **7.** Vertex: $(-3, 1)$; axis of symmetry: $x = -3$ **9.** The value of k shifts the graph of $f(x) = x^2$ vertically.

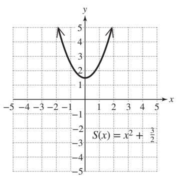
11.



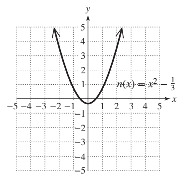
13.



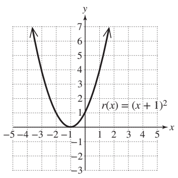
15.



17.



19.



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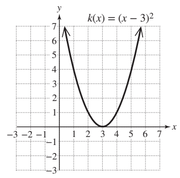

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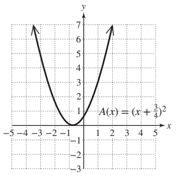
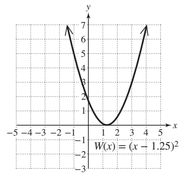
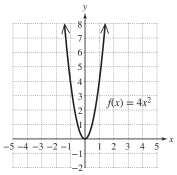


25.



27. The value of a vertically stretches or shrinks the graph of $f(x) = x^2$.

29.



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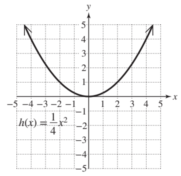

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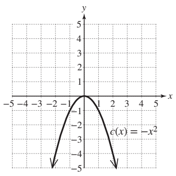

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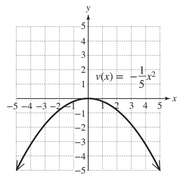
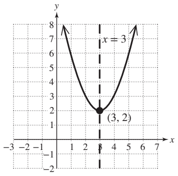


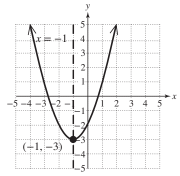
37. d **39.** g **41.** a **43.** e

45.



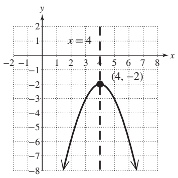
Domain: $(-\infty, \infty)$; range: $[2, \infty)$

47.

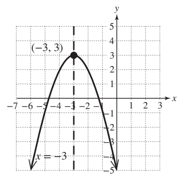


Domain: $(-\infty, \infty)$; range: $[-3, \infty)$

49.

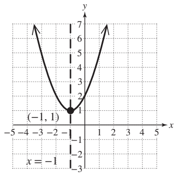


51.



Domain: $(-\infty, \infty)$; range: $(-\infty, -2]$ Domain: $(-\infty, \infty)$; range: $(-\infty, 3]$

53.



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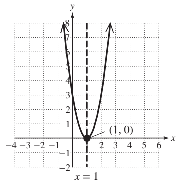
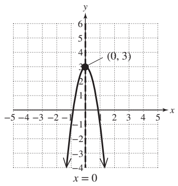
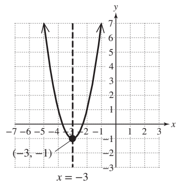


Domain: $(-\infty, \infty)$; range: $[1, \infty)$ Domain: $(-\infty, \infty)$; range: $[0, \infty)$

57.

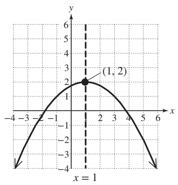


59.

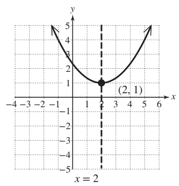


Domain: $(-\infty, \infty)$; range: $(-\infty, 3]$ Domain: $(-\infty, \infty)$; range: $[-1, \infty)$

61.



63.



Domain: $(-\infty, \infty)$; range: $(-\infty, 2]$ Domain: $(-\infty, \infty)$; range: $[1, \infty)$

65. **a.** $y = x^2 + 3$ is $y = x^2$ shifted up 3 units.
b. $y = (x + 3)^2$ is $y = x^2$ shifted left 3 units.
c. $y = 3x^2$ is $y = x^2$ with a vertical stretch.

67. Vertex $(6, -9)$; minimum point; minimum value: -9

69. Vertex $(2, 5)$; maximum point; maximum value: 5

71. Vertex $(-8, 0)$; minimum point; minimum value: 0

73. Vertex $\left(0, \dfrac{21}{4}\right)$; maximum point; maximum value: $\dfrac{21}{4}$

75. Vertex $\left(7, -\dfrac{3}{2}\right)$; minimum point; minimum value: $-\dfrac{3}{2}$

77. Vertex $(0, 0)$; minimum point; minimum value: 0

79. True **81.** False **83.** **a.** $(60, 30)$ **b.** 30 ft **c.** 70 ft

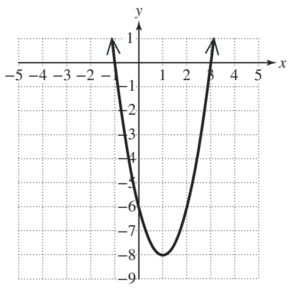
85. **a.** The fireworks will explode at a height of 150 ft.
b. Yes, because the ordered pair $(3, 150)$ is the vertex.

Section 7.5 Activity, pp. 683–684

A.1. Complete the square to write the function in vertex form $f(x) = a(x - h)^2 + k$, or apply the vertex formula.

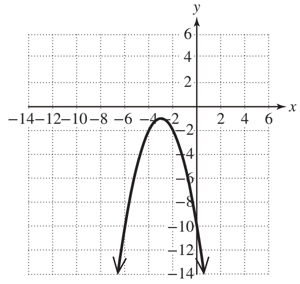
A.2. The x-coordinate of the vertex is given by $\dfrac{-b}{2a}$. The y-coordinate is most easily found by substituting the x-coordinate of the vertex into the function.

A.3. **a.** $f(x) = 2(x^2 - 2x) - 6$ **b.** 1 **c.** $f(x) = 2(x^2 - 2x + 1 - 1) - 6$
d. $f(x) = 2(x^2 - 2x + 1) + 2(-1) - 6$ or equivalently $f(x) = 2(x^2 - 2x + 1) - 8$
e. $f(x) = 2(x - 1)^2 - 8$ **f.** $(1, -8)$ **g.** Upward
h. Minimum value: -8 **i.** $x = 1$
j. x-intercepts: $(-1, 0)$ and $(3, 0)$; y-intercept: $(0, -6)$
k.



A.4. $(1, -8)$. They are the same.

A.5. **a.** $(-3, -1)$ **b.** Downward **c.** Maximum value: -1
d. $x = -3$ **e.** x-intercepts: None; y-intercept: $(0, -10)$
f.



A.6. **a.** $(50, 12{,}250)$
b. The vertex means that 50 sec after launch, the rocket will reach its maximum height of 12,250 m.

Section 7.5 Practice Exercises, pp. 684–687

R.1. $\dfrac{3}{4}$ **R.3.** x-intercept: $(-2, 0)$; y-intercept: $(0, -8)$

R.5. $\{-4, 8\}$ **R.7.** $\{-2, 2\}$ **R.9.** $\{-12\}$ **R.11.** $n = 49$; $(x - 7)^2$

1. **a.** $\dfrac{-b}{2a}$; $\dfrac{-b}{2a}$ **b.** True **c.** True **d.** True **e.** False

3. $g(x) = (x - 4)^2 - 11$; $(4, -11)$

5. $n(x) = 2(x + 3)^2 - 5$; $(-3, -5)$

7. $p(x) = -3(x - 1)^2 - 2$; $(1, -2)$

9. $k(x) = \left(x + \dfrac{7}{2}\right)^2 - \dfrac{89}{4}$; $\left(-\dfrac{7}{2}, -\dfrac{89}{4}\right)$

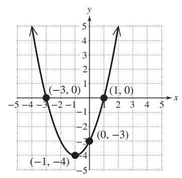
11. $F(x) = 5(x + 1)^2 - 4$; $(-1, -4)$

13. $P(x) = -2\left(x - \dfrac{1}{4}\right)^2 + \dfrac{1}{8}$; $\left(\dfrac{1}{4}, \dfrac{1}{8}\right)$ **15.** $(2, 3)$

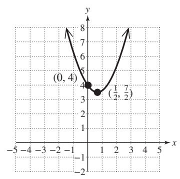
17. $(-1, -2)$ **19.** $(-4, -15)$ **21.** $(-1, 2)$ **23.** $(1, 3)$

25. $\left(\dfrac{3}{2}, \dfrac{3}{4}\right)$ **27.** $(-4, -15)$ **29.** $(-1, 4)$

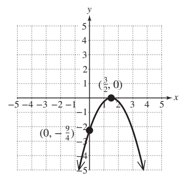
31. **a.** $(-1, -4)$ **b.** $(0, -3)$
c. $(1, 0), (-3, 0)$
d.



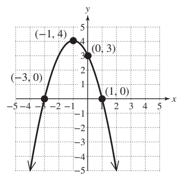
33. **a.** $\left(\frac{1}{2}, \frac{7}{2}\right)$ **b.** $(0, 4)$
c. No x-intercepts
d.



35. **a.** $\left(\frac{3}{2}, 0\right)$ **b.** $\left(0, -\frac{9}{4}\right)$
c. $\left(\frac{3}{2}, 0\right)$
d.



37. **a.** $(-1, 4)$ **b.** $(0, 3)$
c. $(1, 0), (-3, 0)$
d.



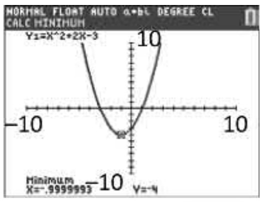
39. **a.** 164.25 ft **b.** 3.125 sec **41.** **a.** 45 mph **b.** 32 mpg

43. **a.** 48 hr **b.** 2 g **45.** $a = -9, b = 5, c = 4$; $y = -9x^2 + 5x + 4$

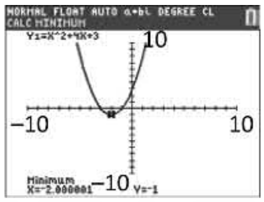
47. $a = 2, b = -1, c = -5$; $y = 2x^2 - x - 5$

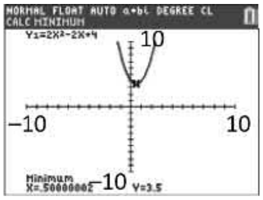
49. $a = -3, b = 4, c = 0$; $y = -3x^2 + 4x$

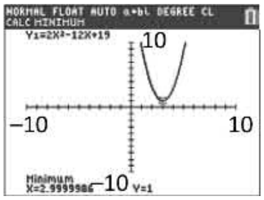
51.

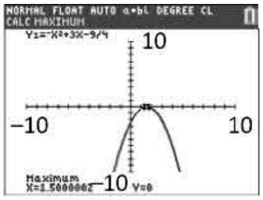


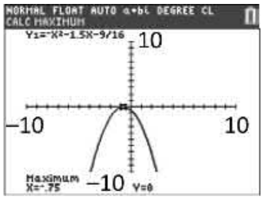
52.



53. 

54. 

55. 

56. 

57. a. The sum of the sides must equal the total amount of fencing. **b.** $A = x(200 - 2x)$ **c.** 50 ft by 100 ft

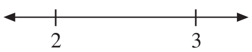
Section 7.6 Activity, pp. 696–697

A.1. a. 0, 5, and -4 **b.** $f(-1) = 0$, $f(4) = 5$, and $f(1) = -4$ **c.** -1 and 3 **d.** $(-1, 3)$ **e.** $(-\infty, -1) \cup (3, \infty)$

A.2. a. $\{-1, 3\}$ **b.** 

c. $(-\infty, -1)$ and $(3, \infty)$ **d.** excluded **e.** $(-\infty, -1) \cup (3, \infty)$ **f.** $(-1, 3)$ **g.** The solutions to the related equation $x^2 - 2x - 3 = 0$ correspond to the x-intercepts of the quadratic function. The solutions to the inequality $x^2 - 2x - 3 > 0$ are the values of x for which the function is positive (above the x-axis). The solutions to the inequality $x^2 - 2x - 3 < 0$ are the values of x for which the function is negative (below the x-axis).

A.3. a. $\{3\}$ **b.** Undefined for $x = 2$; no

c. 

d. $(-\infty, 2)$ and $[3, \infty)$ **e.** Yes **f.** $(-\infty, 2) \cup [3, \infty)$

g. The solution set would not include the boundary $x = 3$. The solution set would be $(-\infty, 2) \cup (3, \infty)$ **h.** $(2, 3]$

A.4. a. $(x - 3)^2$ **b.** $(x - 3)^2 < 0$; no **c.** $\{\ \}$. The expression $x^2 - 6x + 9$ is a perfect square trinomial. Thus, the inequality is equivalent to $(x - 3)^2 < 0$. The square of any real number is greater than or equal to zero. This means that there are no real numbers that make the expression $(x - 3)^2$ negative. **d.** The inequality $x^2 - 6x + 9 \le 0$ allows for the expression $x^2 - 6x + 9$, or equivalently, $(x - 3)^2$ to be equal to zero. This occurs for $x = 3$. The solution set is $\{3\}$. **e.** $(-\infty, \infty)$ **f.** $(-\infty, 3) \cup (3, \infty)$

Section 7.6 Practice Exercises, pp. 697–700

R.1. $\{-7, -2\}$ **R.3.** $\left\{-\frac{1}{2}\right\}$ **R.5.** $\left\{\frac{1}{2}, 6\right\}$ **R.7.** $\{5\}$
R.9. $\left\{\frac{3}{4} \pm \frac{\sqrt{23}}{4}i\right\}$

1. quadratic **3.** test point **5.** rational **7.** $\{\ \}$; $(-\infty, \infty)$ **9. a.** $(-\infty, -2) \cup (3, \infty)$ **b.** $(-2, 3)$ **c.** $[-2, 3]$ **d.** $(-\infty, -2] \cup [3, \infty)$ **11. a.** $(-2, 0) \cup (3, \infty)$ **b.** $(-\infty, -2) \cup (0, 3)$ **c.** $(-\infty, -2) \cup [0, 3]$ **d.** $[-2, 0] \cup [3, \infty)$ **13. a.** $\{4, -\frac{1}{2}\}$ **b.** $(-\infty, -\frac{1}{2}) \cup (4, \infty)$ **c.** $(-\frac{1}{2}, 4)$ **15. a.** $\{-10, 3\}$ **b.** $(-10, 3)$ **c.** $(-\infty, -10) \cup (3, \infty)$
17. a. $\left\{-\frac{1}{2}, 3\right\}$ **b.** $\left[-\frac{1}{2}, 3\right]$ **c.** $\left(-\infty, -\frac{1}{2}\right] \cup [3, \infty)$
19. $(1, 7)$ **21.** $(-\infty, -2) \cup (5, \infty)$ **23.** $[-5, 0]$ **25.** $[4, 8]$
27. $\left(-\infty, \frac{1 - \sqrt{6}}{5}\right) \cup \left(\frac{1 + \sqrt{6}}{5}, \infty\right)$

29. $\left[\frac{-3 - \sqrt{33}}{2}, \frac{-3 + \sqrt{33}}{2}\right]$ **31.** $(-11, 11)$
33. $\left(-\infty, -\frac{1}{3}\right] \cup [3, \infty)$ **35.** $(-\infty, -3] \cup [0, 4]$
37. $(-2, -1) \cup (2, \infty)$ **39. a.** $\{7\}$ **b.** $(-\infty, 5) \cup (7, \infty)$ **c.** $(5, 7)$ **41. a.** $\{4\}$ **b.** $[4, 6)$ **c.** $(-\infty, 4] \cup (6, \infty)$ **43.** $(1, \infty)$ **45.** $(-\infty, -4) \cup (4, \infty)$
47. $\left(2, \frac{7}{2}\right)$ **49.** $(5, 7]$ **51.** $(-\infty, 0) \cup \left[\frac{1}{2}, \infty\right)$
53. $(0, \infty)$ **55.** $(-\infty, \infty)$ **57.** $\{\ \}$ **59.** $\{0\}$ **61.** $\{\ \}$ **63.** $\{\ \}$ **65.** $(-\infty, \infty)$
67. $(-\infty, -11) \cup (-11, \infty)$ **69.** $\left\{\frac{3}{2}\right\}$ **71.** Quadratic; $[-4, 4]$ **73.** Quadratic; $(-\infty, \infty)$ **75.** Linear; $(-\frac{5}{2}, \infty)$
77. Rational; $(-\infty, -1)$ **79.** Polynomial (degree > 2); $(0, 5) \cup (5, \infty)$ **81.** Polynomial (degree > 2); $(2, \infty)$ **83.** Polynomial (degree > 2); $(-\infty, 5]$ **85.** Quadratic; $(-\sqrt{2}, \sqrt{2})$
87. Quadratic; $\left(-\infty, \frac{-5 - \sqrt{33}}{2}\right] \cup \left[\frac{-5 + \sqrt{33}}{2}, \infty\right)$
89. Rational; $(-\infty, -2] \cup (5, \infty)$ **91.** Linear; $(-\infty, 5]$ **93.** Quadratic; $(-\infty, \infty)$ **95.** The fuses should be set for between 1.3 sec and 5.4 sec after launch.

Chapter 7 Problem Recognition Exercises, p. 700

1. a. Equation quadratic in form and polynomial equation **b.** $\{\pm 2\sqrt{2}, \pm 1\}$ **2. a.** Absolute value inequality **b.** $(-\frac{1}{2}, 1)$ **3. a.** Polynomial inequality **b.** $[-\frac{1}{2}, 5]$ **4. a.** Radical equation **b.** $\{1\}$
5. a. Absolute value equation **b.** $\{9, -1\}$
6. a. Rational equation **b.** $\{4 \pm 2\sqrt{6}\}$
7. a. Polynomial inequality **b.** $[-5, -2] \cup [2, \infty)$
8. a. Compound inequality **b.** $(-\infty, 1)$
9. a. Linear inequality **b.** $[-23, \infty)$
10. a. Absolute value equation **b.** $\{9, 1\}$
11. a. Rational inequality **b.** $(-\infty, 2) \cup [5, \infty)$
12. a. Absolute value inequality **b.** $(-\infty, -13) \cup (5, \infty)$
13. a. Radical equation **b.** $\{-4\}$ (The value -7 does not check.)
14. a. Quadratic equation **b.** $\left\{\frac{3}{4} \pm \frac{\sqrt{10}}{4}i\right\}$
15. a. Compound inequality **b.** $(-\infty, -6) \cup (4, \infty)$
16. a. Linear equation and rational equation **b.** $\left\{-\frac{11}{2}\right\}$
17. a. Polynomial inequality **b.** $\{5\}$
18. a. Rational inequality **b.** $\{\ \}$
19. a. Radical equation and equation quadratic in form **b.** $\{16, 81\}$ **20. a.** Polynomial equation and equation quadratic in form **b.** $\{\pm 2, \pm 3\}$

Chapter 7 Review Exercises, pp. 706–709

1. $\{\pm\sqrt{5}\}$ **2.** $\{\pm 2i\}$ **3.** $\{\pm 9\}$ **4.** $\left\{\pm\frac{\sqrt{57}}{3}i\right\}$
5. $\{2 \pm 6\sqrt{2}\}$ **6.** $\left\{\frac{5}{2} \pm \frac{3}{2}i\right\}$ **7.** $\left\{\frac{1 \pm \sqrt{3}}{3}\right\}$
8. $\{4 \pm \sqrt{5}\}$ **9.** $5\sqrt{3}$ in. ≈ 8.7 in. **10.** 9 in.
11. $5\sqrt{6}$ in. ≈ 12.2 in. **12.** $n = 64$; $(x + 8)^2$
13. $n = \frac{81}{4}$; $\left(x - \frac{9}{2}\right)^2$ **14.** $n = \frac{1}{16}$; $\left(y + \frac{1}{4}\right)^2$
15. $n = \frac{1}{25}$; $\left(z - \frac{1}{5}\right)^2$ **16.** $\{-2 \pm 3i\}$ **17.** $\{1 \pm \sqrt{6}\}$
18. $\{3 \pm 2\sqrt{3}\}$ **19.** $\{4 \pm 3i\}$ **20.** $\left\{\frac{1}{3}, -1\right\}$
21. $\left\{\frac{1}{2}, -4\right\}$ **22.** $r = \sqrt{\frac{V}{\pi h}}$ or $r = \frac{\sqrt{V\pi h}}{\pi h}$
23. $s = \sqrt{\frac{A}{6}}$ or $s = \frac{\sqrt{6A}}{6}$

24. There will be two imaginary solutions.
25. Two rational solutions **26.** Two rational solutions
27. Two irrational solutions **28.** Two imaginary solutions
29. One rational solution **30.** Two imaginary solutions

31. $\{2 \pm \sqrt{3}\}$ **32.** $\left\{\frac{5}{2} \pm \frac{5\sqrt{3}}{2}i\right\}$ **33.** $\left\{2, -\frac{5}{6}\right\}$

34. $\left\{2, \frac{4}{3}\right\}$ **35.** $\left\{\frac{4}{5}, -\frac{1}{5}\right\}$ **36.** $\left\{\frac{1}{10}, -\frac{1}{2}\right\}$

37. $\{2 \pm 2i\sqrt{7}\}$ **38.** $\{4 \pm \sqrt{6}\}$ **39.** $\left\{\frac{2 \pm \sqrt{22}}{3}\right\}$

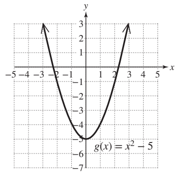
40. $\{8, -2\}$ **41.** $\{-7 \pm \sqrt{3}\}$ **42.** $\{-1 \pm \sqrt{11}\}$
43. a. 1822 ft **b.** 115 ft/sec **44. a.** $\approx$ 53,939 thousand **b.** 2021
45. The dimensions are approximately 3.1 ft by 7.2 ft.
46. The distance between Lincoln and Omaha is approximately 50 mi.
47. $\{49\}$ (The value 9 does not check.)

48. $\{4, 16\}$ **49.** $\{\pm 3, \pm\sqrt{2}\}$ **50.** $\left\{\pm\frac{\sqrt{6}}{2}, \pm i\right\}$

51. $\{-243, 32\}$ **52.** $\{32, 1\}$ **53.** $\{3 \pm \sqrt{10}\}$

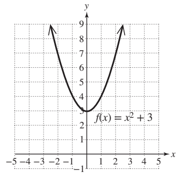
54. $\left\{\frac{1 \pm \sqrt{181}}{6}\right\}$ **55.** $\{\pm 3i, \pm i\sqrt{3}\}$ **56.** $\{\pm\sqrt{7}, \pm 2\}$

57.



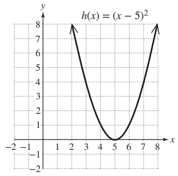
Domain: $(-\infty, \infty)$; range: $[-5, \infty)$

58.



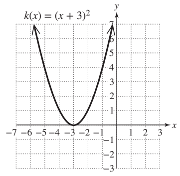
Domain: $(-\infty, \infty)$; range: $[3, \infty)$

59.



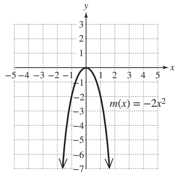
Domain: $(-\infty, \infty)$; range: $[0, \infty)$

60.



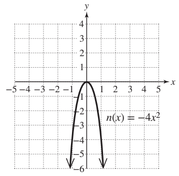
Domain: $(-\infty, \infty)$; range: $[0, \infty)$

61.



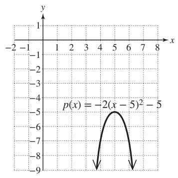
Domain: $(-\infty, \infty)$; range: $(-\infty, 0]$

62.



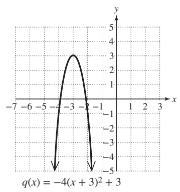
Domain: $(-\infty, \infty)$; range: $(-\infty, 0]$

63.



Domain: $(-\infty, \infty)$; range: $(-\infty, -5]$

64.



Domain: $(-\infty, \infty)$; range: $(-\infty, 3]$

65. $\left(4, \frac{5}{3}\right)$ is the minimum point. The minimum value is $\frac{5}{3}$.

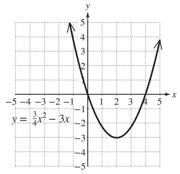
66. $\left(1, -\frac{1}{7}\right)$ is the maximum point. The maximum value is $-\frac{1}{7}$.

67. $x = -\frac{2}{11}$ **68.** $x = \frac{3}{16}$

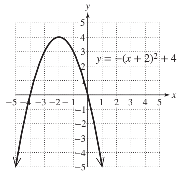
69. $z(x) = (x - 3)^2 - 2$; $(3, -2)$
70. $b(x) = (x - 2)^2 - 48$; $(2, -48)$
71. $p(x) = -5(x + 1)^2 - 8$; $(-1, -8)$
72. $q(x) = -3(x + 4)^2 - 6$; $(-4, -6)$ **73.** $(1, -15)$

74. $(-1, 7)$ **75.** $\left(\frac{1}{2}, \frac{41}{4}\right)$ **76.** $\left(-\frac{1}{3}, -\frac{22}{3}\right)$

77. a. $(2, -3)$ **b.** $(0, 0)$, $(4, 0)$
c.



78. a. $(-2, 4)$ **b.** $(0, 0)$, $(-4, 0)$
c.



79. a. 3 sec **b.** 144 ft **80. a.** 150 meals **b.** \$1200
81. $a = 1, b = 4, c = -1$; $y = x^2 + 4x - 1$
82. $a = 1, b = -1, c = 6$; $y = x^2 - x + 6$
83. a. $\{-2, 2\}$; The x-intercepts are $(-2, 0)$ and $(2, 0)$.
b. On the interval $(-2, 2)$ the graph is below the x-axis.
c. On the intervals $(-\infty, -2)$ and $(2, \infty)$ the graph is above the x-axis.
84. a. $x = 2$ **b.** $\{0\}$ **c.** $(-\infty, 0] \cup (2, \infty)$ **d.** $[0, 2)$
85. $(-2, 6)$ **86.** $(-\infty, \infty)$ **87.** $(-\infty, -2) \cup [0, \infty)$
88. $(-\infty, -1] \cup (1, \infty)$ **89.** $(-2, 0) \cup (5, \infty)$

90. $(-2, 0) \cup \left(\frac{5}{2}, \infty\right)$

91. $(-\infty, -2 - \sqrt{3}] \cup [-2 + \sqrt{3}, \infty)$
92. $(-\infty, -5) \cup (1, \infty)$ **93.** $(3, \infty)$
94. $(-3, \infty)$ **95.** $\{-5\}$ **96.** $(-\infty, -2) \cup (-2, \infty)$

Chapter 7 Test, pp. 709–711

1. $\{2, -8\}$ **2.** $\{2 \pm 2\sqrt{3}\}$ **3.** $\{-1 \pm i\}$

4. $n = \frac{121}{4}$; $\left(d + \frac{11}{2}\right)^2$ **5.** $\{-3 \pm 3\sqrt{3}\}$

6. $\left\{\frac{3}{4} \pm \frac{\sqrt{47}}{4}i\right\}$ **7. a.** $x^2 - 3x + 12 = 0$
b. $a = 1, b = -3, c = 12$ **c.** -39 **d.** Two imaginary solutions
8. a. $y^2 - 2y + 1 = 0$ **b.** $a = 1, b = -2, c = 1$ **c.** 0
d. One rational solution

9. $\left\{1, \frac{1}{3}\right\}$ **10.** $\left\{\frac{-7 \pm \sqrt{5}}{2}\right\}$

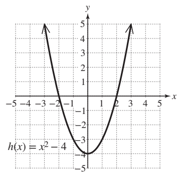
11. The height is approximately 4.6 ft and the base is 6.2 ft.
12. The radius is approximately 12.0 ft.
13. $\{9\}$ (The value 4 does not check.)

14. $\{8, -64\}$ **15.** $\left\{\frac{11}{3}, 6\right\}$

16. $\{\pm\sqrt{6}, \pm 3\}$ **17.** $\left\{\frac{5 \pm \sqrt{57}}{2}\right\}$ **18.** $\left\{5, -\frac{1}{2}\right\}$

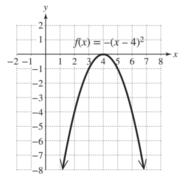
19. $\{4 \pm \sqrt{15}\}$ **20.** $\{-7 \pm 2i\sqrt{6}\}$ **21.** $\{6 \pm \sqrt{23}\}$
22. The vertex is $(3, -17)$.

23.



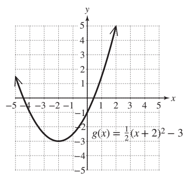
Domain: $(-\infty, \infty)$; range: $[-4, \infty)$

24.



Domain: $(-\infty, \infty)$; range: $(-\infty, 0]$

25.



Domain: $(-\infty, \infty)$; range: $[-3, \infty)$

26. The rocket will hit the ground 256 ft away.
27. a. 1320 million **b.** 1999 **28.** The graph of $y = x^2 - 2$ is the graph of $y = x^2$ shifted down 2 units. **29.** The graph of $y = (x + 3)^2$ is the graph of $y = x^2$ shifted 3 units to the left.

30. The graph of $y = -4x^2$ is the graph of $y = 4x^2$ opening downward instead of upward.
31. a. (4, 2) **b.** Downward **c.** Maximum point
d. The maximum value is 2. **e.** $x = 4$
32. a. $g(x) = 2(x - 5)^2 + 1$; (5, 1) **b.** (5, 1)
33. a. $f(x) = (x + 2)^2 - 16$. **b.** Vertex: $(-2, -16)$
c. x-intercepts: $(-6, 0)$ and $(2, 0)$; y-intercept: $(0, -12)$
d. The minimum value is -16. **e.** $x = -2$
34. a. 200 ft **b.** 20,000 ft^2 **35.** $\left[\frac{1}{2}, 6\right)$ **36.** $(-5, 5)$
37. $(-\infty, -3) \cup (-2, 2)$ **38.** $\left(-3, -\frac{3}{2}\right)$
39. { } **40.** $\{-11\}$

Chapter 8

Section 8.1 Activity, p. 718

A.1. a. $f(0) = -4$ **b.** $g(0) = 2$ **c.** $f(0) + g(0) = -2$
d. $(f + g)(1) = 1$ **e.** $(f + g)(-2) = -2$
A.2. a. $(f - g)(x) = f(x) - g(x)$ **b.** $(f \cdot g)(x) = f(x) \cdot g(x)$
c. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
A.3. a. Substitute the quantity $x - 4$ for all values of x in function f. Then simplify the result.
b. $f(x - 4) = x^2 - 9x + 8$ **c.** $g(x^2 - x - 12) = x^2 - x - 16$
A.4. a. $(f \circ g)(x) = f(g(x))$; $(g \circ f)(x) = g(f(x))$;
b. $(f \circ g)(x) = x^2 - 9x + 8$ **c.** $(g \circ f)(x) = x^2 - x - 16$
A.5. a. $(h \circ k)(4) = h[k(\boxed{4})] = h(\boxed{-1}) = -7$
b. $(k \circ h)(0) = k[h(\boxed{0})] = k(\boxed{-4}) = -\frac{1}{9}$
A.6. a. $(h \circ k)(6) = -1$
b. $(h \circ k)(5)$ is undefined because $k(5)$ is undefined.
c. $(k \circ h)(1) = -\frac{1}{6}$
d. $(k \circ h)(3)$ is undefined because $h(3) = 5$, but 5 is not in the domain of k.
A.7. a. $(f \cdot g)(-1) = 0$ **b.** $(g - f)(-3) = -9$ **c.** $\left(\frac{f}{g}\right)(0) = -2$
d. The value $f(-2) = 0$. Therefore, the quotient $\frac{g(-2)}{f(-2)}$ is undefined because division by 0 is undefined.
e. $(f \circ g)(-1) = -4$ **f.** $(g \circ f)(-2) = 2$

Section 8.1 Practice Exercises, pp. 718–721

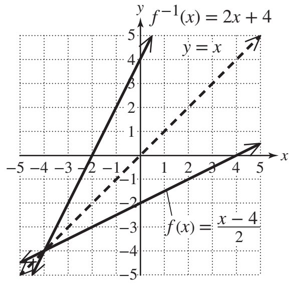
R.1. $9x^2 - 3x - 8$ **R.3.** $-2x^4 + 5x^2 - 6$ **R.5.** $\frac{4}{x - 1}$
R.7. $-x$ **R.9.** $(-\infty, \infty)$ **R.11.** $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
R.13. $(-\infty, \infty)$ **R.15.** $(-\infty, 2]$ **R.17.** 8 **R.19.** -6

1. a. $f(x)$; $g(x)$ **b.** $g(x)$; $g(x)$ **c.** $f(g(x))$
3. $(f + g)(x) = 2x^2 + 5x + 4$
5. $(g - f)(x) = 2x^2 + 3x - 4$
7. $(f \cdot h)(x) = x^3 + 4x^2 + x + 4$
9. $(g \cdot f)(x) = 2x^3 + 12x^2 + 16x$
11. $\left(\frac{h}{f}\right)(x) = \frac{x^2 + 1}{x + 4}, x \neq -4$
13. $\left(\frac{f}{g}\right)(x) = \frac{x + 4}{2x^2 + 4x}, x \neq 0, x \neq -2$
15. $(f \circ g)(x) = 2x^2 + 4x + 4$
17. $(g \circ f)(x) = 2x^2 + 20x + 48$ **19.** $(k \circ h)(x) = \frac{1}{x^2 + 1}$
21. $(k \circ g)(x) = \frac{1}{2x^2 + 4x}, x \neq 0, x \neq -2$ **23.** No
25. $(f \circ g)(x) = 25x^2 - 15x + 1$; $(g \circ f)(x) = 5x^2 - 15x + 5$
27. $(f \circ g)(x) = |x^3 - 1|$; $(g \circ f)(x) = |x|^3 - 1$
29. $(h \circ h)(x) = 25x - 24$ **31.** 0 **33.** -64 **35.** 2

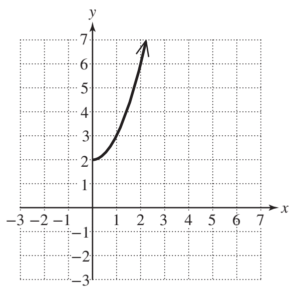
37. 1 **39.** $\frac{1}{64}$ **41.** 0 **43.** Not a real number
45. $\frac{4}{9}$ **47.** -2 **49.** 2 **51.** 0 **53.** 1 **55.** 0
57. Undefined **59.** -1 **61.** 2 **63.** 2 **65.** -1
67. -2 **69.** 3 **71.** -6 **73.** -1 **75.** 4
77. 0 **79.** 2 **81. a.** $P(x) = 3.78x - 1$ **b.** \$188
83. a. $F(t) = 0.2t + 6.4$; $F(t)$ represents the amount of child support (in billion dollars) not paid for year t. **b.** $F(4) = 7.2$ means that in year 4, \$7.2 billion of child support was not paid.
85. a. $(D \circ r)(t) = 560t$; This function represents the total distance Joe travels as a function of time that he rides. **b.** 5600 ft

Section 8.2 Activity, p. 728

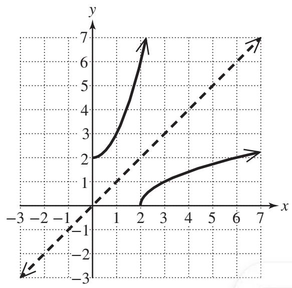
A.1. one-to-one
A.2. horizontal
A.3. Yes
A.4. No; the ordered pairs $(5, -3)$ and $(-1, -3)$ have different x-coordinates, but the same y-coordinate. These ordered pairs fail the *horizontal* line test.
A.5. a. $\{(-4, 2), (1, -3), (-2, 0), (3, -4)\}$ **b.** Yes
A.6. a. $\{(1, 4), (-3, 5), (-3, -1), (5, -2)\}$ **b.** No; the ordered pairs $(-3, 5)$ and $(-3, -1)$ have the same x-coordinate but different y-coordinates (these ordered pairs fail the *vertical* line test).
A.7. a. The graph of f is a line with slope $\frac{1}{2}$ and y-intercept $(0, -2)$.
b. Yes; yes
A.8. a. $y = \frac{x - 4}{2}$ **b.** $x = \frac{y - 4}{2}$ **c.** $y = 2x + 4$ **d.** $f^{-1}(x) = 2x + 4$
e. $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$
A.9. a.



b. The graphs of a function and its inverse are symmetric with respect to the line $y = x$.
A.10. a.



b. Yes **c.** $f^{-1}(x) = \sqrt{x - 2}$
d. Yes



Section 8.2 Practice Exercises, pp. 729–733

R.1. Yes **R.3.** $x = -\frac{y+4}{3}$ or $x = -\frac{1}{3}y - \frac{4}{3}$ **R.5.** $x = y^3 + 1$

R.7. **a.** $f(1) = 6$ **b.** $f(-2) = 0$ **c.** $f(t) = 2t + 4$ **d.** $f\left(\frac{x-4}{2}\right) = x$

R.9. $(f \circ g)(x) = 4x^2 + 17$

1. **a.** $\{(2, 1), (3, 2), (4, 3)\}$ **b.** one-to-one; y **c.** is not **d.** is **e.** $y = x$

3. $g^{-1} = \{(5, 3), (1, 8), (9, -3), (2, 0)\}$ **5.** $r^{-1} = \{(3, a), (6, b), (9, c)\}$

7. The function is not one-to-one. **9.** Yes **11.** No **13.** Yes

15. **a.** $(f \circ g)(x) = 6\left(\frac{x-1}{6}\right) + 1 = x$ **b.** $(g \circ f)(x) = \frac{(6x+1)-1}{6} = x$

17. **a.** $(f \circ g)(x) = \frac{\sqrt[3]{8x^3}}{2} = x$ **b.** $(g \circ f)(x) = 8\left(\frac{\sqrt[3]{x}}{2}\right)^3 = x$

19. **a.** $(f \circ g)(x) = (\sqrt{x-1})^2 + 1 = x$ **b.** $(g \circ f)(x) = \sqrt{(x^2+1)-1} = x$

21. subtracts; $x - 4$ **23.** divides; $\frac{x}{5}$

25. **a.** $f(x) = 9x - 2$ **b.** $f^{-1}(x) = \frac{x+2}{9}$ **27.** $h^{-1}(x) = x - 4$

29. $m^{-1}(x) = 3(x + 2)$ **31.** $p^{-1}(x) = -x + 10$ **33.** $n^{-1}(x) = \frac{5x-2}{3}$

35. $h^{-1}(x) = \frac{3x+1}{4}$ **37.** $f^{-1}(x) = \sqrt[3]{x-1}$

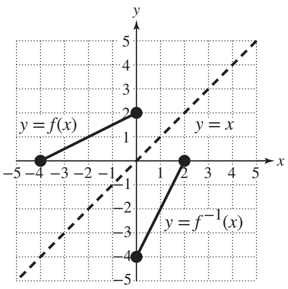
39. $g^{-1}(x) = \frac{x^3+1}{2}$ **41.** $g^{-1}(x) = \sqrt{x-9}$

43. **a.** 1.2192 m, 15.24 m **b.** $f^{-1}(x) = \frac{x}{0.3048}$ **c.** 4921.3 ft

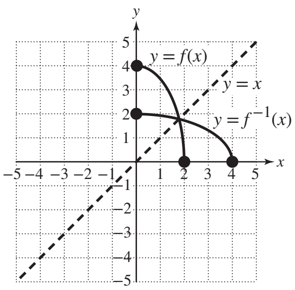
45. False **47.** True **49.** False **51.** True **53.** $(b, 0)$

55. **a.** Domain: $[1, \infty)$, range: $[0, \infty)$ **b.** Domain: $[0, \infty)$, range: $[1, \infty)$

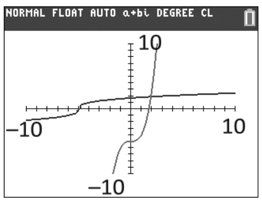
57. **a.** $[-4, 0]$ **b.** $[0, 2]$ **c.** $[0, 2]$ **d.** $[-4, 0]$



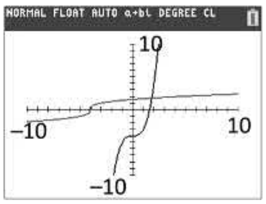
59. **a.** $[0, 2]$ **b.** $[0, 4]$ **c.** $[0, 4]$ **d.** $[0, 2]$



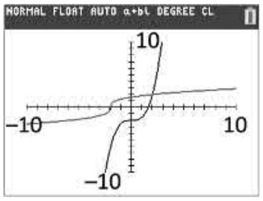
61. $f^{-1}(x) = x^3 - 5$



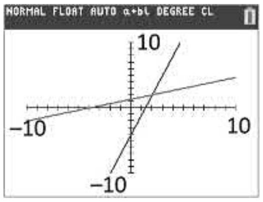
62. $k^{-1}(x) = \sqrt[3]{x+4}$



63. $g^{-1}(x) = \sqrt[3]{2x+4}$



64. $m^{-1}(x) = \frac{x+4}{3}$



65. $q^{-1}(x) = x^2 - 4, x \geq 0$ **67.** $z^{-1}(x) = x^2 - 4, x \leq 0$

69. $f^{-1}(x) = \frac{x+1}{1-x}$ **71.** $t^{-1}(x) = \frac{x+2}{x}$ **73.** $n^{-1}(x) = -\sqrt{x-9}$

Section 8.3 Activity, pp. 739–740

A.1. exponential **A.2.** b, c

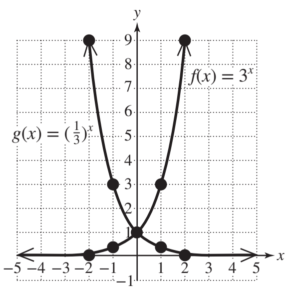
A.3. **a.**

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
-1	$\frac{1}{3}$	3
-2	$\frac{1}{9}$	9

b. For each 1-unit increase in x, the function values (y values) increase by a factor of 3.

c. For each 1-unit increase in x, the function values (y values) decrease by a factor of 3.

d.



e. Exponential growth **f.** Exponential decay **g.** No

h. Domain: $(-\infty, \infty)$; range: $(0, \infty)$

A.4. **a.** $A(8) = 32$ and means that after 8 days, 32 units of ^{131}I is left.

b. $A(16) = 16$ and means that after 16 days, 16 units of ^{131}I is left.

c. $A(24) = 8$ and means that after 24 days, 8 units of ^{131}I is left.

d. After each half-life (each increment of 8 days), the amount of substance is cut in half.

Section 8.3 Practice Exercises, pp. 740–743

R.1. **a.** 16 **b.** 1 **c.** $\frac{1}{16}$ **R.3.** **a.** $\frac{1}{16}$ **b.** 1 **c.** 16

1. **a.** b^x **b.** is not; is **c.** increasing **d.** decreasing **e.** $(-\infty, \infty)$; $(0, \infty)$ **f.** $(0, 1)$ **g.** $y = 0$

3. 5.8731 **5.** 1385.4557 **7.** 0.0063 **9.** 0.8950

11. **a.** $x = 2$ **b.** $x = 3$ **c.** Between 2 and 3

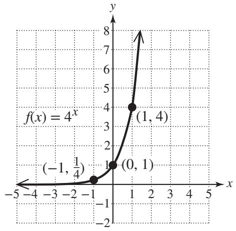
13. **a.** $x = 4$ **b.** $x = 5$ **c.** Between 4 and 5

15. $f(0) = 1, f(1) = \frac{1}{5}, f(2) = \frac{1}{25}, f(-1) = 5, f(-2) = 25$

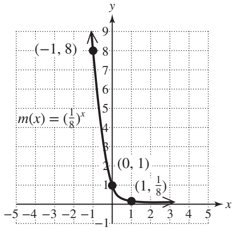
17. $h(0) = 1, h(1) = 3, h(-1) = \frac{1}{3}, h(\sqrt{2}) \approx 4.73, h(\pi) \approx 31.54$

19. If $b > 1$, the graph is increasing. If $0 < b < 1$, the graph is decreasing.

21.



23.



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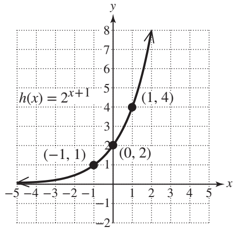

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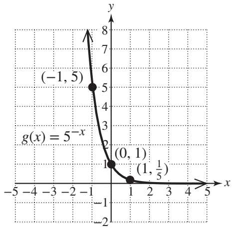
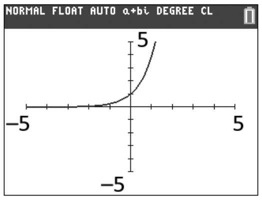


29. a. 0.25 g **b.** ≈ 0.16 g **31. a.** 758,000 **b.** 379,000 **c.** 144,000 **33. a.** $P(t) = 153{,}000{,}000(1.0125)^t$ **b.** $P(41) \approx 255{,}000{,}000$ **35. a.** \$1640.67 **b.** \$2691.80 **c.** $A(0) = 1000$. The initial amount of the investment is \$1000. $A(7) = 2000$. The amount of the investment doubles in 7 years.

37.



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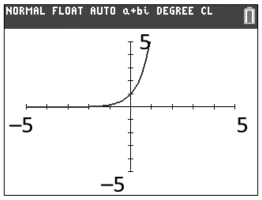

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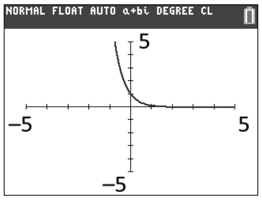

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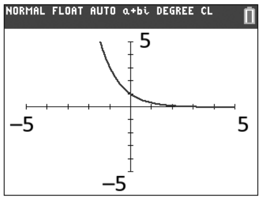

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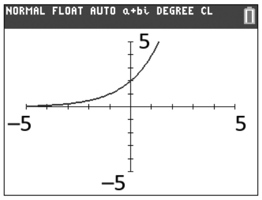

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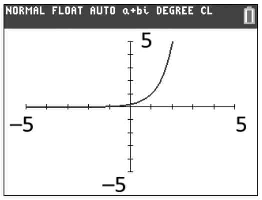

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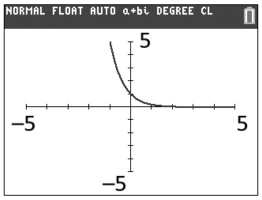

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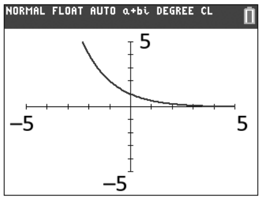


Section 8.4 Activity, pp. 752–753

A.1. a. 8 **b.** 3 **c.** logarithmic; $b^y = x$.

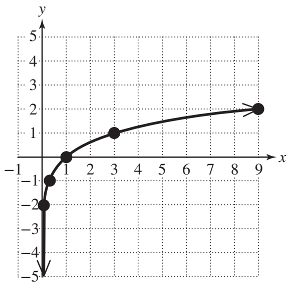
	Exponential Form	Logarithmic Form
A.2.	$4^2 = 16$	$\log_4 16 = 2$
A.3.	$3^4 = 81$	$\log_3 81 = 4$
A.4.	$2^7 = x$	$\log_2 x = 7$
A.5.	$5^y = 125$	$\log_5 125 = y$
A.6.	$6^1 = 6$	$\log_6 6 = 1$
A.7.	$7^0 = 1$	$\log_7 1 = 0$
A.8.	$2^{-4} = \frac{1}{16}$	$\log_2 \frac{1}{16} = -4$
A.9.	$10^{-2} = \frac{1}{100}$	$\log_{10} \frac{1}{100} = -2$

A.10. 5 **A.11.** 2 **A.12.** −2 **A.13.** −3 **A.14.** 0 **A.15.** 1 **A.16. a.** 3 **b.** 4 **c.** Between 3 and 4

A.17. a. $3^y = x$

b.

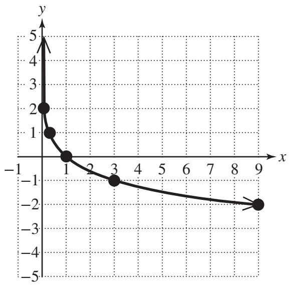
x	y
$\frac{1}{9}$	−2
$\frac{1}{3}$	−1
1	0
3	1
9	2



A.18. a. $\left(\frac{1}{3}\right)^y = x$

b.

x	y
9	−2
3	−1
1	0
$\frac{1}{3}$	1
$\frac{1}{9}$	2



A.19. $(0, \infty)$ **A.20.** $(-\infty, \infty)$ **A.21.** $x = 0$ (the y-axis) **A.22.** (1, 0) **A.23. a.** increases **b.** decreases **A.24.** $(0, \infty)$ **A.25.** $(4, \infty)$ **A.26.** $(-\infty, 4)$ **A.27.** $\left(-\frac{1}{2}, \infty\right)$

Section 8.4 Practice Exercises, pp. 753–757

R.1. 5 **R.3.** 16 **R.5.** $\frac{1}{64}$ **R.7.** 2 **R.9.** −4 **R.11.** −3 **R.13.** −2 **R.15.** $(3, \infty)$ **R.17.** $(-\infty, 6)$

1. a. logarithmic; b **b.** logarithm; base; argument **c.** $(0, \infty)$; $(-\infty, \infty)$ **d.** exponential **e.** common **f.** increasing; decreasing **g.** 2, 3, and 4 **h.** $y = x$

3. $b^y = x$ **5. a.** $5^{\boxed{2}} = 25$ **b.** $\log_5 25 = \boxed{2}$

7. a. $3^{\boxed{3}} = 27$ **b.** $\log_3 27 = \boxed{3}$ **9. a.** $8^{\boxed{1}} = 8$ **b.** $\log_8 8 = \boxed{1}$

11. $5^4 = 625$ **13.** $10^{-4} = 0.0001$ **15.** $6^2 = 36$

17. $b^x = 15$ **19.** $3^x = 5$ **21.** $\left(\frac{1}{4}\right)^{10} = x$ **23.** $\log_3 81 = x$

25. $\log_5 25 = 2$ **27.** $\log_7\left(\frac{1}{7}\right) = -1$ **29.** $\log_b y = x$

31. $\log_e y = x$ **33.** $\log_{1/3} 9 = -2$ **35.** 2 **37.** -1 **39.** $\frac{1}{2}$

41. 0 **43.** 5 **45.** 1 **47.** 3 **49.** $\frac{1}{2}$ **51.** 1 **53.** 3

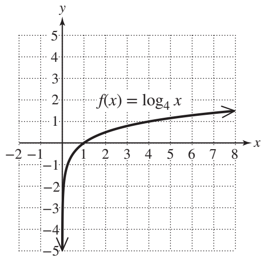
55. 6 **57.** -2 **59.** 0.7782 **61.** 0.4971 **63.** -1.5051

65. -2.2676 **67.** 5.5315 **69.** -7.4202

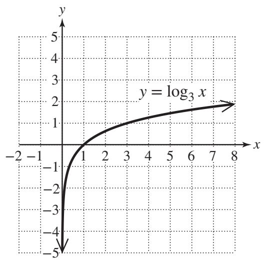
71. a. Slightly less than 2 **b.** Slightly more than 1
c. $\log 93 \approx 1.9685$, $\log 12 \approx 1.0792$

73. a. $f\left(\frac{1}{64}\right) = -3, f\left(\frac{1}{16}\right) = -2, f\left(\frac{1}{4}\right) = -1,$
$f(1) = 0, f(4) = 1, f(16) = 2, f(64) = 3$

b.

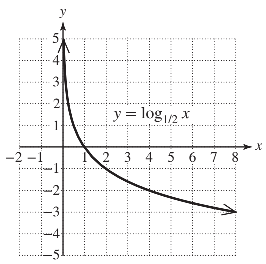


75. $3^y = x$



x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

77. $\left(\frac{1}{2}\right)^y = x$



x	y
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2

79. $(5, \infty)$ **81.** $(-\infty, 2)$ **83.** $\left(\frac{1}{3}, \infty\right)$ **85.** $(-1.2, \infty)$

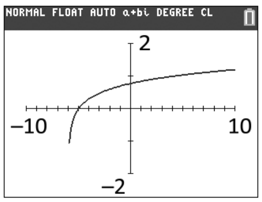
87. $(-\infty, 2)$ **89.** $(-\infty, 0) \cup (0, \infty)$ **91.** ≈ 7.35

93. a.

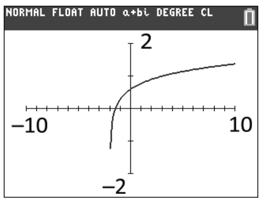
t (months)	0	1	2	6	12	24
$S_1(t)$	91	82.0	76.7	65.6	57.6	49.1
$S_2(t)$	88	83.5	80.8	75.3	71.3	67.0

b. Group 1: 91; Group 2: 88 **c.** Method II

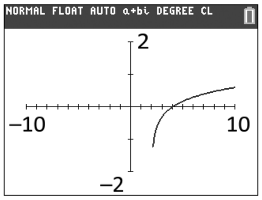
95. Domain: $(-6, \infty)$;
asymptote: $x = -6$



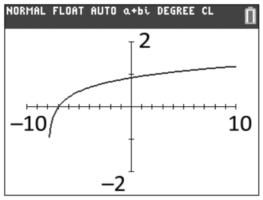
96. Domain: $(-2, \infty)$;
asymptote: $x = -2$



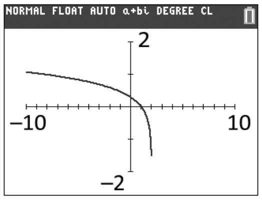
97. Domain: $(2, \infty)$;
asymptote: $x = 2$



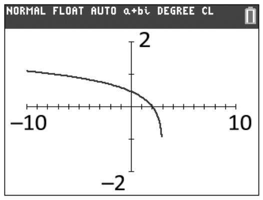
98. Domain: $(-8, \infty)$;
asymptote: $x = -8$



99. Domain: $(-\infty, 2)$;
asymptote: $x = 2$



100. Domain: $(-\infty, 3)$;
asymptote: $x = 3$



Chapter 8 Problem Recognition Exercises, p. 758

1. e **2.** l **3.** j **4.** g **5.** c **6.** a
7. k **8.** b **9.** i **10.** h **11.** f **12.** d

Section 8.5 Activity, p. 764

A.1. a. 1 **b.** 1 **c.** 1 **d.** 1 **A.2. a.** 0 **b.** 0 **c.** 0 **d.** 0

A.3. a. 3 **b.** 4 **c.** 5 **d.** p **A.4. a.** 25 **b.** 16 **c.** $\frac{1}{8}$ **d.** x

A.5. $\boxed{3} = \boxed{2} + \boxed{1}$; $\log_b(x \cdot y) = \log_b x + \log_b y$

A.6. $\boxed{2} = \boxed{3} - \boxed{1}$; $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

A.7. $\boxed{3} = \boxed{3} \cdot \boxed{1}$; $\log_b(x)^p = p \cdot \log_b x$

A.8. $\log_2\left(\frac{x^5\sqrt{z}}{y^2\sqrt[3]{w}}\right)$ **A.9.** $2 + \frac{1}{2}\log x + 3\log y - \log z$

Section 8.5 Practice Exercises, pp. 765–767

R.1. x^6 **R.3.** w^{11} **R.5.** n^{12} **R.7.** $\frac{1}{t^6}$

R.9. $\frac{9b^6}{a^{10}}$ **R.11.** 3 **R.13.** -4 **R.15.** 3

1. a. 1; 0; x; x **b.** $\log_b x + \log_b y$; $\log_b x - \log_b y$ **c.** $p \log_b x$
d. False **e.** False **f.** False

3. a, b, c **5.** 1 **7.** 4 **9.** 11 **11.** 3 **13.** 0

15. 9 **17.** 0 **19.** 3 **21.** 1 **23.** $2x$ **25.** 4 **27.** 0

29. Expressions a and c are equivalent.

31. Expressions a and c are equivalent.

33. $\log_3 x - \log_3 5$ **35.** $\log 2 + \log x$ **37.** $4\log_5 x$

39. $\log_4 a + \log_4 b - \log_4 c$

41. $\frac{1}{2}\log_b x + \log_b y - 3\log_b z - \log_b w$

43. $\log_2(x+1) - 2\log_2 y - \frac{1}{2}\log_2 z$

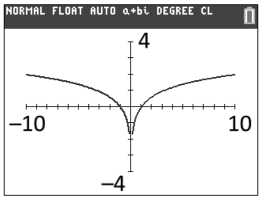
45. $\frac{1}{3}\log a + \frac{2}{3}\log b - \frac{1}{3}\log c$ **47.** $-5\log w$

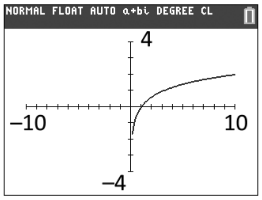
49. $\frac{1}{2}\log_b a - 3 - \log_b c$ **51.** 3 **53.** 2 **55.** $\log_3\left(\frac{x^2 z}{y^3}\right)$

57. $\log_3\left(\frac{a^2 c}{\sqrt[4]{b}}\right)$ **59.** $\log_b x^2$ **61.** $\log_8(8a^5)$ or $\log_8 a^5 + 1$

63. $\log\left[\frac{(x+6)^2\sqrt[3]{y}}{z^5}\right]$ **65.** $\log_b\left(\frac{1}{x-1}\right)$

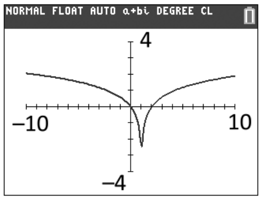
67. **a.** Domain: $(-\infty, 0) \cup (0, \infty)$ **b.** Domain: $(0, \infty)$

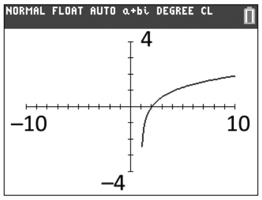




c. They are equivalent for all x in the intersection of their domains, $(0, \infty)$.

68. **a.** Domain: $(-\infty, 1) \cup (1, \infty)$ **b.** Domain: $(1, \infty)$





c. They are equivalent for all x in the intersection of their domains, $(1, \infty)$.

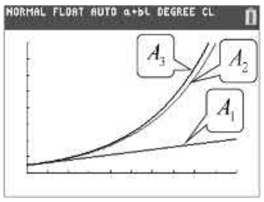
69. 1.792 **71.** 2.485 **73.** 4.396 **75.** 0.916 **77.** 13.812
79. 16.09 **81.** **a.** $B = 10 \log I - 10 \log I_0$ **b.** $10 \log I + 160$

Section 8.6 Activity, pp. 775–776

A.1. **a.** 2.71801 **b.** 2.71825 **c.** 2.71828 **d.** 2.71828

A.2. **a.** 2.71828 **b.** Growth

A.3. **a.** \$17,000; after 30 years of simple interest, \$5000 will grow to \$17,000.
b. \$50,313.28; after 30 years of interest compounded annually, \$5000 will grow to \$50,313.28.
c. \$55,115.88; after 30 years of continuous compounded interest, \$5000 will grow to \$55,115.88.
d. Continuous compounding **e.** \$38,115.88
f.



A.4. logarithmic; $y = \ln x$

A.5. **a.** Domain: $(-\infty, \infty)$; range: $(0, \infty)$
b. Domain: $(0, \infty)$; range: $(-\infty, \infty)$
c. The domain of f is the range of g, and the range of f is the domain of g.

A.6. **a.** 0 **b.** 1 **c.** 4 **d.** 6 **A.7.** $\ln\left(\frac{x+y}{z^3}\right)$

A.8. $2 \ln a - \ln b - \frac{1}{3} \ln c$ **A.9.** **a.** $(-4, \infty)$ **b.** $(-\infty, 6)$

A.10. **a.** $\log_b x = \dfrac{\log_a x}{\log_a b}$ **b.** 4 **c.** 5 **d.** Between 4 and 5
e. 4.3219 **f.** 4.3219 **g.** They are the same.

Section 8.6 Practice Exercises, pp. 776–782

R.1. **a.** 2.717603 **b.** 2.718281

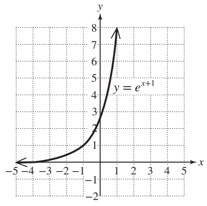
R.3. 0 **R.5.** 1 **R.7.** 5 **R.9.** 3

R.11. $\log_5\left(\dfrac{\sqrt[3]{x} \cdot y}{z^2}\right)$ **R.13.** $\frac{1}{2} \log x - \log y - 4 \log z$

1. **a.** e **b.** e **c.** natural; $\ln x$ **d.** 0; 1; p, x **e.** $\ln x + \ln y$; $\ln x - \ln y$ **f.** $p \ln x$ **g.** $\log_a b$ **3.** e **5.** increasing

7.

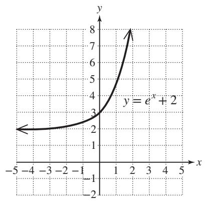
x	y
−4	0.05
−3	0.14
−2	0.37
−1	1
0	2.72
1	7.39



Domain: $(-\infty, \infty)$

9.

x	y
−2	2.14
−1	2.37
0	3
1	4.72
2	9.39
3	22.09



Domain: $(-\infty, \infty)$

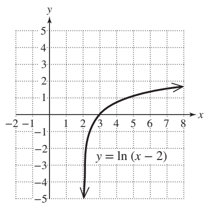
11. **a.** \$12,209.97 **b.** \$13,488.50 **c.** \$14,898.46 **d.** \$16,050.09
An investment grows more rapidly at higher interest rates.

13. **a.** \$12,423.76 **b.** \$12,515.01 **c.** \$12,535.94
d. \$12,546.15 **e.** \$12,546.50
More money is earned at a greater number of compounding periods per year.

15. **a.** \$6920.15 **b.** \$9577.70 **c.** \$13,255.84
d. \$18,346.48 **e.** \$35,143.44
More money is earned over a longer period of time.

17.

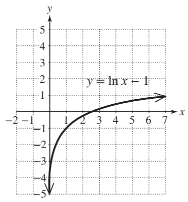
x	y
2.25	−1.39
2.50	−0.69
2.75	−0.29
3	0
4	0.69
5	1.10
6	1.39



Domain: $(2, \infty)$

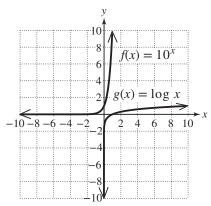
19.

x	y
0.25	−2.39
0.5	−1.69
0.75	−1.29
1	−1.00
2	−0.31
3	0.10
4	0.39



Domain: $(0, \infty)$

21. **a.**



b. Domain: $(-\infty, \infty)$; range: $(0, \infty)$
c. Domain: $(0, \infty)$; range: $(-\infty, \infty)$

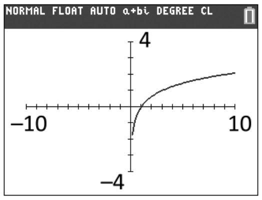
23. 1 **25.** 0 **27.** −6 **29.** $2x + 3$ **31.** $\ln(p^6 \sqrt[3]{q})$

33. $\ln\sqrt{\dfrac{x}{y^3}}$ **35.** $\ln\left(\dfrac{a^2}{b\sqrt[3]{c}}\right)$ **37.** $\ln\left(\dfrac{x^4}{y^3 z}\right)$

39. $2 \ln a - 2 \ln b$ **41.** $2 \ln b + 1$ **43.** $4 \ln a + \frac{1}{2} \ln b - \ln c$

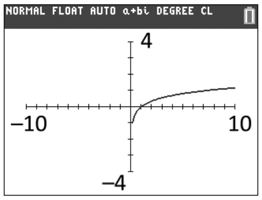
45. $\frac{1}{5} \ln a + \frac{1}{5} \ln b - \frac{2}{5} \ln c$ **47.** $(4, \infty)$ **49.** $\left(-\frac{5}{2}, \infty\right)$

51. $(-\infty, 7)$ **53. a.** 2.9570 **b.** 2.9570 **c.** They are the same.
55. 2.8074 **57.** 1.5283 **59.** −2.1269 **61.** 0
63. −3.3219 **65.** −3.8124 **67. a.** 15.4 years **b.** 6.9 years **c.** 13.8 years **69. a.** 19.8 years **b.** 13.9 years **c.** 27.8 years
71. a–b



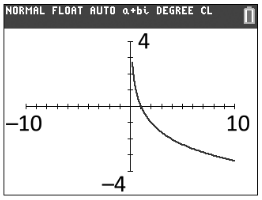
c. They appear to be the same.

72. a–b

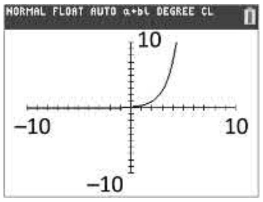


c. They appear to be the same.

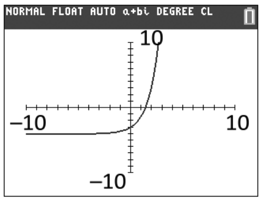
73.



74.



75.



Chapter 8 Problem Recognition Exercises, p. 782

	Exponential Form	Logarithmic Form
1.	$2^5 = 32$	$\log_2 32 = 5$
2.	$3^4 = 81$	$\log_3 81 = 4$
3.	$z^y = x$	$\log_z x = y$
4.	$b^c = a$	$\log_b a = c$
5.	$10^3 = 1000$	$\log 1000 = 3$
6.	$10^1 = 10$	$\log 10 = 1$
7.	$e^a = b$	$\ln b = a$
8.	$e^q = p$	$\ln p = q$
9.	$(\frac{1}{2})^2 = \frac{1}{4}$	$\log_{1/2}(\frac{1}{4}) = 2$
10.	$(\frac{1}{3})^{-2} = 9$	$\log_{1/3} 9 = -2$
11.	$10^{-2} = 0.01$	$\log 0.01 = -2$
12.	$10^x = 4$	$\log 4 = x$
13.	$e^0 = 1$	$\ln 1 = 0$
14.	$e^1 = e$	$\ln e = 1$
15.	$25^{1/2} = 5$	$\log_{25} 5 = \frac{1}{2}$
16.	$16^{1/4} = 2$	$\log_{16} 2 = \frac{1}{4}$
17.	$e^t = s$	$\ln s = t$
18.	$e^r = w$	$\ln w = r$
19.	$15^{-2} = \frac{1}{225}$	$\log_{15}\left(\frac{1}{225}\right) = -2$
20.	$3^{-1} = p$	$\log_3 p = -1$

Section 8.7 Activity, pp. 791–792

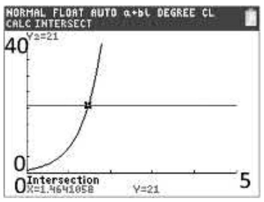
A.1. a. $x = y$ **b.** $5^{x+1} = 5^3$ **c.** $\{2\}$ **d.** $\{5\}$
A.2. a. $e^{2x+1} = 10$ **b.** $\ln e^{2x+1} = \ln 10$ **c.** $(2x+1)\ln e = \ln 10$
d. $\left\{\dfrac{-1+\ln 10}{2}\right\}$; or approximately $\{0.6513\}$
A.3. a. $\log 3^{x-4} = \log 8^x$ **b.** $(x-4)\log 3 = x\log 8$
c. $x\log 3 - x\log 8 = 4\log 3$ **d.** $x = \dfrac{4\log 3}{\log 3 - \log 8}$
e. $\left\{\dfrac{4\log 3}{\log 3 - \log 8}\right\}$; or approximately $\{-4.4803\}$
A.4. a. $x = y$ **b.** $x = 3$ **c.** { }; (The value 3 does not check.)
d. The domain of logarithmic functions is the set of positive real numbers. Therefore, any potential solution to a logarithmic equation that makes the argument to a logarithmic expression negative or zero must be discarded.
A.5. a. $\log_5 x + \log_5(x-20) = 3$ **b.** $\log_5[x(x-20)] = 3$
c. $x(x-20) = 5^3$ **d.** $x = 25, x = -5$
e. {25}; (The value −5 does not check.)
A.6. a. 1.3 μg **b.** 27 days

Section 8.7 Practice Exercises, pp. 792–796

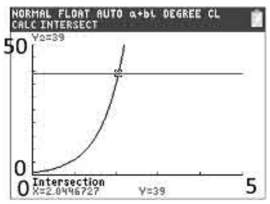
R.1. $\log_2[(x-1)(x+2)]$ **R.3.** $\log\left(\dfrac{x}{1-x}\right)$ **R.5.** 1
R.7. 6 **R.9.** $(2x+1)\cdot\log 5$ **R.11.** 4^{2x} **R.13.** 2^{5x-15}
R.15. $\{-5\}$ **R.17.** $\{-16\}$ **R.19.** $\{-6, -2\}$

1. a. $x = y$ **b.** $x = y$ **3. a.** {2} **b.** {25}
5. a. $\{-1\}$ **b.** $\left\{\dfrac{1}{10}\right\}$ **7.** {9} **9.** $\{10^{42}\}$ **11.** $\{e^{0.08}\}$
13. $\{10^{-9.2} - 40\}$ **15.** {5} **17.** {10} **19.** {25}
21. {59} **23.** {1} **25.** {0} **27.** $\left\{-\dfrac{37}{9}\right\}$
29. {3} (The value −3 does not check.) **31.** {4} **33.** {3}
35. {2} **37.** { } (The value −3 does not check.) **39.** {4}
41. $\{-6\}$ **43.** $\left\{\dfrac{1}{2}\right\}$ **45.** {2} **47.** $\left\{\dfrac{11}{12}\right\}$ **49.** {1}
51. $\left\{\dfrac{4}{19}\right\}$ **53.** $\{-2, 1\}$ **55.** $\left\{\dfrac{\ln 21}{\ln 8}\right\}$ **57.** $\{\ln 8.1254\}$
59. $\{\log 0.0138\}$ **61.** $\left\{\dfrac{\ln 15}{0.07}\right\}$ **63.** $\left\{\dfrac{\ln 3}{1.2}\right\}$
65. $\left\{\dfrac{\ln 3}{\ln 5 - \ln 3}\right\}$ **67.** $\left\{\dfrac{2\ln 2}{\ln 6 - \ln 2}\right\}$ **69.** $\left\{\dfrac{\ln 4}{0.04}\right\}$
71. $\left\{3\ln\left(\dfrac{125}{6}\right)\right\}$ **73.** $\left\{\dfrac{\ln 20}{\ln 5} + 2\right\}$
75. a. ≈ 1285 million (or 1,285,000,000) people
b. ≈ 1466.5 million (or 1,466,500,000) people
c. The year 2049 ($t \approx 50.8$)
77. a. 500 bacteria **b.** ≈ 660 bacteria **c.** ≈ 25 min
79. It will take 9.9 years for the investment to double. **81.** 5 years
83. a. 7.8 g **b.** 18.5 days **85.** The intensity of sound of heavy traffic is $10^{-3.07}$ W/m^2. **87.** It will take 38.4 years.

89.



90.



91. a. 1.42 kg **b.** No **93.** $\{10^5, 10^{-3}\}$ **95.** $\left\{\dfrac{1}{27}, \dfrac{1}{9}\right\}$

Chapter 8 Review Exercises, pp. 802–806

1. $(f-g)(x)=2x^3+9x-7$ **2.** $(f+g)(x)=-2x^3-7x-7$

3. $(f\cdot n)(x)=\dfrac{x-7}{x-2}, x\neq 2$ **4.** $(f\cdot m)(x)=x^3-7x^2$

5. $\left(\dfrac{f}{g}\right)(x)=\dfrac{x-7}{-2x^3-8x}, x\neq 0$ **6.** $\left(\dfrac{g}{f}\right)(x)=\dfrac{-2x^3-8x}{x-7}, x\neq 7$

7. $(m\circ f)(x)=(x-7)^2$ or $x^2-14x+49$ **8.** $(n\circ f)(x)=\dfrac{1}{x-9}, x\neq 9$

9. 100 **10.** $\dfrac{1}{8}$ **11.** -167 **12.** -10

13. a. $(2x+1)^2$ or $4x^2+4x+1$ **b.** $2x^2+1$ **c.** No, $f\circ g\neq g\circ f$

14. $\dfrac{1}{4}$ **15.** -3 **16.** 0 **17.** -1 **18.** 4 **19.** 1

20. No **21.** Yes **22.** $\{(5, 3), (9, 2), (-1, 0), (1, 4)\}$

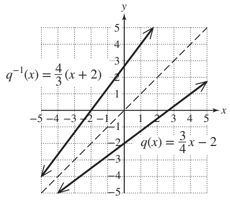
23. $q^{-1}(x)=\dfrac{4}{3}(x+2)$ **24.** $g^{-1}(x)=(x-3)^5$

25. $f^{-1}(x)=\sqrt[3]{x}+1$ **26.** $n^{-1}(x)=\dfrac{2x+4}{x}$

27. $(f\circ g)(x)=5\left(\dfrac{1}{5}x+\dfrac{2}{5}\right)-2=x+2-2=x$

$(g\circ f)(x)=\dfrac{1}{5}(5x-2)+\dfrac{2}{5}=x-\dfrac{2}{5}+\dfrac{2}{5}=x$

28. The graphs are symmetric about the line $y=x$.

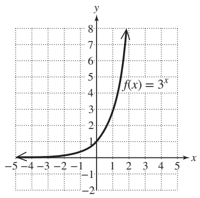


29. a. Domain: $[-1, \infty)$; range: $[0, \infty)$ **b.** Domain: $[0, \infty)$; range: $[-1, \infty)$ **30.** $p^{-1}(x)=(x-2)^2, x\geq 2$

31. 1024 **32.** $\dfrac{1}{36}\approx 0.028$ **33.** 2 **34.** 10

35. 8.825 **36.** 16.242 **37.** 1.627 **38.** 0.681

39.



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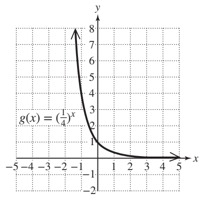

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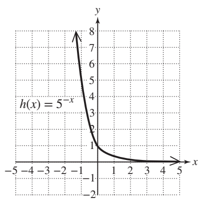

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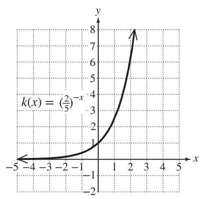
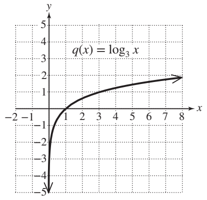


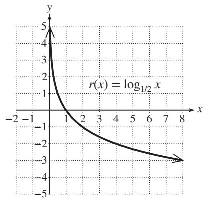
43. a. Horizontal **b.** $y=0$ **44. a.** 15,000 mrem **b.** 3750 mrem **c.** Yes **45.** -3 **46.** 0 **47.** 1

48. 8 **49.** 4 **50.** 4 **51.** 5 **52.** -1

53.



54.



55. a. Vertical asymptote **b.** $x=0$ **56. a.** 2.5 **b.** 9.5

57. 1 **58.** 6 **59.** 0 **60.** 7

61. a. $\log_b x+\log_b y$ **b.** $\log_b\left(\dfrac{x}{y}\right)$ **c.** $p\log_b x$

62. $\log_b\left(\dfrac{\sqrt[4]{x^3y}}{z}\right)$ **63.** $\log_3\left(\dfrac{\sqrt{ab}}{c^2d^4}\right)$ **64.** $\log 5$

65. 0 **66.** b **67.** a **68.** 148.4132 **69.** 14.0940

70. 32.2570 **71.** 57.2795 **72.** 1.7918 **73.** -2.1972

74. 1.3424 **75.** 1.3029 **76.** 3.3219 **77.** 1.9943

78. -0.8370 **79.** -3.6668 **80. a.** \$33,361.92 **b.** \$33,693.90 **c.** \$33,770.48 **d.** \$33,809.18

81. a. $S(0)=95$; the student's score is 95 at the end of the course. **b.** $S(6)\approx 23.7$; the student's score is 23.7 after 6 months. **c.** $S(12)\approx 20.2$; the student's score is 20.2 after 1 year.

82. $(-\infty, \infty)$ **83.** $(-\infty, \infty)$ **84.** $(-\infty, \infty)$

85. $(0, \infty)$ **86.** $(-5, \infty)$ **87.** $(7, \infty)$ **88.** $\left(\dfrac{4}{3}, \infty\right)$

89. $(-\infty, 5)$ **90.** $\{125\}$ **91.** $\left\{\dfrac{1}{49}\right\}$ **92.** $\{216\}$

93. $\{3^{1/12}\}$ **94.** $\left\{\dfrac{1001}{2}\right\}$ **95.** $\{9\}$

96. $\{5\}$ (The value -2 does not check.) **97.** $\left\{\dfrac{190}{321}\right\}$

98. $\{-1\}$ **99.** $\left\{\dfrac{4}{7}\right\}$ **100.** $\left\{\dfrac{\ln 21}{\ln 4}\right\}$ **101.** $\left\{\dfrac{\ln 18}{\ln 5}\right\}$

102. $\{-\ln 0.1\}$ **103.** $\left\{-\dfrac{\ln 0.06}{2}\right\}$ **104.** $\left\{\dfrac{\log 1512}{2}\right\}$

105. $\left\{\dfrac{\log 821}{3}\right\}$ **106.** $\left\{\dfrac{3\ln 2}{\ln 7-\ln 2}\right\}$

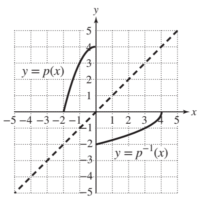
107. $\left\{\dfrac{5\ln 14}{\ln 14-\ln 6}\right\}$ **108. a.** 1.09 μg **b.** 0.15 μg **c.** 16.08 days **109. a.** 150 bacteria **b.** ≈ 185 bacteria **c.** ≈ 99 min **110. a.** $V(0)=15{,}000$; the initial value of the car is \$15,000. **b.** $V(10)=3347$; the value of the car after 10 years is \$3347. **c.** 7.3 years

Chapter 8 Test, pp. 806–808

1. A function is one-to-one if it passes the horizontal line test.

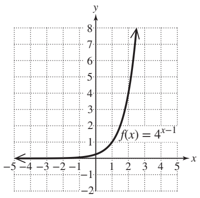
2. b **3.** $f^{-1}(x)=4x-12$ **4.** $g^{-1}(x)=\sqrt{x}+1$

5.



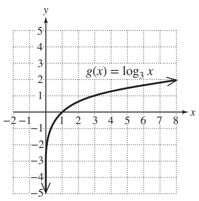
6. a. 4.6416 **b.** 32.2693 **c.** 687.2913

7.



8. a. $\log_{16} 8=\dfrac{3}{4}$ **b.** $x^5=31$

9.



10. $\dfrac{\log_a n}{\log_a b}$

11. a. 1.3222 **b.** 1.8502 **c.** -2.5850

12. a. $1+\log_3 x$ **b.** -5 **13. a.** $\log_b(\sqrt{x}y^3)$ **b.** $\log\left(\dfrac{1}{a^3}\right)$ or $-\log a^3$ **14. a.** 1.6487 **b.** 0.0498 **c.** -1.0986 **d.** 1 **15. a.** $y=\ln x$ **b.** $y=e^x$

16. a. $p(4) \approx 59.8$; 59.8% of the material is retained after 4 months. **b.** $p(12) \approx 40.7$; 40.7% of the material is retained after 1 year. **c.** $p(0) = 92$; 92% of the material is retained at the end of the course. **17. a.** 9762 thousand (or 9,762,000) people **b.** The year 2020 ($t \approx 20.4$)

18. a. $P(0) = 300$; there are 300 bacteria initially. **b.** 35,588 bacteria **c.** 1,120,537 bacteria **d.** 1,495,831 bacteria

19. $\{25\}$ (The value -4 does not check.)

20. $\{32\}$ **21.** $\{e^{2.4} - 7\}$ **22.** $\{-7\}$ **23.** $\left\{\frac{\ln 50}{\ln 4}\right\}$

24. $\left\{\frac{\ln 250}{2.4}\right\}$ **25.** $\left\{\frac{\ln 58}{\ln 2} + 3\right\}$ **26.** $\left\{\frac{7 \ln 4}{\ln 5 - \ln 4}\right\}$

27. a. $P(2500) = 560.2$; at 2500 m the atmospheric pressure is 560.2 mm Hg. **b.** 760 mm Hg **c.** 1498.8 m

28. a. \$2909.98 **b.** 9.24 years to double

29. $\left(\frac{f}{g}\right)(x) = \frac{x-4}{x^2+2}$ **30.** $(h \cdot g)(x) = \frac{x^2+2}{x}, x \neq 0$

31. $(g \circ f)(x) = x^2 - 8x + 18$ **32.** $(h \circ f)(x) = \frac{1}{x-4}, x \neq 4$

33. -48 **34.** $-\frac{3}{2}$ **35.** $\frac{1}{18}$ **36.** 18

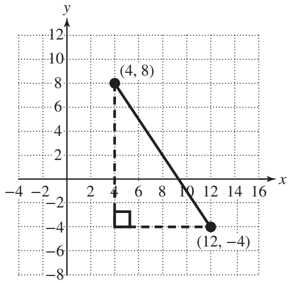
37. $\left(\frac{g}{f}\right)(x) = \frac{x^2+2}{x-4}, x \neq 4$

Chapter 9

Section 9.1 Activity, pp. 815–816

A.1. a. Cyclist 1: (4, 8); Cyclist 2: (12, -4)

b.



c. Horizontal distance: 8 mi; vertical distance: 12 mi

d. $4\sqrt{13}$ mi $\approx$ 14.4 mi

A.2. a. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **b.** $4\sqrt{13}$ mi

c. horizontal; vertical **d.** $4\sqrt{13}$ mi; No

A.3. a. Add the numbers and divide by 2.

b. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ **c.** (8, 2)

A.4. a. center; radius **b.** $(x - h)^2 + (y - k)^2 = r^2$

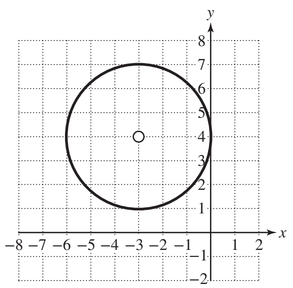
A.5. Center: (5, -2); radius: 2

A.6. Center: $\left(-\frac{3}{2}, 0\right)$; radius: $2\sqrt{5}$

A.7. a. $x^2 + 6x + 9 + y^2 - 8y + 16 = -16 + 9 + 16$

b. $(x + 3)^2 + (y - 4)^2 = 9$ **c.** Center: $(-3, 4)$; radius: 3

d.



A.8. $x^2 + (y + 4)^2 = 49$

Section 9.1 Practice Exercises, pp. 816–821

R.1. $5\sqrt{2}$ **R.3.** $\frac{3}{4}$ **R.5.** $\frac{7}{8}$

R.7. $x^2 + 16x + 64$ **R.9.** $n = 36$; $(x - 6)^2$

1. a. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **b.** circle; center **c.** radius

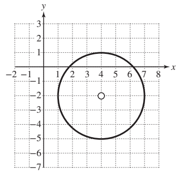
d. $(x - h)^2 + (y - k)^2 = r^2$ **e.** $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

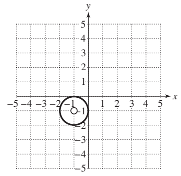
3. $3\sqrt{5}$ **5.** $\sqrt{34}$ **7.** $\frac{\sqrt{73}}{8}$ **9.** 4 **11.** 8

13. $4\sqrt{2}$ **15.** $\sqrt{42}$ **17.** Subtract 5 and -7. This becomes $5 - (-7) = 12$. **19.** $y = 13, y = 1$

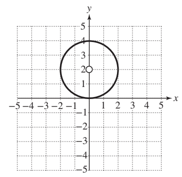
21. $x = 0, x = 8$ **23.** Yes **25.** No

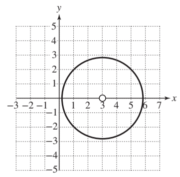
27. Center $(4, -2)$; $r = 3$ **29.** Center $(-1, -1)$; $r = 1$



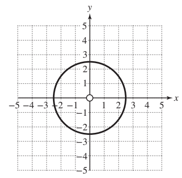


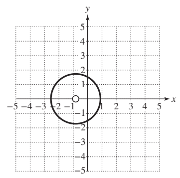
31. Center $(0, 2)$; $r = 2$ **33.** Center $(3, 0)$; $r = 2\sqrt{2}$



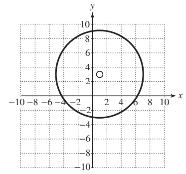


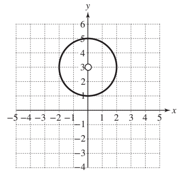
35. Center $(0, 0)$; $r = \sqrt{6}$ **37.** Center $\left(-\frac{4}{5}, 0\right)$; $r = \frac{8}{5}$



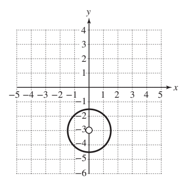


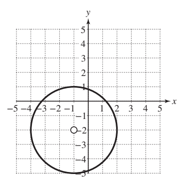
39. Center $(1, 3)$; $r = 6$ **41.** Center: $(0, 3)$; $r = 2$



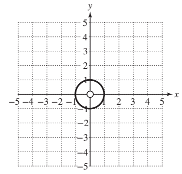


43. Center $(0, -3)$; $r = \frac{4}{3}$ **45.** Center: $(-1, -2)$; $r = 3$





47. Center $(0, 0)$; $r = 1$



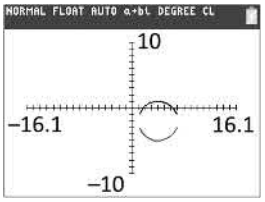
49. $x^2 + y^2 = 4$ **51.** $x^2 + (y - 2)^2 = 4$

53. $(x + 2)^2 + (y - 2)^2 = 9$ **55.** $x^2 + y^2 = 49$

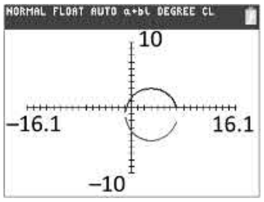
57. $(x+3)^2+(y+4)^2=36$ **59.** $(x-5)^2+(y-3)^2=2.25$
61. $(1, 2)$ **63.** $(-1, 0)$ **65.** $(-1, 6)$
67. $(0, 3)$ **69.** $\left(-\frac{1}{2}, \frac{3}{2}\right)$ **71.** $(-0.4, -1)$

73. $(40, 7\frac{1}{2})$; they should meet 40 mi east, $7\frac{1}{2}$ mi north of the warehouse. **75. a.** $(1, 3)$ **b.** $(x-1)^2+(y-3)^2=5$
77. a. $(0, 3)$ **b.** $x^2+(y-3)^2=4$

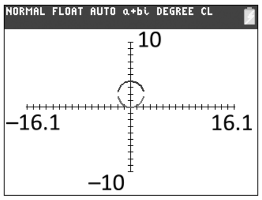
79.



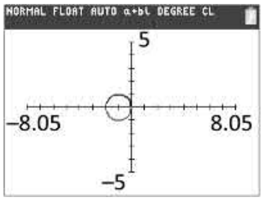
80.



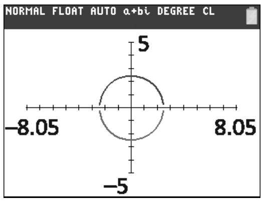
81.



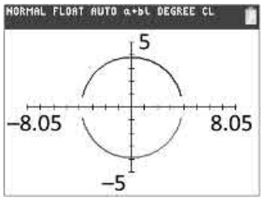
82.



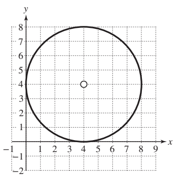
83.



84.



85. $(x-4)^2+(y-4)^2=16$ **87.** $(x-1)^2+(y-1)^2=29$

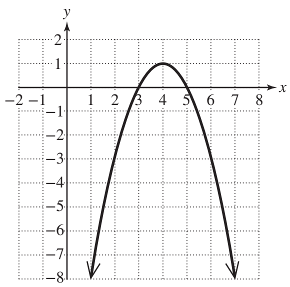


Section 9.2 Activity, p. 828

A.1. a. The graph of $y=a(x-h)^2+k$ is a parabola with vertex (h, k) and axis of symmetry $x=h$. The parabola opens upward if $a>0$ and opens downward if $a<0$.

b. The graph of $x=a(y-k)^2+h$ is a parabola with vertex (h, k) and axis of symmetry $y=k$. The parabola opens to the right if $a>0$ and opens to the left if $a<0$.

A.2.



A.3. The graph would be a parabola with vertex $(4, 1)$ but opening to the left.

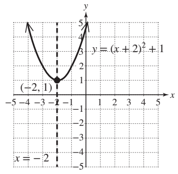
A.4. a. $a=1, b=4, c=1$ **b.** To the right
c. $x=(y+2)^2-3$ **d.** $(-3, -2)$ **e.** $y=-2$

Section 9.2 Practice Exercises, pp. 829–831

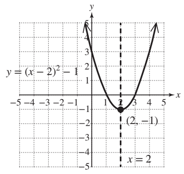
R.1. a. $(1, 4)$ **b.** $(0, 3)$ **c.** $(-1, 0)$ and $(3, 0)$
R.3. a. $(-2, 0)$ **b.** $(0, 4)$ **c.** $(-2, 0)$ **R.5.** -2

1. a. conic; plane **b.** parabola; directrix; focus **c.** vertex; symmetry **d.** $y=k$

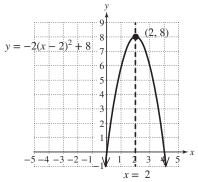
3.



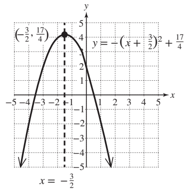
5.



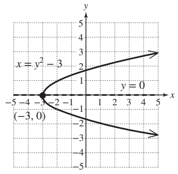
7.



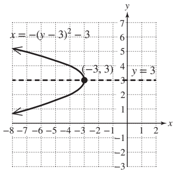
9.



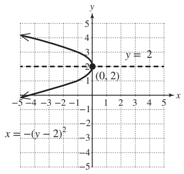
11.



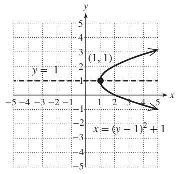
13.



15.



17.



19. $(2, -1)$ **21.** $(5, -1)$ **23.** $\left(2, \frac{7}{4}\right)$ **25.** $\left(\frac{3}{2}, -\frac{1}{4}\right)$

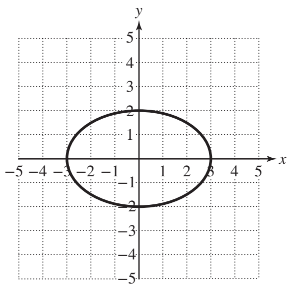
27. $(10, -1)$ **29.** The maximum height of the water is 22 ft.
31. A parabola whose equation is in the form $y=a(x-h)^2+k$ has a vertical axis of symmetry. A parabola whose equation is in the form $x=a(y-k)^2+h$ has a horizontal axis of symmetry.
33. Vertical axis of symmetry; opens upward
35. Vertical axis of symmetry; opens downward
37. Horizontal axis of symmetry; opens right
39. Horizontal axis of symmetry; opens left
41. Vertical axis of symmetry; opens downward
43. Horizontal axis of symmetry; opens right

Section 9.3 Activity, p. 837

A.1. a. The graph of $x^2+y^2=4$ is a circle centered at the origin with radius 2.

b. The graph of $\frac{x^2}{9}+\frac{y^2}{4}=1$ is an ellipse centered at the origin with x-intercepts $(3, 0)$ and $(-3, 0)$ and y-intercepts $(0, 2)$ and $(0, -2)$.

c.



A.2. The graph of $\frac{(x-1)^2}{9}+\frac{(y+3)^2}{4}=1$ is the same graph as $\frac{x^2}{9}+\frac{y^2}{4}=1$ except that the center is $(1, -3)$.

A.3. a. The graph of $\frac{x^2}{25}+\frac{y^2}{49}=1$ is an ellipse centered at the origin with x-intercepts $(5, 0)$ and $(-5, 0)$ and y-intercepts $(0, 7)$ and $(0, -7)$.

b. The graph of $\frac{x^2}{25} - \frac{y^2}{49} = 1$ is a hyperbola centered at the origin with a horizontal transverse axis (the branches of the hyperbola open to the left and right). The vertices of the reference rectangle are (5, 7), (5, −7), (−5, 7), and (−5, −7).

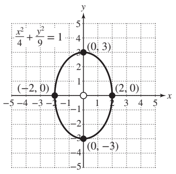
c. The graph of $\frac{y^2}{49} - \frac{x^2}{25} = 1$ is a hyperbola centered at the origin with a vertical transverse axis (the branches of the hyperbola open upward and downward). The vertices of the reference rectangle are (5, 7), (5, −7), (−5, 7), and (−5, −7).

Section 9.3 Practice Exercises, pp. 837–841

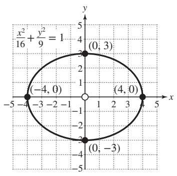
R.1. Center: (0, 0); radius: 4 **R.3.** Center: (3, −2); radius: $2\sqrt{3}$

1. a. ellipse; foci **b.** ellipse **c.** hyperbola; foci

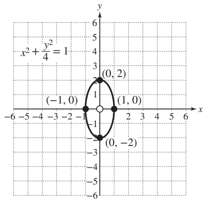
3.



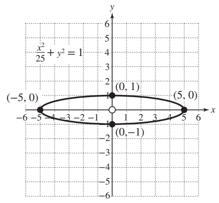
5.



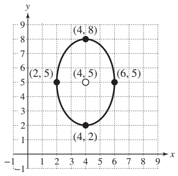
7.



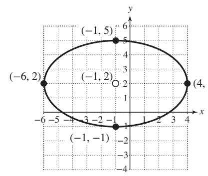
9.



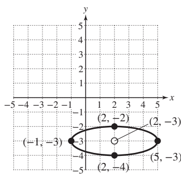
11. Center: (4, 5)



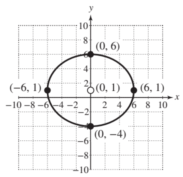
13. Center: (−1, 2)



15. Center: (2, −3)

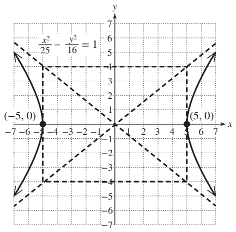


17. Center: (0, 1)

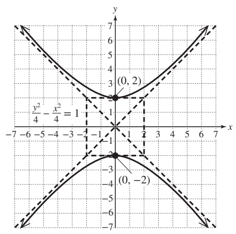


19. Vertical **21.** Horizontal **23.** Horizontal **25.** Vertical

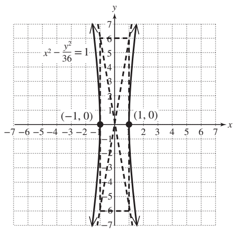
27.



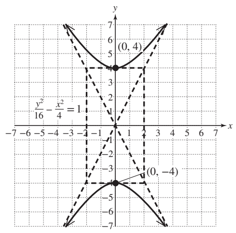
29.



31.

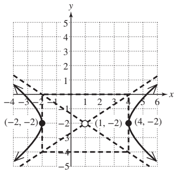


33.

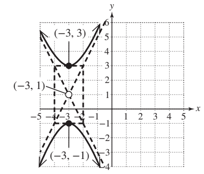


35. Hyperbola **37.** Ellipse **39.** Ellipse
41. Hyperbola **43.** Ellipse **45.** Hyperbola
47. The height 10 ft from the center is approximately 49 ft.

49.



51.

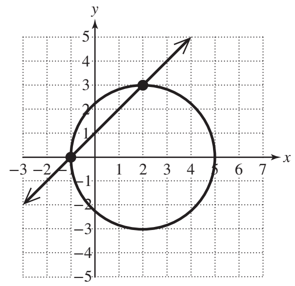


Chapter 9 Problem Recognition Exercises, pp. 841–842

1. Standard equation of a circle **2.** Ellipse centered at the origin **3.** Distance between two points
4. Midpoint between two points **5.** Parabola with vertical axis of symmetry **6.** Hyperbola with horizontal transverse axis
7. Hyperbola with vertical transverse axis
8. Parabola with horizontal axis of symmetry
9. Parabola **10.** Hyperbola **11.** Circle **12.** Circle
13. Hyperbola **14.** Ellipse **15.** None of these
16. Circle **17.** Parabola **18.** Hyperbola **19.** Parabola
20. Circle **21.** None of these **22.** Parabola
23. Ellipse **24.** Circle **25.** Parabola **26.** Ellipse
27. Hyperbola **28.** Circle **29.** Ellipse **30.** Parabola

Section 9.4 Activity, p. 846

A.1. a.



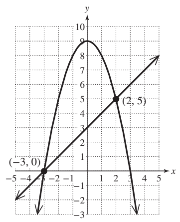
b. (−1, 0) and (2, 3) **c.** {(−1, 0), (2, 3)}

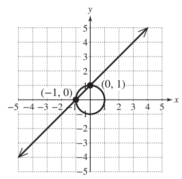
A.2. a. {(1, 3), (−1, 3), (1, −3), (−1, −3)}
b. The points of intersection of the graphs match the solutions found in part (a).

Section 9.4 Practice Exercises, pp. 846–849

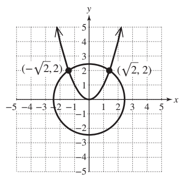
R.1. {(5, 1)} **R.3.** {(1, 2)} **R.5.** Circle
R.7. Ellipse **R.9.** Parabola

1. a. nonlinear **b.** intersection **3.** Zero, one, or two
5. Zero, one, or two **7.** Zero, one, two, three, or four
9. {(−3, 0), (2, 5)} **11.** {(0, 1), (−1, 0)}



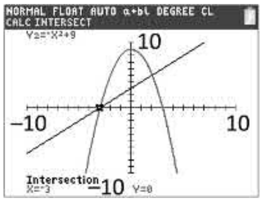
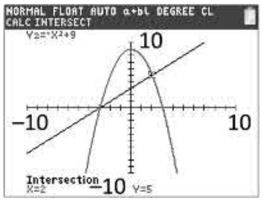


13. $\{(\sqrt{2}, 2), (-\sqrt{2}, 2)\}$



15. {(1, 1)} **17.** {(0, 0)} **19.** $\left\{\left(\frac{3}{2}, \frac{9}{4}\right)\right\}$

21. {(0, 0), (−2, −8)} **23.** {(1, 4)}
25. {(1, 0), (−1, 0)} **27.** {(2, 0), (−2, 0)}
29. {(3, 2), (−3, 2), (3, −2), (−3, −2)}
31. {(8, 6), (−8, −6)} **33.** {(0, −2), (0, 2)}
35. {(2, 0), (0, 1)} **37.** {(4, 5), (−4, −5)}
39. $\left\{\left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right)\right\}$

41.



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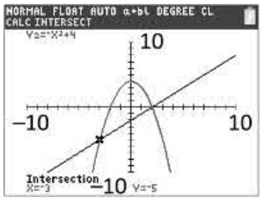
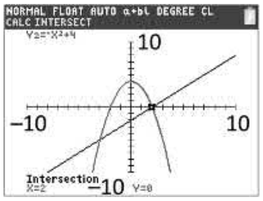

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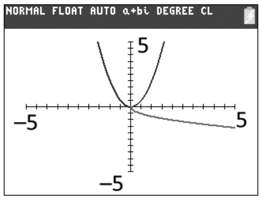

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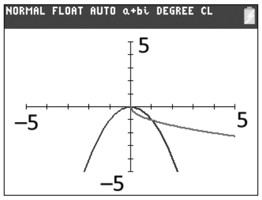
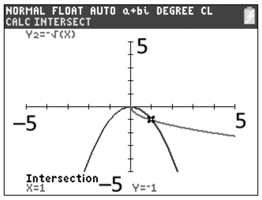
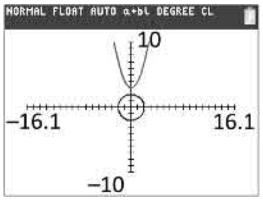
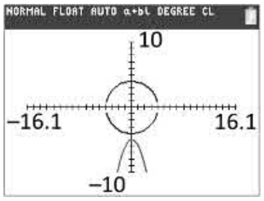
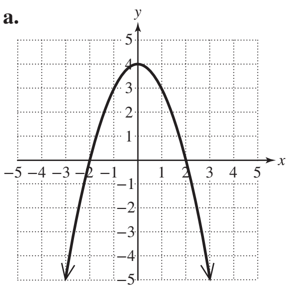
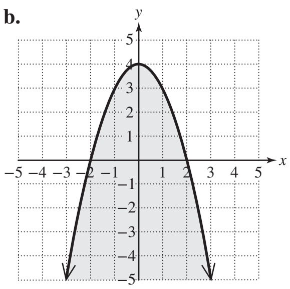



45. { } **46.** { }



47. The numbers are 3 and 4. **49.** The numbers are 5 and $\sqrt{7}$, −5 and $\sqrt{7}$, 5 and $-\sqrt{7}$, or −5 and $-\sqrt{7}$.

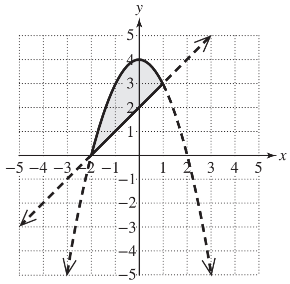
Section 9.5 Activity, p. 853

A.1. a.


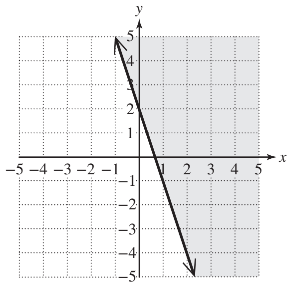
b.


c. The solution set would not contain the boundary $y = -x^2 + 4$. To denote this, draw the parabola as a dashed curve.

d. The solution set would contain all points on and above the parabola.

A.2.


Section 9.5 Practice Exercises, pp. 853–858

R.1.


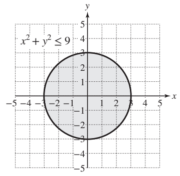
R.3. a. The solution set would be the same except that the bounding line would be dashed.

b. The solution set would contain points on and *below* the line rather than above the line.

R.5. Parabola **R.7.** Line

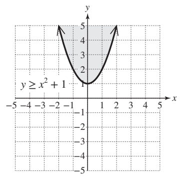
R.9. Hyperbola

1. True **3.** False

5. a.


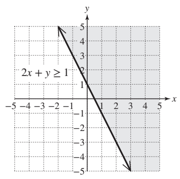
b. The set of points on and "outside" the circle $x^2 + y^2 = 9$

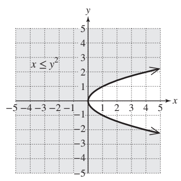
c. The set of points on the circle $x^2 + y^2 = 9$

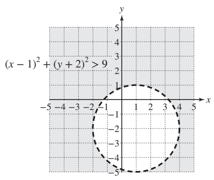
7. a.


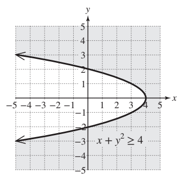
b. The parabola $y = x^2 + 1$ would be drawn as a dashed curve.

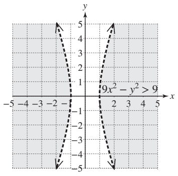
9. $(x - 3)^2 + (y + 4)^2 \le 625$

11.


13.


15.


17.


19.


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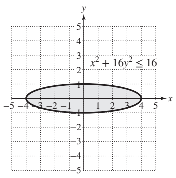
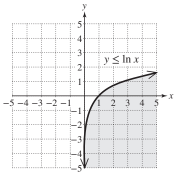


23.


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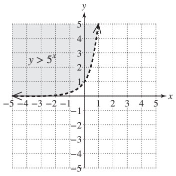

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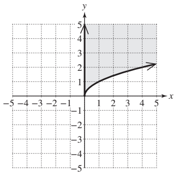

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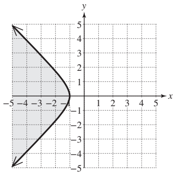
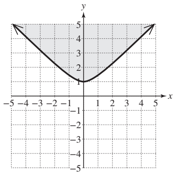
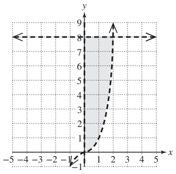
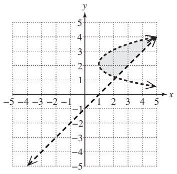
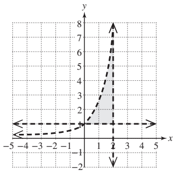


31.


33.


35. { }

37.


39.


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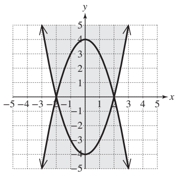
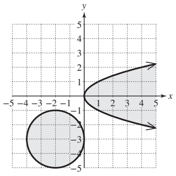


43.


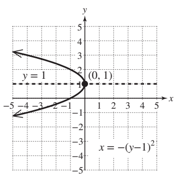
Chapter 9 Review Exercises, pp. 864–867

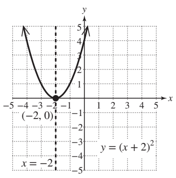
1. $2\sqrt{10}$ **2.** $\sqrt{89}$ **3.** $x = 5$ or $x = -1$
4. $x = -3$ **5.** Center $(12, 3)$; $r = 4$
6. Center $(-7, 5)$; $r = 9$ **7.** Center $(-3, -8)$; $r = 2\sqrt{5}$
8. Center $(1, -6)$; $r = 4\sqrt{2}$ **9. a.** $x^2 + y^2 = 64$
b. $(x - 8)^2 + (y - 8)^2 = 64$ **10.** $(x + 6)^2 + (y - 5)^2 = 10$

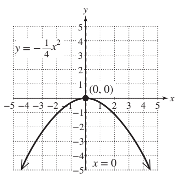
11. $(x + 2)^2 + (y + 8)^2 = 8$ **12.** $\left(x - \frac{1}{2}\right)^2 + (y - 2)^2 = 4$

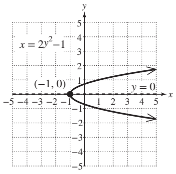
13. $(x - 3)^2 + \left(y - \frac{1}{3}\right)^2 = 9$ **14.** $x^2 + y^2 = \frac{49}{4}$

15. $x^2 + (y - 2)^2 = 9$ **16.** $(-4, -2)$ **17.** $(-1, 8)$
18. Vertical axis of symmetry; parabola opens downward
19. Horizontal axis of symmetry; parabola opens right
20. Horizontal axis of symmetry; parabola opens left
21. Vertical axis of symmetry; parabola opens upward

22.


23.


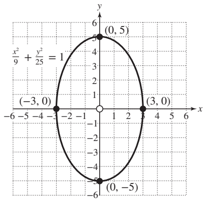
24.


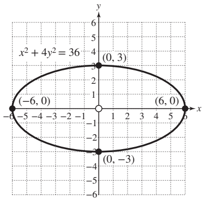
25.


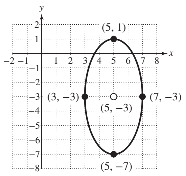
26. $y = (x - 3)^2 - 4$; vertex: $(3, -4)$; axis of symmetry: $x = 3$
27. $x = (y + 2)^2 - 2$; vertex: $(-2, -2)$; axis of symmetry: $y = -2$

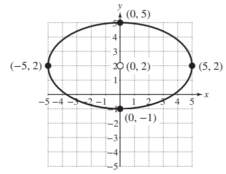
28. $x = -4\left(y - \frac{1}{2}\right)^2 + 1$; vertex: $\left(1, \frac{1}{2}\right)$; axis of symmetry: $y = \frac{1}{2}$

29. $y = -2\left(x + \frac{1}{2}\right)^2 + \frac{1}{2}$; vertex: $\left(-\frac{1}{2}, \frac{1}{2}\right)$; axis of symmetry: $x = -\frac{1}{2}$

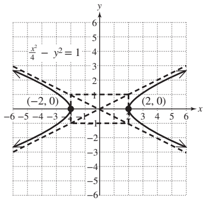
30.


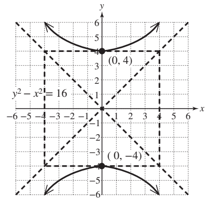
31.


32. Center: $(5, -3)$


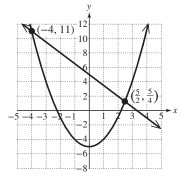
33. Center: $(0, 2)$


34. Horizontal **35.** Vertical **36.** Vertical **37.** Horizontal

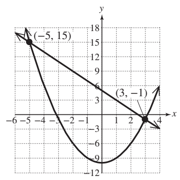
38.


39.


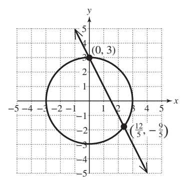
40. Hyperbola **41.** Ellipse **42.** Ellipse **43.** Hyperbola

44. a. Line and parabola
b.


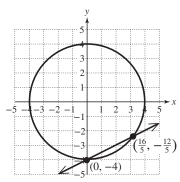
c. $\left\{\left(\frac{5}{2}, \frac{5}{4}\right), (-4, 11)\right\}$

45. a. Line and parabola
b.


c. $\{(-5, 15), (3, -1)\}$

46. a. Circle and line
b.


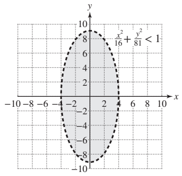
c. $\left\{(0, 3), \left(\frac{12}{5}, -\frac{9}{5}\right)\right\}$

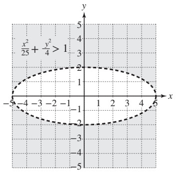
47. a. Circle and line
b.


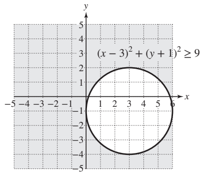
c. $\left\{(0, -4), \left(\frac{16}{5}, -\frac{12}{5}\right)\right\}$

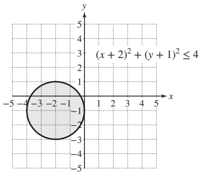
48. $\left\{(0, -2), \left(\frac{16}{9}, \frac{14}{9}\right)\right\}$ **49.** $\left\{\left(-\frac{7}{5}, \frac{13}{5}\right), (-5, -1)\right\}$

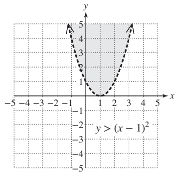
50. $\{(8, 4), (2, -2)\}$ **51.** $\{(2, 4), (-2, 4), (\sqrt{2}, 2), (-\sqrt{2}, 2)\}$
52. $\{(3, 1), (3, -1), (-3, 1), (-3, -1)\}$
53. $\{(6, 5), (-6, 5), (6, -5), (-6, -5)\}$

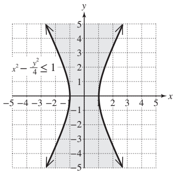
54. 

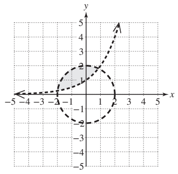
55. 

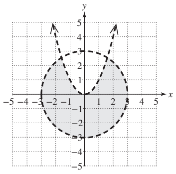
56. 

57. 

58. 

59. 

60. 

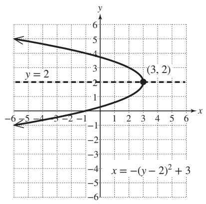
61. 

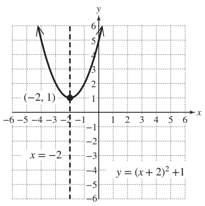
Chapter 9 Test, pp. 867–868

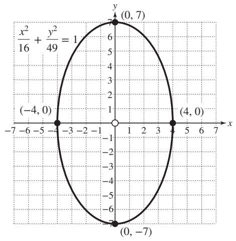
1. $4\sqrt{2}$ **2.** Center: $\left(\frac{5}{6}, -\frac{1}{3}\right)$; $r = \frac{5}{7}$

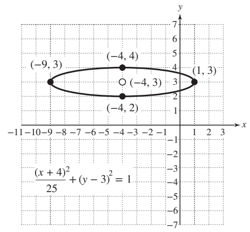
3. Center: (0, 2); $r = 3$ **4. a.** $\sqrt{5}$ **b.** $x^2 + (y-4)^2 = 5$

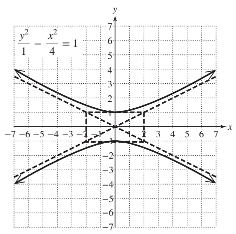
5. The center is the midpoint (3.8, 1.95).

6. 

7. 

8. 

9. 

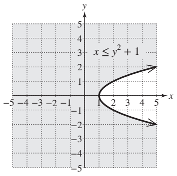
10. 

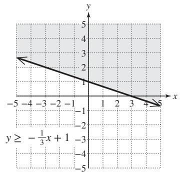
11. {(−3, 0), (0, 4)}; graph b

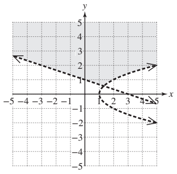
12. { }; graph a

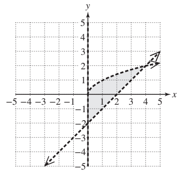
13. The addition method can be used if the equations have corresponding *like* terms.

14. {(2, 0), (−2, 0)}

15. 

16. 

17. 

18. 

Chapter 10

Section 10.1 Activity, p. 875

A.1. 1 **A.2.** $a + b$ **A.3.** $a^2 + 2ab + b^2$

A.4. $a^3 + 3a^2b + 3ab^2 + b^3$

A.5. a. 1 4 6 4 1 **b.** $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

A.6. a. 1 **b.** 4 **c.** 6 **d.** 4 **e.** 1

A.7. a. $[5x + (-y^2)]^4$ **b.** $625x^4 - 500x^3y^2 + 150x^2y^4 - 20xy^6 + y^8$

Section 10.1 Practice Exercises, pp. 876–877

R.1. 15 **R.3. a.** $16x^2 - 24x + 9$ **b.** $64x^3 - 144x^2 + 108x - 27$

1. a. $a^2 + 2ab + b^2$; $a^3 + 3a^2b + 3ab^2 + b^3$; binomial **b.** $n(n-1)(n-2)\cdots(2)(1)$; factorial **c.** 6; 2; 1; 1 **d.** binomial **e.** Pascal's **3.** $a^3 + 3a^2b + 3ab^2 + b^3$

5. $1 + 4g + 6g^2 + 4g^3 + g^4$

7. $p^7 + 7p^6q^2 + 21p^5q^4 + 35p^4q^6 + 35p^3q^8 + 21p^2q^{10} + 7pq^{12} + q^{14}$

9. $s^5 - 5s^4t + 10s^3t^2 - 10s^2t^3 + 5st^4 - t^5$

11. $625 - 500u^3 + 150u^6 - 20u^9 + u^{12}$

13. $x^6 - 12x^4 + 48x^2 - 64$ **15.** 120 **17.** 1 **19.** False

21. True **23.** $6! = 6 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 6 \cdot 5!$ **25.** 1680

27. 6 **29.** 56 **31.** 1 **33.** $m^{11} + 11m^{10}n + 55m^9n^2$

35. $u^{24} - 12u^{22}v + 66u^{20}v^2$ **37.** 9 terms

39. $s^6 + 6s^5t + 15s^4t^2 + 20s^3t^3 + 15s^2t^4 + 6st^5 + t^6$

41. $b^3 - 9b^2 + 27b - 27$ **43.** $16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

45. $c^{14} - 7c^{12}d + 21c^{10}d^2 - 35c^8d^3 + 35c^6d^4 - 21c^4d^5 + 7c^2d^6 - d^7$

47. $\frac{1}{32}a^5 - \frac{5}{16}a^4b + \frac{5}{4}a^3b^2 - \frac{5}{2}a^2b^3 + \frac{5}{2}ab^4 - b^5$

49. $x^4 + 16x^3y + 96x^2y^2 + 256xy^3 + 256y^4$

51. $-462m^6n^5$ **53.** $495u^{16}v^4$ **55.** g^9

Section 10.2 Activity, p. 883

A.1. a. $1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}$ **b.** $-1, \frac{2}{3}, -\frac{1}{2}, \frac{2}{5}, -\frac{1}{3}$ **c.** $1, -\frac{2}{3}, \frac{1}{2}, -\frac{2}{5}, \frac{1}{3}$

A.2. a. $\frac{29}{10}$ **b.** $-\frac{23}{30}$ **c.** $\frac{23}{30}$

A.3. a. 100, 200, 400, 800, 1600 **b.** $s_n = 100 \cdot 2^{n-1}$

c. $r_n = 10(100 \cdot 2^{n-1})$ or $r_n = 1000 \cdot 2^{n-1}$ **d.** $\sum_{i=1}^{5} 1000 \cdot 2^{i-1}$

e. 1000 + 2000 + 4000 + 8000 + 16,000 **f.** $31,000

Section 10.2 Practice Exercises, pp. 884–886

R.1. a. −1 **b.** 1 **c.** −1 **d.** 1 **e.** −1 **f.** 1

R.3. 47 **R.5.** $\frac{16}{25}$ **R.7.** $\frac{23}{60}$ **R.9.** 56

1. a. infinite; finite **b.** terms; nth **c.** alternating **d.** series **e.** summation **f.** index; $(3)^2 + (4)^2 + (5)^2$

3. 22, 25, 28 **5.** $-18, 21, -24$ **7.** 4, 7, 10, 13, 16

9. $\sqrt{3}, 2, \sqrt{5}, \sqrt{6}$ **11.** $-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}$

13. $0, -3, 8$ **15.** 0, 2, 6, 12, 20, 30 **17.** $-3, 9, -27, 81$

19. When n is odd, the term is negative. When n is even, the term is positive. **21.** $a_n = 2n$ **23.** $a_n = 2n - 1$

25. $a_n = \frac{1}{n^2}$ **27.** $a_n = (-1)^{n+1}$ **29.** $a_n = (-1)^n 2^n$

31. $a_n = \frac{3}{5^n}$ **33.** \$60, \$58.80, \$57.62, \$56.47

35. 25,000; 50,000; 100,000; 200,000; 400,000; 800,000; 1,600,000

37. A sequence is an ordered list of terms. A series is the sum of the terms of a sequence.

39. 90 **41.** $\frac{31}{16}$ **43.** 30 **45.** 10 **47.** $\frac{73}{12}$ **49.** 38

51. -1 **53.** 55 **55.** $\sum_{n=1}^{6} n$ **57.** $\sum_{i=1}^{5} 4$ **59.** $\sum_{j=1}^{5} 4j$

61. $\sum_{k=1}^{4} (-1)^{k+1} \frac{1}{3^k}$ **63.** $\sum_{i=1}^{6} \frac{i+4}{11i}$ **65.** $\sum_{n=1}^{5} x^n$

67. $-3, 2, 7, 12, 17$ **69.** 5, 21, 85, 341, 1365

71. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

Section 10.3 Activity, p. 891

A.1. a.

Year	Base Salary Plus Raise(s)	Total Salary
Year 1	\$55,000 + 0(\$2000)	\$55,000
Year 2	\$55,000 + 1(\$2000) (Joyce has had 1 raise.)	\$57,000
Year 3	\$55,000 + 2(\$2000) (Joyce has had 2 raises.)	\$59,000
Year 4	\$55,000 + 3(\$2000) (Joyce has had 3 raises.)	\$61,000
Year 5	\$55,000 + 4(\$2000) (Joyce has had 4 raises.)	\$63,000
Year n	\$55,000 + $(n - 1)$(\$2000) (Joyce has had $n - 1$ raises.)	\$53,000 + \$2000n

b. \$103,000 **A.2. a.** arithmetic **b.** $a_n = a_1 + (n - 1)d$

A.3. a. 17 **b.** -6 **c.** $a_n = 17 + (n - 1)(-6)$ or $a_n = 23 - 6n$

d. 54 **A.4. a.** $\sum_{i=1}^{25} (53{,}000 + 2000i)$ **b.** \$1,975,000

Section 10.3 Practice Exercises, pp. 891–893

R.1. $\{21\}$ **R.3.** $\{-2\}$ **R.5. a.** $f(1) = 1$ **b.** $f(2) = 5$ **c.** $f(3) = 9$ **d.** $f(4) = 13$

R.7. $-3 + (-7) + (-11) + (-15) + (-19)$; -55

1. a. arithmetic **b.** difference **c.** $a_1 + (n - 1)d$; d **d.** series **e.** $\frac{n}{2}(a_1 + a_n)$ **3.** $-1, 3, 7, 11$ **5.** $\frac{1}{2}, 2, \frac{7}{2}, 5$

7. 3, 11, 19, 27, 35 **9.** 80, 60, 40, 20, 0 **11.** $3, \frac{15}{4}, \frac{9}{2}, \frac{21}{4}, 6$

13. 2 **15.** -3 **17.** -2 **19.** 3, 8, 13, 18, 23 **21.** $2, \frac{5}{2}, 3, \frac{7}{2}, 4$

23. $2, -2, -6, -10, -14$ **25.** $a_n = -5 + 5n$

27. $a_n = -2n$ **29.** $a_n = \frac{3}{2} + \frac{1}{2}n$ **31.** $a_n = 25 - 4n$

33. $a_n = -14 + 6n$ **35.** $a_6 = 17$ **37.** $a_9 = 47$

39. $a_7 = -30$ **41.** $a_{11} = -48$ **43.** 19 **45.** 22

47. 23 **49.** 11 **51.** $a_1 = -2, a_2 = -5$ **53.** 670

55. 290 **57.** -15 **59.** 95 **61.** 924 **63.** 300

65. -210 **67.** 5050 **69.** 980 seats; \$14,700

Section 10.4 Activity, p. 899

A.1. a.

Year	Salary		Simplified
Year 1	\$50,000$(1.04)^0$		\$50,000
Year 2	\$50,000$(1.04)^1$	(Kevin has 1 raise of 4% of \$50,000 or equivalently 104% of his first year's salary.)	\$52,000
Year 3	\$50,000$(1.04)^2$	(Kevin has 2 raises of 4%. This is 104% of 104% of his first year's salary.)	\$54,080
Year 4	\$50,000$(1.04)^3$	(Kevin has 3 raises of 4%.)	\$56,243.20
Year 5	\$50,000$(1.04)^4$	(Kevin has 4 raises of 4%.)	\$58,492.93
Year n	\$50,000$(1.04)^{\boxed{n-1}}$	(Kevin has $n - 1$ raises of 4%.)	\$50,000$(1.04)^{n-1}$

b. \$128,165.21

c. Kevin's salary increases each year, and his raise is a percentage of this increasing salary. Therefore, his raises also increase each year.

d. $\sum_{i=1}^{25} 50{,}000(1.04)^{i-1}$ **e.** \$2,082,295.41 **f.** \$107,295.41

A.2. a. Geometric; the ratio of two consecutive terms is a constant. In this case, each term is $\frac{1}{2}$ of its predecessor.

b. $a_n = 16\left(\frac{1}{2}\right)^{n-1}$ **c.** $\frac{1}{8}$ **d.** 9

A.3. a. Arithmetic; the difference between consecutive terms is constant. In this case, each term after the first is 8 less than its predecessor.

b. $b_n = 16 + (n - 1)(-8)$ or $b_n = 24 - 8n$ **c.** -72 **d.** 48

A.4. Neither **A.5.** $\frac{511}{16}$ or $31\frac{15}{16}$ **A.6.** 32 **A.7.** 8176

A.8. Does not exist; this is a geometric series with common ratio 2. Because the common ratio is greater than 1, the terms of the series increase without bound and the sum becomes infinitely large.

Section 10.4 Practice Exercises, pp. 900–903

R.1. a. $f(1) = 20$ **b.** $f(2) = 8$ **c.** $f(3) = \frac{16}{5}$ **d.** $f(4) = \frac{32}{25}$

R.3. 32 **R.5.** 150 **R.7.** 28

1. a. geometric **b.** ratio **c.** $a_1 r^{n-1}$ **d.** $\frac{a_1(1 - r^n)}{1 - r}$ **e.** $\frac{a_1}{1 - r}$; 1 **3.** 4, 8, 16, 32 **5.** $1, -5, 25, -125$

7. 1, 10, 100, 1000 **9.** 64, 32, 16, 8

11. $8, -2, \frac{1}{2}, -\frac{1}{8}$ **13.** 2 **15.** $-\frac{1}{4}$

17. -2 **19.** $-3, 6, -12, 24, -48$ **21.** $6, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}$

23. $-1, -6, -36, -216, -1296$ **25.** $a_n = 3(4)^{n-1}$

27. $a_n = -5(-3)^{n-1}$ **29.** $a_n = \frac{1}{2}(4)^{n-1}$ **31.** $\frac{1}{64}$

33. $-\frac{243}{8}$ **35.** -48 **37.** -9 **39.** $\frac{1}{8}$ **41.** 4

43. A geometric sequence is an ordered list of numbers in which the ratio between each term and its predecessor is constant. A geometric series is the sum of the terms of such a sequence.

45. a. $3 + 12 + 48 + 192$ **b.** 255

47. $\frac{1562}{125}$ **49.** $-\frac{11}{8}$ **51.** $\frac{3124}{27}$ **53.** $\frac{665}{243}$

55. -172 **57. a.** \$1050.00, \$1102.50, \$1157.63, \$1215.51 **b.** $a_{10} =$ \$1628.89; $a_{20} =$ \$2653.30; $a_{40} =$ \$7039.99

59. $r = \frac{1}{6}$; sum is $\frac{6}{5}$ **61.** $r = -\frac{1}{4}$; sum is $\frac{4}{5}$

63. $r = -\frac{3}{2}$; sum does not exist **65.** The sum is \$800 million.

67. The total vertical distance traveled is 28 ft.

69. a. $\frac{7}{10}$ **b.** $\frac{1}{10}$ **c.** $\frac{7}{9}$ **71. a.** \$1,429,348 **b.** \$1,505,828 **c.** \$76,480

Chapter 10 Problem Recognition Exercises, p. 904

1. Geometric, $r = -\frac{1}{2}$ **2.** Arithmetic, $d = \frac{1}{2}$ **3.** Geometric, $r = 2$ **4.** Neither **5.** Arithmetic, $d = \frac{2}{3}$ **6.** Geometric, $r = -\frac{3}{2}$ **7.** Neither **8.** Geometric, $r = -1$ **9.** Arithmetic, $d = -2$ **10.** Neither **11.** Neither **12.** Arithmetic, $d = 4$ **13.** Arithmetic, $d = \frac{1}{4}\pi$ **14.** Geometric, $r = -1$ **15.** Neither **16.** Arithmetic, $d = \frac{1}{3}e$ **17.** Neither **18.** Geometric, $r = 3$

Chapter 10 Review Exercises, pp. 908–909

1. 40,320 **2.** 120 **3.** 66 **4.** 84 **5.** $x^{10} + 20x^8 + 160x^6 + 640x^4 + 1280x^2 + 1024$ **6.** $c^4 - 12c^3d + 54c^2d^2 - 108cd^3 + 81d^4$ **7.** $a^{11} - 22a^{10}b + 220a^9b^2$ **8.** $1512x^{10}y^3$ **9.** $-15{,}000x^3y^{21}$ **10.** $160a^3b^3$ **11.** $1, -2, -5, -8, -11$ **12.** $-2, -16, -54$ **13.** $\frac{1}{3}, -\frac{1}{2}, \frac{3}{5}, -\frac{2}{3}$ **14.** $-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}$ **15.** $a_n = \frac{n}{n+1}$ **16.** $a_n = (-1)^n \cdot \frac{3}{2^n}$ **17.** k **18.** 5 **19.** 5 **20.** -22 **21.** $\sum_{i=1}^{7} \frac{i+3}{i}$ **22.** $a_n = n^2 + \left(\frac{n}{2}\right)h$ **23.** $-12, -10.5, -9, -7.5, -6$ **24.** $\frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2$ **25.** $a_n = 10n - 9$ **26.** $a_n = -14n + 20$ **27.** $a_{17} = \frac{9}{2}$ **28.** $a_{25} = -155$ **29.** 24 terms **30.** 19 terms **31.** $d = 10$ **32.** $a_n = -5n + 19$ **33.** -620 **34.** 92.5 **35.** 294 **36.** 5989 **37.** $r = 3$ **38.** $r = 1.2$ **39.** $-1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}$ **40.** 10, 20, 40, 80 **41.** $a_n = -4(2)^{n-1}$ **42.** $a_n = 6\left(-\frac{1}{3}\right)^{n-1}$ **43.** $a_6 = \frac{128}{243}$ **44.** $a_4 = 270$ **45.** $a_1 = 4$ **46.** $a_1 = 1$ **47.** 1275 **48.** $-\frac{182}{3}$ **49.** 18 **50.** $\frac{25}{2}$ **51. a.** \$10,700, \$11,449, \$12,250.43 **b.** The account is worth \$12,250.43 after 3 years. **c.** \$19,671.51

Chapter 10 Test, pp. 909–910

1. 1 **2.** 210 **3.** $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ **4.** $81y^4 - 216y^3x^2 + 216y^2x^4 - 96yx^6 + 16x^8$ **5.** $-56a^3c^{15}$ **6.** $-1, -\frac{3}{4}, -\frac{3}{5}, -\frac{1}{2}$ **7.** 65 **8. a.** 8, 9.5, 11, 12.5, 14 **b.** $a_n = 1.5n + 6.5$ **9.** $\sum_{n=1}^{4} x^{3n}$ **10.** $d = \frac{1}{4}$ **11.** $r = \frac{1}{3}$ **12. a.** $-15, -18, -21, -24$ **b.** $a_n = -3n - 12$ **c.** $a_{40} = -132$ **13. a.** $4, -8, 16, -32$ **b.** $a_n = 4(-2)^{n-1}$ **c.** $a_{10} = -2048$ **14.** $a_n = \frac{3}{5} \cdot \left(\frac{1}{2}\right)^{n-1}$ **15.** $a_n = -2n + 22$ **16.** 31 terms **17.** 7 terms **18.** 10,900 **19.** $\frac{1023}{64}$ **20.** 8 **21.** $a_1 = \frac{1}{27}$ **22.** $a_{18} = -202$ **23. a.** \$2007.50 **b.** \$2087.80 **c.** \$6260.69 **d.** \$112,590.51

Additional Topics Appendix

Section A.1 Practice Exercises, pp. A-9–A-12

1. a. determinant; $ad - bc$ **b.** minor **c.** a_2; $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

3. 16 **5.** 1 **7.** 4 **9.** 6 **11.** 26 **13.** 6 **15.** 46 **17. a.** 30 **b.** 30 **19.** Choosing the row or column with the most zero elements simplifies the arithmetic when evaluating a determinant. **21.** 12 **23.** -15 **25.** 0 **27.** $8a - 2b$ **29.** $4x - 3y + 6z$ **31.** 0 **33.** $\mathbf{D} = 16$; $\mathbf{D}_x = -63$; $\mathbf{D}_y = 66$ **35.** $\{(2, -1)\}$ **37.** $\{(4, -1)\}$ **39.** $\left\{\left(\frac{23}{13}, \frac{9}{13}\right)\right\}$ **41.** $x = 1$ **43.** $z = \frac{1}{2}$ **45.** $y = \frac{16}{41}$ **47.** Cramer's rule does not apply when the determinant $\mathbf{D} = 0$. **49.** No solution; { }; inconsistent system **51.** $\{(0, 0)\}$ **53.** Infinitely many solutions; $\{(x, y) \mid x + 5y = 3\}$; dependent equations **55.** $\{(3, 2, 1)\}$ **57.** No solution; { }; inconsistent system **59.** $x = -19$ **61.** $w = 1$ **63.** 36 **65. a.** 2 **b.** -2 **c.** $x = -1$ **67.** The measures are 37.5° and 52.5°. **69.** 900 tickets in section A, 1500 tickets in section B, and 600 tickets in section C were sold. **71.** There were 560 women and 440 men in the survey.

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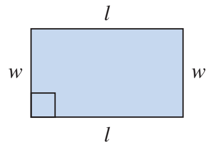
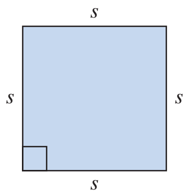
Perimeter and Circumference



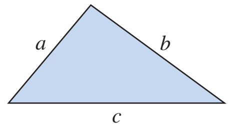
Rectangle

$P = 2l + 2w$



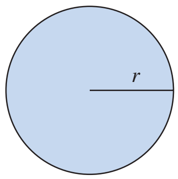
Square

$P = 4s$



Triangle

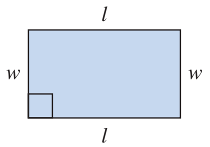
$P = a + b + c$



Circle

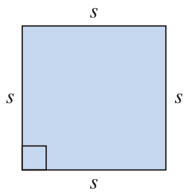
Circumference: $C = 2\pi r$

Area



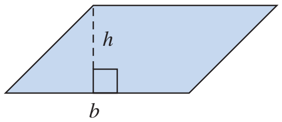
Rectangle

$A = lw$



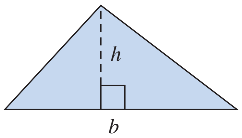
Square

$A = s^2$



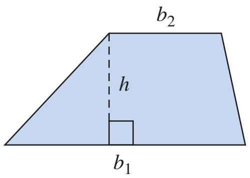
Parallelogram

$A = bh$



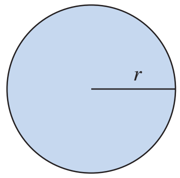
Triangle

$A = \frac{1}{2}bh$



Trapezoid

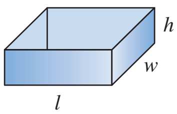
$A = \frac{1}{2}(b_1 + b_2)h$



Circle

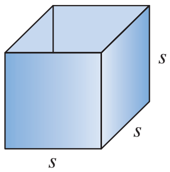
$A = \pi r^2$

Volume



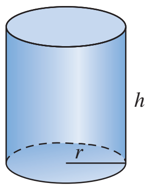
Rectangular Solid

$V = lwh$



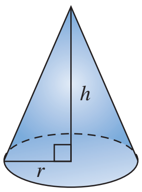
Cube

$V = s^3$



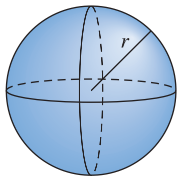
Right Circular Cylinder

$V = \pi r^2 h$



Right Circular Cone

$V = \frac{1}{3}\pi r^2 h$

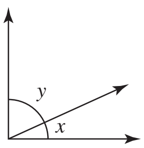


Sphere

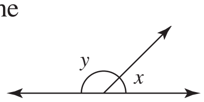
$V = \frac{4}{3}\pi r^3$

Angles

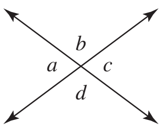
- Two angles are **complementary** if the sum of their measures is 90°.



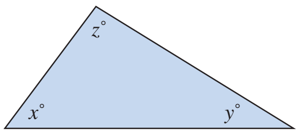
- Two angles are **supplementary** if the sum of their measures is 180°.



- $\angle a$ and $\angle c$ are vertical angles, and $\angle b$ and $\angle d$ are vertical angles. The measures of vertical angles are equal.



- The sum of the measures of the angles of a triangle is 180°.



$x^\circ + y^\circ + z^\circ = 180^\circ$

Linear Equations and Slope

The slope, m, of a line between two distinct points (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_2 - x_1 \neq 0$$

Standard form: $Ax + By = C$ (A and B are not both zero.)

Horizontal line: $y = k$

Vertical line: $x = k$

Slope-intercept form: $y = mx + b$

Point-slope formula: $y - y_1 = m(x - x_1)$

Midpoint Formula

Given two points (x_1, y_1) and (x_2, y_2), the midpoint is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Properties and Definitions of Exponents

Let a and b ($b \neq 0$) represent real numbers and m and n represent positive integers.

$$b^m b^n = b^{m+n}; \qquad \frac{b^m}{b^n} = b^{m-n}; \qquad (b^m)^n = b^{mn};$$

$$(ab)^m = a^m b^m; \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}; \qquad b^0 = 1; \qquad b^{-n} = \left(\frac{1}{b}\right)^n$$

The Quadratic Formula

The solutions to $ax^2 + bx + c = 0$ ($a \neq 0$) are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Vertex Formula

For $f(x) = ax^2 + bx + c$ ($a \neq 0$), the vertex is

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) \quad \text{or} \quad \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

The Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) is
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The Standard Form of a Circle

$(x - h)^2 + (y - k)^2 = r^2$ with center (h, k) and radius r

Properties of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties are true.

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b (b^p) = p$
4. $b^{\log_b(x)} = x$
5. $\log_b(xy) = \log_b x + \log_b y$
6. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
7. $\log_b(x^p) = p \log_b x$

Change-of-Base Formula:

$$\log_b x = \frac{\log_a x}{\log_a b} \qquad a > 0, a \neq 1, b > 0, b \neq 1$$

Properties and Definitions of Radicals

Let a be a real number and n be an integer such that $n > 1$. If $\sqrt[n]{a}$ exists, then

$$a^{1/n} = \sqrt[n]{a}; \qquad a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

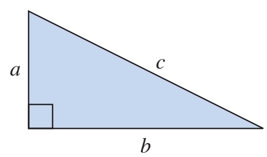
Let a and b represent real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real. Then

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ Multiplication property

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ Division property

The Pythagorean Theorem:

$a^2 + b^2 = c^2$



Transformations, Piecewise-Defined Functions, and Probability

11

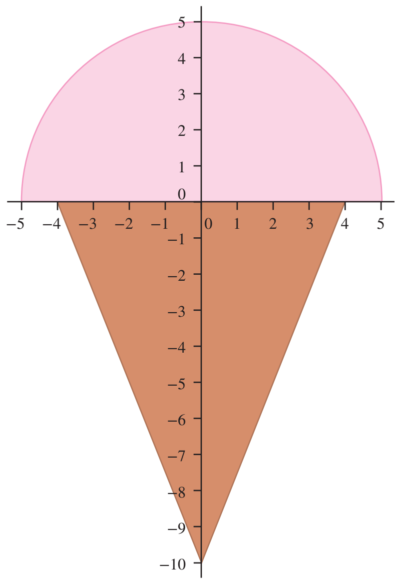
CHAPTER OUTLINE

Mathematics in Animation

Have you ever watched a low-quality animated cartoon in which a character is racing across the screen? Instead of moving the character across the screen, the illusion of movement is accomplished by moving the background elements such as the trees and the clouds in the *opposite* direction. That is, if the character is portrayed to move from left to right, then the background trees and clouds move from right to left. Animators accomplish this by using **transformations** of the curves representing the clouds and trees. Transformations include shifting a curve to the left, right, up, or down, along with shrinking or stretching the curve in the horizontal or vertical direction. In this chapter, we will study the transformations of basic functions.

The concept of a piecewise-defined function is also presented in this chapter. A piecewise-defined function is made up of a series of functions, each defined on an indicated domain. The concept of restricting the domain of a function on an interval is also used in computer gaming to draw figures mathematically. For example, the ice cream cone shown here was rendered from the equations

$$y=\sqrt{25-x^2},\ y=2.5x-10 \text{ for } 0\le x\le 4, \text{ and } y=-2.5x-10 \text{ for } -4\le x\le 0.$$



Section 11.1 Transformations of Graphs and Piecewise-Defined Functions

Concepts

1. **Transformations of Graphs**
2. **Piecewise-Defined Functions**

1. Transformations of Graphs

The graph of a quadratic function of the form $y = a(x - h)^2 + k$ has the shape of a parabola. The values of a, h, and k have the following effects on the graph of the parent function $y = x^2$.

- The value of h shifts the graph of $y = x^2$ to the left or right.
- The value of k shifts the graph of $y = x^2$ upward or downward.
- The value of a will stretch the graph vertically if $|a| > 1$ and shrink the graph vertically if $0 < |a| < 1$.
- Finally, if $a < 0$, the graph is reflected across the x-axis.

Shifting, stretching, shrinking, and reflecting the graph of a function is sometimes called *transforming graphs*. Such transformations can be extended to the graphs of any function. Given a function defined by $y = f(x)$, the graph of $y = a \cdot f(x - h) + k$ can be graphed according to the following guidelines.

FOR REVIEW

Take a minute to review the graphs of six basic functions presented previously in the text.

$f(x) = x$
$f(x) = x^2$
$f(x) = x^3$
$f(x) = |x|$
$f(x) = \sqrt{x}$
$f(x) = \frac{1}{x}$

Shifting and Reflecting the Graph of a Function

Given a function defined by $y = a \cdot f(x - h) + k$, then

If $k > 0$, shift the graph of $y = f(x)$ upward by k units.
If $k < 0$, shift the graph of $y = f(x)$ downward by $|k|$ units.

If $h > 0$, shift the graph of $y = f(x)$ to the right by h units.
If $h < 0$, shift the graph of $y = f(x)$ to the left by $|h|$ units.

If $a < 0$, reflect the graph of $y = f(x)$ across the x-axis.
If $|a| > 1$, stretch the graph of $y = f(x)$ vertically.
If $0 < |a| < 1$, shrink the graph of $y = f(x)$ vertically.

Example 1 Shifting the Graph of a Function

Graph the functions, and determine the domain and range of each.

a. $f(x) = \sqrt{x - 2}$ **b.** $g(x) = |x + 1| - 4$

Solution:

a. The graph of $y = \sqrt{x}$ is shown in black (Figure 11-1). The graph of $f(x) = \sqrt{x - 2}$ is in the form $y = f(x - h)$, where $h = 2$. Since $h > 0$, shift the graph to the right (shown in blue).

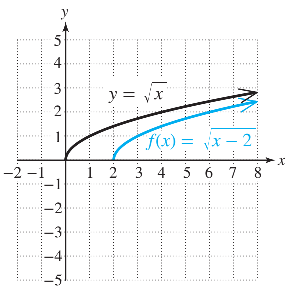


Figure 11-1

The domain of f is: $[2, \infty)$
The range of f is: $[0, \infty)$

b. The graph of $y = |x|$ is shown in black (Figure 11-2). The graph of $g(x) = |x + 1| - 4$ is in the form $y = g(x - h) + k$, where $h = -1$ and $k = -4$. Since $h < 0$, shift the graph to the left. Since $k < 0$, shift the graph downward (shown in red).

The domain of g is: $(-\infty, \infty)$
The range of g is: $[-4, \infty)$

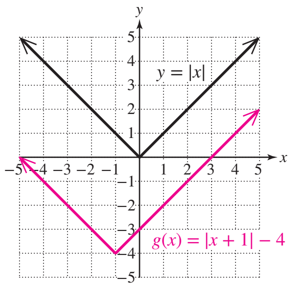


Figure 11-2

Skill Practice Graph the functions and determine the domain and range.

1. $g(x) = |x + 3|$ **2.** $f(x) = \sqrt{x - 3} - 1$

FOR REVIEW

When using interval notation, use parentheses, (or), with ∞ and $-\infty$ and to indicate that an "endpoint" is *not* included in the set. Use square brackets, [or], to indicate that an "endpoint" *is* included in the set.

Example 2 Reflecting the Graph of a Function

Graph the function and determine the domain and range. $g(x) = -x^3$

Solution:

The graph of $y = x^3$ is shown in Figure 11-3. The graph of $g(x) = -x^3$ is the graph of $y = x^3$ reflected across the x-axis (Figure 11-4).

The domain of g is $(-\infty, \infty)$.
The range of g is $(-\infty, \infty)$.

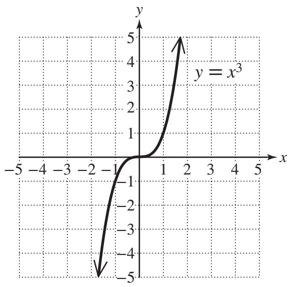


Figure 11-3

Reflect across the x-axis.

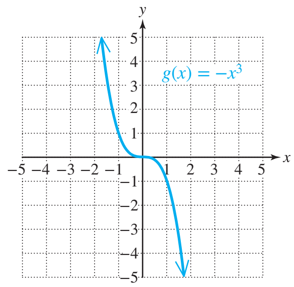


Figure 11-4

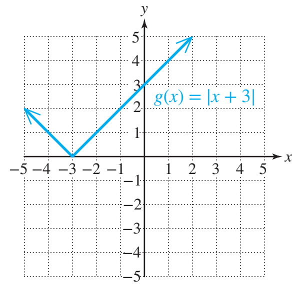
Skill Practice Graph the function and give the domain and range.

3. $h(x) = -|x|$

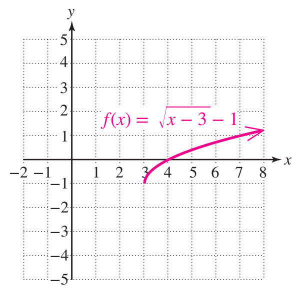
In Examples 3 and 4, we analyze graphs involving a vertical shrink or stretch from the parent function.

Answers

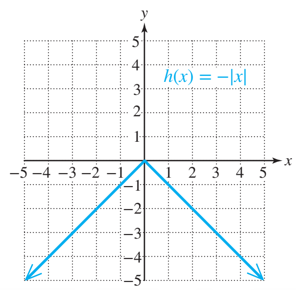
1. Domain: $(-\infty, \infty)$; range: $[0, \infty)$



2. Domain: $[3, \infty)$; range: $[-1, \infty)$



3. Domain: $(-\infty, \infty)$; range: $(-\infty, 0]$



Example 3 Stretching and Shrinking the Graph of a Function

Graph the function and determine the domain and range. $f(x) = 3\sqrt{x}$

Solution:

The graph of $y = \sqrt{x}$ is the basic square root graph (Figure 11-5).

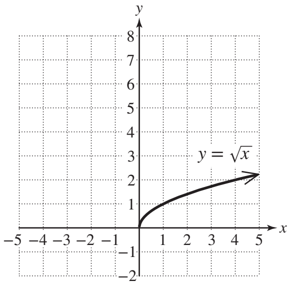


Figure 11-5

The graph of $f(x) = 3\sqrt{x}$ is the graph of $y = a\sqrt{x}$, where $a = 3$. Since $a > 1$, graph of f is the graph of $y = \sqrt{x}$ with a vertical stretch by a factor of 3 (Figure 11-6).

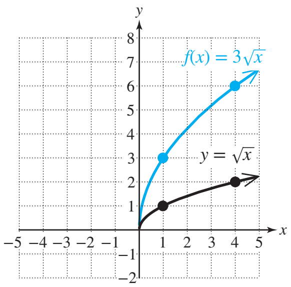


Figure 11-6

Notice that for each point on the graph of $y = \sqrt{x}$, the y-coordinate is multiplied by 3. For example, the point (1, 1) on the graph of $y = \sqrt{x}$ corresponds to the point (1, 3) on the graph of f. The point (4, 2) on the graph of $y = \sqrt{x}$ corresponds to the point (4, 6) on the graph of f.

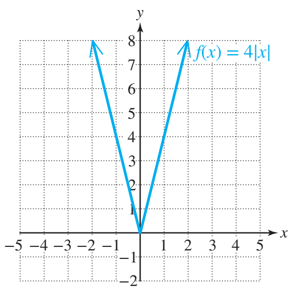
The domain of f is $[0, \infty)$.
The range of f is $[0, \infty)$.

Skill Practice Graph the function and determine the domain and range.

4. $f(x) = 4|x|$

Answer

4. Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



In Example 4, we apply multiple transformations to the graph of $y = f(x)$. In such a case, we apply a horizontal shift to the right or left first. Then we apply any vertical shrink, stretch, or reflection, followed by a vertical shift upward or downward.

Example 4 Applying Multiple Transformations to a Graph

Graph the function and determine the domain and range. $f(x) = -\dfrac{1}{2}|x - 1|$

Solution:

The graph of $y = |x|$ is the parent function (Figure 11-7).

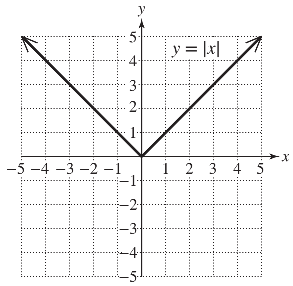


Figure 11-7

Next, the expression $x - 1$ within the absolute value bars indicates a shift to the right 1 unit (Figure 11-8).

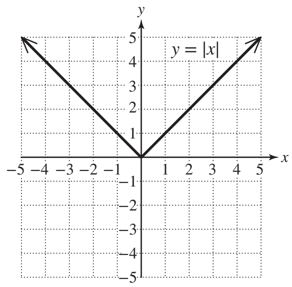


Figure 11-8

Next, the factor of $-\frac{1}{2}$ shrinks the graph vertically and reflects the graph over the x-axis. That is, the y-coordinates of the graph of $y = |x - 1|$ are multiplied by $-\frac{1}{2}$. For example, the point (3, 2) on the graph of $y = |x - 1|$ (shown in green in Figure 11-9) corresponds to the point (3, −1) on the graph of $f(x) = -\frac{1}{2}|x - 1|$ (shown in blue).

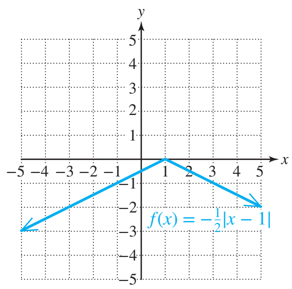


Figure 11-9

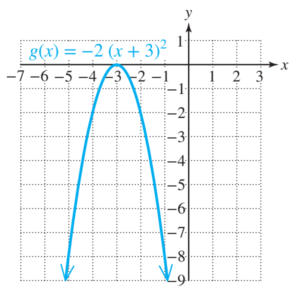
The domain of f is $(-\infty, \infty)$.
The range of f is $(-\infty, 0]$.

Skill Practice Graph the function and determine the domain and range.

5. $g(x) = -2(x + 3)^2$

Answer

5. Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



2. Piecewise-Defined Functions

Sometimes a function may be defined by more than one rule on different intervals within the domain. Such a function is called a **piecewise-defined function**. The function $y = k(x)$ pictured in Figure 11-10 is defined piecewise.

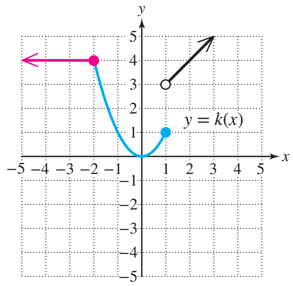


Figure 11-10

$$k(x) = \begin{cases} 4 & \text{if } x \le -2 \\ x^2 & \text{if } -2 < x \le 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

- For x values less than or equal to -2, the function is constant and follows the first rule, $k(x) = 4$ (pictured in red).
- For x values greater than -2 and less than or equal to 1, the function follows the second rule, $k(x) = x^2$ (pictured in blue).
- For x values greater than 1, the function follows the third rule, $k(x) = x + 2$ (pictured in black).

TIP: The open dot indicates that the point (1, 3) is *not* included in the graph.

Example 5 Evaluating a Piecewise-Defined Function

Evaluate the function for the given values of x.

a. $k(0)$
b. $k(-3)$
c. $k(5)$
d. $k(1)$
e. $k(-2)$

$$k(x) = \begin{cases} 4 & \text{if } x \le -2 \\ x^2 & \text{if } -2 < x \le 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

FOR REVIEW

Recall the meanings of the inequality symbols.

- Is less than: $<$
- Is greater than: $>$
- Is less than or equal to: $\le$
- Is greater than or equal to: $\ge$
- Is not equal to: $\ne$

Solution:

To evaluate a piecewise-defined function, the value of x determines which rule to use.

a. To evaluate $k(0)$, note that $x = 0$ is on the interval $-2 < x \le 1$. Use the *second* rule.

$$k(x) = x^2 \quad \text{for } -2 < x \le 1$$
$$k(0) = (0)^2$$
$$= 0$$

b. To evaluate $k(-3)$, note that $x = -3$ is on the interval $x \le -2$. Use the *first* rule.

$$k(x) = 4 \quad \text{for } x \le -2$$
$$k(-3) = 4$$

c. To evaluate $k(5)$, note that $x = 5$ is on the interval $x > 1$. Use the *third* rule.

$$k(x) = x + 2 \quad \text{for } x > 1$$
$$k(5) = (5) + 2$$
$$= 7$$

d. To evaluate $k(1)$, note that $x = 1$ is the right endpoint of the interval $-2 < x \leq 1$. Use the *second* rule.

$$k(x) = x^2 \quad \text{for } -2 < x \leq 1$$
$$k(1) = (1)^2$$
$$= 1$$

e. To evaluate $k(-2)$, we see that $x = -2$ is on the interval $x \leq -2$. Use the *first* rule. Note that $x = -2$ is *not* in the interval $-2 < x \leq 1$.

$$k(x) = 4 \quad \text{for } x \leq -2$$
$$k(-2) = 4$$

TIP: The function values in Example 5 can be confirmed with the graph of $y = k(x)$ given in Figure 11-10.

Skill Practice Evaluate the function for the given values of x.

$$g(x) = \begin{cases} x - 4 & \text{if } x < -1 \\ |x| & \text{if } -1 \leq x < 3 \\ 5 & \text{if } x \geq 3 \end{cases}$$

6. $g(-2)$ **7.** $g(4)$ **8.** $g(-0.5)$ **9.** $g(-1)$ **10.** $g(3)$

To graph a piecewise-defined function, graph each component function, showing only the portion of that function over the indicated domain.

Example 6 Graphing a Piecewise-Defined Function

Graph the function defined by $f(x) = \begin{cases} x + 3 & \text{for } x < -1 \\ |x| & \text{for } x \geq -1 \end{cases}$

Solution:

The first rule $f(x) = x + 3$ defines a line with slope of 1 and y-intercept (0, 3). This line should be graphed only on the interval $x < -1$ (i.e., to the left of $x = -1$). The point $(-1, 2)$ is an open dot, because the point is not part of the graph of the function. See Figure 11-11 (red portion of graph).

The second rule, $f(x) = |x|$ is the absolute value function. Its graph is a "V" shape with vertex at the origin. We sketch this function only for x values of -1 and greater. The point $(-1, 1)$ is a closed dot to show that it is part of the graph of the function. See Figure 11-11 (blue portion of the graph).

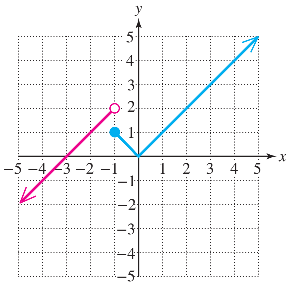


Figure 11-11

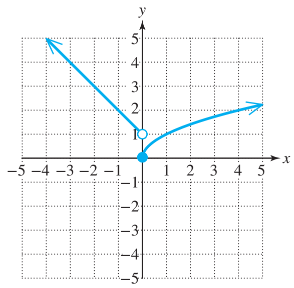
From the graph, we see that the domain of f is $(-\infty, \infty)$. The range is $(-\infty, \infty)$.

Skill Practice

11. Graph the function defined by $f(x) = \begin{cases} -x + 1 & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \geq 0 \end{cases}$

Answers

6. -6 **7.** 5 **8.** 0.5
9. 1 **10.** 5
11.



Section 11.1 Practice Exercises

Vocabulary and Key Concepts

1. Explain how the graph of $g(x) = (x + 2)^2 - 8$ is related to the graph of $y = x^2$.

2. Explain how the graph of $h(x) = -(x - 3)^2 + 5$ is related to the graph of $y = x^2$.

Concept 1: Transformations of Graphs

For Exercises 3–8, match the function with its graph.

3. $f(x) = x$

4. $f(x) = x^2$

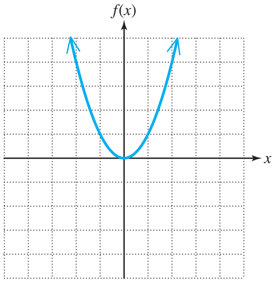
5. $f(x) = x^3$

6. $f(x) = |x|$

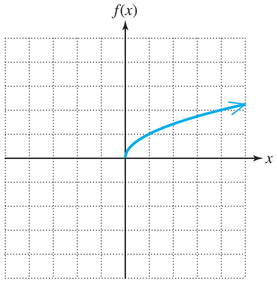
7. $f(x) = \sqrt{x}$

8. $f(x) = \dfrac{1}{x}$

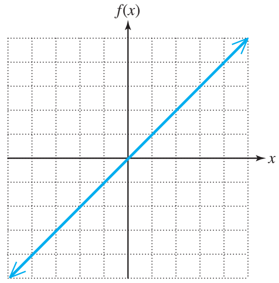
a.



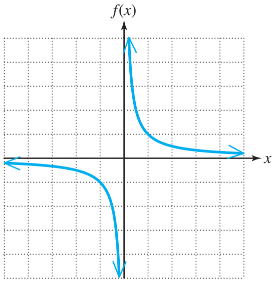
b.



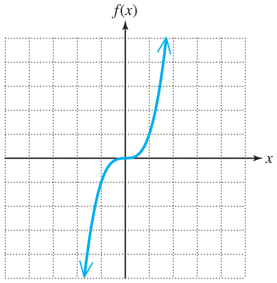
c.



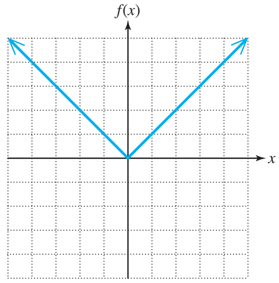
d.



e.

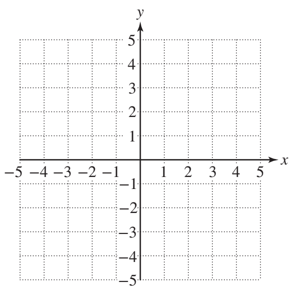


f.

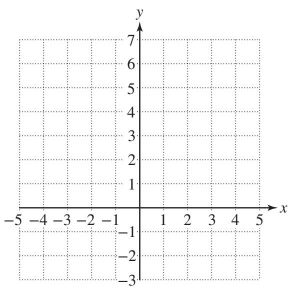


For Exercises 9–29, graph the function. Also determine the domain and range. **(See Examples 1–4.)**

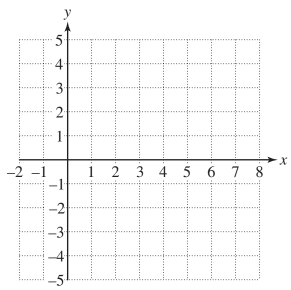
9. $f(x) = |x| - 4$



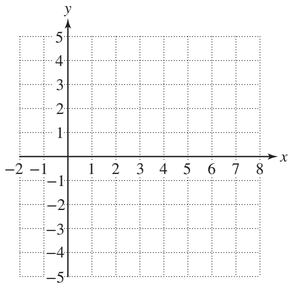
10. $g(x) = \sqrt{x} + 3$



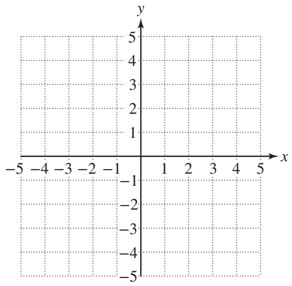
11. $p(x) = \sqrt{x - 3}$



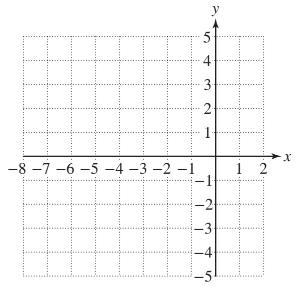
12. $q(x) = |x - 4|$



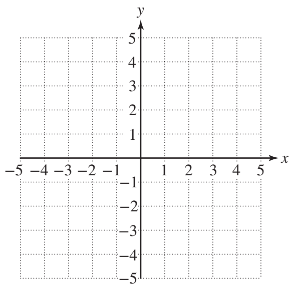
13. $h(x) = (x - 1)^3$



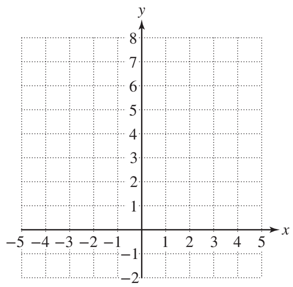
14. $k(x) = (x + 4)^2$



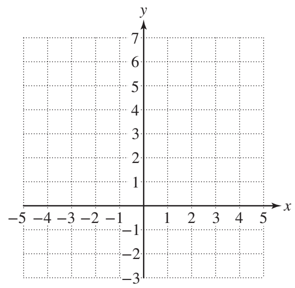
15. $t(x) = x^3 - 1$



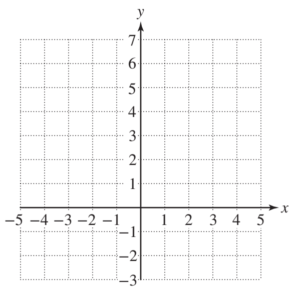
16. $w(x) = x^2 + 4$



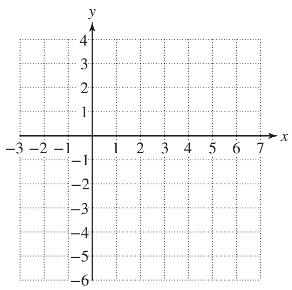
17. $n(x) = 2|x|$



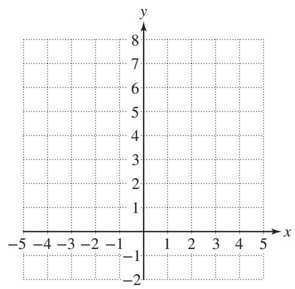
18. $v(x) = 3x^2$



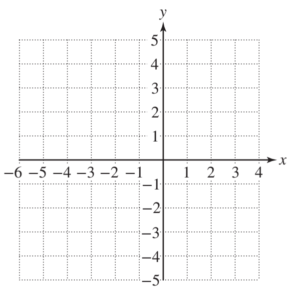
19. $f(x) = \sqrt{x + 1} - 4$



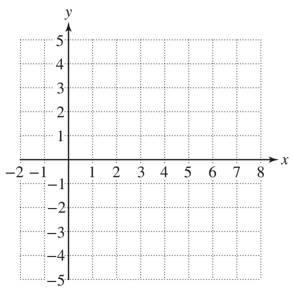
20. $g(x) = |x - 2| + 5$



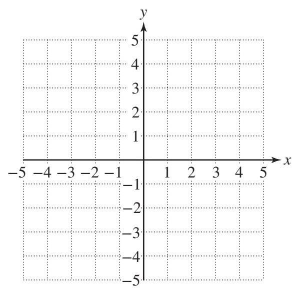
21. $c(x) = |x + 3| + 2$



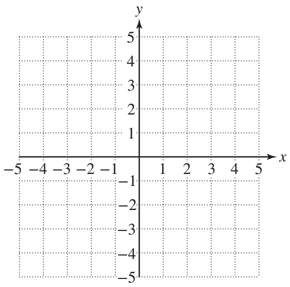
22. $d(x) = \sqrt{x - 4} + 2$



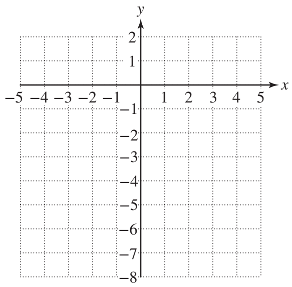
23. $h(x) = -\sqrt{x}$



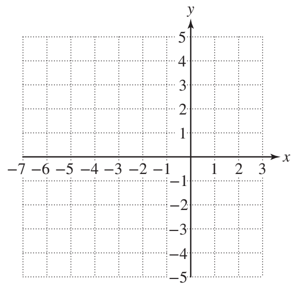
24. $k(x) = -\frac{1}{x}$



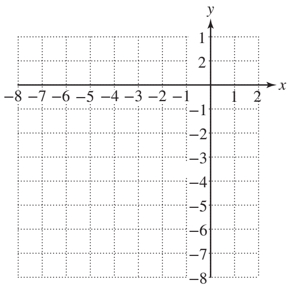
25. $v(x) = -(x - 1)^2 - 3$



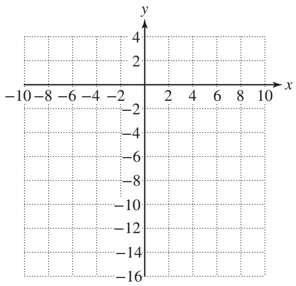
26. $f(x) = -(x + 2)^2 + 4$



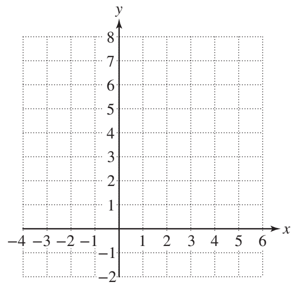
27. $r(x) = -2|x + 3| - 2$



28. $t(x) = -3\sqrt{x + 4} - 1$



29. $t(x) = \frac{1}{2}(x - 2)^2 + 1$



Concept 2: Piecewise-Defined Functions

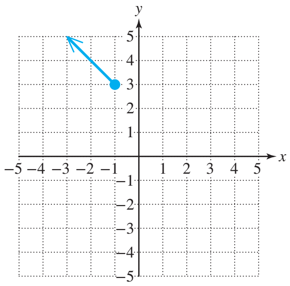
For Exercises 30–32, match the function with its graph.

30. $f(x) = -x + 2; x < -1$

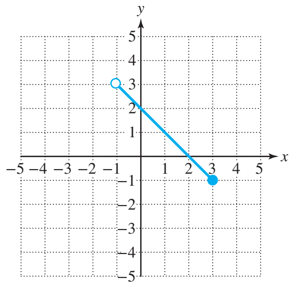
31. $f(x) = -x + 2; x \leq -1$

32. $f(x) = -x + 2; -1 < x \leq 3$

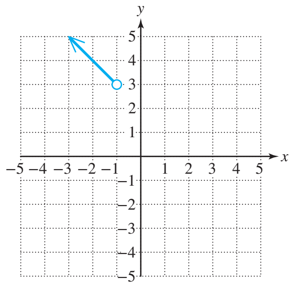
a.



b.



c.



For Exercises 33–36, evaluate the function for the given values of x. **(See Example 5.)**

33. $f(x) = \begin{cases} x^2 - 3 & \text{for } x \leq 1 \\ 2x + 1 & \text{for } x > 1 \end{cases}$

a. $f(3)$

b. $f(0)$

c. $f(1)$

d. $f(\frac{3}{2})$

34. $g(x) = \begin{cases} -|x| + 4 & \text{for } x \leq 2 \\ x - 5 & \text{for } x > 2 \end{cases}$

a. $g(4)$

b. $g(-1)$

c. $g(2)$

d. $g(-3)$

35. $h(x) = \begin{cases} 7 & \text{for } x < -2 \\ |x - 3| & \text{for } -2 \le x \le 1 \\ \sqrt{x} & \text{for } x > 1 \end{cases}$

a. $h(4)$

b. $h(-3)$

c. $h(-1)$

d. $h(-2)$

e. $h(1)$

36. $k(x) = \begin{cases} x + 2 & \text{for } x < -3 \\ 1 & \text{for } -3 \le x \le 3 \\ x^3 & \text{for } x > 3 \end{cases}$

a. $k(4)$

b. $k(-4)$

c. $k(-2)$

d. $k(-3)$

e. $k(3)$

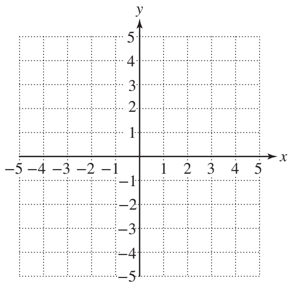
37. **a.** Graph $f(x) = x$ for $x < 0$.

b. Graph $g(x) = \sqrt{x}$ for $x \ge 0$.

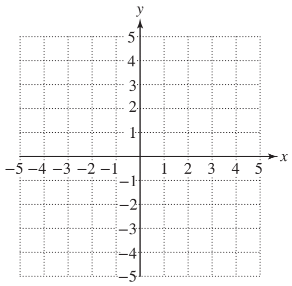
c. Sketch both $y = f(x)$ and $y = g(x)$ on the same graph to represent the piecewise defined function, h. **(See Example 6.)**

$$h(x) = \begin{cases} x & \text{for } x < 0 \\ \sqrt{x} & \text{for } x \ge 0 \end{cases}$$

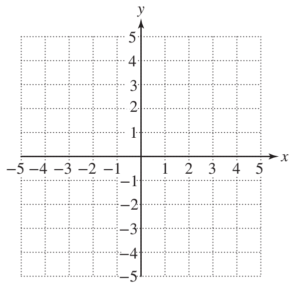
a.



b.



c.



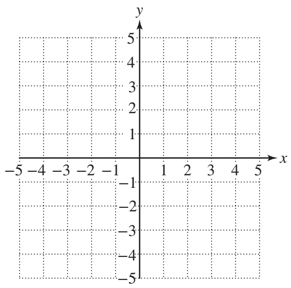
38. **a.** Graph $f(x) = |x|$ for $x \le 1$.

b. Graph $g(x) = 2$ for $x > 1$.

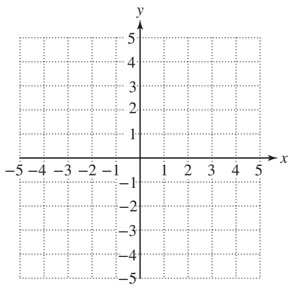
c. Sketch both $y = f(x)$ and $y = g(x)$ on the same graph to represent the piecewise-defined function, h.

$$h(x) = \begin{cases} |x| & \text{for } x \le 1 \\ 2 & \text{for } x > 1 \end{cases}$$

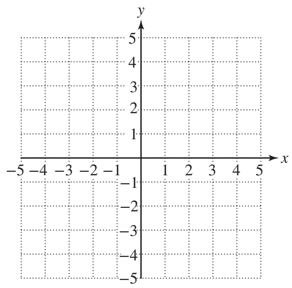
a.



b.

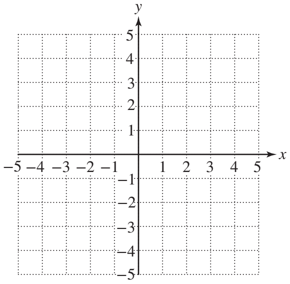


c.

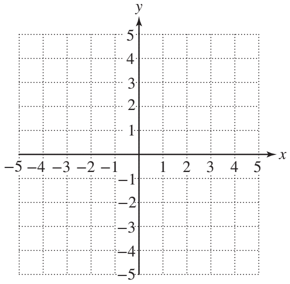


For Exercises 39–45, graph the function. Also determine the domain and range. **(See Example 6.)**

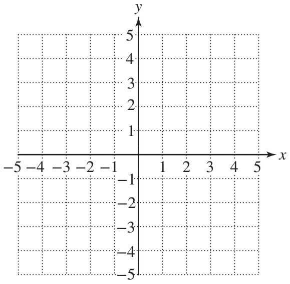
39. $f(x) = \begin{cases} 3 & \text{for } x \leq -1 \\ x^3 & \text{for } x > -1 \end{cases}$



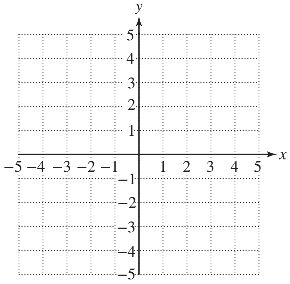
40. $g(x) = \begin{cases} \dfrac{1}{x} & \text{for } x < 0 \\ x + 1 & \text{for } x \geq 0 \end{cases}$



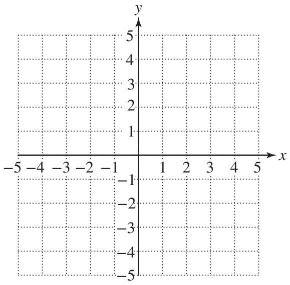
41. $h(x) = \begin{cases} x + 1 & \text{for } x < 2 \\ -x - 1 & \text{for } x \geq 2 \end{cases}$



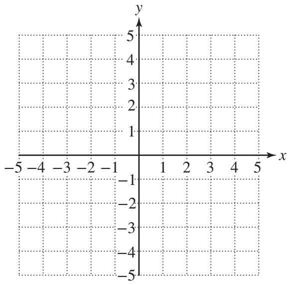
42. $k(x) = \begin{cases} 2x & \text{for } x < 1 \\ -2x & \text{for } x \geq 1 \end{cases}$



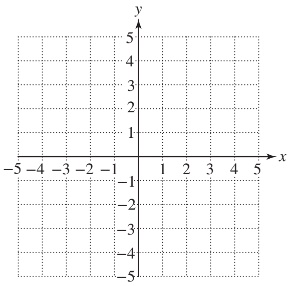
43. $p(x) = \begin{cases} x^2 - 4 & \text{for } x \leq 0 \\ 2x - 4 & \text{for } x > 0 \end{cases}$



44. $q(x) = \begin{cases} -x - 1 & \text{for } x < -1 \\ \sqrt{x + 1} & \text{for } x \geq -1 \end{cases}$



45. $f(x) = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$



46. What basic function resembles the graph of $y = f(x)$ from Exercise 45?

47. A bicyclist rides at a steady speed of 16 mph for 35 min. Then she encounters a hill and her average speed immediately slows to 10 mph for the next 25 min. Then, after reaching the summit, she rides down the hill and her speed increases at a rate of 2 mph per minute until she reaches the finish line 7 min later.

The bicyclist's speed can be approximated by the piecewise-defined function, where t is time measured in minutes and $s(t)$ is speed in mph.

$$s(t) = \begin{cases} 16 & \text{for } 0 \le t \le 35 \\ 10 & \text{for } 35 < t \le 60 \\ 2t - 110 & \text{for } 60 < t \le 67 \end{cases}$$

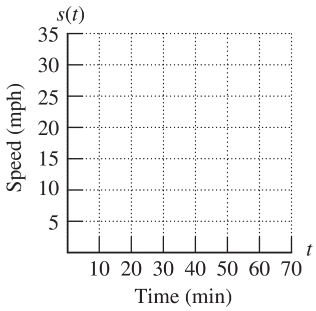
a. Evaluate $s(30)$ and interpret the meaning in the context of this problem.

b. Evaluate $s(40)$ and interpret the meaning.

c. Evaluate $s(65)$ and interpret the meaning.

d. Find the bicyclist's speed at the end of the ride.

e. Sketch $y = s(t)$.



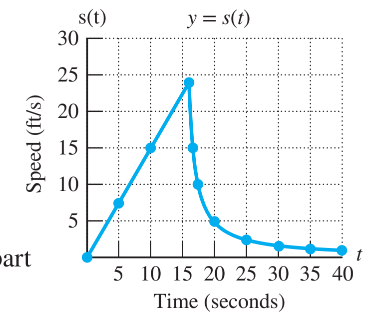
48. A sled accelerates (gains speed) down a hill and then slows down after it reaches a flat portion of ground. The speed of the sled can be approximated by the piecewise-defined function.

$$S(t) = \begin{cases} 1.5t & \text{for } 0 \le t \le 16 \\ \dfrac{24}{t - 15} & \text{for } t > 16 \end{cases}$$

where t is in seconds and $S(t)$ is in feet per second.

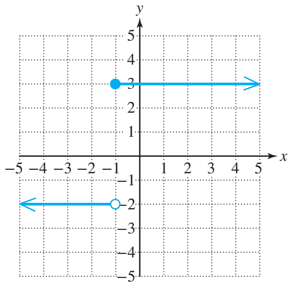
a. Evaluate $S(0)$, $S(10)$, $S(16)$, $S(20)$, and $S(25)$.

b. Locate the points defined by the function values in part (a) on the graph of $y = S(t)$.



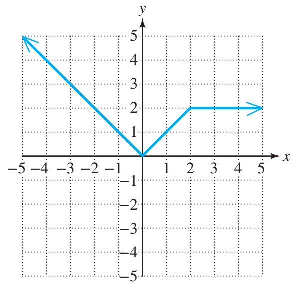
For Exercises 49–52, produce a rule for the function whose graph is shown.

49.



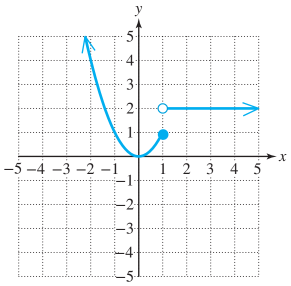
$f(x) = \{$

50.



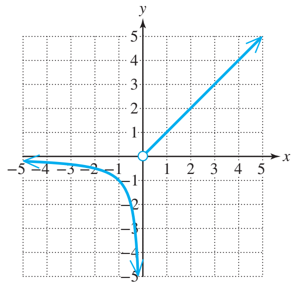
$f(x) = \{$

51.



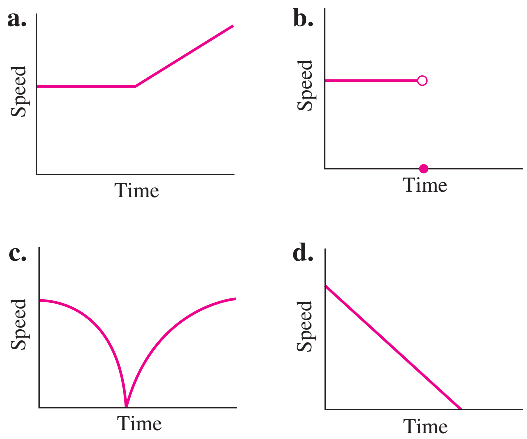
$f(x) = \{$

52.

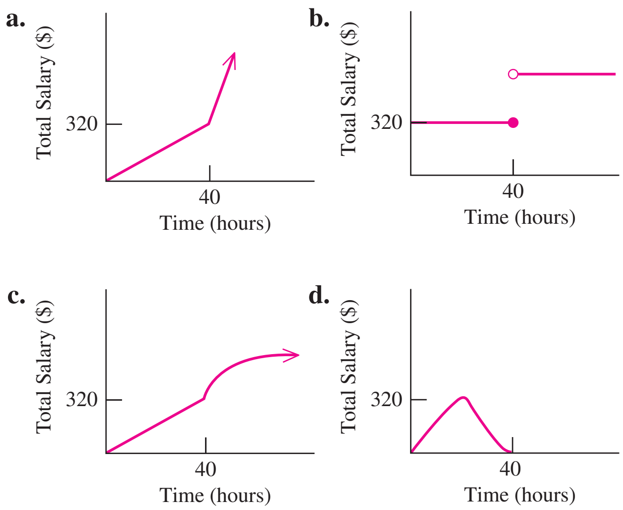


$f(x) = \{$

53. A shopping cart rolls across a parking lot at a *constant* speed and then crashes into a tree and *stops*. Which graph of speed as a function of time represents this scenario? Explain your answer.



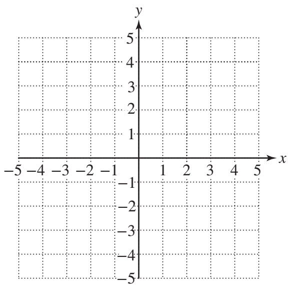
54. A student makes \$8 per hour for the first 40 hr worked in a week. Then she makes time and a half (\$12/hr) for the work exceeding 40 hr. Which graph best depicts her total salary as a function of time? Explain your reasoning.



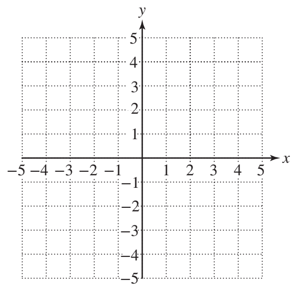
Expanding Your Skills

For Exercises 55–56, graph the function.

55. $f(x) = \begin{cases} -x - 3 & \text{for } x \leq -1 \\ x^2 & \text{for } -1 < x < 2 \\ 4 & \text{for } x \geq 2 \end{cases}$



56. $g(x) = \begin{cases} 3 & \text{for } x < -2 \\ |x| & \text{for } -2 \leq x < 1 \\ -x + 2 & \text{for } x \geq 1 \end{cases}$



Fundamentals of Counting

Section 11.2

Concepts

1. Fundamental Principle of Counting
2. Permutations
3. Combinations
4. Comparing Permutations and Combinations

1. Fundamental Principle of Counting

A child makes an ice cream sundae with one scoop of ice cream plus a syrup. Suppose there are three choices of ice cream (vanilla, chocolate, and strawberry) and two choices of syrup (fudge and caramel). For each of the three ice cream choices there are two possible syrups, yielding $3 \cdot 2 = 6$ possible sundae combinations (Figure 11-12).

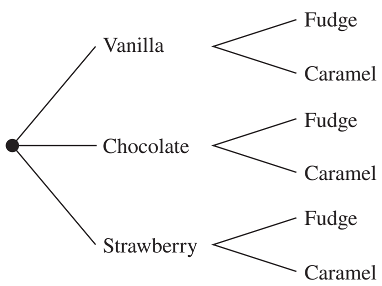


Figure 11-12

This example illustrates the **fundamental principle of counting**.

Fundamental Principle of Counting

If one event can occur in m different ways and a second event can occur in n different ways, then the sequence of both events can occur in $m \cdot n$ different ways.

The fundamental principle of counting can be extended to more than two events as demonstrated in Example 1.

Example 1 Applying the Fundamental Principle of Counting

Suppose Denisha visits Jimmy G's, a Cajun restaurant in Houston, Texas. She opts for a combo-meal in which she may choose from 12 different entrées, 4 different soups, and 3 different desserts. How many different dinners can she choose if she selects one item from each category?

Solution:

Applying the fundamental principle of counting, we have 144 different dinners.

$$\underbrace{12}_{\substack{\text{Number}\\ \text{of}\\ \text{entrées}}} \cdot \underbrace{4}_{\substack{\text{Number}\\ \text{of}\\ \text{soups}}} \cdot \underbrace{3}_{\substack{\text{Number}\\ \text{of}\\ \text{desserts}}} = 144$$

Skill Practice

1. Ten-year-old Max chooses his outfit for school each morning. He has 5 pairs of pants, 8 shirts, and 3 caps. Each day he chooses one pair of pants, one shirt, and one cap. How many different outfits can he choose?

Answer

1. $5 \cdot 8 \cdot 3 = 120$

Sometimes the events in a sequence depend on a preceding event as shown in Example 2.

Example 2 Applying the Fundamental Principle of Counting

Maximus has five different photos that he wants to arrange on a shelf. In how many different ways can he arrange the five photos?

Solution:

Think of the photo arrangement as five different slots on the shelf (Figure 11-13).

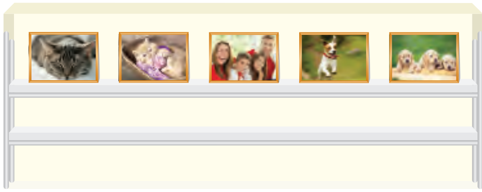


Figure 11-13

(cat nap): Michelle Jefferies/Flickr Open/Getty Images
(kitten): Lilun_Li/iStockphoto/Getty Images
(family): Martin Barraud/Caia Image/Glow Images
(dog): Ammit Jack/Shutterstock
(puppies): DAJ/Amana Images/Getty Images

The first slot can have any of the five pictures. However, once the first picture is in place, there are only four available for the second slot. Similarly, once the first two pictures are in place, there are only three remaining, and so on. The total number of photo arrangements is

$$\underline{\ \ 5\ \ } \cdot \underline{\ \ 4\ \ } \cdot \underline{\ \ 3\ \ } \cdot \underline{\ \ 2\ \ } \cdot \underline{\ \ 1\ \ } = 120.$$

Skill Practice

2. Chuck has seven award-winning photos to arrange in a row on the wall of his gallery. In how many ways can he arrange his photos on the wall?

The solution to Example 2 could have been written using factorial notation.

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$$

Let n be a positive integer. Recall that $n!$ (read as "n factorial") is defined as the product of integers from 1 to n. That is,

$$n! = n(n-1)(n-2) \cdots (1) \text{ and by definition, } 0! = 1.$$

Example 3 Applying the Fundamental Principle of Counting

Suppose eight horses run in a race. How many first-, second-, and third-place arrangements are possible, assuming no ties?

Solution:

Any of the eight horses can come in first place. That leaves seven possibilities for second place, and six possibilities for third. Therefore, there are $8 \cdot 7 \cdot 6 = 336$ possible first-, second-, and third-place arrangements.

Skill Practice

3. Ten performers are finalists in a talent competition that awards first- and second-place prizes. How many first- and second-place arrangements are possible?

Answers

2. $7! = 5040$ **3.** $10 \cdot 9 = 90$

2. Permutations

The scenario presented in Example 3 has three different horses selected in a specified order from a group of eight horses. Each of the 336 arrangements is called a permutation of eight horses taken three at a time. In general, an ordered arrangement of r different items selected from n different items is called a **permutation** of n items taken r at a time. The number of all such arrangements is given by the following formula.

Permutation Rule

If ${}_nP_r$ represents the number of permutations of n items taken r at a time, then

$${}_nP_r = \frac{n!}{(n-r)!}$$

Other commonly used notations for the number of permutations of n items taken r at a time are P_r^n and $P(n, r)$.

From Example 3, we found the number of ways three different horses could be selected from eight horses in a race in a specified order. This value is equivalent to ${}_8P_3$.

$${}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = 8 \cdot 7 \cdot 6 = 336$$

This is consistent with the result of Example 3.

Example 4 Computing Permutations

Compute. **a.** ${}_{12}P_4$ **b.** ${}_7P_7$

Solution:

a. $${}_{12}P_4 = \frac{12!}{(12-4)!} = \frac{12!}{8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot \cancel{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}$$

$$= 12 \cdot 11 \cdot 10 \cdot 9$$

$$= 11{,}880$$

This result implies that there are 11,880 ways to select 4 items from a group of 12 items, where the 4 items are selected in a specified order.

b. $${}_7P_7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 7! = 5040$$

This result implies that there are 5040 ways to select seven out of seven items in a specific order.

Avoiding Mistakes

Remember that by definition, $0! = 1$.

Skill Practice Compute.

4. a. ${}_6P_3$ **b.** ${}_9P_1$

Answer

4. a. 120 **b.** 9

Example 5 will apply the permutation formula to solve a counting problem.

Example 5 Applying Permutations to a Counting Problem

From a group of nine students, four are to be selected to receive cash prizes of \$100, \$50, \$25, and \$10, respectively. In how many ways can the four students be selected?

Solution:

Because four different students are to be selected from a group of nine in a specified order, we have ${}_9P_4$.

$${}_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$$

There are 3024 ways to pick four different students from a group of nine in a specific order.

TIP: The order of selection is important in Example 5, because the four prizes are different. For example, suppose person A is awarded the \$100 prize and B is awarded the \$50 prize. It would be a different outcome if B got the \$100 prize and A received the \$50 prize.

Skill Practice

5. From a group of 10 employees, Ali must choose 3 to receive bonuses of \$1000, \$500, and \$300. In how many ways can three employees be chosen to receive these bonuses?

3. Combinations

In Example 5, we saw that a permutation defines a selection of four distinct items in a specified order from nine items. Now we will compare an event in which the order is important to one in which the order is not important.

Suppose that from a group of four members of student government we want to select two to serve as president and vice-president. If we label the students as A, B, C, and D, we can list all possible selections. The first student listed is designated as president and the second student listed is designated as vice-president. There are 12 possible outcomes shown here.

$$\left.\begin{matrix} \text{AB} & \text{AC} & \text{AD} & \text{BC} & \text{BD} & \text{CD} \\ \text{BA} & \text{CA} & \text{DA} & \text{CB} & \text{DB} & \text{DC} \end{matrix}\right\} \text{12 permutations}$$

These outcomes represent the set of all permutations of four students taken two at a time. Therefore, the number of president/vice-president selections can also be found by computing

$${}_4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot \cancel{2 \cdot 1}}{\cancel{2 \cdot 1}} = 12$$

Now suppose that we want to choose two students from the group of four to form a safety committee. In this case, there is no specific label assigned to the two students selected (in other words, we are not assigning roles to the students such as president or vice-president). In some sense, the two people selected are "equals." They are not holding distinguishable positions. In such a case, we can see that the order of selection is not important. Hence, selecting A first followed by B forms the same committee as selecting B first followed by A. We do not want to count the outcomes AB and BA twice. Therefore, there are only six possible groups as shown here.

$$\left.\begin{matrix} \text{AB} & \text{AC} & \text{AD} & \text{CB} & \text{BD} & \text{CD} \end{matrix}\right\} \text{6 combinations}$$

Answer

5. ${}_{10}P_3 = 720$

In this situation, the order of selection is not important. In general, an *unordered* selection of r different items taken from n different items is called a **combination** of n items taken r at a time. The number of all such groupings is given by the following formula.

TIP: The number of combinations of n items taken r at a time can also be found by computing the number of permutations and dividing the result by $r!$. The factor of $r!$ divides out repeated combinations that result from order within the permutations. Therefore, the combination formula

$${}_nC_r = \frac{n!}{(n-r)! \cdot r!} \text{ can also}$$

be written as ${}_nC_r = \frac{{}_nP_r}{r!}$.

Combination Rule

If ${}_nC_r$ represents the number of combinations of n items taken r at a time, then

$${}_nC_r = \frac{n!}{(n-r)! \cdot r!}$$

To count the number of ways two students can be selected from a group of four without regard to order, we compute ${}_4C_2$.

$${}_4C_2 = \frac{4!}{(4-2)! \cdot 2!} = \frac{4!}{2! \cdot 2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (2 \cdot 1)} = 6$$

Notice that the combination formula differs from the permutation formula by a factor of $r!$ in the denominator. In this example, there is an additional factor of 2! in the denominator. This factor divides out the repeated arrangements within the groups of two that result from order. In other words, the factor of 2! divides the total list of permutations by 2 to divide out the repeated arrangements such as AB and BA, AC and CA, and so on.

Example 6 Computing Combinations

Compute. **a.** ${}_{10}C_2$ **b.** ${}_{12}C_{12}$

Solution:

a. $${}_{10}C_2 = \frac{10!}{(10-2)! \cdot 2!} = \frac{10!}{8! \cdot 2!} = \frac{10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!} \cdot 2 \cdot 1} = 45$$

This result implies that there are 45 ways to select 2 items from a group of 10 when the order of selection is not considered.

b. $${}_{12}C_{12} = \frac{12!}{(12-12)! \cdot 12!} = \frac{12!}{0! \cdot 12!} = \frac{\cancel{12!}}{1 \cdot \cancel{12!}} = 1$$

This result implies that there is one way to select 12 items from a group of 12 without regard to order.

TIP: You might have noticed that the formula to compute ${}_nC_r$ follows the same format as the formula used to compute the coefficients of a binomial expansion.

Skill Practice Compute.

6. a. ${}_8C_2$ **b.** ${}_{10}C_1$

Answer

6. a. 28 **b.** 10

Example 7 **Applying Combinations to a Counting Problem**

Sara picked out 15 different CDs that she likes equally well. However, she only has enough money to purchase 4 CDs. In how many ways can she select 4 CDs from the group of 15?

Solution:

In this example, there is no implied order. Therefore, the number of ways she can select 4 CDs from 15 CDs is given by ${}_{15}C_4$.

$${}_{15}C_4 = \frac{15!}{(15-4)! \cdot 4!} = \frac{15!}{11! \cdot 4!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot \cancel{11!}}{\cancel{11!} \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1365$$

Skill Practice

7. Coach Petersen has 8 girls on her tennis team, but is only able to take 5 of them to tennis camp for the summer. In how many ways can she select 5 girls from the group of 8?

4. Comparing Permutations and Combinations

In this section, three different rules for counting have been presented: the fundamental principle of counting, the permutation rule, and the combination rule. Perhaps the most difficult part of solving a counting problem is to select the most appropriate rule to apply.

- The fundamental principle of counting can always be applied, but it is not always the most convenient method.
- The permutation rule can be applied when r different items are selected from a group of n different items in a specified order.
- The combination rule can be applied when r different items are selected from a group of n different items in no specific order.

Answer

7. ${}_8C_5 = 56$

Section 11.2 Practice Exercises

Concept 1: Fundamental Principle of Counting

For Exercises 1–6, evaluate the factorial expression.

1. 5! **2.** 6! **3.** 8!

4. 4! **5.** 0! **6.** 1!

For Exercises 7–12, apply the fundamental principle of counting. **(See Examples 1–3.)**

7. At a certain hospital the dinner menu consists of 4 choices of entrées, 3 choices of salads, 8 choices of beverages, and 6 choices of desserts. How many different meals can be formed if a patient chooses one item from each category?

8. Mr. Dehili must dress for an important business meeting. He can choose his outfit from 3 different suits, 6 different shirts, and 12 different ties. Assuming that Mr. Dehili has no regard for color combination, how many different outfits can he form, given that he picks one item from each category?

9. In how many different ways can 6 people be seated in a row?

10. In how many different ways can 10 children line up to leave the classroom?

11. In a garden show, the best four flower arrangements receive the honors of first place, second place, third place, and honorable mention. How many ways can the awards be given if there are 16 arrangements to choose from?

12. At the All-State Music Festival, a group of 3 soloists will be awarded a blue ribbon, a red ribbon, and a silver ribbon. How many arrangements of the ribbons are possible if there are 12 soloists at the festival?

Concept 2: Permutations

13. Evaluate ${}_{10}P_3$ and interpret its meaning.

14. Evaluate ${}_{7}P_2$ and interpret its meaning.

For Exercises 15–22, compute the permutation. **(See Example 4.)**

15. ${}_{12}P_5$

16. ${}_{6}P_4$

17. ${}_{7}P_1$

18. ${}_{5}P_1$

19. ${}_{8}P_8$

20. ${}_{4}P_4$

21. ${}_{7}P_6$

22. ${}_{3}P_2$

23. In how many ways can a chairperson and an assistant chairperson be selected for the English department if there are eight faculty members qualified for the positions? **(See Example 5.)**

24. How many five-letter passwords can be formed from the letters in the name *Fermat?*

25. How many six-letter permutations can be made from the word *Euclid*?

26. In how many ways can five different cereal boxes be displayed on a shelf?

Concept 3: Combinations

27. Evaluate ${}_{10}C_3$ and interpret its meaning.

28. Evaluate ${}_{7}C_2$ and interpret its meaning.

For Exercises 29–36, compute the combination. **(See Example 6.)**

29. ${}_{12}C_9$

30. ${}_{6}C_4$

31. ${}_{7}C_1$

32. ${}_{5}C_1$

33. ${}_{8}C_8$

34. ${}_{4}C_4$

35. ${}_{7}C_6$

36. ${}_{3}C_2$

37. How many tests can be made if an instructor selects 12 questions from a bank of 15 questions? **(See Example 7.)**

38. A Shakespeare Festival offers five plays during one season. If Olivia can purchase tickets for three plays, how many combinations of plays will she have to choose from?

39. Nora and Stu decide to order two different desserts and plan to share them. How many different ways can they choose two desserts from five different desserts listed on the menu?

40. A team of 4 officers is selected to investigate a case. How many ways can the team be selected from 16 officers?

Concept 4: Comparing Permutations and Combinations

41. Given the set of elements {W, X, Y, Z},

a. List all permutations of two elements

b. List all combinations of two elements.

42. Given the set of elements {A, B, C},

a. List all permutations of two elements.

b. List all combinations of two elements.

For Exercises 43–50, use the permutation or combination rule.

43. How many first-, second-, and third-place finishes are possible in a dog race containing 10 dogs?

44. In how many ways can a judge award blue, red, and yellow ribbons to 3 films at a film festival if there are 12 films entered in the contest?

45. Heather wants to invite 11 girls from her fifth-grade class to a slumber party. However, her mother will only allow her to invite six people. In how many ways can she invite 6 girls out of the 11 to her party?

46. How many different five-member committees can be formed from 100 U.S. senators?

47. A book club offers 5 books for \$9.99 as an introductory offer. If you can choose from a list of 10 books, in how many ways can you make your selection of 5?

48. A basketball coach must pick 5 players from a roster of 12 to start a game. In how many ways can he choose his staring 5 players assuming that all players have equal ability?

49. Suppose there are eight employees who work at a chain coffee shop in Chicago. The manager wants to select two employees to work at two new shops (shop A and shop B) to help train new employees. If one of the selected employees is to work in shop A and the other is to work in shop B, how many possible selections can the manager make?

50. A disc jockey has five songs that he must play in a half-hour period. How many different ways can he arrange the five songs?

Mixed Exercises

For Exercises 51–66, use an appropriate rule of counting.

51. Most radio stations that were licensed after 1927 have four call letters starting with K or W such as WROD. Assuming no repetitions of letters, how many four-letter sets are possible?

52. A lock on a school locker consists of three different numbers taken from 1 to 39 in a specific order. How many three-number codes are possible?

53. In how many ways can a book buyer select 4 books from a list of 10 different books?

54. How many samples of size 6 can be selected from a population of 30 members?

55. Three men and three women have reserved six seats in a row at a concert. In how many ways can they arrange themselves if the men and women are to alternate seats and a man must sit in the first seat?

56. Three men and three women have reserved six seats in a row at a concert. In how many ways can they arrange themselves if the men must all sit together and the women must all sit together?

57. A musician plans to perform nine selections. In how many ways can she arrange the musical selections?

58. From a pool of 12 candidates, the offices of president, vice-president, treasurer, and secretary must be filled. In how many different ways can the offices be filled?

59. From a jury pool of 40 people, 12 are to be selected. In how many different ways can a jury of 12 be selected?

60. To play the Georgia lottery, a person must choose 6 numbers (in any order) from a list of 49 numbers. How many different choices of 6 numbers are possible?

61. A committee is to be formed from a collection of 10 men and 8 women. How many committees can be made consisting of exactly 3 men and 1 woman?

62. A committee is to be formed from a collection of 10 men and 8 women. How many committees can be made consisting of 2 men and 2 women?

63. If a fair coin is flipped three times, how many different sequences of heads and tails can be formed?

64. If a couple plans to have four children, how many different gender sequences can be formed?

65. In how many ways can a 10-question true or false test be answered assuming that a student answers all questions?

66. At a pizza place, a customer can order a pizza with or without any of the following options: pepperoni, sausage, mushrooms, peppers, onions, olives, or anchovies. How many different pizzas can be formed?

Expanding Your Skills

67. In how many ways can the batting order be determined for a co-ed softball team with six women and three men if

a. There are no restrictions?

b. The first and the last batters must be women?

c. The men must all be after the women?

68. In how many ways can a group of six men and five women be lined up for a photograph if

a. There are no restrictions?

b. The men and women must alternate?

c. There must be a man on each end?

69. Given the set of numbers $\{2, 3, 4, 5, 6, 7, 8\}$,

a. How many different three-digit numbers can be formed?

b. How many different three-digit numbers can be formed if the number cannot have repeated digits?

c. How many different three-digit numbers can be formed if the number is to be even and repetition of digits is allowed?

70. Given the set of numbers $\{1, 2, 3, 4, 5, 6, 7\}$,

a. How many different four-digit numbers can be formed?

b. How many different four-digit numbers can be formed if the number cannot have repeated digits?

c. How many different four-digit numbers can be formed if the number is to be divisible by 5 and repetition of digits is allowed?

For Exercises 71–74, use the following description of a standard deck of 52 cards. A standard deck of cards has four suits (clubs, diamonds, hearts, and spades). Each suit has 13 cards. The hearts and diamonds are red cards, and the clubs and spades are black cards. Assume that five cards are selected from a standard deck. In how many ways can the following occur?

71. All five cards are black.

72. All five cards are hearts.

73. There are three diamonds and two clubs.

74. There are three red cards and two black cards.

Section 11.3 Introduction to Probability

Concepts

1. Basic Definitions
2. Probability of an Event
3. Estimating Probabilities From Observed Data
4. Events Expressed as Alternatives

1. Basic Definitions

The study of probability provides a mathematical means to measure the likelihood of an event occurring. It is of particular interest because of its application to everyday events.

- The National Cancer Institute estimates that a woman has a 1 in 8 chance of developing breast cancer in her lifetime.
- The probability of winning the California Fantasy Five lottery grand prize is

$$\frac{1}{575{,}757}$$

- Genetic DNA analysis can be used to determine the risk that a child will be born with cystic fibrosis. If both parents test positive, the probability is 25% that a child will be born with the disease.

To begin our discussion, we must first understand some basic definitions.

An activity with observable outcomes is called an **experiment**. Each repetition of an experiment is called a **trial**. The result of a trial is called an **outcome** of the experiment. The set of all possible outcomes of an experiment is called the **sample space**, S, of the experiment.

TIP: The word *die* is the singular of the word *dice*. Therefore, we may roll a pair of dice, but we roll a single die.

For example, if a single die is rolled, the sample space is $\{1, 2, 3, 4, 5, 6\}$. If a coin is tossed, the outcomes are heads (H) or tails (T). The sample space is $\{\text{H}, \text{T}\}$.

Any subset of a sample space is called an **event**. For example, if we define event, E_1, as the event that a number greater than 4 is rolled on a die, then, $E_1 = \{5, 6\}$. If event E_2 is the event that a coin lands as a head, then $E_2 = \{\text{H}\}$.

2. Probability of an Event

The number of elements in a sample space is denoted by $n(S)$. The number of elements in the sample space that are also in event E is denoted by $n(E)$. The notation $P(E)$ denotes the probability of event E, defined as follows:

Probability of Event *E*

In a sample space S of equally likely outcomes, the **probability of** $\boldsymbol{E}$ is given by

$$P(E) = \frac{\text{number of elements in the event}}{\text{number of elements in the sample space}} = \frac{n(E)}{n(S)}$$

For the event, E_1, of rolling a number greater than 4 on a die, we have $E_1 = \{5, 6\}$ and $S = \{1, 2, 3, 4, 5, 6\}$. Then

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

For the event, E_2, that a coin will land as a head, we have $E_2 = \{\text{H}\}$ and $S = \{\text{H}, \text{T}\}$. Then

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{1}{2}$$

The value of a probability can be written as a fraction, as a decimal, or as a percent. Therefore, $P(E_2) = \frac{1}{2}$, or 0.5 or 50%. In words, this means that theoretically we expect half (50%) of the outcomes to land as a head. This does not necessarily mean that exactly one out of every two coin tosses will land as a head. Instead, it is a ratio that we expect experimental observations to approach after a large number of trials. For example, if we flip a coin 1000 times, we might get 493 heads for a ratio of $\frac{493}{1000}$, or 0.493. This value is close to the theoretical value of 0.500.

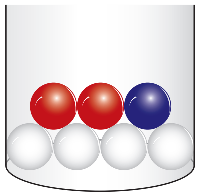
Example 1 Computing Probabilities

A box contains two red, four white, and one blue marble. Suppose one marble is selected at random. Find the probability of selecting

a. A red marble.

b. A white marble.

c. A green marble.



Solution:

Denote the sample space as $S = \{R_1, R_2, W_1, W_2, W_3, W_4, B_1\}$.

a. Let R represent the event of selecting a red marble. Because there are two red marbles, then $R = \{R_1, R_2\}$. Then

$$P(R) = \frac{n(R)}{n(S)} = \frac{2}{7}$$

b. Let W represent the event of selecting a white marble. Because there are four white marbles, then $W = \{W_1, W_2, W_3, W_4,\}$. Then $P(W) = \frac{4}{7}$.

c. Let G represent the event of selecting a green marble. Because there are no green marbles in the box, then G equals the empty set. $G = \{\ \ \}$. Then $P(G) = \frac{0}{7} = 0$.

Skill Practice

1. A bag of M&M's contains 7 red, 10 brown, 8 yellow, and 4 blue candies. Find the probability of choosing:

a. A red M&M.

b. A brown M&M.

c. A blue M&M.

From Example 1(c), the event of selecting a green marble is impossible. The probability of an **impossible event** is 0. An event that is certain to happen is called a **certain event**. Its probability is 1. For example, if a die is tossed, the probability that the die will land as a number between 0 and 7 is certain to happen. Any of the six outcomes 1, 2, 3, 4, 5, 6 will satisfy the event. Therefore, the probability of rolling a number between 0 and 7 is $\frac{6}{6} = 1$. In general, for any event E,

$$0 \le P(E) \le 1$$

The counting rules we have already learned can be helpful in determining the number of elements in an event and in a sample space. This is illustrated in Example 2.

Answer

1. a. $\frac{7}{29}$ **b.** $\frac{10}{29}$ **c.** $\frac{4}{29}$

Example 2 **Applying the Counting Rules to Probability**

Suppose a group of politicians consists of nine men and five women. If three people are selected at random to form a committee, what is the probability that all are women?

Solution:

Let W represent the event that a committee of three women is selected.

Let S represent the sample space consisting of all possible committees of three with no restrictions.

There are 14 people available to form a committee of 3. If no restrictions are imposed, the number of possible committees of three is given by ${}_{14}C_3 = 364$. The number of possible committees of three selected from the group of women is given by ${}_5C_3 = 10$. Therefore, the probability of selecting a committee that consists of all women is given by

$$P(W) = \frac{n(W)}{n(S)} = \frac{{}_5C_3}{{}_{14}C_3} = \frac{10}{364} = \frac{5}{182}$$

Skill Practice

2. A cooler of drinks contains eight bottles of water and four sodas. If two drinks are chosen at random, what is the probability that both are water?

3. Estimating Probabilities From Observed Data

We were able to compute the probabilities in Examples 1 and 2 because the sample space was known. Sometimes we need to collect information from a series of repeated trials to help us estimate probabilities.

Example 3 **Estimating Probabilities From Observed Data**

In a carnival game, Erin will win a prize if she can toss a ring around the neck of a bowling pin. After observing 200 players that had gone before her, she learns that 15 players won a prize. Based on this observation, what is the probability of winning a prize?

Solution:

In this situation, we may think of the outcomes of the 200 trials as the sample space. In this case, 15 trials came out as wins. Therefore, if W represents the event of a win, then

$$P(W) = \frac{15}{200} = \frac{3}{40}, \text{ or } 7.5\%$$

Skill Practice

3. In a class of 88 students, it was observed that 11 were left-handed. Based on this observation, what is the probability of being left-handed?

Answers

2. $\frac{{}_8C_2}{{}_{12}C_2} = \frac{28}{66} = \frac{14}{33}$

3. $\frac{11}{88} = \frac{1}{8}$ or 0.125

Example 4 Determining Probabilities From a Graph

The data in the pie chart (Figure 11-14) categorize the number of accidental deaths in the United States for a recent year.

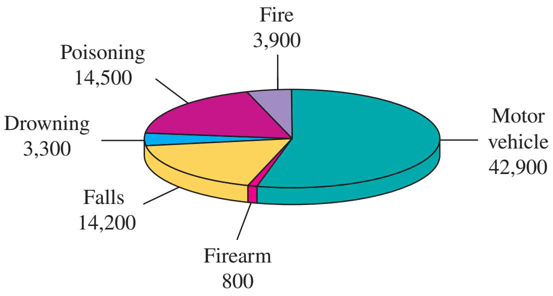


Figure 11-14

a. Based on the chart, what is the probability that an accidental death is caused by fire?

b. What is the probability that an accidental death is caused by a motor vehicle?

Solution:

First note that the total number of accidental deaths depicted in the graph is 79,600.

a. Let F represent the event that the death is caused by fire.

$$\text{Then } P(F) = \frac{3900}{79{,}600} = \frac{39}{796} \text{ or about } 4.9\%.$$

b. Let M represent the event that the death was caused by a motor vehicle.

$$\text{Then } P(M) = \frac{42{,}900}{79{,}600} = \frac{429}{796} \text{ or about } 53.9\%.$$

Skill Practice

4. Based on Figure 11-14 in Example 4, what is the probability that an accidental death is caused by drowning?

4. Events Expressed as Alternatives

A compound event in probability involves two or more events. If two events are expressed as alternatives, they can be considered as a compound event often joined by the word *or*.

Answer

4. $\frac{3300}{79{,}600} = \frac{33}{796} \approx 4.1\%$

Example 5 Finding the Probability of Two or More Events

The safety and security department at a certain college asked a sample of 265 students to respond to the following question.

"Do you think that the College has adequate lighting on campus at night?"

The table shows the results of the survey by gender and response.

	Yes	No	No Opinion	Total
Male	92	7	4	103
Female	36	102	24	162
Total	128	109	28	265

If one student is selected at random from the group, find the probability that

a. The student answered yes or had no opinion.

b. The student answered no or was female.

Solution:

a. Let Y be the event that a student answered yes, and let Z represent the event that the student had no opinion. There are 128 people who answered "yes" and 28 who had "no opinion." The total number of unique elements in the event Y or Z is 156. Therefore,

$$P(Y \text{ or } Z) = \frac{156}{265} \text{ or about } 58.9\%$$

b. Let N be the event that a student answered no, and let F be the event that a student is female. The events N and F "overlap." That is, some people who answered "no" are also "female." We must be careful not to count any element in the sample space twice. There are 109 people who answered "no" (some of whom are female). Counting 109 "no" answers and the remaining females, we have a total of $109 + 36 + 24 = 169$ unduplicated individuals who answered "no" or who are "female." Therefore,

$$P(N \text{ or } F) = \frac{169}{265} \text{ or about } 63.8\%$$

Skill Practice

5. Fifty college students were asked if they owned a PC or Mac computer. The chart shows the results of the survey. If one student is selected at random from this group, what is the probability that:

a. The student has a Mac or no computer?

b. The student is a male or owns a PC?

	PC	Mac	No Computer	Total
Male	18	8	5	31
Female	12	3	4	19
Total	30	11	9	50

Answer

5. a. $\frac{2}{5}$ **b.** $\frac{43}{50}$

Practice Exercises

Section 11.3

Review Exercises

1. In how many ways can 4 songs be selected from 10 different songs without regard to order?

2. Samira wants her father to buy her six toys, but he only has enough money to buy two. In how many ways can Samira's father choose two toys from six?

3. In how many different orders can the five musical notes A, B, C, D, and E be played?

4. In how many ways can Mr. Zahnan rank three movies from a group of seven?

Concept 1: Basic Definitions

5. Which of the values can represent the probability of an event?

a. 57% **b.** $\frac{3}{2}$ **c.** $\frac{1}{4}$

d. 0.8 **e.** 120% **f.** -0.41

6. Which of the values can represent the probability of an event?

a. 0.5 **b.** $\frac{2}{3}$ **c.** $-\frac{7}{5}$

d. 1.00 **e.** 150% **f.** 3.7

7. Which of the values can represent the probability of an event?

a. 1.62 **b.** $-\frac{2}{7}$ **c.** 0.00

d. 200% **e.** 1.00 **f.** 0.87

8. Which of the values can represent the probability of an event?

a. 4.5 **b.** 4.5% **c.** $\frac{5}{4}$

d. -0.6 **e.** 0.00 **f.** 0.02%

Concept 2: Probability of an Event

9. If a single die is rolled, what is the probability that it will come up as an odd number?

10. If a single die is rolled, what is the probability that it will come up as a number divisible by 3?

11. If a single die is rolled, what is the probability that it will come up as a number less than 5?

12. If a single die is rolled, what is the probability that it will come up as a number greater than 5?

13. A jar contains 7 yellow marbles, 3 red marbles, and 6 green marbles. What is the probability of selecting a red marble? **(See Example 1.)**

14. A jar contains 4 yellow marbles, 10 red marbles, and 6 green marbles. What is the probability of selecting a yellow marble?

15. In a group of eight students, three are female and five are male. If a committee of two is to be selected at random, what is the probability that

a. Both members are female? **(See Example 2.)**

b. Both members are male?

16. In a group of 10 students at a community college, 6 are freshmen and 4 are sophomores. If a committee of two is selected at random, what is the probability that

a. Both members are sophomores?

b. Both members are freshmen?

17. In the California Fantasy Five lottery, a player wins the grand prize if the player picks the winning 5 numbers (in any order) out of 39 numbers.

a. What is the probability that a player will win the grand prize?

b. What is the probability that a player will not win the grand prize?

c. Joanne thinks that the probability of winning the grand prize in the lottery is 0.50, because according to her "There's a 50-50 chance of winning, because you either win or lose." What is wrong with Joanne's logic?

18. In the Florida lottery, a player wins a grand prize if the player picks the winning 6 numbers (in any order) out of 53 numbers.

a. What is the probability that a player will win the grand prize?

b. What is the probability that a player will not win the grand prize?

Concept 3: Estimating Probabilities From Observed Data

19. The final exam in a course in contemporary science resulted in the following distribution.

Grade	A	B	C	D	F
Number of Students	8	15	21	10	5

a. What is the probability that a student selected at random received an "A" in the course? (See Example 3.)

b. What is the probability that a student did not pass the course if a passing grade is a "C" or better?

20. A sample of students is taken from a physical education class at a 4-year college. The distribution is given in the table. If one student is selected at random, find the probability that

a. The student is a sophomore.

b. The student is not a senior.

Class	Number of Students
Freshman	15
Sophomore	11
Junior	6
Senior	3

21. The tardy record for a group of second-graders for one school year is given in the table. If one student is picked at random, find the probability that the student was late:

a. Exactly 3 days.

b. Between 1 and 5 days, inclusive.

c. At least 4 days.

d. More than 5 days or fewer than 2 days.

Number of Days Late	Number of Students
0	4
1	2
2	14
3	10
4	16
5	18
6	10
7	6

22. The table displays the length of stay for vacationers at a small motel.

a. What is the probability that a vacationer will stay for 4 days?

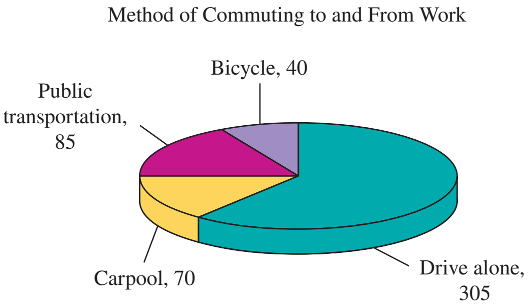
b. What is the probability that a vacationer will stay for less than 4 days?

Length of Stay in Days	Frequency
2	14
3	13
4	18
5	28
6	11
7	30
8	6

23. The data in the pie chart categorize the method by which workers in a college town commute to work. If one working member of the community is selected at random, find the probability that the individual

a. Commutes by bicycle. **(See Example 4.)**

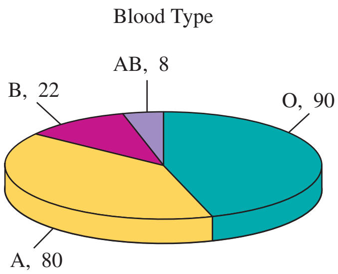
b. Does not use public transportation.



24. The data in the pie chart categorize the blood types in a sample of students. If one student is selected at random, find the probability that the individual

a. Has blood type B.

b. Does not have blood type O.



Concept 4: Events Expressed as Alternatives

For Exercises 25–30, use the following table categorizing a sample of smokers and nonsmokers according to their blood pressure levels. **(See Example 5.)**

	Normal Blood Pressure	Elevated Blood Pressure	Total
Smokers	32	18	50
Nonsmokers	71	9	80
Total	103	27	130

If one person from the sample is selected at random, find the probability that

25. The person has elevated blood pressure.

26. The person is a nonsmoker.

27. The person has normal blood pressure.

28. The person is a smoker or has normal blood pressure.

29. The person is a nonsmoker or has normal blood pressure.

30. The person is a nonsmoker or has elevated blood pressure.

For Exercises 31–36, use the following table categorizing a sample of alcoholics and nonalcoholics according to their cholesterol level.

	Elevated Cholesterol	Normal Cholesterol	Total
Alcoholic	120	30	150
Nonalcoholic	60	240	300
Total	180	270	450

If one person from the sample is selected at random, find the probability that

31. The person is nonalcoholic.

32. The person has elevated cholesterol.

33. The person does not have elevated cholesterol.

34. The person is an alcoholic or has elevated cholesterol.

35. The person is a nonalcoholic or has normal cholesterol.

36. The person is an nonalcoholic or has elevated cholesterol.

A standard deck of cards has 52 cards divided into four suits: clubs, diamonds, hearts, and spades. Each suit has 13 cards consisting of an ace, a king, a queen, a jack, and cards numbered from 2 to 10. Assume that one card is selected from a standard deck. For Exercises 37–48, find the probability of selecting the indicated card.

37. A heart

38. A face card (face cards are jacks, queens, and kings)

39. A red card (hearts and diamonds are red)

40. A red card or a jack

41. A heart or a 6

42. A spade or a heart

43. A club

44. A 5 or a 10

45. A black card (clubs and spades are black)

46. A black card or an ace

47. A diamond or a 2

48. A club or a diamond

Expanding Your Skills

49. The firefighters in one county in the United States consist of 600 members: 120 women and 480 men. Over a 5-year period, 160 firefighters were promoted. The table shows the breakdown of promotions for male and female firefighters.

	Promoted	Not Promoted	Total
Male	140	340	480
Female	20	100	120
Total	160	440	600

If one firefighter is selected at random, what is the probability that

a. The firefighter is a female?

b. The firefighter was promoted?

c. The firefighter is male or was not promoted?

d. The firefighter was promoted, given that he is male?

e. The firefighter was promoted, given that she is female?

f. Write the probabilities in parts (d) and (e) in decimal form (round to 3 decimal places). What does the difference in probabilities mean in the context of this problem?

50. The following table depicts the grade distribution for a college algebra class based on age and grades.

	A	B	C	F	Total
17–26 years	150	220	370	180	920
27–36 years	140	200	220	90	650
37–46 years	150	120	60	40	370
Total	440	540	650	310	1940

If one student is selected at random from the group, find the probability that

a. The student received an “A” in the course.

b. The student was in the 27–36 age group.

c. The student was in the 37–46 age group or received a “B” in the course.

d. The student received an “A” in the course, given that the student is in the 37–46 age group.

e. The student received an “A” in the course given that the student is in the 17–26 age group.

f. Write the probabilities in parts (d) and (e) in decimal form (round to 3 decimal places). What does the difference in probabilities mean in the context of this problem?

Chapter 11 Summary

Section 11.1 Transformation of Graphs and Piecewise-Defined Functions

Key Concepts

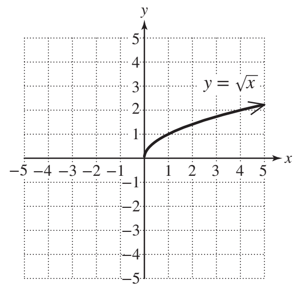
The graph of $y = a \cdot f(x - h) + k$ is the graph of $y = f(x)$ with the following transformations.

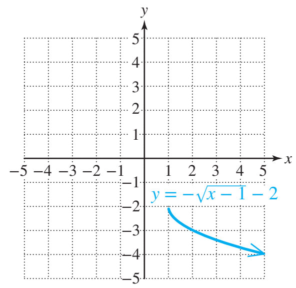
- Horizontal shift h units
- Vertical shift k units
- Vertical shrink if $0 < |a| < 1$
- Vertical stretch if $|a| > 1$
- Reflection across the x-axis if $a < 0$

Examples

Example 1

Graph $y = -\sqrt{x - 1} - 2$ by shifting the graph of $y = \sqrt{x}$ 1 unit to the right, reflecting $y = \sqrt{x}$ over the x-axis, and shifting the graph 2 units downward.



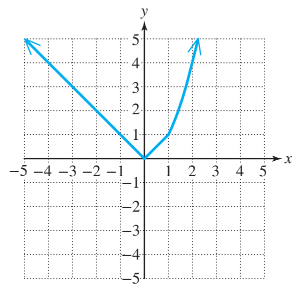


A **piecewise-defined function** is defined by more than one rule on different intervals within the domain.

Example 2

Graph the piecewise-defined function.

$$f(x) = \begin{cases} |x| & \text{for } x < 1 \\ x^2 & \text{for } x \geq 1 \end{cases}$$



Section 11.2 Fundamentals of Counting

Key Concepts

The **fundamental principle of counting** states that if one event can occur in m different ways and a second event can occur in n different way, then the sequence of both events can occur in $m \cdot n$ different ways.

A **permutation** is an ordered arrangement of items. The number of permutations of n items taken r at a time is given by ${}_nP_r = \dfrac{n!}{(n-r)!}$.

A **combination** is an unordered arrangement of items. The number of combinations of n items taken r at a time is given by ${}_nC_r = \dfrac{n!}{(n-r)! \cdot r!}$.

Examples

Example 1

The number of ways six people can stand in line for a picture is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

Example 2

The number of ways that a president and vice president can be selected from a board of five directors is

$${}_5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot \cancel{3}!}{\cancel{3}!} = 20$$

Example 3

The number of ways that a two-person subcommittee can be selected from a five-person committee is

$${}_5C_2 = \frac{5!}{(5-2)! \cdot 2!} = \frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4 \cdot \cancel{3}!}{\cancel{3}! \cdot (2 \cdot 1)} = 10$$

Section 11.3 Introduction to Probability

Key Concepts

The **sample space** S of an experiment is the set of all possible outcomes. An **event** E is a subset of the sample space.

The **probability of an event** E is given by

$$P(E) = \frac{\text{number of elements in the event}}{\text{number of elements in the sample space}}$$

$$= \frac{n(E)}{n(S)}$$

The probability of an event E is a value between 0 and 1, inclusive. That is, $0 \le P(E) \le 1$. The probability of an **impossible event** is 0. The probability of a **certain event** is 1.

Examples

Example 1

A person is holding five cards consisting of two kings, one queen, and two jacks.

a. What is the probability of selecting a jack from this hand?

$$P(\text{jack}) = \frac{n(\text{jacks})}{n(\text{total cards})} = \frac{2}{5}$$

b. What is the probability of selecting a face card?

$$P(\text{face card}) = \frac{n(\text{face cards})}{n(\text{total cards})} = \frac{5}{5} = 1$$

c. What is the probability of selecting an ace?

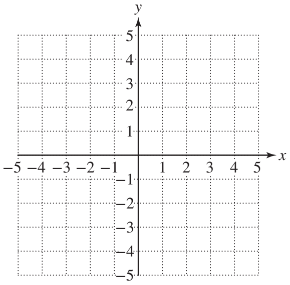
$$P(\text{ace}) = \frac{n(\text{aces})}{n(\text{total cards})} = \frac{0}{5} = 0$$

Chapter 11 Review Exercises

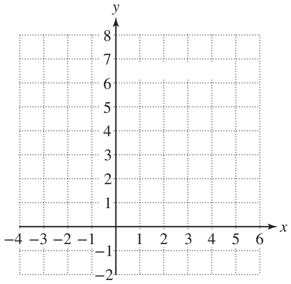
Section 11.1

For Exercises 1–4, graph the function.

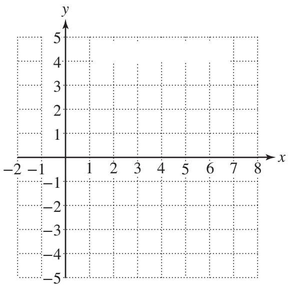
1. $f(x) = x^2 - 2$



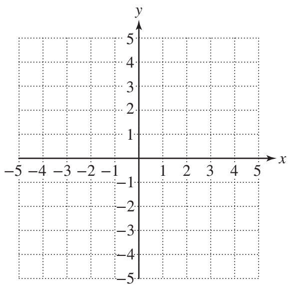
2. $g(x) = 2|x - 2|$



3. $f(x) = \sqrt{x-2} + 1$



4. $h(x) = -x^3 - 1$



For Exercises 5–8, evaluate the function for the given values of x.

5. $h(x) = \begin{cases} x^2 & \text{if } x < -2 \\ x + 4 & \text{if } -2 \le x < 1 \\ \sqrt{x} & \text{if } x \ge 1 \end{cases}$

a. $h(-2)$ **b.** $h(0)$ **c.** $h(1)$ **d.** $h(4)$

6. $g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \le -1 \\ x^3 & \text{if } x > -1 \end{cases}$

a. $g(-2)$ **b.** $g(-1)$ **c.** $g(0)$ **d.** $g(2)$

7. $f(x) = \begin{cases} 2x - 1 & \text{if } x < 4 \\ \sqrt{x-1} & \text{if } x \ge 4 \end{cases}$

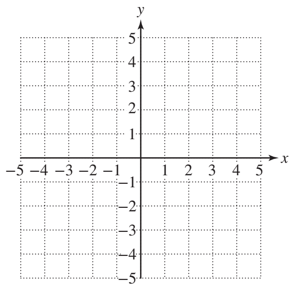
a. $f(-2)$ **b.** $f(0)$ **c.** $f(4)$ **d.** $f(10)$

8. $h(x) = \begin{cases} 4x + 1 & \text{if } x < 0 \\ (x-5)^2 & \text{if } x \ge 0 \end{cases}$

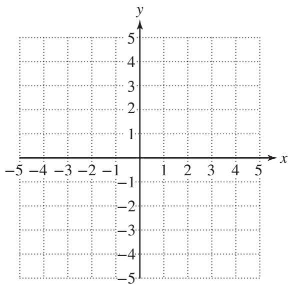
a. $h(0)$ **b.** $h(2)$ **c.** $h(-1)$ **d.** $h(-2)$

For Exercises 9–12, graph the function.

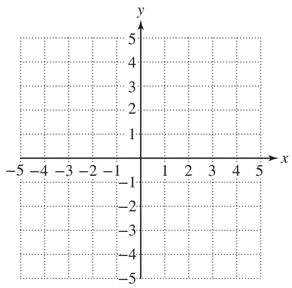
9. $g(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 4 & \text{if } x > 2 \end{cases}$



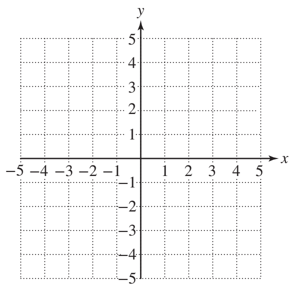
10. $f(x) = \begin{cases} |x| & \text{if } x < 1 \\ x^3 & \text{if } x \ge 1 \end{cases}$



11. $g(x) = \begin{cases} -x - 1 & \text{if } x < -1 \\ x + 2 & \text{if } x \ge -1 \end{cases}$



12. $h(x) = \begin{cases} |x| & \text{if } x \le -3 \\ 2 & \text{if } x > -3 \end{cases}$



Section 11.2

For Exercises 13–18, evaluate the expression.

13. ${}_5P_2$

14. ${}_6P_1$

15. ${}_5C_2$

16. ${}_6C_1$

17. ${}_3P_3$

18. ${}_3C_3$

19. In taking a survey, 5 different people need to be selected at random from group of 30. In how many ways can this be done?

20. In how many ways can four different vases be arranged on a shelf?

21. A combination lock has a 4-digit code. How many different codes can be made from 10 digits (0, 1, 2, 3 . . . 9) with no repetition of any digit?

22. A restaurant offers a complete dinner including one appetizer, one entrée, and one dessert. If the restaurant has three appetizers, four entrées, and three desserts, how many different meals can be selected?

Section 11.3

23. Which of the values can represent the probability of an event?

a. $\frac{1}{6}$ **b.** $0.\overline{3}$ **c.** 0 **d.** 75% **e.** 125% **f.** 0.75%

24. Which of the values can represent the probability of an event?

a. $\frac{1}{2}$ **b.** 150% **c.** 0.185 **d.** 1.00
e. 2.5% **f.** 2.5

25. In a lottery, a player can win a grand prize if the player chooses the correct 6 numbers from 46 numbers. What is the probability that the player wins the grand prize?

26. What is the probability of choosing 3 red cards from a deck of cards that contains 26 red cards and 26 black cards?

27. At a state fair there is a game where a player throws a ball to knock down some pins. If enough pins are knocked down, then the player wins a prize. Richard observes that in 150 games, 12 people won. Based on this observation, what is the probability of winning a prize?

28. Ben realizes that he mixed up four new batteries with two old batteries in a box. If he selects one battery at random, what is the probability that he will get a new battery?

29. A sample was taken of the viewers of a popular cable news network. The distribution of ages is given in the table. Suppose that one person is selected at random from the sample.

Age (yr)	Number of Viewers
<18	12
18 to 34	37
35 to 54	84
55 to 74	60
75 to 94	52

What is the probability that

a. The person is younger than 18?

b. The person is between 18 and 34, inclusive?

c. The person is 55 or older?

A standard deck of cards has 52 cards divided into four suits: clubs, diamonds, hearts, and spades. Each suit has 13 cards consisting of an ace, a king, a queen, a jack, and cards numbered from 2 to 10. Assume that one card is selected from a standard deck. For Exercises 30–33, find the probability of selecting the indicated card.

30. The ace of hearts

31. A 5

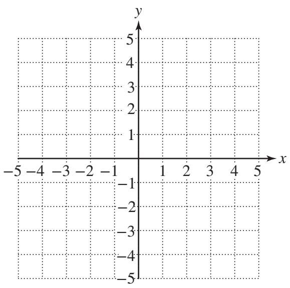
32. An 8, 9, or 10

33. A diamond or a jack

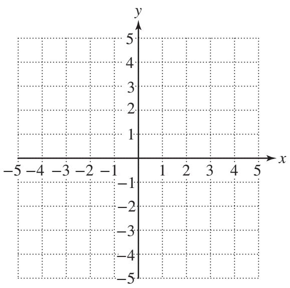
Chapter 11 Test

For Exercises 1–4, graph the function. Also write the domain and range in interval notation.

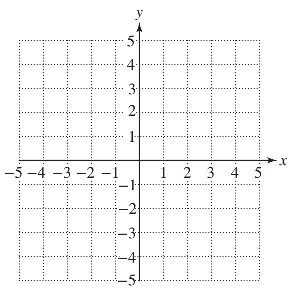
1. a. $h(x) = |x|$



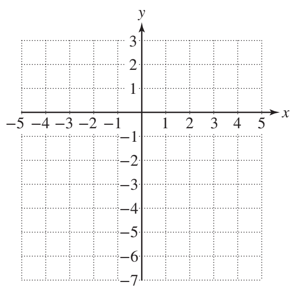
b. $f(x) = |x - 3|$



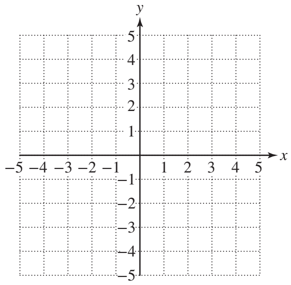
2. a. $f(x) = x^2$



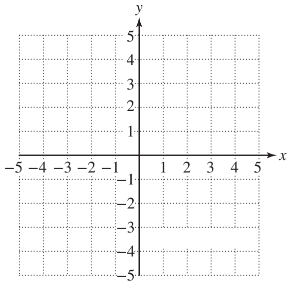
b. $g(x) = -x^2 + 2$



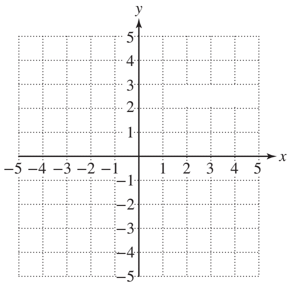
3. **a.** $h(x) = \sqrt{x}$



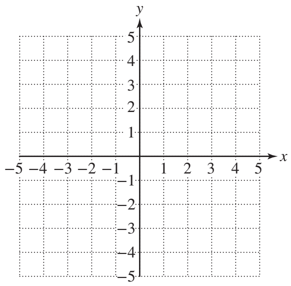
b. $g(x) = \sqrt{x+2} - 3$



4. a. $f(x) = x^3$



b. $g(x) = (x-2)^3 + 1$



For Exercises 5–6, evaluate the function for the given values of x.

5. $f(x) = \begin{cases} x+4 & \text{if } x < 1 \\ -x^2 & \text{if } x \geq 1 \end{cases}$

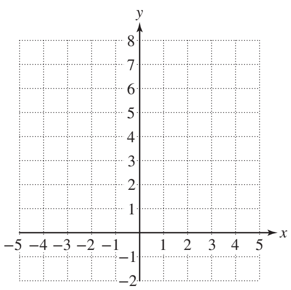
a. $f(-2)$ **b.** $f(0)$ **c.** $f(1)$ **d.** $f(3)$

6. $g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < -1 \\ 0 & \text{if } -1 \leq x \leq 2 \\ \sqrt{x} & \text{if } x > 2 \end{cases}$

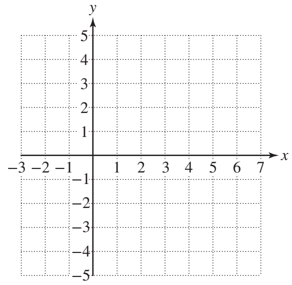
a. $g(9)$ **b.** $g(1)$ **c.** $g(0)$ **d.** $g(-3)$

For Exercises 7–10, graph the function. Also determine the domain and range.

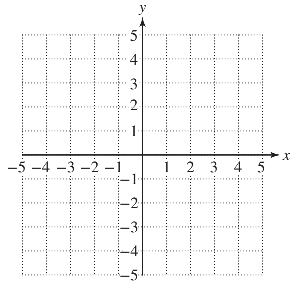
7. $h(x) = \begin{cases} x^2 + 3 & \text{if } x < 0 \\ x + 3 & \text{if } x \geq 0 \end{cases}$



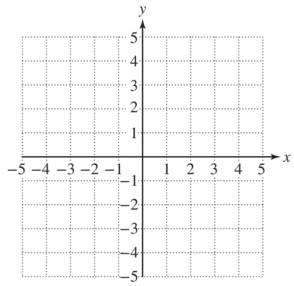
8. $g(x) = \begin{cases} |x| & \text{if } x < 4 \\ \sqrt{x} & \text{if } x \geq 4 \end{cases}$



9. $f(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ \dfrac{1}{x} & \text{if } x > 1 \end{cases}$



10. $h(x) = \begin{cases} x^2 - 2 & \text{if } x \leq 1 \\ 2x & \text{if } x > 1 \end{cases}$



For Exercises 11–15, evaluate the permutation or combination.

11. ${}_4P_2$

12. ${}_4C_2$

13. ${}_7P_5$

14. ${}_8P_8$

15. ${}_8C_8$

16. A shelf can hold two different plants, three different vases, and two different pictures. In how many ways can these items be arranged?

17. There are 18 little league teams in one conference. Among the teams there will be a first-place winner, a second-place winner, and a third-place winner. How many arrangements of these places are possible?

18. How many four-letter passwords can be formed from the letters in the name *Euclid*?

19. A baseball team is selected from a pool of 20 possible players. In how many ways can 9 players be selected? Assume that each player can play any position.

20. There are five parking spaces and 12 cars that have access. How many ways can 5 cars be chosen to park in the spaces?

21. Nick has won an award for basketball, for football, for bowling, and for soccer. He only has room for three awards to be displayed on a shelf. In how many ways can he display his awards?

22. A spinner has six sections of equal size colored red, green, blue, purple, yellow, and pink. What is the probability of the spinner landing on a pink or a red section?

23. Five people at a restaurant ordered drinks. Three ordered soda and two ordered coffee. The server forgot who ordered which drinks. If the server comes to the table with one soda, what is the probability that he will deliver it to a person who ordered it?

24. The grade distribution for a history class is given in the table. A person is selected at random from the class.

Grade	Number of Students
A	12
B	8
C	14
D	5
F	3

What is the probability that

a. The person earned an "A"?

b. The person passed the class with a "C" or better?

25. A psychology student performs a study to compare the attitudes of men and women concerning the violence on television. The table shows the results from a random sample of 200 people who were asked the question, "Is there too much violence on television?"

	Yes	No	Maybe
Male	14	62	22
Female	47	35	20

Suppose that a person is selected at random from the sample. What is the probability that

a. The person is a female?

b. The person thinks that there is too much violence on television?

c. The person answered "no" or "maybe"?

d. The person is a male or the person thinks there is too much violence on television?

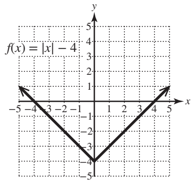
Chapter 11

Section 11.1 Practice Exercises, pp. 11-8–11-14

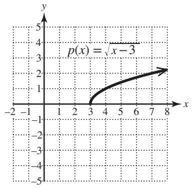
1. The graph of g is the graph of $y = x^2$ shifted to the left 2 units and shifted downward 8 units.

3. c **5.** e **7.** b

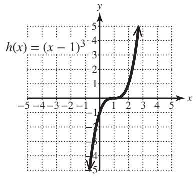
9. Domain: $(-\infty, \infty)$; range: $[-4, \infty)$



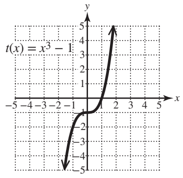
11. Domain: $[3, \infty)$; range: $[0, \infty)$



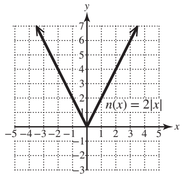
13. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$



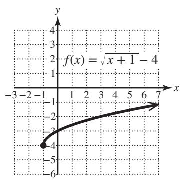
15. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$



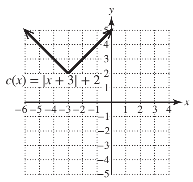
17. Domain: $(-\infty, \infty)$; range: $[0, \infty)$



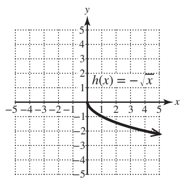
19. Domain: $[-1, \infty)$; range: $[-4, \infty)$



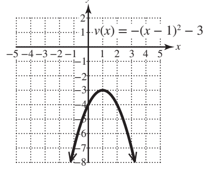
21. Domain: $(-\infty, \infty)$; range: $[2, \infty)$



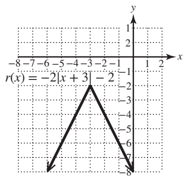
23. Domain: $[0, \infty)$; range: $(-\infty, 0]$



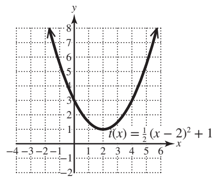
25. Domain: $(-\infty, \infty)$; range: $(-\infty, -3]$



27. Domain: $(-\infty, \infty)$; range: $(-\infty, -2]$



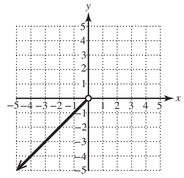
29. Domain: $(-\infty, \infty)$; range: $[1, \infty)$



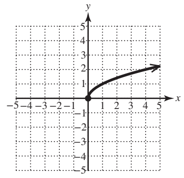
31. a **33. a.** 7 **b.** −3 **c.** −2 **d.** 4

35. a. 2 **b.** 7 **c.** 4 **d.** 5 **e.** 2

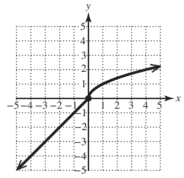
37. a.



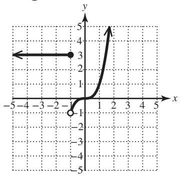
b.



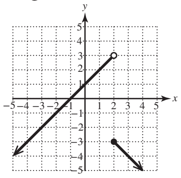
c.



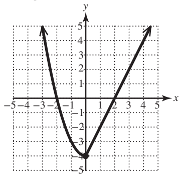
39. Domain: $(-\infty, \infty)$; range: $(-1, \infty)$



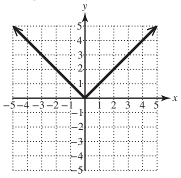
41. Domain: $(-\infty, \infty)$; range: $(-\infty, 3)$



43. Domain: $(-\infty, \infty)$; range: $[-4, \infty)$



45. Domain: $(-\infty, \infty)$; range: $[0, \infty)$;



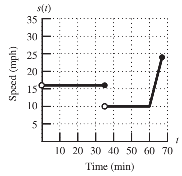
47. a. $s(30) = 16$; This means that 30 min into the ride, the bicyclist was riding 16 mph.

b. $s(40) = 10$; This means that 40 min into the ride, the bicyclist was riding 10 mph.

c. $s(65) = 20$; This means that 65 min into the ride, the bicyclist was riding 20 mph.

d. The ride ended after 67 min. Therefore, since $s(67) = 24$, the cyclist was riding 24 mph at the end of the ride.

e.

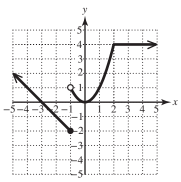


49. $f(x) = \begin{cases} -2 & \text{for } x < -1 \\ 3 & \text{for } x \geq -1 \end{cases}$

51. $f(x) = \begin{cases} x^2 & \text{for } x \leq 1 \\ 2 & \text{for } x > 1 \end{cases}$

53. b. Constant speed is a constant function whose graph is a horizontal line.

55.



Section 11.2 Practice Exercises, pp. 11-20–11-23

1. 120 **3.** 40,320 **5.** 1 **7.** $4 \cdot 3 \cdot 8 \cdot 6 = 576$

9. $6! = 720$ **11.** $16 \cdot 15 \cdot 14 \cdot 13 = 43{,}680$

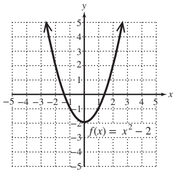
13. 720; There are 720 ways in which 3 items can be selected from 10 in a specific order. **15.** 95,040 **17.** 7 **19.** 40,320 **21.** 5040 **23.** ${}_8P_2 = 56$ **25.** ${}_6P_6 = 720$ **27.** 120; There are 120 ways in which 3 items can be selected from 10 items in *no* specific order. **29.** 220 **31.** 7 **33.** 1 **35.** 7 **37.** ${}_{15}C_{12} = 455$ **39.** ${}_5C_2 = 10$ **41. a.** WX, XW, WY, YW, WZ, ZW, XY, YX, XZ, ZX, YZ, ZY **b.** WX, WY, WZ, XY, XZ, YZ **43.** ${}_{10}P_3 = 720$ **45.** ${}_{11}C_6 = 462$ **47.** ${}_{10}C_5 = 252$ **49.** ${}_8P_2 = 56$ **51.** $2 \cdot 25 \cdot 24 \cdot 23 = 27{,}600$ **53.** ${}_{10}C_4 = 210$ **55.** $3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 36$ **57.** ${}_9P_9 = 362{,}880$ **59.** ${}_{40}C_{12} = 5{,}586{,}853{,}480$ **61.** ${}_{10}C_3 \cdot {}_8C_1 = 960$ **63.** $2^3 = 8$ **65.** $2^{10} = 1024$ **67. a.** $9! = 362{,}880$ **b.** $6 \cdot 5 \cdot 7! = 151{,}200$ **c.** $6! \cdot 3! = 4320$ **69. a.** $7^3 = 343$ **b.** $7 \cdot 6 \cdot 5 = 210$ **c.** $7 \cdot 7 \cdot 4 = 196$ **71.** ${}_{26}C_5 = 65{,}780$ **73.** ${}_{13}C_3 \cdot {}_{13}C_2 = 22{,}308$

Section 11.3 Practice Exercises, pp. 11-29–11-32

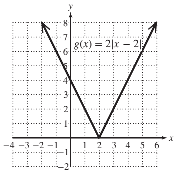
1. ${}_{10}C_4 = 210$ **3.** $5! = 120$ **5.** a, c, d **7.** c, e, f **9.** $\frac{1}{2}$ **11.** $\frac{2}{3}$ **13.** $\frac{3}{16}$ **15. a.** $\frac{{}_3C_2}{{}_8C_2} = \frac{3}{28}$ **b.** $\frac{{}_5C_2}{{}_8C_2} = \frac{5}{14}$ **17. a.** $\frac{1}{575{,}757}$ **b.** $\frac{575{,}756}{575{,}757}$ **c.** The events of winning the grand prize and losing are not equally likely events. **19. a.** $\frac{8}{59}$ **b.** $\frac{15}{59}$ **21. a.** $\frac{1}{8}$ **b.** $\frac{3}{4}$ **c.** $\frac{5}{8}$ **d.** $\frac{11}{40}$ **23. a.** 0.08 or 8% **b.** 0.83 or 83% **25.** $\frac{27}{130}$ **27.** $\frac{103}{130}$ **29.** $\frac{56}{65}$ **31.** $\frac{2}{3}$ **33.** $\frac{3}{5}$ **35.** $\frac{11}{15}$ **37.** $\frac{1}{4}$ **39.** $\frac{1}{2}$ **41.** $\frac{4}{13}$ **43.** $\frac{1}{4}$ **45.** $\frac{1}{2}$ **47.** $\frac{4}{13}$ **49. a.** $\frac{120}{600} = \frac{1}{5}$ **b.** $\frac{160}{600} = \frac{4}{15}$ **c.** $\frac{580}{600} = \frac{29}{30}$ **d.** $\frac{140}{480} = \frac{7}{24}$ **e.** $\frac{20}{120} = \frac{1}{6}$ **f.** $\frac{7}{24} \approx 0.292$; $\frac{1}{6} \approx 0.167$; male firefighters are more likely to be promoted.

Chapter 11 Review Exercises, pp. 11-35–11-36

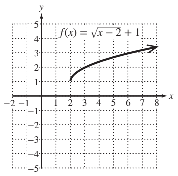
1.



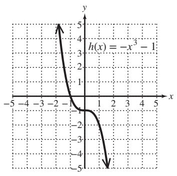
2.



3.

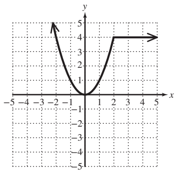


4.

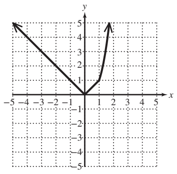


5. a. 2 **b.** 4 **c.** 1 **d.** 2 **6. a.** $-\frac{1}{2}$ **b.** -1 **c.** 0 **d.** 8 **7. a.** -5 **b.** -1 **c.** $\sqrt{3}$ **d.** 3 **8. a.** 25 **b.** 9 **c.** -3 **d.** -7

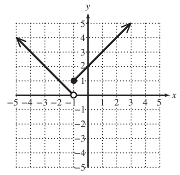
9.



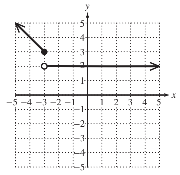
10.



11.



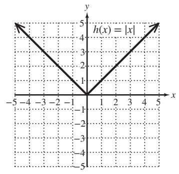
12.



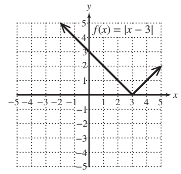
13. 20 **14.** 6 **15.** 10 **16.** 6 **17.** 6 **18.** 1 **19.** ${}_{30}C_5 = 142{,}506$ **20.** ${}_4P_4 = 24$ **21.** ${}_{10}P_4$ or $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ **22.** $3 \cdot 4 \cdot 3 = 36$ **23.** a, b, c, d, f **24.** a, c, d, e **25.** $\frac{{}_6C_6}{{}_{46}C_6} = \frac{1}{9{,}366{,}819}$ **26.** $\frac{{}_{26}C_3}{{}_{52}C_3} = \frac{2600}{22{,}100} = \frac{2}{17}$ **27.** $\frac{2}{25}$ **28.** $\frac{2}{3}$ **29. a.** $\frac{12}{245}$ **b.** $\frac{37}{245}$ **c.** $\frac{16}{35}$ **30.** $\frac{1}{52}$ **31.** $\frac{1}{13}$ **32.** $\frac{3}{13}$ **33.** $\frac{4}{13}$

Chapter 11 Test, pp. 11-36–11-38

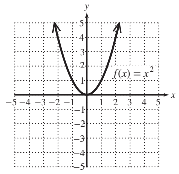
1. a. Domain: $(-\infty, \infty)$; range: $[0, \infty)$



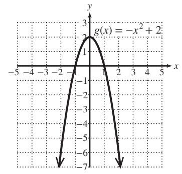
b. Domain: $(-\infty, \infty)$; range: $[0, \infty)$



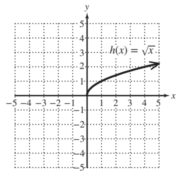
2. a. Domain: $(-\infty, \infty)$; range: $[0, \infty)$



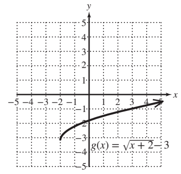
b. Domain: $(-\infty, \infty)$; range: $(-\infty, 2]$



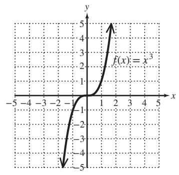
3. a. Domain: $[0, \infty)$; range: $[0, \infty)$



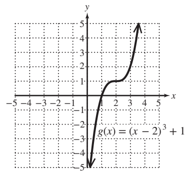
b. Domain: $[-2, \infty)$; range: $[-3, \infty)$



4. a. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

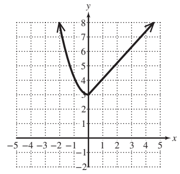


b. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

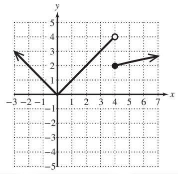


5. a. 2 **b.** 4 **c.** -1 **d.** -9 **6. a.** 3 **b.** 0 **c.** 0 **d.** $-\frac{1}{3}$

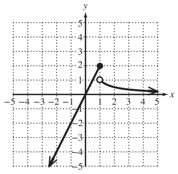
7. Domain: $(-\infty, \infty)$; range: $[3, \infty)$



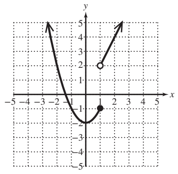
8. Domain: $(-\infty, \infty)$; range: $[0, \infty)$



9. Domain: $(-\infty, \infty)$; range: $(-\infty, 2]$



10. Domain: $(-\infty, \infty)$; range: $[-2, \infty)$



11. 12 **12.** 6 **13.** 2520 **14.** 40,320 **15.** 1

16. $7! = 5040$ **17.** ${}_{18}P_3$ or $18 \cdot 17 \cdot 16 = 4896$

18. ${}_{6}P_4 = 360$ **19.** ${}_{20}C_9 = 167{,}960$ **20.** ${}_{12}C_5 = 792$

21. ${}_{4}P_3 = 24$ **22.** $\frac{2}{6} = \frac{1}{3}$ **23.** $\frac{3}{5}$ **24. a.** $\frac{12}{42} = \frac{2}{7}$ **b.** $\frac{34}{42} = \frac{17}{21}$

25. a. $\frac{102}{200} = \frac{51}{100}$ **b.** $\frac{61}{200}$ **c.** $\frac{139}{200}$ **d.** $\frac{145}{200} = \frac{29}{40}$

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